Kouider Function have Basis a

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Abstract

Josephus function is a new numerical function which presented by Kouider (2019,[1]) by studying Joseph's problem . In this paper we interesting in study of its derived function and some of its related properties for Josephus function. From this point of view we saw that we can define a more comprehensive function than the Josephus function. And we called it the Kouider function with basis a . We have also studied some of its related properties with proof as well.

Keywords: *Josephus function*

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1. Introduction

According to the story of Josephus, which narrates his survival from the persecution of Rome, and which was transmitted by the Roman historian Flavius Josephus himself (for instant , see [1],[2]). Kouider (-) introduced a new function called the Josephus function which is defined as follows.

Let $[x]$ be integer part of real number x .The Josephus function is numeric function define for all $x>0$ to $\mathbb{R}$ where $s=[\ln x/\ln 2]$ by:

$$J(x)=2x-2^{s+1}+1 \qquad (1)$$

In this paper, we'll study some properties of this function, in order to know this function more. Therefore, we study its derivative function and draw some conclusions regarding the derivative function. That is why we saw that we can extend this function to include any basis and we called it the Kouider function and this is what we will explain in detail.

Theorem2: Let J be Josephus function, we have:

1) $J'(x)=\ln 2(J(x)-1)$ (4)

2) $J''(x)=(\ln 2)^2(J(x)-1)$ (5)

3) $J^{(n)}(x)=(\ln 2)^n(J(x)-1)$ (6)

Proof: We have J be Josephus function defined for $x\in\mathbb{R}_+^*$ to $\mathbb{R}$ by:

$$J(x)=2x-2^{\left[\frac{\ln x}{\ln 2}\right]+1}+1$$

Next, if we take

$$\frac{\ln x}{\ln 2}=s+\varepsilon \quad \text{for } 0\le\varepsilon<1 \text{ and } s\in\mathbb{Z} \quad (7)$$

this imply that,

$$\left[\frac{\ln x}{\ln 2}\right]=s \qquad (8)$$

And, we get

$$\ln x=(s+\varepsilon)\ln 2=\ln 2^{(s+\varepsilon)}$$

Then, we have for $0\le\varepsilon<1$ and $s\in\mathbb{Z}$

$$x=2^{(s+\varepsilon)} \qquad (9)$$

Therefore, we can get that

$$2x-2^{\left[\frac{\ln x}{\ln 2}\right]+1}+1=2\times 2^{s+\varepsilon}-2^{s+1}+1=2^{s+\varepsilon+1}-2^{s+1}+1$$

So we have other formula of Josephus function with $0\le\varepsilon<1$ and $s\in\mathbb{Z}$ by

$$J(s)=2^{s+\varepsilon+1}-2^{s+1}+1 \qquad (10)$$

it's easy to get

$$J'(s)=\ln 2\left(2^{s+\varepsilon+1}-2^{s+1}\right)$$

Equivalent, that for (7)-(9) and (1) we have

$$J'(s)=\ln 2(J(x)-1)$$

then we have

$$J'(x)=\frac{J''(s)}{\ln 2}=\frac{\ln 2\left(2^{s+\varepsilon+1}-2^{s+1}\right)'}{\ln 2}=\frac{(\ln 2)^2\left(2^{s+\varepsilon+1}-2^{s+1}\right)}{\ln 2}$$

and,

$$J'(x)=\ln 2\left(2^{s+\varepsilon+1}-2^{s+1}\right)=J'(s)$$

Then we have,

$$J'(x)=\ln 2(J(x)-1)$$

For Proof (2) and (3), by derivate the inequality (4) we have the second derivation of Josephus function

$$J^{(2)}(x)=J''(x)=(\ln 2)^2(J(x)-1)$$

Then, the same procedure we defined the n derivation of the Josephus function by

$$J^{(n)}(x)=(\ln 2)^n(J(x)-1)$$

Theorem3: Let $J^{(1)},J^{(2)},\ldots,J^{(n)}$ sequential derivatives of the Josephus function, we have:

1) $\sum_{i=1}^{n} J^{(i)}(x) = (J(x)-1)\sum_{i=1}^{n}(\ln 2)^{i}$ (11)

