THE THEORY OF CELLS

T. AGAMA

ABSTRACT. In this paper we introduce and develop the notion of universe, induced communities and cells with their corresponding spots. We study the concept of the density, the mass of communities, the concentration of spots in a typical cell, connectedness and the rotation of communities. In any case we establish the connection that exist among these notions. We also formulate the celebrated union-close set conjecture in the language of density of spots and the mass of a typical community.

1. Introduction and motivation

The union-closed set conjecture - roughly speaking - is the assertion that in any collection $\Im_{\mathbb{U}}$ of subsets of a set $\mathbb{U}$ closed under union, it is possible to find an element of $\mathbb{U}$ that lives in as many sets in the collection $\Im_{\mathbb{U}}$. The conjecture was first formulated in 1979 by Peter Frankl, in the equivalent form

Conjecture 1.1. *For any intersection-closed family of sets containing more than one set, there exists an element that lives in at most half of the sets in the family.*

It is easy to see that the above conjecture is equivalent to the conjecture:

Conjecture 1.2 (Union-closed set conjecture)**.** *For any union-closed family of sets containing more that one set, there exist an element that lives in at least half of the sets in the family.*

The union-closed set conjecture remains open despite considerable efforts by many authors and several papers just devoted to study the problem. Regardless, the substantial progress with inputs and tools brought to bear are noteworthy. In fact the conjecture is proven for a few special cases. The conjecture is known to hold for families of at **most** forty-six sets [4]. It is also known to hold for families whose union has at most eleven elements [1]. It is also known to hold for families whose smallest set has only one or two elements [2]. The conjecture (see [3]) is also known to hold for family with $(\frac{1}{2}-\epsilon)2^n$ subsets of n elements for some $\epsilon > 0$.
The union-closed set conjecture also has several analogous versions and establishing the truth in those variant would imply the truth of the conjecture. The conjecture has a lattice-theoretic twist, which appeared in [5] with only special cases resolved. A well-known graph-theoretic version can also be found in [6]. Any of these variants could yet be a good terrain for the resolution of this simple-sounding conjecture and an appeal to any will certainly depend on that which seems amenable with the

Date: March 10, 2021.
2010 *Mathematics Subject Classification.* Primary 11Pxx, 11Bxx; Secondary 11Axx, 11Gxx.
Key words and phrases. concentration; mass; stability; rotation.

tools available.
In this paper, we construct a new terrain within which the union-close set conjecture and its variants could easily be studied. This domain could even provide the leeway to obtain other results along these lines. As such we introduce the language of density of spots in a cell, the mass of induced communities and concentration of spots within communities and establish some subtle connections. It turns out the union-close set conjecture can be stated in the language of density of spots in a cell and the mass of a typical community in the following manner:

Conjecture 1.3. *Let $\mathbb{U}$ be a finite universe with an induced community $\mathcal{M}_{\mathbb{U}}$. Then there exist some spot $a_i \in \mathbb{U}$ such that*
$$\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i) \geq \frac{1}{2}.$$

Conjecture 1.4 (Union-close set conjecture)**.** *Let $\mathbb{U}$ be a finite universe such $|\mathbb{U}| = n$ with an induced community $\mathcal{M}_{\mathbb{U}}$. Then we have*
$$\frac{\mathbb{M}(\mathcal{M}_{\mathbb{U}})}{n} \geq (1+o(1))\frac{1}{2}.$$

2. **The notion of universe, community and cells**

In this section we introduce the notion of cells, community and their corresponding universe. We study some elementary properties of this notion.

Definition 2.1. Let $\mathbb{U}$ be a set and consider the collection
$$\mathcal{M} := \bigcup_{i=1}^{n} \{\mathbb{A}_i |\ \mathbb{A}_i \cup \mathbb{A}_j \subseteq \mathbb{U},\ i \neq j\}.$$
Then we say the collection $\mathcal{M}$ is a **community** induced by set $\mathbb{U}$ if and only if for any $\mathbb{A}_i, \mathbb{A}_j \in \mathcal{M}$ then $\mathbb{A}_i \cup \mathbb{A}_j \in \mathcal{M}$. We call $\mathbb{U}$ the **universe** of the community. We call each $\mathbb{A}_j$ in the community a **cell** and each $a \in \mathbb{A}_j$ a **spot** in the cell. We say a cell $\mathbb{A}_i$ in the community admits an **embedding** in the community if there exists another different cell $\mathbb{A}_j$ in the same community such that $\mathbb{A}_j \subset \mathbb{A}_i$.

