A theorem on the number of distinct eigenvalues

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Abstract

A theorem on the number of distinct eigenvalues of diagonalizable matrices is obtained. Some applications related to matrices with simple eigenvalues, triangular defective matrices, adjacency matrices and graphs are discussed. Other ideas and examples are provided.

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1 Preliminaries

Most of the work done in the recent years about the number of distinct eigenvalues of matrices revolves around the relationship between graphs and matrices. In particular, the concept of minimum number of distinct eigenvalues of graph has been studied trough some articles such as [1], [8], [4], [6], [2]. Other recent articles about the number of distinct eigenvalues are related to low-rank perturbation of matrices, [7], [11], [14] and [15].
The way we go in this paper is different in the sense that our main result is stated, proved and applied within matrix analysis. Then, in the last section, we establish a connection with graph theory. The question about the distinct eigenvalues is formulated in this work as follows.

Is there a way to find the number of distinct eigenvalues of a given complex matrix without calculating the eigenvalues themselves?
We provide an affirmative answer to this question in the general case of diagonalizable matrices. A major tool to achieve this goal, is the next lemma.
Let v_1, v_2 and v_3 be three vectors in $\mathbb{C}^n$, $n \geq 2$, such that v_1 and v_2 are linearly independent. It can be easily verified that there is always at least one unit vector w_1 in the subspace spanned by v_1 and v_2 such that w_1 is orthogonal to v_3, i.e, $w_1^* v_3 = 0$. This idea can be generalized as follows.

Lemma 1.1. *Let V be a vector space of dimension $n \geq 2$ and let H be a subspace of V of dimension k with $2 \leq k \leq n$. For every vector $v \in V$, there exists an orthonormal basis $B = \{w_1, \dots, w_k\}$ of H such that al least $(k-1)$ elements of B are orthogonal to v.*

Proof. We prove the lemma for $\mathbb{C}^n$ and the proof extends naturally to every vector space endowed with an inner product and having dimension ≥ 2.
The trivial case is where v is orthogonal to H. If $v \in H$, then the existence of B is ensured by Gram-Schmidt process. Suppose that $v \notin H$ and v is not orthogonal to H. Then there exists a vector $v_1 \in H$ such that $v_1 * v \neq 0$. Let $v_1, \dots, v_k$ be a basis of H. For $i = 2, 3, \dots, k$, we use v, v_1 and v_i to form the unit vector

$$u_i = \frac{-\dfrac{v^* v_i}{v^* v_1} v_1 + v_i}{\|-\dfrac{v^* v_i}{v^* v_1} v_1 + v_i\|}, \tag{1}$$

which is orthogonal to v. Moreover, $u_2, u_3, \dots, u_k$ are linearly independent, belong to H and span a subspace $G \subset H$ of dimension $(k-1)$. Therefore, Gram-Schmidt process is applied to obtain an orthogonal basis $\{w_2, \dots, w_k\}$ of G from $u_2, u_3, \dots, u_k$. Note that the vectors $w_2, \dots, w_k$ are orthogonal to v since G itself is orthogonal to v. Let

$$w_1 = \frac{v_1 - \sum_{i=2}^{k} (w_i^* v_1) w_i}{\|v_1 - \sum_{i=2}^{k} (w_i^* v_1) w_i\|}. \tag{2}$$

Then $w_1 \in H$, $\|w_1\| = 1$ and $\{w_1, \dots, w_n\}$ form an orthonormal basis of H. The assertion of the lemma is satisfied by this basis. $\square$

Some other key ideas on which our analysis is based are about matrix product and spectral decomposition of matrices. If $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are vectors in $\mathbb{C}^n$, then

$$\begin{bmatrix} x_1 \dots x_n \end{bmatrix} \begin{bmatrix} y_1^* \\ \vdots \\ y_n^* \end{bmatrix} = \sum_{i=1}^{n} x_i y_i^* . \tag{3}$$

Moreover, if $\lambda_1, \dots, \lambda_n$ are complex numbers, then

$$\begin{bmatrix} x_1 \dots x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1^* \\ \vdots \\ y_n^* \end{bmatrix} = \sum_{i=1}^{n} \lambda_i x_i y_i^* . \tag{4}$$

