Theorems on the transfer function of first–order RC–circuits with either an ideal or a non–ideal capacitor

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Abstract

In this letter two theorems are stated, the first one on the ratio of an electrical output voltage signal $y(t)$ to an electrical input voltage signal $x(t)$ of a circuit with an ideal impedance and the second one on the ratio of an electrical output voltage signal $y(t)$ to an electrical input voltage signal $x(t)$ of a circuit with a non–ideal impedance. In the latter case, the change of the ratio $y(t)/x(t)$ is a measurable quantity of the change of the resistive part of the output impedance and therefore a measure of its quality.

1 Theorems on the transfer function of first order circuits

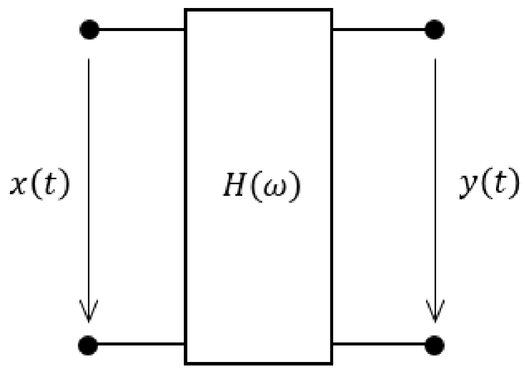

Figure 1: The transfer function $H(\omega)$ describing the linear relationship between the input signal $x(t)$ and the output signal $y(t)$.

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For a given input signal $u = x(t) = r_x \cdot \cos(\omega \cdot t + \varphi_x)$ and a given transfer function $H(\omega)$ the output signal $y(t) = r_y \cdot \cos(\omega \cdot t + \varphi_y)$ can be determined from

$$r_y \cdot e^{\jmath \cdot \varphi_y} = H(\omega) \cdot r_x \cdot e^{\jmath \cdot \varphi_x}$$

Motivated by practical applications, we will confine our studies to the class of transfer functions

$$H(\omega) = \rho \cdot \cos(\varphi) \cdot e^{\jmath \cdot \varphi}, \;\; \varphi \in \langle -\frac{\pi}{2}, +\frac{\pi}{2} \rangle, \;\; \rho > 0$$

We then have the following result:

Lemma

Let

$$x(t) = r_x \cdot \cos(\omega \cdot t + \varphi_x), \;\; \varphi \in \langle -\frac{\pi}{2}, +\frac{\pi}{2} \rangle, \;\; \rho > 0 \text{ and}$$
$$y(t) = \rho \cdot r_x \cdot \cos(\varphi) \cdot \cos(\omega \cdot t + \varphi_x + \varphi).$$

Then

$$(1) \qquad y(t) - \rho \cdot x(t) = \frac{\tan(\varphi)}{\omega} \cdot \dot{y}(t)$$
$$(2) \qquad \text{Let } z(t) = x(t) - y(t). \text{ Then:}$$
$$\dot{z}(t) - (1-\rho) \cdot \dot{x}(t) = \omega \cdot \tan(\varphi) \cdot (x(t) - z(t))$$

Proof

$$(1) \qquad \dot{y} = -\rho \cdot \omega \cdot r_x \cdot \cos(\varphi) \cdot \sin(\omega \cdot t + \varphi_x + \varphi) \Leftrightarrow$$
$$\frac{\dot{y}}{\omega \cdot \cos(\varphi)} = -\rho \cdot r_x \cdot \sin(\omega \cdot t + \varphi_x + \varphi)$$
$$y - \rho \cdot x = \sin(\varphi) \cdot (-\rho \cdot r_x \cdot \sin(\omega \cdot t + \varphi_x + \varphi)) =$$
$$= \sin(\varphi) \cdot \frac{\dot{y}}{\omega \cdot \cos(\varphi)} = \frac{\tan(\varphi)}{\omega} \cdot \dot{y}$$

$$(2) \qquad \ddot{y} = -\omega^2 \cdot y, \qquad y = x - z$$
$$\dot{y} - \rho \cdot \dot{x} = \frac{\tan(\varphi)}{\omega} \cdot \ddot{y} = \frac{\tan(\varphi)}{\omega} \cdot \left(-\omega^2 \cdot y\right) = -\omega \cdot \tan(\varphi) \cdot y$$
$$(\dot{x} - \dot{z}) - \rho \cdot \dot{x} = -\omega \cdot \tan(\varphi) \cdot (x - z)$$
$$\dot{z} - (1-\rho) \cdot \dot{x} = \omega \cdot \tan(\varphi) \cdot (x - z) \qquad \square$$

1.1 Theorem

$$\text{Let } x(t) = r_x \cdot \cos(\omega \cdot t + \varphi_x), \;\; \varphi \in \langle -\frac{\pi}{2}, +\frac{\pi}{2} \rangle \backslash \{0\} \text{ and}$$
$$y(t) = r_x \cdot \cos(\varphi) \cdot \cos(\omega \cdot t + \varphi_x + \varphi)$$