Proposition 2.2. *Let $\mathbb{U}$ be a universe with $|\mathbb{U}| = n$ and $\mathcal{M}_{\mathbb{U}}$ be a community induced by the universe. Then we have*
$$|\mathcal{M}_{\mathbb{U}}| \leq 2^n.$$

Proof. Let $\mathbb{U}_{\mathfrak{I}}$ be the power set induced by the universe $\mathbb{U}$. It is easy to see that $\mathbb{U}_{\mathfrak{I}}$ is the largest community induced by the universe with size
$$|\mathbb{U}_{\mathfrak{I}}| = 2^n$$
so that $|\mathcal{M}_{\mathbb{U}}| \leq 2^n$. □

Proposition 2.3. *The communities induced by a finite universe are **totally ordered**.*

Proof. Let $\mathbb{U}$ be a universe with $|\mathbb{U}| = n$ and let $\mathcal{M}_{i_{\mathbb{U}}}$ and $\mathcal{M}_{j_{\mathbb{U}}}$ be any two of distinct communities induced by the universe. Then it follows that the communities must differ by at least one cell so that without loss of generality with $|\mathcal{M}_{j_{\mathbb{U}}}| \leq |\mathcal{M}_{i_{\mathbb{U}}}|$ we can write
$$|\mathcal{M}_{j_{\mathbb{U}}}| \leq |\mathcal{M}_{i_{\mathbb{U}}}| < |\mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}|.$$

We claim that the collection $\mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}$ is also a community. Let us pick arbitrarily two cells $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}$. We consider three sub-cases: The case $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{M}_{i_{\mathbb{U}}}$ so that $\mathbb{A}_1 \cup \mathbb{A}_2 \in \mathcal{M}_{i_{\mathbb{U}}}$ since $\mathcal{M}_{i_{\mathbb{U}}}$ is a community.
For the case $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{M}_{j_{\mathbb{U}}}$ it must certainly be that $\mathbb{A}_1 \cup \mathbb{A}_2 \in \mathcal{M}_{j_{\mathbb{U}}}$ since $\mathcal{M}_{j_{\mathbb{U}}}$ is also a community. For the last case, where $\mathbb{A}_1 \in \mathcal{M}_{i_{\mathbb{U}}}$ and $\mathbb{A}_2 \in \mathcal{M}_{j_{\mathbb{U}}}$ then

$$\mathbb{A}_1 \cup \mathbb{A}_2 \in \mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}.$$

By choosing a community $\mathcal{M}_{k_{\mathbb{U}}} \neq \mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}$ with $k \neq i, j$ and $|\mathcal{M}_{k_{\mathbb{U}}}| < |\mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}} \cup \mathcal{M}_{k_{\mathbb{U}}}|$ we obtain a five-term inequality by inserting $|\mathcal{M}_{k_{\mathbb{U}}}|$ into the a priori chain. Repeating the argument in this manner establishes the claim. □

3. **Density of spots in a cell**

In this section we introduce the notion of density of spots contained within a **cell**. We launch the following languages.

Definition 3.1. Let $\mathbb{U}$ be a finite universe with $|\mathbb{U}| = n$ and $a_i \in \mathbb{U}$. Let $\mathcal{M}_{\mathbb{U}}$ be the community induced by the universe $\mathbb{U}$. Then we denote the density of the spot a_i in cells in the community $\mathcal{M}_{\mathbb{U}}$ with

$$\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i) = \lim_{n \longrightarrow \infty} \frac{\#\left\{\mathbb{A} \in \mathcal{M}_{\mathbb{U}} |\ a_i \in \mathbb{A}\right\}}{|\mathcal{M}_{\mathbb{U}}|}.$$

Equivalently we can write

$$\frac{\#\left\{\mathbb{A} \in \mathcal{M}_{\mathbb{U}} |\ a_i \in \mathbb{A}\right\}}{|\mathcal{M}_{\mathbb{U}}|} \sim \mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i)$$

or with the use of the little oh notation

$$\frac{\#\left\{\mathbb{A} \in \mathcal{M}_{\mathbb{U}} |\ a_i \in \mathbb{A}\right\}}{|\mathcal{M}_{\mathbb{U}}|} = (1 + o(1))\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i).$$

Remark 3.2. Next we investigate some properties of the notion of density of spots in a cell. The following properties will be useful in the sequel.