And for $n \to +\infty$,

2) $\frac{\sum_{i=1}^{n} J^{(i)}(x)}{J(x)-1} = \frac{-\ln 2}{\ln 2 - 1}$ (12)

3) $\frac{\sum_{i=1}^{n} J^{(i)}(x)}{J^{(1)}(x)} = \frac{-1}{\ln 2 - 1}$ (13)

Proof: we have $J^{(1)}, J^{(2)}, \ldots, J^{(n)}$ sequential derivatives of the Josephus function which defined by

$$J^{(1)}(x) = \ln 2(J(x)-1)$$
$$J^{(2)}(x) = (\ln 2)^{2}(J(x)-1)$$
$$\vdots$$
$$J^{(n)}(x) = (\ln 2)^{n}(J(x)-1)$$

We have

$J^{(1)}(x) + J^{(2)}(x) + \cdots + J^{(n)}(x) =$

$\ln 2(J(x)-1) + (\ln 2)^{2}(J(x)-1) + \cdots + (\ln 2)^{n}(J(x)-1)$

then, we find that

$$\sum_{i=1}^{n} J^{(i)}(x) = (J(x)-1)\sum_{i=1}^{n}(\ln 2)^{i}$$

Let the sequence geometries be defined on $\mathbb{N}^{*}$ by $v_n = (\ln 2)^{n}$ with basis $\ln 2$ and the first limit $v_1 = \ln 2$ then,

$$\sum_{i=1}^{n}(\ln 2)^{i} = \frac{\ln 2}{\ln 2 - 1}\left((\ln 2)^{n} - 1\right) \quad (14)$$

Then we substitute it in the previous equation, we find

$$\frac{\sum_{i=1}^{n} J^{(i)}(x)}{(J(x)-1)} = \frac{\ln 2}{\ln 2 - 1}\left((\ln 2)^{n} - 1\right)$$

and for $J^{(1)}(x) = \ln 2(J(x)-1)$

$$\frac{\sum_{i=1}^{n} J^{(i)}(x)}{J^{(1)}(x)} = \frac{(\ln 2)^{n} - 1}{\ln 2 - 1}$$

While n tend to $+\infty$ we have

$$\frac{\sum_{i=1}^{n} J^{(i)}(x)}{(J(x)-1)} = \frac{-\ln 2}{\ln 2 - 1} = 2.258891353....$$

And

$$\frac{\sum_{i=1}^{n} J^{(i)}(x)}{J^{(1)}(x)} = \frac{-1}{\ln 2 - 1} = 3.258891353....$$

Propertie1: Let $J^{(1)}, \ldots, J^{(n)}$ sequential derivatives of the Josephus function, and from (12) and (13) we get:

1) $\frac{\sum_{i=1}^{n} J^{(i)}(x)}{J^{(1)}(x)} - \frac{\sum_{i=1}^{n} J^{(i)}(x)}{J(x)-1} = 1$ (15)

2) $\frac{\sum_{i=1}^{n} J^{(i)}(x)}{J^{(1)}(x)} + \frac{\sum_{i=1}^{n} J^{(i)}(x)}{J(x)-1} = \frac{-(\ln 2 + 1)}{(\ln 2 - 1)}$ (16)

Proof: By subtracting the formula (13) to (12)

$$\frac{\sum_{i=1}^{n} J^{(i)}(x)}{J^{(1)}(x)} - \frac{\sum_{i=1}^{n} J^{(i)}(x)}{J(x)-1} = \frac{-1}{\ln 2 - 1} - \frac{-\ln 2}{\ln 2 - 1} = \frac{\ln 2 - 1}{\ln 2 - 1} = 1$$

And by adding formula (13) to (12) we find

$$\frac{\sum_{i=1}^{n} J^{(i)}(x)}{J^{(1)}(x)} + \frac{\sum_{i=1}^{n} J^{(i)}(x)}{J(x)-1} = \frac{-(\ln 2 + 1)}{\ln 2 - 1} = 5.517782707....$$

