On h-open sets and h-continuous functions

Fadhil Hussein Abbas

fadhilhaman@gmail.com

Johannes Kepler University, Linz-Austria

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Abstract

In this paper, we introduce a new class of open sets in a topological space (X, τ) called h-open sets. Also, introduce and study topological properties of h-interior, h-closure, h-limit points, h-derived, h-interior points, h-border, h-frontier and h-exterior by using the concept of h-open sets. Moreover introduce the notion of h-continuous functions, h-open functions, h-irresolute functions, h-totally continuous functions, h-contra-continuous functions, h-homeomorphism and investigate some properties of these functions and study some properties, remarks related to them.

Keywords: h-open sets; h-interior; h-closure; h-limit points; h-border; h-frontier; h-exterior; h-continuous functions;h-open functions; h-irresolute functions; h-homeomorphism; h-totally continuous functions; h-contra-continuous functions.

1 Introduction and Preliminaries

The concept of open sets is an important concepts in topology and its applications. Levine [7] introduced semi-open set and semicontinuous function, Njastad [8] introduced α-open set, Askander [15] introduced iopen set, i-irresolute mapping and i-homeomorphism, Biswas [6] introduced semi-open

ORCID: https://orcid.org/0000-0002-2308-5292

functions, Mashhour, Hasanein, and El-Deeb [1] introduced α-continuous and α-open mappings, Noiri [16] introduced totally (perfectly) continuous function, Crossley [11] introduced irresolute function, Maheshwari [14] introduced α-irresolute mapping, Beceren [13] introduced semi α-irresolute functions, Donchev [4] introduced contra continuous functions, Donchev and Noiri [5] introduced contra semi continuous functions, Jafari and Noiri [12] introduced Contra-α-continuous functions, Ekici and Caldas [3] introduced clopen-T_1, Staum [10] introduced, ultra hausdorff, ultra normal, clopen regular and clopen normal, Ellis [9] introduced ultra regular, Maheshwari [13] introduced s-normal space, Arhangel [2] introduced α-normal space.
For a subset A of a topological space (X, τ), the closure of A, the interior of A with respect to τ are denoted by Cl(A) and Int(A) respectively. The complement of A is denoted by A^c. A subset A of a topological space (X, τ) is said to be clopen set, if A is open and closed. This work consists of two sections. In section one we will introduce and study a new class of open sets which is called h-open set and introduce the notions of h-interior, h-closure, h-limit points, h-derived, h-interior points, h-border, h-frontier and h-exterior by using the concept of h-open sets, and study their topological properties. In section two we will present the notion of h-continuous functions, h-open functions, h-irresolute functions, h-totally continuous functions, h-contra-continuous functions, h-homeomorphism and investigate some properties of these functions and study some properties, remarks related to them.

Definition 1.1. *A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be*

1. *totally-continuous if $f^{-1}(U)$ is clopen set in X for every open set U in Y.*
2. *contra-continuous if $f^{-1}(U)$ is closed set in X for every open set U in Y.*

2 h-open sets

In this section, we introduce a new class of open sets which is called h-open set and introduce the notions of h-interior, h-closure, h-limit points, h-derived, h-interior points, h-border, h-frontier and h-exterior by using the concept of h-open sets, and study their topological properties.

Definition 2.1. *A subset A of the topological space (X, τ) is said to be h-open set if for every non-empty set U in X, $U \neq X$ and $U \in \tau$, such that $A \subseteq Int(A \cup U)$. The complement of the h-open set is called h-closed. We denote the family of all h-open sets of a topological space (X, τ) by τ^h.*

Example 2.1. *Let* $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. *Then* $\tau^h = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$.

Example 2.2. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. *Then* $\tau^h = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Remark 2.1. *From Example.2.1, and Example.2.2. Note that* $\tau \subseteq \tau^h$.