Proposition 3.3. *Let $\mathbb{U}$ be a finite universe with $|\mathbb{U}| = n$ and $a_i \in \mathbb{U}$. Let $\mathcal{M}_{\mathbb{U}}$ and $\mathcal{N}_{\mathbb{U}}$ be any two communities induced by the universe $\mathbb{U}$. Then the following properties hold*

(i) $\mathcal{D}_{\mathcal{M}_{\mathbb{U}} \cup \mathcal{N}_{\mathbb{U}}}(a_i) \leq \mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i) + \mathcal{D}_{\mathcal{N}_{\mathbb{U}}}(a_i)$.

(ii) $\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i) \leq 1 - \mathcal{D}_{\mathcal{M}^c_{\mathbb{U}}}(a_i)$, *where $\mathcal{M}^c_{\mathbb{U}}$ denotes the complement of the collection $\mathcal{M}_{\mathbb{U}}$ in the power set $\mathbb{U}_{\mathfrak{I}}$ induced by the universe $\mathbb{U}$.*

Proof. For (i) we notice that by appealing to Definition 3.1, we can write

$$\begin{aligned}\mathcal{D}_{\mathcal{M}_{\mathbb{U}}\cup\mathcal{N}_{\mathbb{U}}}(a_i) &= \lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{A}\in\mathcal{M}_{\mathbb{U}}\cup\mathcal{N}_{\mathbb{U}}|\ a_i\in\mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}\cup\mathcal{N}_{\mathbb{U}}|}\\ &= \lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{A}\in\mathcal{M}_{\mathbb{U}}|\ a_i\in\mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}\cup\mathcal{N}_{\mathbb{U}}|}+\lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{A}\in\mathcal{N}_{\mathbb{U}}|\ a_i\in\mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}\cup\mathcal{N}_{\mathbb{U}}|}\\ &-\lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{A}\in\mathcal{M}_{\mathbb{U}}\cap\mathcal{N}_{\mathbb{U}}|\ a_i\in\mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}\cup\mathcal{N}_{\mathbb{U}}|}\\ &\leq \lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{A}\in\mathcal{M}_{\mathbb{U}}|\ a_i\in\mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}\cup\mathcal{N}_{\mathbb{U}}|}+\lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{A}\in\mathcal{N}_{\mathbb{U}}|\ a_i\in\mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}\cup\mathcal{N}_{\mathbb{U}}|}\\ &\leq \lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{A}\in\mathcal{M}_{\mathbb{U}}|\ a_i\in\mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}|}+\lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{A}\in\mathcal{N}_{\mathbb{U}}|\ a_i\in\mathbb{A}\}}{|\mathcal{N}_{\mathbb{U}}|}\\ &= \mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i)+\mathcal{D}_{\mathcal{N}_{\mathbb{U}}}(a_i).\end{aligned}$$

For (ii) it follows similarly that

$$\begin{aligned}\mathcal{D}_{\mathbb{U}_{\mathfrak{I}}}(a_i) &= \lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{A}\in\mathbb{U}_{\mathfrak{I}}|\ a_i\in\mathbb{A}\}}{|\mathbb{U}_{\mathfrak{I}}|}\\ &= \lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{A}\in\mathcal{M}_{\mathbb{U}}\cup\mathcal{M}_{\mathbb{U}}^c|\ a_i\in\mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}\cup\mathcal{M}_{\mathbb{U}}^c|}\\ &= \mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i)+\mathcal{D}_{\mathcal{M}_{\mathbb{U}}^c}(a_i)\end{aligned}$$

by leveraging the property (i) and noting that $\mathcal{M}_{\mathbb{U}}\cap\mathcal{M}_{\mathbb{U}}^c=\emptyset$. By observing that

$$\lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{A}\in\mathbb{U}_{\mathfrak{I}}|\ a_i\in\mathbb{A}\}}{|\mathbb{U}_{\mathfrak{I}}|}\leq 1$$

the second part also follows. □

It turns out that the union close set conjecture can pretty much be stated in the language of density in the following manner.

Conjecture 3.4 (Union-close set conjecture). *Let $\mathbb{U}$ be a finite universe with an induced community $\mathcal{M}_{\mathbb{U}}$. Then there exist some spot $a_i\in\mathbb{U}$ such that*

$$\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i)\geq\frac{1}{2}.$$

4. **The mass of a community**

In this section we introduce and study the notion of **mass** of a community. We launch formally the following language.