Corollary1: We have J be Josephus function which defined in (1). Then we get

1) $\lim_{x \overset{>}{\to} 0} J(x) = 1$

2) $\lim_{x \to +\infty} J(x) = +\infty$

Proof:

$$\lim_{x \overset{>}{\to} 0} J(x) = \lim_{x \overset{>}{\to} 0}\left(2x - 2^{\left[\frac{\ln x}{\ln 2}\right]+1} + 1\right) = \lim_{x \overset{>}{\to} 0}\left(2x - e^{\left(\left[\frac{\ln x}{\ln 2}\right]+1\right)\ln 2} + 1\right)$$

we have that $\lim_{x \overset{>}{\to} 0}\left[\frac{\ln x}{\ln 2}\right] = -\infty$ and $\lim_{x \to -\infty} e^{x} = 0$ so we find $\lim_{x \overset{>}{\to} 0} J(x) = 1$

$$\lim_{x \to +\infty} J(x) = \lim_{x \to +\infty}\left(2x - 2^{\left[\frac{\ln x}{\ln 2}\right]+1} + 1\right) = \lim_{x \to +\infty}\left(2x - e^{\left(\left[\frac{\ln x}{\ln 2}\right]+1\right)\ln 2} + 1\right)$$

we get that $\lim_{x \to +\infty}\left(\left[\frac{\ln x}{\ln 2}\right]+1\right)\ln 2 = +\infty$ and $\lim_{x \to +\infty} e^{x} = +\infty$

then we find $\lim_{x \to +\infty} J(x) = +\infty - \infty$ which is

Under (7) we have for $0 \le \varepsilon < 1$ and $s \in \mathbb{Z}$ $s = \frac{\ln x}{\ln 2} - \varepsilon$, then if $x \to +\infty$ we find $s \to +\infty$. And with (9) we get,

$\lim_{x \to +\infty}\left(2x - 2^{\left[\frac{\ln x}{\ln 2}\right]+1} + 1\right) = \lim_{s \to +\infty}\left(2^{s+\varepsilon+1} - 2^{s+1} + 1\right)$. Hence,

$\lim_{s \to +\infty}\left(2^{s+\varepsilon+1} - 2^{s+1} + 1\right) = \lim_{s \to +\infty} 2^{s+1}\left(2^{\varepsilon} - 1\right) + 1$. Since $0 \le \varepsilon < 1$ we have $1 \le 2^{\varepsilon} < 2$ then $0 \le 2^{\varepsilon} - 1 < 1$ therefore, $2^{\varepsilon} - 1 > 0$ then we get

$$\lim_{s \to +\infty} 2^{s+1}\left(2^{\varepsilon} - 1\right) + 1 = +\infty$$

2. Kouider Function with basis a

Now, we define a new numerical function with basis a which we symbolize by $K_a(x)$ and defined for all $x > 0$ to $\mathbb{R}$ by:

$$K_a(x) = ax - a^{s+1} + 1 \quad (18)$$

where $s = [\ln x / \ln a]$.

Obviously, for $a = 2$ the function $K_a(x)$ becomes the Josephus function which defined in (1) by

$$J(x) = K_2(x) = 2x - 2^{\left[\frac{\ln x}{\ln 2}\right]+1} + 1$$

It is clear that $a \in]0;1[\cup]1;+\infty[$ because $s = [\ln x / \ln a]$ defined for all $a > 0$ with $a \ne 1$.

Definition1: The Kouider function with basis a is every defined function on $\mathbb{R}_{+}^{*}$ to $\mathbb{R}$ by

$$K_a(x) = ax - a^{\left[\frac{\ln x}{\ln a}\right]+1} + 1 \quad (19)$$

with $a \in]0;1[\cup]1;+\infty[$

Theorem3: Let K_a be Kouider function have basis a. Then we have:

1) $K_a(a^{n}) = 1$ where $n \in \mathbb{Z}$,

For $0 < a < 1$ and $a > 1$ we find

2) $\lim_{x \overset{>}{\to} 0} K_a(x) = 1$ (20)

3) $\lim_{x\to+\infty} K_a(x) = \begin{cases} -\infty & for \quad 0<a<1 \\ +\infty & for \quad a>1 \end{cases}$ (21)

Proof:

(1) $K_a(a^n) = a\times a^n - a^{\left[\frac{\ln a^n}{\ln a}\right]+1} + 1 = a^{n+1} - a^{[n]+1} + 1 = 1$.