Theorem 2.1. *Every open set in any topological space* (X, τ) *is h-open set.*

Proof. Let (X, τ) be any topological space and let $A \subseteq X$ be any open set. Therefore, $A = Int(A) \subseteq Int(A \cup U)$ for every non-empty set $U \neq X$ and $U \in \tau$. Thus, A is h-open set. □

Remark 2.2. *The converse of the Theorem.2.1, need not be true as shown in the following example.*

Example 2.3. *In Example.2.1,* $\{b\}, \{c\}, \{b, c\}$ *and* $\{b, c, d\}$ *are h-open sets but not open sets.*

Theorem 2.2. *Let* (X, τ) *be a topological space and let A, B be two h-open sets. Then*

1. $A \cap B$ *is h-open set.*
2. $A \cup B$ *is h-open set.*

Proof. 1) Let A and B be two h-open sets. Then from Definition.2.1, $A \subseteq Int(A \cup U)$ and $B \subseteq Int(B \cup U)$ for every non-empty set $U \neq X, U \in \tau$. Then $A \cup B \subseteq Int(A \cup U) \cup Int(B \cup U) \subseteq Int((A \cup U) \cup (B \cup U)) = Int((A \cup B) \cup U)$. Therefore $A \cup B$ is h-open set.
2) Let A and B be two h-open sets. Then from Definition.2.1, $A \subseteq Int(A \cup U)$ and $B \subseteq Int(B \cup U)$ for every non-empty set $U \neq X$, $U \in \tau$. Then $A \cap B \subseteq Int(A \cup U) \cap Int(B \cup U) = Int((A \cup U) \cap (B \cup U)) = Int(((A \cup U) \cap B) \cup ((A \cup U) \cap U)) \subseteq Int((A \cap B) \cup U)$. Therefore $A \cap B$ is h-open set. □

Definition 2.2. *Let* (X, τ) *be a topological space and let* $A \subseteq X$. *The h-interior of A is defined as the union of all h-open sets in X content in A, and is denoted by* $Int_h(A)$. *It is clear that* $Int_h(A)$ *is h-open set for any subset A of X.*

Proposition 2.1. *Let* (X, τ) *be a topological space and let* $A \subseteq B \subseteq X$. *Then*

1. $Int_h(A) \subseteq Int_h(B)$.

2. $Int_h(A) \subseteq A$.

3. A is h-open if and only if $A = Int_h(A)$.

Definition 2.3. *Let (X, τ) be a topological space and let $A \subseteq X$. The h-closure of A is defined as the intersection of all h-closed sets in X containing A, and is denoted by $Cl_h(A)$. It is clear that $Cl_h(A)$ is h-closed set for any subset A of X.*

Proposition 2.2. *Let (X, τ) be a topological space and let $A \subseteq B \subseteq X$. Then*

1. $Cl_h(A) \subseteq Cl_h(B)$.

2. $A \subseteq Cl_h(A)$.

3. A is h-closed if and only if $A = Cl_h(A)$.

Definition 2.4. *Let (X, τ) be a topological space and let $A \subseteq X$. A point $x \in X$ is said to be h-limit point of A if it satisfies the following assertion:*

$$(\forall G \in \tau^h)(x \in G \Rightarrow G \cap (A \backslash \{x\}) \neq \emptyset).$$

The set of all h-limit points of A is called the h-derived set of A and is denoted by $D_h(A)$.
Note that for a subset A of X, a point $x \in X$ is not a h-limit point of A if and only if there exists a h-open set G in X such that $x \in G$ and $G \cap (A \backslash \{x\}) = \emptyset$ or, equivalently, $x \in G$ and $G \cap A = \emptyset$ or $G \cap A = \{x\}$ or, equivalently, $x \in G$ and $G \cap A \subseteq \{x\}$.

Theorem 2.3. *Let (X, τ) be a topological space and let A be a subset of X. Then the following are equivalent*

1. $(\forall G \in \tau^h)(x \in G \Rightarrow A \cap G \neq \emptyset)$.

2. $x \in Cl_h(A)$.