Remark 1.2. It follows from (3) and (4) that the complex matrix $M = \sum_{i=1}^{n} \lambda_i x_i y_i^*$ has spectrum $\{\lambda_1, \dots, \lambda_n\}$ if $\sum_{i=1}^{n} x_i y_i^* = I_n$, the $n \times n$ identity matrix. In this case, M is diagonalizable, x_i and y_i are, respectively, right and left eigenvectors of M associated with λ_i. If all the eigenvalues of M are simple, then there is a unique Jordan decomposition given by the left side of (4), in which all the eigenvectors $x_1, \dots, x_n$ are unit vectors. An important fact used in our analysis is that this uniqueness does not hold if some of the eigenvalues of M are nonsimple. In other words, if λ is a nonsimple eigenvalue of M, then every basis of the eigenspace of M associated with λ can be used to produce a Jordan decomposition of M. This idea is illustrated by the following example.

Example 1.3. Look at the two Jordan decompositions:

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 1 & -2 & 1 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 0 & 1 & 1 & 3 \\ -1 & -2 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & -1 & 4 & -3 \\ 1 & 0 & -1 & 1 \\ -3 & 0 & 4 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} .$$

Not only M and N are similar to each other, but they represent the same matrix. In fact, simple calculations shows that

$$M = N = \begin{bmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 2 & -1 \\ -1 & 0 & 3 & -1 \\ -1 & 0 & 1 & 1 \end{bmatrix} .$$

The above equality holds because the first two (last two) eigenvectors in the first decomposition span the same eigenspace associated with $\lambda_1 = 1$ ($\lambda_2 = 2$) and spanned by the first two (last two) eigenvectors in the second decomposition.

2 Main results

2.1 Case of diagonalizable matrices

In this subsection, M is an $n \times n$ complex diagonalizable matrix with $n \geq 2$. It has a Jordan canonical decomposition of the form

$$M = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_n \end{bmatrix} \begin{bmatrix} y_1^* \\ \vdots \\ y_n^* \end{bmatrix}, \tag{5}$$

where $\beta_1, \dots, \beta_n$ are the n eigenvalues of M (not necessarily distinct), It follows from (3) and (4) that

$$M = \sum_{i=1}^{n} \beta_i x_i y_i^* \tag{6}$$

and

$$I_n = \sum_{i=1}^{n} x_i y_i^*, \tag{7}$$

where I_n is the identity matrix. To take account of the multiplicity of eigenvalues of M, we use the following notation:

1. $\lambda_1, \dots, \lambda_r$ are the distinct eigenvalues of M for some $r \leq n$.
2. g_i is the multiplicity of λ_i.
3. The right and left eigenspaces of λ_i are denoted, respectively, H_i and L_i.
4. In the set $\{x_1, \dots, x_n\}$, the eigenvectors of M associated with λ_i are denoted $\{x_{i_1}, \dots, x_{i_{g_i}}\}$ and form a basis of the eigenspace H_i.
5. In the set $\{y_1, \dots, y_n\}$, the left eigenvector of M associated with λ_i are denoted $\{y_{i_1}, \dots, y_{i_{g_i}}\}$ and form a basis of the left eigenspace L_i.

Then (6) can be written as

$$\begin{aligned} M &= \sum_{i=1}^{r} \sum_{j=1}^{g_i} \lambda_i x_{i_j} y_{i_j}^* \\ &= \sum_{i=1}^{r} \lambda_i \sum_{j=1}^{g_i} x_{i_j} y_{i_j}^* \end{aligned} \tag{8}$$

and (7) as

$$\sum_{i=1}^{r}\sum_{j=1}^{g_i} x_{i_j} y_{i_j} = I_n, \tag{9}$$

Note also that

$$y_{i_j}^* x_{s_t} = \begin{cases} 1 & \text{if } i = s \text{ and } j = t, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Using (8) and (10), we obtain

$$M^k = \sum_{i=1}^{r} \lambda_i^k \sum_{j=1}^{g_i} x_{i_j} y_{i_j}^* \quad \text{for every} \quad k \in \mathbb{N}. \tag{11}$$

Let $t \in \{1, \dots, r\}$ and v be a nonzero vector in $\mathbb{C}^n$ not orthogonal to H_i for $i = 1, \dots, t$ and orthogonal to H_i for $i = t+1, \dots, r$. Then (11) implies

$$\begin{aligned} v^* M^k &= \sum_{i=1}^{r} \lambda_i^k \sum_{j=1}^{g_i} (v^* x_{i_j}) y_{i_j}^* \\ &= \sum_{i=1}^{t} \lambda_i^k \sum_{j=1}^{g_i} (v^* x_{i_j}) y_{i_j}^*, \quad k \in \mathbb{N}. \end{aligned} \tag{12}$$