Definition 4.1. Let $\mathbb{U}$ be a universe and $\mathcal{M}_{\mathbb{U}}$ be a community induced by the universe. Then by the **mass** of the community, denoted $\mathbb{M}(\mathcal{M}_{\mathbb{U}})$, we mean the quantity

$$\mathbb{M}(\mathcal{M}_{\mathbb{U}})=\sum_{j\geq 1}\frac{\#\{\mathbb{A}\in\mathcal{M}_{\mathbb{U}}|\ a_j\in\mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}|}.$$

For any finite universe $\mathbb{U}$ with $|\mathbb{U}| = n$, we can write by appealing to the notion of density of spots

$$\mathbb{M}(\mathcal{M}_{\mathbb{U}}) = \sum_{j=1}^{n} \frac{\#\{\mathbb{A} \in \mathcal{M}_{\mathbb{U}} |\ a_j \in \mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}|}$$
$$\sim \sum_{j=1}^{n} \mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_j).$$

It follows from this relation that we can in some way recover the mass of a typical community by asymptotically averaging density of each spot in the same universe within cells in the community. More formally we obtain the proposition

Proposition 4.2. *Let $\mathbb{U}$ be a finite universe such $|\mathbb{U}| = n$ with an induced community $\mathcal{M}_{\mathbb{U}}$. Then we have*

$$\mathbb{M}(\mathcal{M}_{\mathbb{U}}) \sim \sum_{j=1}^{n} \mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_j).$$

Theorem 4.3. *Let $\mathbb{U}$ be a finite universe such $|\mathbb{U}| = n$ with an induced community $\mathcal{M}_{\mathbb{U}}$. If $\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_j) \geq \delta$ for each $1 \leq j \leq n$ for some $\delta > 0$, then*

$$\mathbb{M}(\mathcal{M}_{\mathbb{U}}) \geq (1 + o(1))\delta n.$$

Proof. Appealing to Proposition 4.2 we can write

$$\mathbb{M}(\mathcal{M}_{\mathbb{U}}) = (1 + o(1)) \sum_{j=1}^{n} \mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_j)$$

so that under the requirement $\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_j) \geq \delta$ for each $1 \leq j \leq n$ for some $\delta > 0$, then

$$\mathbb{M}(\mathcal{M}_{\mathbb{U}}) = (1 + o(1)) \sum_{j=1}^{n} \mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_j)$$
$$\geq (1 + o(1))\delta n.$$

□

It follow from Theorem 4.3, we can interpret the average mass as the average density of spots originating from a finite universe $\mathbb{U}$ with $|\mathbb{U}| = n$. That is to say, If $\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_j) \geq \delta$ for each $1 \leq j \leq n$ for some $\delta > 0$ then we can write

$$\frac{\mathbb{M}(\mathcal{M}_{\mathbb{U}})}{n} \geq (1 + o(1))\delta. \tag{4.1}$$

Conversely, it will essentially follow that in any finite universe $\mathbb{U}$ with $|\mathbb{U}| = n$ and an induced community $\mathcal{M}_{\mathbb{U}}$ tied with (4.1) there must exists at least one spot $a_k \in \mathbb{U}$ such that

$$\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_k) \geq \delta.$$

Thus the union close set conjecture can also be stated in the language of the mass of a community as follows

Conjecture 4.4 (Union-close set conjecture)**.** *Let $\mathbb{U}$ be a finite universe such $|\mathbb{U}| = n$ with an induced community $\mathcal{M}_{\mathbb{U}}$. Then we have*

$$\frac{\mathbb{M}(\mathcal{M}_{\mathbb{U}})}{n} \geq (1+o(1))\frac{1}{2}.$$

Proposition 4.5. *Let $\mathbb{U}$ be a finite universe with all of its induced communities $\{\mathcal{M}_{i_{\mathbb{U}}}\}_{i=1}^{l}$. Then we have*

$$\mathbb{M}(\cup_{i=1}^{l}\mathcal{M}_{i_{\mathbb{U}}}) \leq \sum_{i=1}^{l}\mathbb{M}(\mathcal{M}_{i_{\mathbb{U}}}) + \sum_{j\geq 1}\sum_{k=2}^{l}\sum_{i=1}^{k}(-1)^{k+1}\frac{\#\left\{\mathbb{A}\in \cap_{i=1}^{k}\mathcal{M}_{i_{\mathbb{U}}}|\ a_j\in \mathbb{A}\right\}}{|\cup_{s=1}^{l}\mathcal{M}_{s_{\mathbb{U}}}|}.$$