(2) $\lim_{x\overset{>}{\to}0} K_a(x) = \lim_{x\overset{>}{\to}0}\left(ax - a^{\left[\frac{\ln x}{\ln a}\right]+1} + 1\right) = \lim_{x\overset{>}{\to}0}\left(ax - e^{\left(\left[\frac{\ln x}{\ln a}\right]+1\right)\ln a} + 1\right)$

We have that $\lim_{x\overset{>}{\to}0}\left(\left[\frac{\ln x}{\ln a}\right]+1\right)\ln a = -\infty$ if $\ln a>0$ where $a>1$ .Then, $\lim_{x\to-\infty} e^x = 0$. And if $\ln a<0$ where $0<a<1$, we have $\lim_{x\overset{>}{\to}0}\left(\left[\frac{\ln x}{\ln a}\right]+1\right)\ln a = -\infty$. Then $\lim_{x\to-\infty} e^x = 0$. Therefore, we find for all $a\in\left]0;1\right[\cup\left]1;+\infty\right[$ $\lim_{x\overset{>}{\to}0} K_a(x) = 1$. Next proof (3) ,

$$\lim_{x\to+\infty} K_a(x) = \lim_{x\to+\infty}\left(ax - a^{\left[\frac{\ln x}{\ln a}\right]+1} + 1\right) = \lim_{x\to+\infty}\left(ax - e^{\left(\left[\frac{\ln x}{\ln a}\right]+1\right)\ln a} + 1\right)$$

we get that for $a\in\left]0;1\right[\cup\left]1;+\infty\right[$

$$\lim_{x\to+\infty}\left(\left[\frac{\ln x}{\ln a}\right]+1\right)\ln a = +\infty$$

Next we find $\lim_{x\to+\infty} e^x = +\infty$. Then for $a\in\left]0;1\right[\cup\left]1;+\infty\right[$ $\lim_{x\to+\infty} K_a(x) = +\infty - \infty$ which is

We take for $0\le\varepsilon<1$ and $s\in\mathbb{Z}$

$$\frac{\ln x}{\ln a} = s+\varepsilon \qquad (22)$$

Then we have,

$$x = a^{(s+\varepsilon)} \qquad (23)$$

Since,

$$s = \left[\frac{\ln x}{\ln a}\right] \qquad (24)$$

If $x\to+\infty$ then $s\to+\infty$ if $a>1$ and $s\to-\infty$ if $0<a<1$.
Next, under (23) and (24), we find

$$\lim_{x\to+\infty}\left(ax - a^{\left[\frac{\ln x}{\ln a}\right]+1} + 1\right) = \lim_{s\to-\infty}\left(a^{s+\varepsilon+1} - a^{s+1} + 1\right) = \lim_{s\to-\infty}\left(a^{s+1}\left(a^{\varepsilon}-1\right)+1\right).$$

Since if $0<a<1$ we have $1\ge a^{\varepsilon}>a$ where $0\le\varepsilon<1$ implies that $a-1<a^{\varepsilon}-1\le 0$. Then we find

$$\lim_{x\to+\infty}\left(ax - a^{\left[\frac{\ln x}{\ln a}\right]+1} + 1\right) = \lim_{s\to-\infty}\left(a^{s+1}\left(a^{\varepsilon}-1\right)+1\right) = -\infty$$

And if $a>1$ with (23) and (24), we find

$$\lim_{x\to+\infty}\left(ax - a^{\left[\frac{\ln x}{\ln a}\right]+1} + 1\right) = \lim_{s\to+\infty}\left(a^{s+\varepsilon+1} - a^{s+1} + 1\right) = \lim_{s\to+\infty}\left(a^{s+1}\left(a^{\varepsilon}-1\right)+1\right)$$

we have for $0\le\varepsilon<1$, $1\le a^{\varepsilon}<a$ implies that $0\le a^{\varepsilon}-1<a-1$ with $s>0$. Then we find $a^{\varepsilon}-1>0$, with $s>0$

$$\lim_{x\to+\infty}\left(ax - a^{\left[\frac{\ln x}{\ln a}\right]+1} + 1\right) = \lim_{s\to+\infty}\left(a^{s+1}\left(a^{\varepsilon}-1\right)+1\right) = +\infty$$