Moreover, (9) implies

$$\begin{aligned} v^* &= v^* I_n \\ &= \sum_{i=1}^{r} \sum_{j=1}^{g_i} (v^* x_{i_j}) y_{i_j}^* \\ &= \sum_{i=1}^{t} \sum_{j=1}^{g_i} (v^* x_{i_j}) y_{i_j}^*. \end{aligned} \tag{13}$$

By Lemma 1.1, the eigenvectors $x_{i_1}, \dots, x_{i_{g_i}}$ of λ_i can be chosen to form an orthonormal set in which $v^* x_{i_1} \neq 0$ and $v^* x_{i_j} = 0$ for $j = 2, \dots, g_i$ (if $g_i \geq 2$). Then, with this choice of the right eigenvectors $\{x_{i_j}\}$, the existence and uniqueness of the corresponding left eigenvectors $\{y_{i_j}\}$ are ensured by the Jordan normal form (5). We emphasize that this choice of eigenvectors preserves the matrix M as explained in Remark 1.2 and Example 1.3. It follows from (12) that

$$v^* M^k = \sum_{i=1}^{t} \lambda_i^k (v^* x_{i_1}) y_{i_1}^T, \quad k \in \mathbb{N} \tag{14}$$

and from (13) that

$$v^* = \sum_{i=1}^{t} (v^* x_{i_1}) y_{i_1}^*. \tag{15}$$

Now, let $a_0,\dots,a_{t-1}$ be complex numbers such that

$$a_0v^*+a_1v^*M+\cdots+a_{t-1}v^*M^{t-1}=0. \tag{16}$$

From (14), (15) and (16) we have

$$\sum_{k=0}^{t-1}a_k\left[\sum_{i=1}^{t}\lambda_i^k(v^*x_{i_1})y_{i_1}^T\right]=0 \tag{17}$$

or equivalently,

$$\sum_{i=1}^{t}(v^*x_{i_1})\left[\sum_{k=0}^{t-1}a_k\lambda_i^k\right]y_{i_1}^*=0. \tag{18}$$

Since $v^*x_{i_1}\neq 0$ and the eigenvectors $\{y_{i_1}\mid 1\leq i\leq t\}$ are linearly independent, (18) implies

$$\sum_{k=0}^{t-1}a_k\lambda_i^k=0 \ \text{ for } \ i=1,\dots,t. \tag{19}$$

If some of the coefficient (a_i) are different than zero, then (19) implies that the polynomial $f(x)=\sum_{k=0}^{t-1}a_kz^k$ has t distinct roots $\lambda_1,\dots,\lambda_t$, while its degree does not exceed $(t-1)$. This is a contradiction. Hence, $a_0=a_1=\cdots=a_{t-1}=0$. Then we deduce from (16) that the vectors $v,\ M^*v,\ \dots,\ (M^*)^{t-1}v$ are linearly independent. Since there are t of them, those vectors span the subspace spanned by the set of vectors $\{y_{i_1}\mid 1\leq i\leq t\}$ as it can be seen from (14) and (15). We denote this subspace by $L_M(v)$. The vector v as well as all the vectors of the form $(M^*)^kv,\ k\in\mathbb{N}$ are linear combinations of $\{y_{i_1}\mid 1\leq i\leq t\}$ as it can be seen from (14) and (15). Therefore, they belong to $L_M(v)$.
The right and left eigenspaces associated with any eigenvalue of M have the same dimension. Therefore, we could replace M by M^* in the above analysis to obtain the following theorem.