Proof. Since the universe $\mathbb{U}$ is finite, it follows that the mass $\mathbb{M}(\cup_{i=1}^{l}\mathcal{M}_{i_{\mathbb{U}}})$ exists and is finite. Let us now compute the mass

$$\mathbb{M}(\cup_{i=1}^{l}\mathcal{M}_{i_{\mathbb{U}}}) = \sum_{j\geq 1}\frac{\#\left\{\mathbb{A}\in \cup_{i=1}^{l}\mathcal{M}_{i_{\mathbb{U}}}|\ a_j\in \mathbb{A}\right\}}{|\cup_{s=1}^{l}\mathcal{M}_{i_{\mathbb{U}}}|}.$$

It follows that we can write by an application of the inclusion-exclusion principle

$$\begin{aligned}
\sum_{j\geq 1}\frac{\#\left\{\mathbb{A}\in \cup_{i=1}^{l}\mathcal{M}_{i_{\mathbb{U}}}|\ a_j\in \mathbb{A}\right\}}{|\cup_{s=1}^{l}\mathcal{M}_{s_{\mathbb{U}}}|} &= \sum_{j\geq 1}\sum_{i=1}^{l}\frac{\#\left\{\mathbb{A}\in \mathcal{M}_{i_{\mathbb{U}}}|\ a_j\in \mathbb{A}\right\}}{|\cup_{s=1}^{l}\mathcal{M}_{s_{\mathbb{U}}}|}\\
&+\sum_{j\geq 1}\sum_{k=2}^{l}\sum_{i=1}^{k}(-1)^{k+1}\frac{\#\left\{\mathbb{A}\in \cap_{i=1}^{k}\mathcal{M}_{i_{\mathbb{U}}}|\ a_j\in \mathbb{A}\right\}}{|\cup_{s=1}^{l}\mathcal{M}_{s_{\mathbb{U}}}|}\\
&\leq \sum_{j\geq 1}\sum_{i=1}^{l}\frac{\#\left\{\mathbb{A}\in \mathcal{M}_{i_{\mathbb{U}}}|\ a_j\in \mathbb{A}\right\}}{|\mathcal{M}_{i_{\mathbb{U}}}|}\\
&+\sum_{j\geq 1}\sum_{k=2}^{l}\sum_{i=1}^{k}(-1)^{k+1}\frac{\#\left\{\mathbb{A}\in \cap_{i=1}^{k}\mathcal{M}_{i_{\mathbb{U}}}|\ a_j\in \mathbb{A}\right\}}{|\cup_{s=1}^{l}\mathcal{M}_{s_{\mathbb{U}}}|}\\
&= \sum_{i=1}^{l}\mathbb{M}(\mathcal{M}_{i_{\mathbb{U}}}) + \sum_{j\geq 1}\sum_{k=2}^{l}\sum_{i=1}^{k}(-1)^{k+1}\frac{\#\left\{\mathbb{A}\in \cap_{i=1}^{k}\mathcal{M}_{i_{\mathbb{U}}}|\ a_j\in \mathbb{A}\right\}}{|\cup_{s=1}^{l}\mathcal{M}_{s_{\mathbb{U}}}|}
\end{aligned}$$

thereby ending the proof. □

5. The concentration of spots in a community

In this section we introduce the notion of concentration of spots in a **community**. We launch the following language.

Definition 5.1. Let $\mathbb{U}$ be a finite universe and $\{\mathcal{M}_{j_{\mathbb{U}}}\}_{j=1}^{k}$ be all communities induced by the universe. Then by the concentration of spots a in the community, denoted $\mathcal{C}_{\mathbb{U}}(a)$, we mean the quantity

$$\mathcal{C}_{\mathbb{U}}(a) = \sum_{j=1}^{k}\frac{\#\left\{\mathbb{A}\in \mathcal{M}_{j_{\mathbb{U}}}|\ a\in \mathbb{A}\right\}}{|\mathcal{M}_{j_{\mathbb{U}}}|}.$$

