6 Paradoxes in Naive Set Theory

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Abstract.-This chapter discusses Cantor's paradox of the set all cardinals, and proves that in Cantor's set theory every set of cardinal C originates at least 2^C inconsistent infinite sets.

Keywords: Cantor Paradox, Burali-Forti Paradox, hypothesis of the actual infinity, inconsistent totalities.

Paradoxes in naive set theory

P106 The so-called Cantor Paradox is not a paradox but a true inconsistency, a pair of contradictory results deduced from an infinite set: from the set of all cardinals (or from the universal set, the set of all sets). For this reason, these sets are rejected in modern axiomatic set theories. This chapter demonstrates, however, the existence of an uncountable infinitude of inconsistent infinite sets. It will be proved that, within the framework of the naive set theory, each set with a cardinal number C gives rise to at least 2^C inconsistent infinite sets.

P107 Although Burali-Forti was the first to publish [2] the proof of a paradox related to an infinite set (the set of all ordinals) [1, 11], Cantor was the first to discover one of those paradoxes, now known as Paradox of the Maximum Cardinal, or Cantor Paradox [11, 7, 10], though the discovery was not published. There is no agreement regarding the date Cantor discovered his paradox [11] (the proposed dates range from 1883 [14] to 1896 [12]). There is also no agreement on whether he discovered one paradox or more than one paradox, or even on the precise content of the paradoxe(s). Fortunately, the goal of this chapter is not to uncover the history of those discoveries. The main objective of this chapter is to prove, within the framework of the naive set theory, the existence of a non-denumerable infinitude of inconsistent infinite sets. Although before developing this objective, it is convenient to recall those first paradoxes in set theory, which were discovered almost at the same time that set theory itself was beginning to develop. And two of the best known of them are Burali-Forti Paradox of the Maximum Ordinal and Cantor Paradox of the Maximum Cardinal.

P108 Burali-Forti Paradox of the Set of All Ordinals and Cantor Paradox of the Set of All Cardinals are both related to the size of the considered totalities, perhaps too big as to be consistent, according to Cantor. At this stage of his life, Cantor followed a direction in set theory more theoplatonic than logic [10], so that an inconsistent totality for him would be a totality that cannot be considered as a (human) set due to its divine nature. Although for other reasons more theological than logical, Cantor was following the same strategy that the axiomatization of set theory would later follow: putting restrictions on the existence of sets.

P109 At the beginning of the development of set theory, the so-called Principle of Comprehension was used indiscriminately to define sets. This principle states that given a condition expressible by a formula $f(x)$, it is possible to form a set with all the elements x that satisfy that formula f, the set $\{x \mid f(x)\}$. Under these conditions it was possible to define sets as the universal set: $\{x \mid x = x\}$. And once the concepts of cardinal and ordinal were defined, the respective sets of all cardinals and all ordinals were also possible. A possibility that, almost immediately, led respectively to Cantor Paradox and to Burali-Forti Paradox.

P110 On the other hand, it is worth noting the euphemism of calling paradox what really is an inconsistency, i.e. a pair of contradictory terms that surely derive from a common precedent hypothesis. From which precedent hypothesis? Perhaps from the only previous hypothesis (explicitly recognized or not) that establishes the existence of Dedekind's infinite sets as complete totalities? Indeed, the simplest explanation of both paradoxes is that they are inconsistencies derived from the hypothesis of the actual infinity, i.e. from assuming the existence of the infinite sets as complete totalities. But no one has dared to analyze this alternative. As is well known, and has just been indicated, the infinitist alternative was to restrict the existence of sets by means of the appropriate axioms, in such a way that the above conflicting sets, and many others, can no longer be considered legal sets.

