Division by Zero Calculus and Hyper Exponential Functions by K. Uchida

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Abstract: In this paper, we will consider the basic relations of the normal solutions (hyper exponential functions by K. Uchida) of ordinary differential equations and the division by zero calculus. In particular, by the concept of division by zero calculus, we extend the concept of Uchida's hyper exponential functions by considering the equations and solutions admitting singularities. Surprisingly enough, by this extension, any analytic functions with any singularities may be considered as Uchida's hyper exponential functions. Here, we will consider very concrete examples as prototype examples.

David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Oliver Heaviside:

Mathematics is an experimental science, and definitions do not come first, but later on.

Key Words: Division by zero, division by zero calculus, normal solutions of ordinary differential equations, Uchida's hyper exponential functions,

isolated singular point, analytic function, Laurent expansion, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0, [(z^n)/n]_{n=0} = \log z$, $[e^{(1/z)}]_{z=0} = 1$.

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1 Introduction

K. Uchida ([23]) has a long love for the solutions of the differential equations

$$\frac{d^n y}{dx} = f(x)y$$

and he are appointing the importance of the solutions. He called the solutions *hyper exponential functions* (*Uchida's hyper exponential functions*). He considered the solutions for some functions $f(x)$ and derived many beautiful computer graphics with their elementary properties ([24]). We see the few concrete solutions from [17] and [24]. Of course, the case $n = 1$ is trivial and the $n > 3$ cases are rare examples and the case $n = 2$ is important.

Meanwhile, we introduced the concept of division by zero calculus in [21] that we can consider analytic functions and their derivatives even at isolated singular points. Therefore, we can consider the Uchida's exponential functions for analytic functions $f(x)$ with singularities. Surprisingle enough, then any analytic functions with any singular points may be considered as the Uchida's hyper exponential functions. As one typical example, we will consider the simplest case of

$$f(x) = \frac{1}{(x-a)^m} \tag{1.1}$$

for the general real number m of $m \neq 0$ and for $n = 2$.

2 Division by zero calculus

We will simply introduce the division by zero calculus.

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{2.1}$$

we will define

$$f(a) = C_0. \tag{2.2}$$

We define the value of the function $f(z)$ at the singular point $z = a$ by ignoring the singular parts of the Laurent expansion.

For the correspondence (2.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (2.1), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

With this assumption (definition), we can obtain many new results and new concepts. See [21] and the references in this paper.

We shall note that for real valued functions we can extend the concept of the division by zero calculus by means of the Laplace transform.

For the Laplace transform of the function

$$\frac{t^{n-1}e^{-at}}{(n-1)!}, \quad n = 1, 2, 3, ...,$$

we have

$$\frac{1}{(s+a)^n}.$$

Then, for $s = -a$, by the division by zero calculus (DBZC), we have

$$\frac{1}{(s+a)^n}(-a) = 0,$$

for $s = -a$. Then, how will be the corresponding Laplace transform

$$\int_0^\infty \frac{t^{n-1}e^{-at}}{(n-1)!} e^{at} dt = \int_0^\infty \frac{t^{n-1}}{(n-1)!} dt$$

? Note that this integral is zero, because infinity may be represented by 0. Conversely, from this argument for the general function for any positive k

$$\frac{\Gamma(k)}{(s+a)^k}$$

that is the Laplace transform of the function

$$t^{k-1}e^{-at},$$

we can derive the result

$$\frac{\Gamma(k)}{(s+a)^k}(-a) = 0.$$

Indeed, since this result is not defined by DBZC for general positive k, this result now was derived here, by this logic. This means that the function (1.1) is zero at $x = a$ and the function is differentiable and its value at $x = a$ is zero for any real number m except $m = 0$.

3 Second order differential equations

We recall the general result that for the second order differential equation of the homogeneous equation

$$y'' + f_1(x)y' + f_0(x)y = 0 \tag{3.1}$$

and for a non-trivial solution $y_1 = y_1(x)$, the general solution is given by

$$y = y_1\left(C_1 + C_2\int \frac{\exp\left(-F(x)\right)}{y_1(x)^2}dx\right); \quad F(x) = \int f_1(x)dx \tag{3.2}$$

(see, for example, [17], page 21).

Now, for any analytic function $f(x)$ with arbitrary singularities, of course, it satisfies the normal equation

$$\frac{d^2y}{dx^2} = \frac{f''(x)}{f(x)}y. \tag{3.3}$$

Furthermore, it is a nontrivial and simple solution for non-constant function case of the homogeneous equation

$$y'' + \frac{f''(x)}{f'(x)}y' - \frac{2f''(x)}{f(x)}y = 0. \tag{3.4}$$

This type equation has the simple and non-trivial solution $f(x)$, and its structure may be checked with for the coefficient $f_1(x)$ of y'

$$f_1(x) = \frac{f''(x)}{f'(x)};$$

that is,

$$f'(x) = \exp\left(\int f_1(x)dx\right)$$

and

$$-\frac{2f''(x)}{f(x)} = \frac{-2f_1(x)\exp\left(\int f_1(x)dx\right)}{\int\left(\exp\int f_1(x)dx\right)dx}$$

should be the coefficient f_2 of y.