Theorem 2.1. *Let M be an $n\times n$ complex diagonalizable matrix with $n\geq 2$ and let $\lambda_1,\dots,\lambda_r$ be the distinct eigenvalues of M with $r\in\{1,\dots,n\}$. For every nonzero vector $v\in\mathbb{C}^n$, let $R_M(v)$ and $L_M(v)$ be the subspaces of $\mathbb{C}^n$ defined by*

$$R_M(v)=span\ \left\{v,\ Mv,\ M^2v,\dots\right\} \tag{20}$$

and

$$L_M(v)=span\ \left\{v,\ M^*v,\ (M^*)^2v,\dots\right\} \tag{21}$$

1. *If v is not orthogonal to t eigenspaces of M and orthogonal to the $(r-t)$ remaining ones for some $t\in\{1,\dots,r\}$. Then the vectors $v,M^*v,\dots,(M^*)^{t-1}v$ are linearly independent and span $L_M(v)$.*

2. *If v is not orthogonal to t left eigenspaces of M and orthogonal to the $(r-t)$ remaining ones for some $t \in \{1,\dots,r\}$. Then the vectors $v, Mv, \dots, M^{t-1}v$ are linearly independent and span $R_M(v)$.*

Clearly, it is always possible to find a vector v that is not orthogonal to any eigenspace (left eigenspace) of M. Therefore, we have the following corollary of Theorem 2.1.

Corollary 2.2. Let M be an $n \times n$ complex diagonalizable matrix M and $q(M)$ be the number of its distinct eigenvalues. Then

$$q(M) = \max\left\{\operatorname{rank}\left(\begin{bmatrix} v & Mv & \dots & M^{n-1}v \end{bmatrix}\right) \;\middle|\; v \in \mathbb{C}^n\right\}. \tag{22}$$

Remark 2.3. The inequality

$$q(M) \le rank(M) + 1 \tag{23}$$

holds for all complex matrices including the diagonalizable matrix M. It follows from Corollary 2.2 and (23) that for every $v \in \mathbb{C}^n$,

$$\operatorname{rank}\left(\begin{bmatrix} v & Mv & \dots & M^{n-1}v \end{bmatrix}\right) \;\; \le \;\; \operatorname{rank}(M) + 1. \tag{24}$$

The practical aspect of Theorem 2.1 consists of Corollary 2.2 and the following corollary which are considered together with the theorem as our main results.

Corollary 2.4. An $n \times n$ complex diagonalizable matrix M has at least k distinct eigenvalues if and only if there exists an nonzero vector $v \in \mathbb{C}^n$ such that the matrix

$$A = \begin{bmatrix} v & Mv & \dots & M^{n-1}v \end{bmatrix} \tag{25}$$

has rank k.

Proof. If M has at least k distinct eigenvalues, then by Corollary 2.2, there exists a vector v such as A has rank k. Conversely, Suppose that the rank of A is k. If M has less than k distinct eigenvalues, say k' with $k' < k$, then it has exactly k' left eigenspaces. Hence, the largest possible number of left eigenspaces of M that could be not orthogonal to v is k'. It follows by Theorem 2.1 that the dimension of the subspace $L_M(v)$ spanned by v, Mv, $M^2v, \dots$ has dimension at most equal to k'. This is in contradiction with our assumption that the rank of A is k. Hence, $k' \ge k$. □

Remark 2.5. Corollary 2.2 tells us that, for every nonzero vector $v \in \mathbb{C}$ and for every $k \in \mathbb{N}$, the rank of a matrix of the form (25) is a lower bound for the number of distinct eigenvalue of M given that M is diagonalizable. Furthermore, if M is nonsingular, let

$$B = \left[M^h v \;\; M^{h+1} v \;\; \dots \;\; M^{h+k-1} v\right], \tag{26}$$

with $h \in \mathbb{N}$. Then B can be written as

$$B = M^h \left[v \;\; Mv \;\; \dots \;\; M^{k-1} v\right] = BA, \tag{27}$$

where A is as in (25). Since M^h is nonsingular, the rank of B is same as the rank of $\left[v \;\; Mv \;\; \dots \;\; M^{k-1}v\right]$. By Corollary 2.2, we conclude that the rank of the matrix B is also a lower bound of the nubmber of distinct eigenvalues of the matrix M given that M is diagonalizable and nonsingular.