Proof. (1) $\Rightarrow$(2) If $x \notin Cl_h(A)$, then there exists a h-closed set F such that $A \subseteq F$and $x \notin F$. Hence $G = X - F$ is a h-open set such that $x \in G$ and $G \cap A = \emptyset$. This is a contradiction, and hence (2) is valid.
(2) $\Rightarrow$(1) Straightforward. □

Theorem 2.4. *Let (X, τ) be a topological space and let $A \subseteq B \subseteq X$. Then*

1. $Cl_h(A) = A \cup D_h(A)$.
2. A is h-closed if and only if $D_h(A) \subseteq A$.
3. $D_h(A) \subseteq D_h(B)$.
4. $D_h(A) \subseteq D(A)$.
5. $Cl_h(A) \subseteq Cl(A)$.

Proof. 1) Let $x \notin Cl_h(A)$. Then there exists a h-closed set F in X such that $A \subseteq F$ and $x \notin F$. Hence $G = X - F$ is a h-open set such that $x \in G$ and $G \cap A = \emptyset$. Therefore $x \notin A$ and $x \notin D_h(A)$, then $x \notin A \cup D_h(A)$. Thus $A \cup D_h(A) \subseteq Cl_h(A)$. On the other hand, $x \notin A \cup D_h(A)$ implies that there exists a h-open set G in X such that $x \in G$ and $G \cap A = \emptyset$. Hence $F = X - G$ is a h-closed set in X such that $A \subseteq F$ and $x \notin F$. Hence $x \notin Cl_h(A)$. Thus $Cl_h(A) \subseteq A \cup D_h(A)$. Therefore $Cl_h(A) = A \cup D_h(A)$. For (2), (3), (4) and (5) the proof is easy. □

Example 2.4. *Let $X = \{a, b, c\}$ with topology, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then we have the followings*

1. $\tau \subseteq \tau^h = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$.
2. $If A = \{a, c\}$*, then* $D(A) = \{c\}$ *and* $D_h(A) = \emptyset$.
3. $If B = \{a, b\}$*, then* $D(B) = \{b, c\}$ *and* $D_h(B) = \{c\}$.

Theorem 2.5. *Let τ_1 and τ_2 be topologies on X such that $\tau_1^h \subseteq \tau_2^h$. For any subset A of X, every h-limit point of A with respect to τ_2 is a h-limit point of A with respect to τ_1.*

Proof. Let x be a h-limit point of A with respect to τ_2. Then $G \cap (A \backslash \{x\}) \neq \emptyset$ for every $G \in \tau_2^h$ such that $x \in G$. But $\tau_1^h \subseteq \tau_2^h$ so, in particular, $G \cap (A \backslash \{x\}) \neq \emptyset$ for every $G \in \tau_1^h$ such that $x \in G$. Hence x is a h-limit point of A with respect to τ_1. □

Remark 2.3. *The converse of the Theorem.2.5, need not be true as shown in the following example.*

Example 2.5. *$X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}, \{a, b\}$. Then $\tau_1^h = \{\emptyset, X, \{a\}, \{b, c\}$ and $\tau_2^h = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}$. Not that $\tau_1^h \subseteq \tau_2^h$ and b is a h-limit point of $A = \{a, b\}$ with respect to τ_1, but it is not a h-limit point of A with respect to τ_2.*

Theorem 2.6. *If τ is the indiscrete (resp. discrete) topology on a set X, then τ^h is indiscrete (resp. discrete) topology on X.*

Proof. Straightforward. $\square$

Theorem 2.7. *If A is a subset of a discrete topological space (X, τ), then $D_h(A) = \emptyset$.*

Proof. Let $x \in X$. Recall that every subset of X is open, and so h-open. In particular, the singleton set $G = \{x\}$ is h-open. But $x \in G$ and $G \cap A = \{x\} \cap A \subseteq \{x\}$. Hence x is not a h-limit point of A, and so $D_h(A) = \emptyset$. $\square$

Theorem 2.8. *Let (X, τ) be a topological space and let A, B subsets of X. If A is h-closed, then $Cl_h(A \cap B) \subseteq A \cap Cl_h(B)$.*