Cantor and Burali-Forti Paradoxes

P111 The following is a short version of Cantor Paradox (for a detailed analysis see [11, p. 66-74], [10]): In Cantor's naive set theory, let U be the set of all sets, the so called universal set, and $P(U)$ its power set, the set of all its subsets. Let us denote by $|U|$ and $|P(U)|$ their respective cardinals. Being U the set of *all* sets it must contain all sets and its cardinal must be the maximum cardinal. Then we can write:

$$P(U) \subseteq U \tag{1}$$

$$|P(U)| \leq |U| \tag{2}$$

On the other hand, and according to Cantor's Theorem on the Power Set [4], it holds:

$$|U| < |P(U)| \tag{3}$$

which contradicts (2). Equations (2)-(3) represent Cantor Paradox, which is a true contradiction, i.e. a couple of contradictory conclusions:

$$\text{Cantor Paradox} \begin{cases} |P(U)| \leq |U| \\ |P(U)| > |U| \end{cases} \tag{4}$$

P112 As is well known, Cantor gave no importance to that inconsistency [9] and clinched the argument by assuming the existence of two types of infinite totalities, the consistent and the inconsistent ones [3]. As noted above, in Cantor's opinion the inconsistency of those inconsistent infinite totalities would be due to their excessive infinitude as well as to its divine nature. In fact, we would be in the face of the mother of all infinities, the absolute infinity which, according to Cantor, leads directly to God, being just the divine nature of this absolute infinitude what makes it inconsistent for our poor human minds [3].

P113 Burali-Forti Paradox is similar, although it is deduced from the set $\mathcal{O}$ of all ordinals. According to the description given in [11] (taken from [6]), the paradox results from the following argument. The set $\mathcal{O}$ of all ordinals is well-ordered, so it has a defined ordinal Ω. Therefore, $\Omega \in \mathcal{O}$. On the other hand, any ordinal $a \in \mathcal{O}$ satisfies:

$$\exists (a+1) \in \mathcal{O} \tag{5}$$

$$a \leq \Omega \tag{6}$$

$$a < a+1 \tag{7}$$

and since Ω is an element of $\mathcal{O}$, it must satisfy (5)-(7). Hence, if we replace a with Ω in (5) we get:

$$\exists (\Omega+1) \in \mathcal{O} \tag{8}$$

Now by replacing a with $\Omega + 1$ in (6); and a with Ω in (7), we can write:

$$\Omega + 1 \leq \Omega \tag{9}$$

$$\Omega < \Omega + 1 \tag{10}$$

And we come to Burali-Forti Paradox:

$$\text{Burali-Forti Paradox} \begin{cases} \Omega + 1 \leq \Omega \\ \Omega + 1 > \Omega \end{cases} \tag{11}$$

Which is another undoubted contradiction, a new pair of contradictory results.

P114 Finally, we could recall the well-known Russell's Paradox, of the set R of all sets that do not belong to themselves [11]. In this case we will obtain a true paradox, a self-contradictory statement: a part of a statement denies the other part of the statement, and vice versa: it is clear that if R belongs to R, then it does not belong to R; and if it dos not belongs to R, then it belongs to R.

P115 The three set theoretical paradoxes we have just recalled have one word in common, the word "all":

- Set of *all* cardinals.
- Set of *all* ordinals.
- Set of *all* sets.
- Set of *all* sets that do not belong to themselves.

where the word "all" refers to the elements of particular infinite totalities, and in order to be able to consider all of its elements, those totalities have to be considered as complete totalities. Totalities whose infinitude is actual, not potential. In the case of finite totalities, the only legitimate totalities according to the alternative hypothesis of the potential infinity, none of the above paradoxes (contradictions) occurs. From the next chapter, it will be shown over and over again that the only consistent totalities are the finite totalities.

P116 In the next section we will see that, within the same framework of the Cantorian set theory, it is possible to extend Cantor's Paradox to other sets much more modest than the set of all sets, or the set of all cardinals. And it will be shown that the number of inconsistent infinite totalities is infinitely greater than the number of consistent ones: each denumerable set gives rise to nothing less than $2^{\aleph_o}$ inconsistent infinite sets. That is, an uncountable infinity of inconsistent infinite sets. We will always be in doubt about what would have happened with the development of set theory and infinitist mathematics, if that uncountable infinitude of inconsistent infinite sets had been discovered when the theory was beginning its development.

An extension of Cantor's Paradox

P117 To illustrate what could have been but was not, the following discussion will take place within the framework of the Cantorian (naive) set theory. To begin with, let us define two types of disjoint sets:

a) *Sets relatively disjoints.* Two sets are said relatively disjoint if they have no common element, but at least one element of one of them is part of the definition of at least one element of the other.

b) *Sets absolutely disjoints.* Two sets are said absolutely disjoint if they have no common element, and no element of any of them is part of the definition of any element of the other.