2.2 Normal matrices

By definition, an $n \times n$ complex matrix M is normal if $MM^* = M^*M$, where M^* is the conjugate transpose of M. It is well known that M is normal if and only if it has a spectral decomposition of the form

$$M = \sum_{i=1}^{n} \beta_i w_i w_i^*, \tag{28}$$

where $\{(\beta_i, w_i)\}$ is a complete set of eigenpairs of M. Look at equivalent condition (15) and its proof in [9]. The vectors $w_1, \dots, w_n$ form an orthonormal basis of $\mathbb{C}^n$ which implies by (3) that

$$\sum_{i=1}^{n} w_i w_i^* = I_n, \tag{29}$$

where I_n is the $n \times n$ identity matrix. Complex normal matrices are diagonalizable. Therefore, Theorem 2.1 and its corollaries apply to them. As it can be seen from (28), the right and left eigenspaces associated with every eigenvalue β_i of M are the same. Hence, for every nonzero vector $v \in \mathbb{C}^n$, the matrices $\left[v,\, Mv,\, \dots,\, M^{n-1}v\right]$ and $\left[v^*,\, v^*M,\, \dots,\, v^*M^{n-1}\right]^*$ span the same space. In other words, the subspaces $L_M(v)$ and $R_M(v)$ given, respectively, by (20) and (21) are actually the same subspace if M is normal. It follows from Theorem 2.1 that the dimension of this subspace is a lower bound of the number of distinct eigenvalues of M.

Remark 2.6. Theorem 2.1 and its corollaries apply, in particular, to symmetric matrices which form an important subclass of normal matrices. As we shall explain in the third sections, there are some applications in graph theory related to the adjacency matrix of simple undirected graph which is a symmetric matrix.

2.3 Case of defective matrices

A matrix M is said to be defective if it has at least one eigenvalue for which the algebraic and geometric multiplicities are not equal. Theorem 2.1 and its corollaries do not hold in the general case of non-diagonalizable matrices. Here is a counterexample.

Example 2.7. Consider the following matrix

$$M = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 1 & -1 & 3 \end{bmatrix}.$$

If we choose the canonical vector $v = [1\ \ 0\ \ 0]^T$, then the matrix

$$A = [v\ \ Mv\ \ M^2v]\ =\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

is nonsingular since its determinant is nonzero, $det(A) = 1$. According to Corollary 3.1, matrix M should have 3 simple eigenvalues. However this is not the case since M is defective and has Jordan decomposition

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

from which we see that M has one distinct eigenvalue 1 with different algebraic and geometric multiplicities, respectively, equal to 3 and 1.

Note that an $n \times n$ complex matrix has a Jordan decomposition of the form

$$M = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \delta_{12} & & \\ & \ddots & \ddots & \\ & & \lambda_{n-1} & \delta_{n-1,n} \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1^* \\ \vdots \\ y_n^* \end{bmatrix}, \tag{30}$$

where $\delta_{i-1,i} \in \{0, 1\}$, x_i and y_i are respectively right and left eigenvector or generalized eigenvector associated with λ_i. Equation in (30) can be written in another way:

$$M = \sum_{i=1}^{n} \lambda_i x_i y_i^* + \sum_{i=2}^{n} \delta_{i-1,i}\, x_{i-1}\, y_i^*. \tag{31}$$

The second sum on the right side of (31) is equal to 0 if and only if M is diagonalizable. It is responsible for the fact that Theorem 2.1 and its corollaries do not hold in the general case of defective matrices.

2.4 Fields other than $\mathbb{C}$ and $\mathbb{R}$

If $\mathbb{K}$ is an algebraically closed field, then every matrix M in $M_n(\mathbb{K})$, the set of $n\times n$ matrices with elements in $\mathbb{K}$, have a Jordan normal decomposition of the form

$$M = SAS^{-1}, \tag{32}$$

where $S,\ A \in M_n(\mathbb{K}),\ A$ is upper triangular and S is nonsingular.
Let $x_1,\dots,x_n$ be the columns of S and $y_1^T,\dots,y_n^T$ be the rows of S^{-1}. If M is diagonalizable, i.e., A is diagonal, then (32) is equivalent to

$$M = \sum_{i=1}^{n} \lambda_i x_i y_i^T, \tag{33}$$

where $\lambda_1,\dots,\lambda_n$ are the diagonal elements of A. Replacing y_i^* by y_i^T in the previous reasoning done for Theorem 2.1 in the case of complex matrices, we see that this theorem holds for the general case of diagonalizable matrices over algebraically closed field $\mathbb{K}$.

3 Applications in matrix analysis

3.1 A criterion for diagonalizable matrices to have all simple eigenvalues

It is possible to check that all the eigenvalues of a given diagonalizable matrix are simple without computing the eigenvalues themselves. This can be done by using the following corollary.