Proof. If A is h-closed, then $Cl_h(A) = A$ and so $Cl_h(A \cap B) \subseteq Cl_h(A) \cap Cl_h(B) \subseteq A \cap Cl_h(B)$. $\square$

Lemma 2.1. *Let (X, τ) be a topological space and let A subset of X. Then A is h-open if and only if there exists an open set U in X such that $A \subseteq U \subseteq Cl(A)$.*

Proof. Straightforward. $\square$

Lemma 2.2. *The intersection of an open set and a h-open set is a h-open set.*

Proof. Let A be an open set in X and B a h-open set in X. Then there exists an open set U in X such that $B \subseteq U \subseteq Cl(B)$. It follows that $A \cap B \subseteq A \cap U \subseteq A \cap Cl(B) \subseteq Cl(A \cap B)$. Now since $A \cap U$ is open, it follows from Lemma.2.1, that $A \cap B$ is h-open. $\square$

Definition 2.5. *Let (X, τ) be a topological space and let $A \subseteq X$. Then $b_h(A) = A \backslash Int_h(A)$ is called the h-border of A, and the set $Fr_h(A) = Cl_h(A) \backslash Int_h(A)$ is called the h-frontier of A.*
Note that if A is a h-closed subset of X, then $b_h(A) = Fr_h(A)$.

Example 2.6. *Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{b\}, \{b, c\}\}$, $\tau^h = \{\emptyset, X, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. If $A = \{a, b\}$, then $Int_h(A) = \{b\}$, $b_h(A) = \{a\}$ and so $Cl_h(A) = \{a, b\}$, $Fr_h(A) = \{a\}$. If we take $A = \{b, c\}$, then $Int_h(A) = \{b, c\}$, $b_h(A) = \emptyset$ and so $Cl_h(A) = X$, $Fr_h(A) = \{a\}$.*

Theorem 2.9. *Let (X, τ) be a topological space and let $A \subseteq X$. Then*

1. $A = Int_h(A) \cup b_h(A)$.

2. $Int_h(A) \cap b_h(A) = \emptyset$.

3. A *is a h-open set if and only if* $b_h(A) = \emptyset$.

4. $b_h(Int_h(A)) = \emptyset$.

5. $Int_h(b_h(A)) = \emptyset$.

6. $b_h(b_h(A)) = b_h(A)$.

7. $b_h(A) = A \cap Cl_h(X \backslash A)$.

8. $b_h(A) = A \cap D_h(X \backslash A)$.

Proof. (1) and (2). Straightforward.
(3) Since $Int_h(A) \subseteq A$, it follows from Proposition.2.1(3) that A is h-open $\Leftrightarrow A = Int_h(A) \Leftrightarrow b_h(A) = A \backslash Int_h(A) = \emptyset$.
(4) Since $Int_h(A)$ is h-open, it follows from (3) that $b_h(Int_h(A)) = \emptyset$.
(5) If $x \in Int_h(b_h(A))$, then $x \in b_h(A) \subseteq A$ and $x \in Int_h(A)$. Since $Int_h(b_h(A)) \subseteq Int_h(A)$. Thus $x \in b_h(A) \cap Int_h(A) = \emptyset$, which is a contradiction. Hence $Int_h(b_h(A)) = \emptyset$.
(6) Using (5), we get $b_h(b_h(A)) = b_h(A) \backslash Int_h(b_h(A)) = b_h(A)$.
(7) $b_h(A) = A \backslash Int_h(A) = A \backslash (X \backslash Cl_h(X \backslash A)) = A \cap Cl_h(X \backslash A)$.
(8) Applying (7) and Theorem.2.4 (1), we have $b_h(A) = A \cap Cl_h(X \backslash A) = A \cap ((X \backslash A) \cup D_h(X \backslash A)) = A \cap D_h(X \backslash A)$. □

Lemma 2.3. *Let (X, τ) be a topological space and let $A \subseteq X$. Then A a h-closed if and only if $Fr_h(A) \subseteq A$.*