For example, we have the differential equations

$$y'' - (\tan x)y' + 2y = 0,$$

$$y'' - \left(2x + \frac{1}{x}\right)y' - 4(2x^2+1)y = 0,$$

$$y'' + \frac{x^2-4x+1}{x(x^2+1)}y' + \frac{4(x^2-4x+1)}{(x^2+1)^2}y = 0,$$

$$y'' - \frac{1}{x}y' + \frac{2}{x^2\log x}y = 0,$$

and

$$y'' + \left(\frac{-1}{x} + \frac{1}{x\log x}\right)y' + \left(\frac{4}{x^2\log x} - \frac{1}{x^2(\log x)^2}\right)y = 0.$$

For $y_1(x) = f(x)$, when there exist the integrals, for

$$F(x) = \int f_1(x)dx = \log f'(x)$$

$$y_2(x) = y_1\int\frac{\exp(-F)}{y_1^2}dx = f(x)\int\frac{1}{f(x)^2f'(x)}dx, \tag{3.5}$$

the function y_2 is an independing solution of the equation (3.1).

Then, we know that the general solution is given by

$$y = C_1y_1 + C_2y_2 \tag{3.6}$$

$$+y_2\int y_1g\frac{dx}{W} - y_1\int y_2g\frac{dx}{W}.$$

Here W is the Wronskian determinant

$$W(x) = y_1y_2' - y_2y_1'$$

and it is, in general, given by Liouville's formula

$$W(x) = W(x_0) \exp\left[-\int_{x_0}^{x} f_1(t)dt\right] = \frac{1}{f'(x)}.$$

See [17], pages 21-23.

Then, for any continuous function g that is integrable in the following integrals, we obtain the general solution of the inhomogeneous differential equation

$$y'' + \frac{f''(x)}{f'(x)}y' - \frac{2f''(x)}{f(x)}y = g(x), \tag{3.7}$$

$$y = C_1y_1 + C_2y_2 \tag{3.8}$$

$$+y_2 \int f(x)f'(x)g(x)dx - y_1 \int y_2(x)f'(x)g(x)dx.$$

In connection with (3.3) and (3.4), we have the homogeneous equation, for $\alpha + \beta = -1$

$$y'' + \frac{\alpha f''(x)}{f'(x)}y' + \frac{\beta f''(x)}{f(x)}y = 0. \tag{3.9}$$

Therefore, for this case, we obtain the similar results.

4 For the simple case of $y = (x-a)^\nu$

For the function

$$y = (x-a)^\nu, \tag{4.1}$$

we obtain the normal equation

$$y'' = \frac{\nu(\nu-1)}{(x-a)^2}y. \tag{4.2}$$

Then, we see that the function

$$y = (x-a)^{\nu-1/2} \tag{4.3}$$

is a special solution of the differential equation

$$y'' + \frac{1}{x-a}y' - \frac{\nu(\nu-1)+(1/4)}{(x-a)^2}y = 0. \tag{4.4}$$

Therefore, by the above general formula, we obtain the general solution of (4.4) for $\nu \neq 1/2$,

$$y = C_1(x-a)^{\nu-(1/2)} + C_2(x-a)^{-\nu+(1/2)}. \tag{4.5}$$

For the case $\nu = 1/2$, the result and the situation are trivial.

For $\nu \neq 1/2$, since we obtain a fundamental system of the solutions

$$y_1(x) = (x-a)^{\nu-1/2}, \quad y_2(x) = (x-a)^{-\nu+(1/2)}, \tag{4.6}$$

of the homogeneous equation (4.4), we can obtain easily the general solution of the inhomogeneous equation

$$y'' + f_1(x)y' + f_0(x)y = g(x). \tag{4.7}$$

Here we assume that $a < x_0 < x_2, x_1 < x_0$. In our situation, we obtain the result:

For the inhomogeneous equation

$$y'' + \frac{1}{x-a}y' - \frac{\nu(\nu-1)+(1/4)}{(x-a)^2}y = g(x), \tag{4.8}$$

we obtain the general solution, for $\nu \neq 1/2$

$$y = C_1y_1 + C_2y_2 \tag{4.9}$$

$$+\frac{1}{-2\nu+1}\left[y_2\int(x-a)^{\nu+(1/2)}g(x)dx - y_1\int(x-a)^{(3/2)-\nu}g(x)dx\right],$$

if the integrals exist.

5 Thomas-Fermi equation and open problems

In connection with the normal equations, we recall the important Thomas-Fermi equation

$$y'' = \frac{1}{\sqrt{x}}y^{(3/2)}$$

in the Thomas-Fermi model. We find one special solution

$$y = \frac{144}{x^3}.$$

However, in connection with the theory of Uchida and with our new idea in this paper, the problems are all open.

See [17], pages 306-314, for many concrete examples of

$$y'' = Ax^n y^m$$

and their special solutions.

In [1], page 362, we see the following interesting equations and concrete special solutions:

$$y'' = -\left(\lambda^2 - \frac{\nu^2 - (1/4)}{z^2}\right) y,$$

$$y'' = -\left(\frac{\lambda^2}{4z} - \frac{\nu^2 - 1}{4z^2}\right) y,$$

$$y'' = -\lambda^2 z^{p-1} y,$$

$$y'' = -(\lambda^2 \exp(2z) - \nu^2) y,$$

and

$$y^{2n} = (-1)^n \lambda^{2n} z^{-n} y.$$

6 Remark

For the hyper exponential functions of the second order, note that:

For any C^2 function $f(x)$ on a closed interval that is a non-vanishing $f(x) \neq 0$, it is a solution of the normal equation (3.3) and conversely, for any C^2 function $h(x)$ on the closed interval, the normal differential equation

$$\frac{d^2 y}{dx^2} = h(x) y \tag{6.1}$$

has a non-vanishing solution $f(x)$ of C^2 functions.

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