Corollary 3.1. Let M be an $n\times n$ complex diagonalizable matrix. Then all the eigenvalues of M are simple if and only if there exist a vector $v\in\mathbb{C}^n$ such that $v, Mv,\dots,M^{n-1}v$ are linearly independent.

Proof. Follows immediately from Corollary 2.4. □

Example 3.2. Consider the 3×3 diagonalizable matrix

$$M = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

We need to show that all the eigenvalues of M are simple by using Corollary 3.1.

For this purpose, we choose v to be the canonical vector $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

and we form the matrix

$$A = \begin{bmatrix} v & Mv & M^2v \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 5 \\ 0 & 1 & -2 \end{bmatrix}.$$

Since $\det(A) = 5 \neq 0$, we conclude, by Corollary 3.1, that all the eigenvalues of M are simple.

Remark 3.3. In the previous example the conclusion that the eigenvalues of M are simple wouldn't be made if the vector $u = [0 \ 1 \ 0]^T$ is chosen instead of v. This is because u is orthogonal to one of the eigenspaces of M (can be checked by the reader).
In general, the vector v could be chosen to have a simple structure to reduce the amount of computation. For example canonical vectors, scalar multiples of canonicals vector, the all 1's vector, etc.

3.2 Separation of close eigenvalues

Practically speaking, one big advantage of Corollary 3.1 is that the vector v can be chosen randomly and at the same time it is highly expected to be not orthogonal to any of the eigenspaces of M even when those eigenspaces are unknown to us. The numerical method used to compute eigenvalues of large diagonalizable matrices might produce uncertainty about multiplicities of eigenvalues because of computation errors related to eigenvalues that are close to each other in value. In this case, Corollary 3.1 can be used to conclude with certainty that all the eigenvalues of a given matrix are simple. Here is a demonstration with a small 3×3 diagonalizable matrix.

Example 3.4. Consider the diagonalizable matrix

$$M = \begin{bmatrix} -8.98 & -9.99 & -10.09 \\ 12 & 13.01 & 12.11 \\ -2 & -2 & -1 \end{bmatrix}.$$

Let $v = [0 \ 0 \ 3]^T$ and

$$\begin{aligned} A &= [v \ Mv \ M^2v] \\ &= \begin{bmatrix} 0 & -30.27 & -60.8421 \\ 0 & 36.33 & 73.0833 \\ 3 & -3 & -9.12 \end{bmatrix}. \end{aligned}$$

If we consider approximation to one decimal digit, then the eigenvalues of M are going to look equal to each other, $\lambda_1 \approx \lambda_2 \approx \lambda_3 \approx 1.0$. However, matrix A has a determinant neatly different than zero; $det(A) = -5.514$, which implies, by Corollary

3.1, that the eigenvalues of M are distinct without any doubt. In fact the eigenvalues of M are exactly $\lambda_1 = 1,\ \ \lambda_2 = 1.01$ and $\lambda_3 = 1.02$. This example shows us that Corollary 3.1 can be used as a supporting algorithm for the separation of close eigenvalues of large diagonalizable matrices computed by another algorithm. The computation cost of such combination of algorithms is beyond the scope of this work and can be the subject of further research.

3.3 Criterion for a triangular matrix to be defective

We have seen in the previous section that Theorem 2.1 and its corollaries do not hold in the general case of defective matrices. However, even in this case they are of some use as shown in the following example.

Example 3.5. Consider the upper-diagonal matrix

$$M = \begin{bmatrix} 1 & -2 & 5 & 14 \\ 0 & 2 & -1 & -5 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Apparently, the eigenvalue $\lambda = 2$ has algebraic multiplicity 2 but it's geometric multiplicity is hidden so we don't know if M is diagonalizable or not. We choose, for example, $v = [0\ 0\ 0\ 1]$ to form the matrix

$$A = [v\ \ Mv\ \ M^2v\ \ M^3v] = \begin{bmatrix} 0 & 14 & 40 & 106 \\ 0 & -5 & -11 & -33 \\ 0 & 6 & 6 & 18 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

which is nonsingular since its determinant is nonzero, $det(A) = 504$. If M is diagonalizable, then by Corollary 3.1, all its eigenvalues have to be simple which is not the case. Therefore M is defective and the geometric multiplicity of the eigenvalue $\lambda = 2$ is equal to 1. In fact M has Jordan decomposition

$$M = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 & -7 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The following corollary summarizes the idea illustrated in the above example.