Theorem: Let $K_a^{(1)},\ldots,K_a^{(n)}$ sequential derivatives of the Kouider function(19), we have

$$K_a^{(n)}(x) = (\ln a)^n\left(K_a(x)-1\right) \qquad (25)$$

where $K_a^{(n)}$ is the function derived from rank n

Proof: We have K_a be Kouider function with basis a defined for $x\in\mathbb{R}_+^*$ to $\mathbb{R}$ by:

$$J(x) = ax - a^{\left[\frac{\ln x}{\ln a}\right]+1} + 1$$

where $a\in\left]0;1\right[\cup\left]1;+\infty\right[$

Next, Under (22) and (23), we can get that

$$ax - a^{\left[\frac{\ln x}{\ln a}\right]+1} + 1 = a\times a^{s+\varepsilon} - a^{s+1} + 1 = a^{s+\varepsilon+1} - a^{s+1} + 1$$

So we have other formula of Kouider function have a basis a with $0\le\varepsilon<1$ and $s\in\mathbb{Z}$ by

$$K_a(s) = a^{s+\varepsilon+1} - a^{s+1} + 1 \qquad (26)$$

it's easy to get

$$K_a^{'}(s) = \ln a\left(a^{s+\varepsilon+1} - a^{s+1}\right)$$

Equivalent, that for (23)-(24) and (19) we have

$$K_a^{'}(s) = \ln a\left(K_a(x)-1\right)$$

then we have

$$K_a^{'}(x) = \frac{K_a^{''}(s)}{\ln a} = \frac{\ln a\left(a^{s+\varepsilon+1}-a^{s+1}\right)^{'}}{\ln a} = \frac{(\ln a)^2\left(a^{s+\varepsilon+1}-a^{s+1}\right)}{\ln a}$$

and,

$$K_a^{'}(x) = \ln a\left(a^{s+\varepsilon+1}-a^{s+1}\right) = K_a^{'}(s)$$

Then we have,

$$K_a^{'}(x) = \ln a\left(K_a(x)-1\right)$$

Then the same procedure for $K_a^{(1)},K_a^{(2)},\ldots,K_a^{(n)}$ sequential derivatives of the Kouider function with basis a defined by

$$K_a^{(1)}(x) = \ln a\left(K_a(x)-1\right)$$
$$K_a^{(2)}(x) = (\ln a)^2\left(K_a(x)-1\right)$$
$$K_a^{(2)}(x) = (\ln 3)^2\left(K_a(x)-1\right)$$
$$\vdots$$
$$K_a^{(n)}(x) = (\ln a)^n\left(K_a(x)-1\right)$$

Consequentially, we defined the n derivation of the Kouider function with basis a by

$$K_a^{(n)}(x) = (\ln a)^n\left(K_a(x)-1\right)$$

Corollary1: Solve the following differential equation

$$\begin{cases} f^{(n)}(x) = (\ln a)^n\left(f(x)-1\right) \\ f(a^m) = 1 \quad for \quad m\in\mathbb{Z} \end{cases}$$

where $f^{(n)}$ is the function derived from rank n of f , is

$$f(x) = K_a(x)$$

Theorem3: Let $K_a^{(1)},K_a^{(2)},\ldots,K_a^{(n)}$ sequential derivatives of the Josephus function, we have:

1) $\sum_{i=1}^{n} K_a^{(i)}(x) = \left(K_a(x)-1\right)\sum_{i=1}^{n}(\ln a)^i$ (27)

And with $1<a<e$ for $n\to+\infty$,

2) $\dfrac{\sum_{i=1}^{n} K_a^{(i)}(x)}{K_a(x)-1} = \dfrac{-\ln a}{\ln a-1}$ (28)

3) $\dfrac{\sum_{i=1}^{n} K_a^{(i)}(x)}{K_a^{(1)}(x)} = \dfrac{-1}{\ln a-1}$ (29)

Proof: we have $K_a^{(n)}$ sequential derivatives of the Josephus function which defined by

$$K_a^{(n)}(x) = (\ln a)^n\left(K_a(x)-1\right)$$

We have

$$K_a^{(1)}(x) + K_a^{(2)}(x) + \cdots + K_a^{(n)}(x) =$$
$$\ln a\left(K_a(x)-1\right) + (\ln a)^2\left(K_a(x)-1\right) + \cdots + (\ln a)^n\left(K_a(x)-1\right)$$

then, we find that

$$\sum_{i=1}^{n} K_a^{(i)}(x) = (K_a(x)-1)\sum_{i=1}^{n}(\ln a)^i$$

Let the sequence geometries be defined on $\mathbb{N}^*$ by $v_n = (\ln a)^n$ with basis $\ln a$ and the first limit $v_1 = \ln a$ then,

$$\sum_{i=1}^{n}(\ln a)^i = \frac{\ln a}{\ln a - 1}\left((\ln a)^n - 1\right) \quad (30)$$