Proof. Assume that A is h-closed. Then $Fr_h(A) = Cl_h(A) \backslash Int_h(A) = A \backslash Int_h(A) \subseteq A$. Conversely suppose that $Fr_h(A) \subseteq A$ Then $Cl_h(A) \backslash Int_h(A) \subseteq A$ and so $Cl_h(A) \subseteq A$ Since $Int_h(A) \subseteq A$. Noticing that $A \subseteq Cl_h(A)$, we have $A = Cl_h(A)$. □

Definition 2.6. *Let (X, τ) be a topological space and let $A \subseteq X$. Then $Ext_h(A) = Int_h(X \backslash A)$ is called the h-exterior of A.*

Example 2.7. *Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\tau^h = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. If $A = \{a, c\}$, then we have $Ext_h(A) = \{b\}$.*

Theorem 2.10. *Let (X, τ) be a topological space and let $A \subseteq B \subseteq X$. Then*

1. $Ext_h(A)$ *is h-open.*

2. $Ext_h(A) = X \backslash Cl_h(A)$.

3. *If* $A \subseteq B$, *then* $Ext_h(B) \subseteq Ext_h(A)$.

4. $Ext_h(A \cup B) \subseteq Ext_h(A) \cap Ext_h(B)$.

5. $Ext_h(A \cap B) \supseteq Ext_h(A) \cup Ext_h(B)$.

6. $Ext_h(X) = \emptyset, Ext_h(\emptyset) = X$.

7. $Ext_h(A) = Ext_h(X \backslash Ext_h(A))$.

8. $X = Int_h(A) \cup Ext_h(A) \cup Fr_h(A)$.

Proof. (1) and (2) straightforward.
(3) Assume that $A \subseteq B$. Then $Ext_h(B) = Int_h(X \backslash B) \subseteq Int_h(X \backslash A) = Ext_h(A)$.
(4) $Ext_h(A \cup B) = Int_h(X \backslash (A \cup B)) = Int_h((X \backslash A) \cap (X \backslash B)) \subseteq Int_h(X \backslash A) \cap Int_h(X \backslash B) = Ext_h(A) \cap Ext_h(B)$.
(5) $Ext_h(A \cap B) = Int_h(X \backslash (A \cap B)) = Int_h((X \backslash A) \cup (X \backslash B)) \supseteq Int_h(X \backslash A) \cup Int_h(X \backslash B) = Ext_h(A) \cup Ext_h(B)$.
(6) Straightforward.
(7) $Ext_h(X \backslash Ext_h(A)) = Ext_h(X \backslash Int_h(X \backslash A)) = Int_h(X \backslash A) = Ext_h(A)$.
(8)Straightforward. □

3 h-continuous functions and h-Homeomorphism

In this section, we introduce new classes of functions called h-continuous functions, h-open functions, h-irresolute functions, h-totally continuous functions, h-contra-continuous functions, h-homeomorphism and study some properties of these functions.

Definition 3.1. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is said to be h-continuous, if* $f^{-1}(U)$ *is h-open set in X for every open set U in Y.*

Example 3.1. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$, $\tau^h = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ *and* $\sigma = \{\emptyset, Y, \{a, c\}\}$. *Clearly, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is h- continuous.*

Theorem 3.1. *Every continuous function is h-continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be continuous function and U be any open subset in Y. Since, f is continuous, then $f^{-1}(U)$ is open set in X. Since, every open set is h-open set by Theorem.2.1, then $f^{-1}(U)$ is h-open set in X. Therefore, f is h-continuous. □

Remark 3.1. *The converse of the Theorem.3.1, need not be true as shown in the following example.*

Example 3.2. *Let* $X = \{a, b, c\}$ *and* $Y = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{b\}\}$, $\tau^h = \{\emptyset, X, \{b\}, \{a, c\}\}$, $\sigma = \{\emptyset, Y, \{1\}, \{2, 3\}\}$. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is defined by* $f(\{a\}) = \{2\}, f(\{b\}) = \{1\}, f(\{c\}) = \{3\}$. *Clearly, f is a h-continuous, but f is not continuous.*