Corollary 3.6. Let M be an $n \times n$ triangular complex matrix and suppose that some of the diagonal elements of M repeat. If there exists a vector $v \in \mathbb{C}^n$ such that the matrix $\begin{bmatrix} v & Mv & \dots & M^{n-1}v \end{bmatrix}$ is nonsingular, then M is defective.

Proof. Follows from Corollary 3.1 and the fact that the diagonal elements of a triangular matrix are its own eigenvalues. $\square$

3.4 Connection with the inverse eigenvalue problem

Theorem 2.1 impose a condition for diagonalizable matrices to be realizable by a given set of complex numbers.

Corollary 3.7. Let $\lambda_1, \dots, \lambda_k$ be k distinct complex numbers and let n be an integer such that $n \geq k$. If M is a diagonalizable matrix realizable by $\{\lambda_1, \dots, \lambda_k\}$, then for every vector $v \in \mathbb{C}^n$, the subspace $L_M(v)$ spanned by $v, Mv, M^2v, \dots$ has dimension at most equal to k. Moreover, if v is not orthogonal to any of the eigenspaces of M, then $L_M(v)$ has dimension exactly equal to k and is spanned by $v,\ Mv,\ \dots,\ M^{k-1}$.

Proof. The first assertion of the corollary follows from Corollary 2.4 and the second one from Corollary 2.2. $\square$

4 Applications in graph theory

Let G be a simple undirected graph with n vertices $a_1, \dots, a_n$ and let A be its adjacency matrix. By $q(A)$ we denote the number of distinct eigenvalues of A. A class of matrices that contains A is the set $S(G)$ of $n \times n$ real symmetric matrices $M = [m_{ij}]$ compatible with G, i.e., matrices that satisfy the condition: every off-diagonal element m_{ij} of M is different than 0 if and only if there is an edge between the vertices a_i and a_j of G. The minimum number of distinct eigenvalues q(G) of graph G is defined to be the minimum number of distinct eigenvalues a matrix M in $S(G)$ can have.
The following theorem is a well-known result on the adjacency matrix of undirected graph. It has an elegant proof based, among other ideas, on the fact that the roots of minimal polynomial of symmetric matrix are distinct, look at [3, Theorem 2.2.1]. Here, we provide a new proof of this theorem based on Corollary 2.4, then we make a comparison between the two proofs.

Theorem 4.1. *The number of distinct eigenvalues $q(A)$ of A is at least one more than the diameter d of G. That is,*

$$q(A) \geq d + 1. \tag{34}$$

Proof 1: new proof based on Corollary 2.4

Proof. Let $\{a_1, \dots, a_n\}$ be the set of vertices of G. Denote by e_1 It is known that the position (i, j) of A^t is equal to the number of paths of length t between a_i and

a_j for $t = 1,2,\ldots$. Without loss of generality, we assume that the diameter of G occurs between a_1 and a_{d+1} for some $d \in \{2,3,\ldots,n\}$ and consists of the edges: $a_1a_2, a_2a_3, \ldots, a_da_{d+1}$. Hence, for $j = 2,\ldots,d+1$, the shortest path between a_1 and a_j has length $j-1$. It is well known that the (i,j)th and (j,i) entries of A^t are equal to the the number of paths of length t between the vertices a_i and a_j. Consequently, for $t = 1,2,\ldots,$

$$(A^t)_{j1} = \begin{cases} t & \text{if} \quad j = t+1, \\ 0 & \text{if} \quad t+1 < j \leq d+1. \end{cases} \tag{35}$$

Denote by $(A^t)_1$ the first column of the matrix A^t and by e_1 the first column of the identity matrix I_n, i.e., the canonical vector $e_1 = [1,0,\ldots,0]^T$. We readily deduce from 35 that the columns $e_1, A_1, (A^2)_1, \ldots, (A^d)_1$ are linearly independent. Notice that $(A^t)_1 = (A^t)e_1$ and construct the matrix

$$\begin{aligned} B = [b_{ij}] &= \begin{bmatrix} e_1 & A_1 & (A^2)_1 & \ldots & (A^d)_1 \end{bmatrix} \\ &= \begin{bmatrix} e_1 & Ae_1 & A^2e_1 & \ldots & (A^d)e_1 \end{bmatrix}. \end{aligned} \tag{36}$$

As it is explained, the columns of B are linearly independent and it follows by Corollary 2.4 that the matrix A has at least $d+1$ distinct eigenvalues. □

Proof 2: old proof using minimal polynomial (look at [3]).