Consider, for example, the following three sets:

$$A = \{\{a, \{b\}\}, c, d, \{e\}, f\} \tag{12}$$

$$B = \{1, 2, b\} \tag{13}$$

$$C = \{11, 22, 33\} \tag{14}$$

According to the above definitions, A and B are relatively disjoint because they have no common element, but the element b of the set B is part of the definition of the element $\{a, \{b\}\}$ of the set A. On the other hand, A and C are absolutely disjoint because they have no common element and no element of any of them is part of the definition of any element of the other. For the same reason, B and C are also absolutely disjoint.

P118 Consider also the recursive sequence $\langle S_i(X) \rangle$ of the successor sets of a given set X, whose first term is X and whose nth ($n > 1$) term is the set whose elements are the elements of the $(n-1)$th term plus a new element which is the set whose unique element is the $(n-1)$th term:

$$S_1(X) = X \tag{15}$$

$$S_2(X) = \{X,\ \{X\}\} \tag{16}$$

$$S_3(X) = \{X,\ \{X\},\ \{X, \{X\}\}\} \tag{17}$$

$$S_4(X) = \{X,\ \{X\},\ \{X, \{X\}\},\ \{X, \{X\}, \{X, \{X\}\}\}\} \tag{18}$$

$$\ldots$$

If X is the empty set, the above sequence is the well-known sequence used to define the successive finite cardinals and ordinals (see Chapter 4).

P119 Let X be any non empty set; Y any of its subsets; and D_Y the set of all sets absolutely disjoint with the set Y. If Y is the empty set, then D_Y would be the universal set, which is inconsistent according to (2)-(3).

In any other case, it is immediate to prove that D_Y is infinite. In fact, let n be any natural, and then finite, number and assume the cardinal $|D_Y|$ of D_Y satisfies $|D_Y| = n$. Let A be any element of D_Y. Since A is absolutely disjoint with Y, the successor sets $S_1(A)$, $S_2(A)\dots$, $S_{n+1}(A)$ of the set A are also absolutely disjoint with Y, and they are elements of D_Y. Therefore, the cardinal $|D_Y|$ is greater than any natural number n. In consequence D_Y cannot be finite but infinite.

P120 Consider now the set $P(D_Y)$ of all subsets of D_Y, i.e. the power set of D_Y. The elements of $P(D_Y)$ are all of them subsets of D_Y and therefore sets of sets that are absolutely disjoint with the set Y. Consequently, it holds:

$$\forall A \in P(D_Y): \ A \in D_Y \tag{19}$$

And then:

$$P(D_Y) \subseteq D_Y \tag{20}$$

Accordingly, we can write:

$$|P(D_Y)| \leq |D_Y| \tag{21}$$

P121 On the other hand, and in accordance with Cantor's Theorem of the Power Set it holds:

$$|P(D_Y)| > |D_Y| \tag{22}$$

Again a contradiction. But now X is any non empty set, and Y any of its subsets. Therefore, and taking into account that every set of cardinal C has 2^C different subsets, we have proved the following:

Theorem P121, of Cantor Paradox. *In Cantor's set theory, every set whose cardinal is C gives rise to at least 2^C inconsistent infinite sets.*

Each of the sets of that uncountable infinitude of inconsistent infinite sets could only be an absolute and divine infinity, according to Cantor. Or simply a proof of the inconsistency of a concept, the concept of the actual infinity.