Similarly by appealing to the notion of density of spots in cells within a community, we can write the concentration of spots in a finite universe as

$$\mathcal{C}_{\mathbb{U}}(a)=\sum_{j=1}^{k}\frac{\#\left\{\mathbb{A}\in\mathcal{M}_{j_{\mathbb{U}}}|\ a\in\mathbb{A}\right\}}{|\mathcal{M}_{j_{\mathbb{U}}}|}$$

$$\sim\sum_{j=1}^{k}\mathcal{D}_{\mathcal{M}_{j_{\mathbb{U}}}}(a)$$

so that the concentration of a fixed spot in induced communities is essentially asymptotically averaging the density over each induced community by the finite universe. In this framework we can write

Proposition 5.2. *Let $\mathbb{U}$ be a finite universe and $\{\mathcal{M}_{j_{\mathbb{U}}}\}_{j=1}^{k}$ be all communities induced by the universe. Then we have*

$$\mathcal{C}_{\mathbb{U}}(a)\sim\sum_{j=1}^{k}\mathcal{D}_{\mathcal{M}_{j_{\mathbb{U}}}}(a).$$

Let $\mathbb{U}$ be a finite universe with $|\mathbb{U}|=n$ and all induced communities $\{\mathcal{M}_{j_{\mathbb{U}}}\}_{j=1}^{k}$. Then we can write

$$\sum_{1\leq s\leq n}\mathcal{C}_{\mathbb{U}}(a_s)=\sum_{1\leq s\leq n}\sum_{j=1}^{k}\frac{\#\left\{\mathbb{A}\in\mathcal{M}_{j_{\mathbb{U}}}|\ a_s\in\mathbb{A}\right\}}{|\mathcal{M}_{j_{\mathbb{U}}}|}$$

so that by interchanging the order of summation we can write

$$\sum_{1\leq s\leq n}\mathcal{C}_{\mathbb{U}}(a_s)=\sum_{j=1}^{k}\sum_{1\leq s\leq n}\frac{\#\left\{\mathbb{A}\in\mathcal{M}_{j_{\mathbb{U}}}|\ a_s\in\mathbb{A}\right\}}{|\mathcal{M}_{j_{\mathbb{U}}}|}$$

$$=\sum_{j=1}^{k}\mathbb{M}(\mathcal{M}_{j_{\mathbb{U}}})$$

so that the process of averaging the concentration of all spots originating from a common universe is the same as averaging the mass over all induced communities of a finite universe. In other words, we can essentially cover the total mass of all induced communities of a finite universe with total concentration of all spots originating from a same universe.

Proposition 5.3. *Let $\mathbb{U}$ be a finite universe with $|\mathbb{U}|=n$ and all induced communities $\{\mathcal{M}_{j_{\mathbb{U}}}\}_{j=1}^{k}$. Then we have*

$$\sum_{1\leq s\leq n}\mathcal{C}_{\mathbb{U}}(a_s)=\sum_{j=1}^{k}\mathbb{M}(\mathcal{M}_{j_{\mathbb{U}}}).$$

Proposition 5.4. *Let $\mathbb{U}$ be a finite universe with $|\mathbb{U}|=n$ and all induced communities $\{\mathcal{M}_{j_{\mathbb{U}}}\}_{j=1}^{k}$. Then we have*

$$\sum_{1\leq s\leq n}\sum_{j=1}^{k}\mathcal{D}_{\mathcal{M}_{j_{\mathbb{U}}}}(a_s)\sim\sum_{j=1}^{k}\mathbb{M}(\mathcal{M}_{j_{\mathbb{U}}}).$$

Proof. The result follows by appealing to Proposition 5.2 and Proposition 5.3. □

Remark 5.5. In the sequel we state an inequality that relates the mass of a typical community with the concentration of spots originating from the underlying finite universe. This inequality is paramount to our investigations and will be exploited in our further studies along these lines.

Theorem 5.6 (The mass-concentration inequality)**.** *Let $\mathbb{U}$ be a finite universe such that $|\mathbb{U}| = n$ with all of its induced communities $\{\mathcal{M}_{i_{\mathbb{U}}}\}_{i=1}^{l}$. Then we have*

$$\mathbb{M}(\cup_{i=1}^{l}\mathcal{M}_{i_{\mathbb{U}}}) \leq \sum_{1\leq s\leq n}\mathcal{C}_{\mathbb{U}}(a_s) + \sum_{j\geq 1}\sum_{k=2}^{l}\sum_{i=1}^{k}(-1)^{k+1}\frac{\#\left\{\mathbb{A}\in\cap_{i=1}^{k}\mathcal{M}_{i_{\mathbb{U}}} |\ a_j\in\mathbb{A}\right\}}{|\cup_{s=1}^{l}\mathcal{M}_{s_{\mathbb{U}}}|}.$$

Proof. The inequality is easily established by appealing to Proposition 5.3 and Proposition 4.5. □

6. Connected and disconnected cells and communities

In this section we introduce and study the notion of **connectedness** and **disconnectedness** of cells and their associated communities induced by a typical universe. We launch the following languages.