Proof. Using the same reasoning as in Proof 1, The first columns of $I, A, A^2, \ldots, A^d$ are linearly independent. Hence, the matrices $I, A, A^2, \ldots, A^d$ themselves are linearly independent. It follows that A^d is not a linear combination of $I_n, A, \ldots, A^{d-1}$ which means that the minimal polynomial of A has degree at least equal to $d+1$. Since A is symmetric, all the roots of its minimal polynomial are distinct and consequently there are at least $d+1$ of them. They are distinct eigenvalues of A. □

Theorem 4.1 is extended to all nonnegative matrices in $S(G)$, by the same reasoning as in Proof 2. There are other results related to $S(G)$ and $q(G)$ obtained by a similar reasoning. For example, [1, Theorem 3.2] is another extension of Theorem 4.1 obtained by using the same technique as in Proof 2 and therefore can be proved by using the ideas of Proof 1. In fact, using Proof 2 with the minimal polynomial and linearity independence of $I, A, \ldots, A^d$ is the same as using Proof 1 and Corollary 2.4 with the canonical vector $v = e_1$ which is the first column of the identity matrix. Here appears the advantage of Corollary 2.4. It offers the possibility of using other vector v instead of the canonical vectors to produce possible better lower bounds for $q(A), q(G)$ and $q(M)$ for $M \in S(G)$. The question now is: for a given graph G,

how to find a vector v that leads to a better lower bound for $q(A)$ or $q(G)$?
Actually, this question follows from (22) which implies

$$q(A) = \max\left\{\operatorname{rank}\left(\begin{bmatrix} v & Av & \dots & A^{n-1}v \end{bmatrix}\right) \;\middle|\; v \in \mathbb{C}^n\right\} \tag{37}$$

and

$$q(G) = \min_{M\in S(G)} \left\{ \max_{v\in\mathbb{C}^n} \left\{\operatorname{rank}\left(\begin{bmatrix} v & Mv & \dots & M^{n-1}v \end{bmatrix}\right)\right\}\right\}. \tag{38}$$

4.1 The walk matrix and main eigenvalues of graph

Let G be an undirected graph with n vertices $a_i,\dots,a_n$ and let A be its adjacency matrix. Denote by $q(A)$ the number of distinct eigenvalues of A. Let $e = [1,\dots,1]^T$, the all 1's vector of n components. An eigenvalue λ of A is said to be main eigenvalue of A if it is associated with at least one eigenvector v of A that is not orthogonal to e. Some work done in this matter can be found in [5], [10], [12], [13] and [16].
The walk matrix W of the graph G is given by

$$W = [w_{ij}] = \begin{bmatrix} e & Ae & \dots & A^{n-1} \end{bmatrix}. \tag{39}$$

The walk matrix acquires its importance from the fact that the entry w_{ij} of W is equal to the number of paths of lengths $j-1$ that starts at vertex a_i with $1 \le i \le n$ and $2 \le j \le n$. It follows from Corollary 2.4 that

$$q(A) \;\ge\; \operatorname{rank}(W). \tag{40}$$

The following Theorem that relates between the walk matrix and main eigenvalues is obtained in [10].

Theorem 4.2. *[10, Theorem 2.1] The rank of W is equal to the number of main eigenvalues of A.*

Saying that λ is associated with an eigenvector v that is not orthogonal to e is same as saying that e is not orthogonal to the eigenspace of A associated with λ. Since A is symmetric, the right and left eigenspaces of M associated with the same eigenvalue are equal. Therefore the above theorem can be obtained as a consequence of Theorem 2.1.

5 Conclusion

We have shown how the rank of $\begin{bmatrix} M & Mv & \dots & Mv^{n-1} \end{bmatrix}$, $v \in \mathbb{C}^n$, can tell about the number of distinct eigenvalues of diagonalizable matrix M. Some applications have been discussed briefly throughout the paper and it seems that there are more applications out there in linear algebra, combinatorics and numerical analysis of matrices. We believe that the main results of this work deserve further exploration.

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