Then:

$$(1) \qquad \dot{y}(t) = 0 \Leftrightarrow y(t) = x(t)$$
$$(2) \qquad \text{Let } z(t) = x(t) - y(t). \text{ Then:}$$
$$\dot{z}(t) = 0 \Leftrightarrow z(t) = x(t)$$

Proof

(1) Using part (1) of the previous lemma for $\rho = 1$:

$$\dot{y} = 0 \Leftrightarrow 0 = \frac{\tan(\varphi)}{\omega} \cdot \dot{y} = y - x \Leftrightarrow y = x$$

(2) Using part (2) of the previous lemma for $\rho = 1$:

$$\dot{z} = 0 \Leftrightarrow 0 = \dot{z} = \omega \cdot \tan(\varphi) \cdot (x - z) \Leftrightarrow z = x$$

From part (1) of the previous lemma we have the following theorem:

1.2 Theorem

Let $x(t) = r_x \cdot \cos(\omega \cdot t + \varphi_x)$, $\varphi \in \langle -\frac{\pi}{2}, +\frac{\pi}{2} \rangle, \rho > 0$ and
$y(t) = \rho \cdot r_x \cdot \cos(\varphi) \cdot \cos(\omega \cdot t + \varphi_x + \varphi)$. Then:
$\dot{y}(t) = 0 \Leftrightarrow y(t) = \rho \cdot x(t)$

Proof

Using part (1) of the lemma:

$$\dot{y} = 0 \Leftrightarrow 0 = \frac{\tan(\varphi)}{\omega} \cdot \dot{y} = y - \rho \cdot x \Leftrightarrow y = \rho \cdot x \quad \square$$

2 Application to an RC–circuit with an ideal capacitor

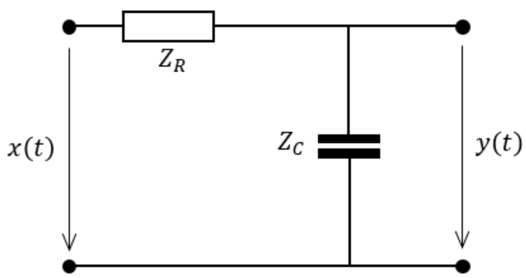

Figure 2: The $RC-$circuit with $x(t)$ as input voltage signal and $y(t)$ as output voltage signal. This circuit is a special case of the circuit in Figure 5, as the latter converges to the former for $\widetilde{R} \to \infty$.

Applying Theorem 1.1

$$r_x = 405 \text{ V}, \; \varphi_x = -\frac{\pi}{2} \text{ rad}, \; \omega = 100\pi \text{ rad/s}$$

$$Z_R = R = 15 \text{ k}\Omega, \; Z_C = \frac{1}{\jmath\omega C} = -10\jmath \text{ k}\Omega$$

$$H(\omega) = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{\jmath\omega C}}{R + \frac{1}{\jmath\omega C}} = \frac{1}{1 + \jmath\omega RC} := \cos(\varphi) \cdot e^{\jmath \cdot \varphi}$$

$$\varphi := -\arg(1 + \jmath\omega RC)$$

Remark: in the case of an ideal capacitor, the graph of the signal $u = x(t)$ intersects the graph of the signal $u = y(t)$ at its extremum. Accordingly, the graph of the signal $u = x(t)$ intersects the graph of the difference signal $u = z(t) = x(t) - y(t)$ at its extremum. These results can be used as a didactic aid to visually recognize the fact that a capacitor is ideal, in the graphs of both signals.

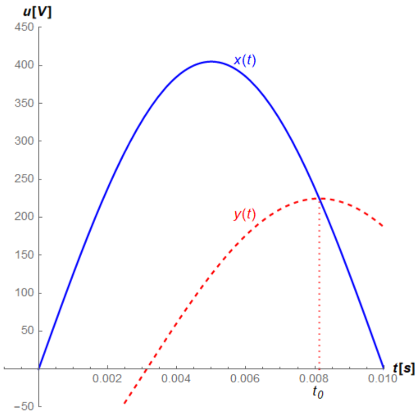

Figure 3: Application of the theorem to an RC–circuit with an ideal capacitor: the graph of the input signal $u = x(t)$ in blue intersects the graph of the output signal $u = y(t)$ in red at its extremum, i.e. $y(t_0) = x(t_0)$.

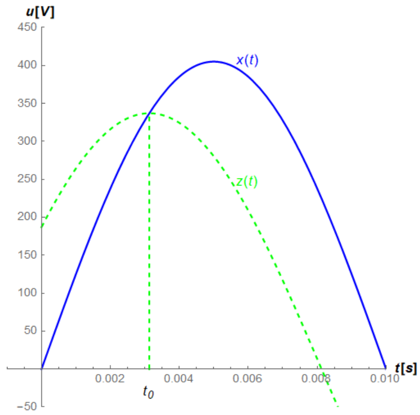

Figure 4: The graph of the signal $u = x(t)$ intersects the graph of the difference signal $u = z(t) = x(t) - y(t)$ at its extremum.