P122 The above argument not only proves the number of inconsistent infinite totalities is infinitely greater than the number of consistent ones, it also suggests the excessive size of the sets could not be the cause of the inconsistency. Consider, for example, the set X of all sets whose elements are exclusively defined by means of the natural number 1:

$$X = \{1, \mathbf{\{}1\mathbf{\}},\ \mathbf{\{}1, \{1\}, \{1, \{1\}\}\mathbf{\}},\ \mathbf{\{}\{\{1\}\}\mathbf{\}},\ \mathbf{\{}\{1, \{1\}\}\mathbf{\}} \dots\ \} \tag{23}$$

An argument similar to P119-P121 would immediately prove it is an inconsistent infinite totality, although compared with the universal set (which contains X as a tiny part of its elements) it is an insignificant totality. As a comparative reference, let us remember that, for example, between any two real numbers an uncountable infinitude ($2^{\aleph_o}$) of other different reals numbers do exist. What makes one feel dizzy, as Wittgenstein would surely say [16, p. 110]

P123 Notice that the sets as the set X defined by (23) are inconsistent only when considered from the perspective of the actual infinity, i.e. when considered as *complete* totalities. And recall that from the potential infinite point of view those sets make no sense because from this perspective the only *complete* totalities are the finite totalities, as large as wished but always finite.

P124 Had we known the existence of so many inconsistent infinite sets, and not necessarily as gigantic as the absolute infinity, and perhaps Cantor transfinite set theory would have been received in a different way. Perhaps the very notion of the actual infinity would have been put into question just in set theoretical terms; and perhaps we would have found the way to prove it is an inconsistent notion. But, as we know, this was not the case. The case was the platonic infinitism, increasingly intolerant of disagreement.

P125 The history of the reception of set theory and the way to deal with its inconsistencies (most of them promoted by the actual infinity hypothesis and by self-reference) is well known. From the beginnings of the XX century a great deal of effort has been carried out to found set theory on a formal basis free of inconsistencies. Although the objective could only be accomplished with the aid of the appropriate axiomatic patching. At least half a dozen of axiomatic set theories have been developed ever since. There are also some contemporary attempts to recover naive set theory [13]. Some hundreds of pages are needed to explain in detail all axiomatic restrictions of contemporary axiomatic set theories. Just the contrary one could expect from the axiomatic foundation of a formal science as set theory.

P126 As noted above, the simplest explanation of Cantor and Burali-Forti inconsistencies is that they are true contradictions derived from the inconsistency of the hypothesis of the actual infinity. The same applies to the set of all sets that are not member of themselves (Russell Paradox). All sets involved in the paradoxes of naive set theory were finally removed from the theory by the opportune axiomatic restrictions. No one dared to suggest the possibility that some of those paradoxes were in fact contradictions derived from the hypothesis of the actual infinity; i.e. from assuming the existence of infinite sets as complete totalities.

P127 What is really true is that Cantor set of *all* cardinals, Burali-Forti set of *all* ordinals, the set of *all* sets, and Russell set of *all* sets that are not members of themselves, are all of them inconsistent totalities when considered from the perspective of the actual infinity hypothesis. Even Turing's famous halting problem is related to the hypothesis of the actual infinity because it also assumes the existence of all pairs programs-inputs as a complete infinite totality [15]. Under the hypothesis of the potential infinity, on the other hand, none of those totalities makes sense because from this perspective only finite totalities can be considered, indefinitely extensible, but always finite.

P128 As indicated above, Cantor Paradox and Burali-Forti Paradox are not paradoxes but inconsistencies, i.e. two couples of contradictory results:

$$\text{Cantor Paradox} \begin{cases} |U| \geq |P(U)| \\ |U| < |P(U)| \end{cases} \tag{24}$$

$$\text{Burali-Forti Paradox} \begin{cases} \Omega + 1 \leq \Omega \\ \Omega + 1 > \Omega \end{cases} \tag{25}$$

Recall that we are discussing within the framework of Cantor's naive set theory, where axiomatic restrictions had not yet been established. In those conditions, the contradictory terms of (24) and (25) can only derive from some previous inconsistent assumption. And the only assumption to get (24) and (25) is the hypothesis of the actual infinity, implicitly assumed by Cantor when he established the existence of the set of all finite cardinals [5, pgs. 103-104] (italic is mine):

> The first example of a transfinite aggregate is given by the *totality* of finite cardinal numbers v; we call its cardinal number Aleph-zero and denote it by $\aleph_o$ [...]

His theoplatonic convictions "as firm as a rock" [8, p.283] prevented him from considering the possibility that his statement about the totality finite cardinals could only be a hypothesis. And much less the possibility that this hypothesis were the cause of the contradiction derived from the set of all cardinals, or from the set of all sets, found by himself.