Definition 6.1. Let $\mathbb{U}$ be a universe with the set of all induced communities $\{\mathcal{M}_{i_{\mathbb{U}}}\}_{i=1}^{\infty}$. Let $\mathbb{A},\mathbb{B}\in\mathcal{M}_{i_{\mathbb{U}}}$ for $i\geq 1$ be any two cells of the community. Then we say $\mathbb{A}$ and $\mathbb{B}$ are connected if $\mathbb{A}\cap\mathbb{B}\neq\emptyset$. Similarly, we say two arbitrary communities $\mathcal{M}_{i_{\mathbb{U}}}$ and $\mathcal{M}_{j_{\mathbb{U}}}$ are connected if $\mathcal{M}_{i_{\mathbb{U}}}\cap\mathcal{M}_{j_{\mathbb{U}}}\neq\{\emptyset\}$ and there exists some cell $\mathbb{A}\neq\emptyset$ such that $\mathbb{A}\in\mathcal{M}_{i_{\mathbb{U}}}\cap\mathcal{M}_{j_{\mathbb{U}}}$.

Remark 6.2. Next we relate the notion of density of spots originating from a finite universe to the notion of connectedness of cells in a typical community. The connection works pretty well in one direction but fails unfortunately in the other. Indeed passing from the notion of density of spots to the notion of connectedness is fairly easy but the converse would require additional conditions to hold.

Proposition 6.3. *Let $\mathbb{U}$ be a finite universe such that $|\mathbb{U}| = n$ with an induced community $\mathcal{M}_{\mathbb{U}}$. If there exists at least a spot $a_i\in\mathbb{U}$ such that $\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i)\geq\delta$ for some $\delta>0$, then the community $\mathcal{M}_{\mathbb{U}}$ contains at least two connected cells.*

Proof. Let us suppose there exists at least a spot $a_i\in\mathbb{U}$ such that $\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i)\geq\delta$ for some $\delta>0$. Let us suppose to the contrary the community $\mathcal{M}_{\mathbb{U}}$ contains no connected cells. Then it follows that for any spot $a_j\in\mathbb{U}$ then

$$\#\left\{\mathbb{A}\in\mathcal{M}_{\mathbb{U}} |\ a_j\in\mathbb{A}\right\} = 1$$

so that for the spot $a_i\in\mathbb{U}$, we have

$$\begin{aligned}\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i) &= \lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{A}\in\mathcal{M}_{\mathbb{U}} |\ a_i\in\mathbb{A}\right\}}{|\mathcal{M}_{\mathbb{U}}|}.\\ &= \lim_{n\longrightarrow\infty}\frac{1}{|\mathcal{M}_{\mathbb{U}}|} = 0.\end{aligned}$$

This contradicts the **minimality** of the density of the spot a_i. □

7. Rotation of cells in a community

In this section we introduce and develop the notion of **rotation** of cells in a typical community. We launch the following languages.

Definition 7.1. Let $\mathbb{U}$ be a universe with induced communities $\{\mathcal{M}_{i_{\mathbb{U}}}\}_{i\geq 1}$. By the rotation of communities we mean the map

$$\varpi : \mathcal{M}_{k_{\mathbb{U}}} \longrightarrow \mathcal{M}_{j_{\mathbb{U}}}$$

so that for any cell $\mathbb{A} \in \mathcal{M}_{k_{\mathbb{U}}}$ then there exist some cell $\mathbb{B} \in \mathcal{M}_{j_{\mathbb{U}}}$ such that

$$\varpi(\mathbb{A}) = \mathbb{B}.$$

We say the community $\mathcal{M}_{k_{\mathbb{U}}}$ is a **stable** community under the rotation if $\mathcal{M}_{k_{\mathbb{U}}} = \mathcal{M}_{j_{\mathbb{U}}}$.