Then we substitute it in the previous equation, we find

$$\frac{\sum_{i=1}^{n} K_a^{(i)}(x)}{(K_a(x)-1)} = \frac{\ln a}{\ln a - 1}\left((\ln a)^n - 1\right)$$

and for $K_a^{(1)}(x) = \ln a\,(K_a(x)-1)$

$$\frac{\sum_{i=1}^{n} K_a^{(1)}(x)}{K_a^{(1)}(x)} = \frac{(\ln a)^n - 1}{\ln a - 1}$$

While with n tend to $+\infty$ and if $0 < \ln a < 1$ implie $1 < a < e$ $(e = 2.71828\underline{1828}...)$ we have

$$\frac{\sum_{i=1}^{n} K_a^{(i)}(x)}{(K_a(x)-1)} = \frac{-\ln a}{\ln a - 1} \quad \text{and} \quad \frac{\sum_{i=1}^{n} K_a^{(1)}(x)}{K_a^{(1)}(x)} = \frac{-1}{\ln a - 1}$$

Propertie2: Let $K_a^{(1)},\ldots,K_a^{(n)}$ sequential derivatives of the Kouider function(19), and from (28) and (29) we get:

1) $$\frac{\sum_{i=1}^{n} K_a^{(1)}(x)}{K_a^{(1)}(x)} - \frac{\sum_{i=1}^{n} K_a^{(i)}(x)}{(K_a(x)-1)} = 1 \quad (31)$$

2) $$\frac{\sum_{i=1}^{n} K_a^{(1)}(x)}{K_a^{(1)}(x)} + \frac{\sum_{i=1}^{n} K_a^{(i)}(x)}{(K_a(x)-1)} = \frac{-(\ln a + 1)}{(\ln a - 1)} \quad (32)$$

for $a \in]1;e[$ with $e = \exp(e)$

Proof: By subtracting the formula (29) to (28)

$$\frac{\sum_{i=1}^{n} K_a^{(1)}(x)}{K_a^{(1)}(x)} - \frac{\sum_{i=1}^{n} K_a^{(i)}(x)}{(K_a(x)-1)} = \frac{-1}{\ln a - 1} - \frac{-\ln a}{\ln a - 1} = \frac{\ln a - 1}{\ln a - 1} = 1$$

And by adding formula (29) to (28) we find

$$\frac{\sum_{i=1}^{n} K_a^{(1)}(x)}{K_a^{(1)}(x)} + \frac{\sum_{i=1}^{n} K_a^{(i)}(x)}{(K_a(x)-1)} = \frac{-(\ln a + 1)}{\ln a - 1}$$

The following figure(1) represent the graph of the Kouider function with basis 2 (the blue) and with the basis $\frac{1}{2}$ (the green)

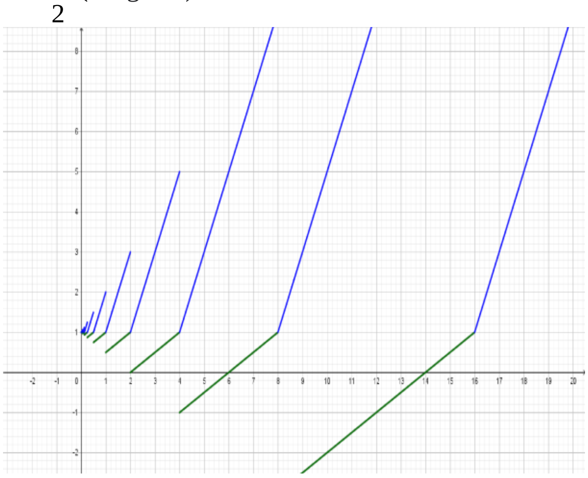

Figure 1. The graph of K_2 and $K_{1/2}$

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