3 Application to an RC–circuit with a non–ideal capacitor

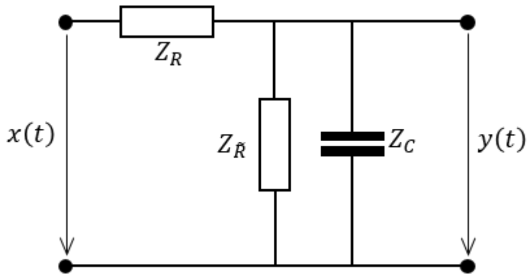

Figure 5: The $RC-$circuit of Figure 2 now with a resistive impedance $Z_{\widetilde{R}}$ added in parallel to impedance Z_C.

Applying Theorem 1.2

$$Z_R = R = 15\ \text{k}\Omega, \qquad Z_C = \frac{1}{\jmath\omega C} = -10\jmath\ \text{k}\Omega, \qquad Z_{\widetilde{R}} = \widetilde{R} = 30\ \text{k}\Omega$$

$$Z_{\widetilde{R},C} = Z_{\widetilde{R}} \,||\, Z_C = \frac{Z_{\widetilde{R}} \cdot Z_C}{Z_{\widetilde{R}} + Z_C} = \frac{\widetilde{R} \cdot \frac{1}{\jmath\omega C}}{\widetilde{R} + \frac{1}{\jmath\omega C}} = \frac{\widetilde{R}}{1 + \jmath\omega \widetilde{R} C}$$

$$H(\omega) = \frac{Z_{\widetilde{R},C}}{Z_R + Z_{\widetilde{R},C}} = \frac{\frac{\widetilde{R}}{1+\jmath\omega\widetilde{R}C}}{R + \frac{\widetilde{R}}{1+\jmath\omega\widetilde{R}C}} = \frac{\widetilde{R}}{R + \widetilde{R} + \jmath\omega R\widetilde{R}C}$$

$$= \frac{\widetilde{R}}{R+\widetilde{R}} \cdot \frac{1}{1 + \jmath\omega C \frac{R\widetilde{R}}{R+\widetilde{R}}} := \rho \cdot \cos(\varphi) \cdot e^{\jmath \cdot \varphi}$$

$$\rho := \frac{\widetilde{R}}{R+\widetilde{R}} \quad \text{i.e., independent of the capacitance } C$$

$$\varphi := -\arg\left(1 + \jmath\omega C \frac{R\widetilde{R}}{R+\widetilde{R}}\right)$$

$$\rho = \frac{\widetilde{R}}{R+\widetilde{R}} = \frac{30\ \text{k}\Omega}{15\ \text{k}\Omega + 30\ \text{k}\Omega} = \frac{2}{3}$$

$$\dot{y}(t) = 0 \Leftrightarrow y(t) = \rho \cdot x(t) = \frac{2}{3} \cdot x(t)$$

Remark: In the case of a non–ideal capacitor, the graph of the signal $u = \rho \cdot x(t)$ intersects the graph of the signal $u = y(t)$ at its extremum. In the extremum it therefore holds that the ratio of the signal values $y(t)$ and $x(t)$ is equal to the ratio of resistance values $\widetilde{R}$ and $R + \widetilde{R}$. These results can be used as a didactic aid to visually recognize the fact that a capacitor is non-ideal, in the graphs of both signals.

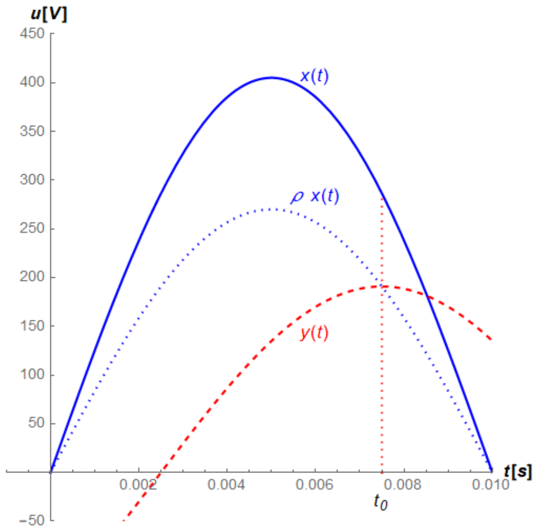

Figure 6: Application of the theorem on an RC–circuit with a non–ideal capacitor: the output signal $u = y(t)$ depicted in red is at its extremum, i.e. $y(t_0) = \rho \cdot x(t_0) \Leftrightarrow \rho = y(t_0)/x(t_0)$ with the input signal $u = x(t)$ drawn in blue.

4 Acknowledgement

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