Proposition 7.2. *Let $\mathbb{U}$ be a universe with induced communities $\{\mathcal{M}_{i_{\mathbb{U}}}\}_{i\geq 1}$. If the communities $\mathcal{M}_{k_{\mathbb{U}}}$ and $\mathcal{M}_{j_{\mathbb{U}}}$ are stable under the rotation ϖ, then $\mathcal{M}_{k_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}$ is also stable under the rotation ϖ.*

Proof. Under the assumption that the communities $\mathcal{M}_{k_{\mathbb{U}}}$ and $\mathcal{M}_{j_{\mathbb{U}}}$ are stable under the rotation ϖ, it follows that $\varpi : \mathcal{M}_{k_{\mathbb{U}}} \longrightarrow \mathcal{M}_{k_{\mathbb{U}}}$ and $\varpi : \mathcal{M}_{j_{\mathbb{U}}} \longrightarrow \mathcal{M}_{j_{\mathbb{U}}}$. First let $\mathbb{A} \in \varpi[\mathcal{M}_{k_{\mathbb{U}}}] \cup \varpi[\mathcal{M}_{j_{\mathbb{U}}}]$. Without loss of generality let us suppose $\mathbb{A} \in \varpi[\mathcal{M}_{k_{\mathbb{U}}}]$, then it follows that there exists some cell $\mathbb{B} \in \mathcal{M}_{k_{\mathbb{U}}}$ such that $\varpi(\mathbb{B}) = \mathbb{A}$. Under the cover $\mathcal{M}_{k_{\mathbb{U}}} \subset \mathcal{M}_{k_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}$, it must be that $\varpi(\mathbb{B}) \in \varpi[\mathcal{M}_{k_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}]$ so that we

$$\varpi[\mathcal{M}_{k_{\mathbb{U}}}] \cup \varpi[\mathcal{M}_{j_{\mathbb{U}}}] \subseteq \varpi[\mathcal{M}_{k_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}].$$

Also let $\mathbb{A} \in \varpi[\mathcal{M}_{k_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}]$ then there must exists some cell $\mathbb{B} \in \mathcal{M}_{k_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}$ such that $\varpi(\mathbb{B}) = \mathbb{A}$. Without loss of generality let us assume that $\mathbb{B} \in \mathcal{M}_{k_{\mathbb{U}}}$, then it must be that $\varpi(\mathbb{B}) \in \varpi[\mathcal{M}_{k_{\mathbb{U}}}] \subseteq \varpi[\mathcal{M}_{k_{\mathbb{U}}}] \cup \varpi[\mathcal{M}_{j_{\mathbb{U}}}]$ so that

$$\varpi[\mathcal{M}_{k_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}] \subseteq \varpi[\mathcal{M}_{k_{\mathbb{U}}}] \cup \varpi[\mathcal{M}_{j_{\mathbb{U}}}].$$

The upshot is the equality

$$\begin{aligned}\varpi[\mathcal{M}_{k_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}] &= \varpi[\mathcal{M}_{k_{\mathbb{U}}}] \cup \varpi[\mathcal{M}_{j_{\mathbb{U}}}] \\ &= \mathcal{M}_{k_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}\end{aligned}$$

thereby establishing the claim. □

References

1. Ivica, Bošnjak and Petar, Marković *The 11-element case of Frankl's conjecture*, Electronic Journal of Combinatorics, 2008.
2. Sarvate, DG and Renaud, JC *On the union-closed sets conjecture*, Ars Combinatoria, vol. 27, CHARLES BABBAGE, 1989, pp. 149–154.
3. Karpas, Ilan *Two results on union-closed families*, arXiv preprint arXiv:1708.01434, 2017.
4. Roberts, Ian T and Simpson, Jamie *A note on the union-closed sets conjecture*, Australasian Journal of Combinatorics, vol. 47, Centre for Discrete Mathematics & Computing, 2010, pp. 265–267.
5. Abe, Tetsuya *Strong semimodular lattices and Frankl's conjecture*, Algebra Universalis, vol. 44(3), Springer, 2000, pp. 379–382.
6. Bruhn, Henning and Charbit, Pierre and Schaudt, Oliver and Telle, Jan Arne *The graph formulation of the union-closed sets conjecture*, European Journal of Combinatorics, vol. 43, Elsevier, 2015, pp. 210–219.

Department of Mathematics, African Institute for mathematical sciences, Ghana.
E-mail address: Theophilus@aims.edu.gh/emperordagama@yahoo.com