P129 What is extraordinary about this case is that for more than a century no one has questioned Cantor's claim of the existence of "the totality of the finite cardinal numbers." No one has seriously considered that Cantor's or Burali-Forti's inconsistencies were consequences of that initial Cantor statement. Instead, it was converted in one of the fundamental axioms of set theory. But if that axiom is finally proved to be inconsistent, it will have set back the progress of humanity for more than a century. Convictions as firm as a rock could be valid for religions, not for science. Science is the place for hypotheses, errors and corrections, not for dogmas.

P130 In any case (24) and (25) are not paradoxes but true inconsistencies. And tracing their origins, we come to the only hypothesis that supports them: the hypothesis of the actual infinity. But instead of considering the possible inconsistency of that hypothesis, Cantor's successors chose another path: to set the foundation of set theory in such a way that it were possible to avoid all conflicting sets as U, while subsuming the hypothesis the actual infinity into the Axiom of Infinity. By the way, an axiom not sufficiently transparent with respect to that hypothesis. Certainly, it would have been more transparent to explicitly declare the infinity involved in the axiom is the actual infinity, so that the infinite sets exist as complete totalities. Maybe an explicit reference to the completion of incompletable could have motivated the criticism of the actual infinity: completing what cannot be completed does not seem very reasonable. Or maybe human reason is not reasonable enough: The idea that the exotic and incomprehensible adds value to scientific theories has been gaining ground since the last century. Consideration should be given to the possibility that such eccentricities were symptoms of a bad foundation of some areas of science.

Chapter References

[1] R. Bunn, *Los desarrollos en la fundamentación de la matemática desde 1870 a 1910*, Del cálculo a la teoría de conjuntos, 1630-1910. Una introducción histórica (I. Grattan-Guinness, ed.), Alianza, Madrid, 1984, pp. 283–327.

[2] Cesare Burali-Forti, *Una questione sui numeri transfiniti*, Rendiconti del Circolo Matematico di Palermo **11** (1897), 154–164.

[3] Georg Cantor, *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, Mathematishen Annalen **21** (1883), 545 – 591.

[4] ______, *Über Eine elementare frage der mannigfaltigkeitslehre*, Jahresberich der Deutschen Mathematiker Vereiningung **1** (1891), 75–78.

[5] ______, *Contributions to the founding of the theory of transfinite numbers*, Dover, New York, 1955.

[6] Irving M. Copi, *The burali-forti paradox*, Philosophy of Science **25** (1958), no. 4, 281–286.

[7] Josep W. Dauben, *Georg Cantor. His mathematics and Philosophy of the Infinite*, Princeton University Press, Princeton, N. J., 1990.

[8] William Dunham, *Journey through genius. the great theorems of mathematics*, John John Wiley and Sons, New York, 1990.

[9] José Ferreirós, *El nacimiento de la teoría de conjuntos*, Universidad Autónoma de Madrid, Madrid, 1993.

[10] ______, *Fundamentos para una teoría general de conjuntos*, 1 ed., Clásicos de la ciencia y la tecnología, ch. Introducción, pp. 9–78, Editorial Crítica, Barcelona, 2006.

[11] Alejandro R. Garciadiego Dantan, *Bertrand Rusell y los orígenes de las paradojas de la teoría de conjuntos*, Alianza, Madrid, 1992.

[12] I. Grattan-Guinness, *Are there paradoxes of the set of all sets?*, International Journal of Mathematical Education, Science and Technology **12** (1981), 9–18.

[13] M. Randall Holmes, *Alternative Axiomatic Set Theories*, The Stanford Encyclopedia of Philosophy (Edward N. Zalta, ed.), Standford University, 2007.

[14] W. Purkert, *Cantor's view on the foundations of mathematics*, The history of modern mathematics (D. Rowe and J. McClearly, eds.), Academic Press, New York, 1989, pp. 49–64.

[15] Alan M. Turing, *On Computability Numbers, With an Application to the Entscheidungsproblem*, Proc. London Math. Soc. Series 2 **43** (1937), 230 – 265.

[16] Ludwig Wittgenstein, *Observaciones sobre los fundamentos de la matemática*, Alianza Universidad, Madrid, 1987.