Theorem 3.2. *If* $f : (X, \tau) \to (Y, \sigma)$ *is h-continuous and* $g : (Y, \sigma) \to (Z, \eta)$ *is continuous, then* $g \circ f : (X, \tau) \to (Z, \eta)$ *is h-continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be h-continuous and $g : (Y, \sigma) \to (Z, \eta)$ be continuous . Let U be an open set in Z. Since, g is continuous, then $g^{-1}(U)$ is an open set in Y. Since, f is h-continuous, then $f^{-1}((g^{-1}(U)) = (g \circ f)^{-1}(U)$ is h-open set in X. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is h-continuous. □

Definition 3.2. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is said to be h-open, if* $f(U)$ *is h-open set in* Y *for every open set* U *in* X.

Example 3.3. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{a\}\}$ *and* $\sigma^h = \{\emptyset, Y, \{a\}, \{b, c\}\}$. *Clearly, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is h-open.*

Theorem 3.3. *Every open function is h-open.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be open function and U be any open set in X. Since, f is open, then $f(U)$ is open set in Y. Since, every open set is h-open set by Theorem.2.1, then $f(U)$ is h-open set in Y. Therefore, f is h-open. □

Remark 3.2. *The converse of the Theorem.3.3, need not be true as shown in the following example.*

Example 3.4. *In Example.3.3, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is h-open but not open.*

Theorem 3.4. *If* $f : (X, \tau) \to (Y, \sigma)$ *is open and* $g : (Y, \sigma) \to (Z, \eta)$ *is h-open, then* $g \circ f : (X, \tau) \to (Z, \eta)$ *is h-open.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be open and $g : (Y, \sigma) \to (Z, \eta)$ is a h-open. Let U be an open set in X. Since, f is an open, then $f(U)$ is an open set in Y. Since, every open set is h-open set by Theorem.2.1, then $f(U)$ is h-open set in Y. Since, g is a h-open, then $(g \circ f)(U) = g(f(U))$ is a h-open set in Z. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is h-open. □

Definition 3.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be h-irresolute, if $f^{-1}(U)$ is h-open set in X for every h-open set U in Y.*

Example 3.5. *Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{b\}, \{b, c\}\}$, $\tau^h = \{\emptyset, X, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{b\}\}$ and $\sigma^h = \{\emptyset, Y, \{b\}, \{a, c\}\}$. Clearly, the identity function $f : (X, \tau) \to (Y, \sigma)$ is h-irresolute.*

Theorem 3.5. *Every continuous function is h-irresolute.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous function and U be any h-open set in Y. Since, f is a continuous, then Then $f^{-1}(U)$ is open set in X. Hence, h-open set in X by Theorem.2.1. Therefore, f is h-irresolute. □

Remark 3.3. *The converse of the Theorem.3.5, need not be true as shown in the following example.*

Example 3.6. *Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, c\}\}$, $\tau^h = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma^h = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}$. Clearly, the identity function $f : (X, \tau) \to (Y, \sigma)$ is h-irresolute, but f is not continuous function.*

Theorem 3.6. *Every h-irresolute function is h-continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be h-irresolute function and U be any open set in Y. Since, every open set is h-open set by Theorem.2.1. Since, f is h-irresolute, then $f^{-1}(U)$ is h-open set in X. Therefore f is h-continuous. □

Remark 3.4. *The converse of the Theorem.3.6, need not be true as shown in the following example.*

Example 3.7. *Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\tau^h = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{b, c\}\}$ and $\sigma^h = \{\emptyset, Y, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Clearly, the identity function $f : (X, \tau) \to (Y, \sigma)$ is h-continuous, but f is not h-irresolute.*

Theorem 3.7. *The composition of two h-irresolute function is also h-irresolute.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two h-irresolute. Let U be any h-open in Z. Since, g is h-irresolute, then $g^{-1}(U)$ is h-open set in Y. Since, f is h-irresolute, then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is h-open in X. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is h-irresolute. □

Theorem 3.8. *If $f : (X, \tau) \to (Y, \sigma)$ is h-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is h-continuous, then $gof : (X, \tau) \to (Z, \eta)$ is h-irresolute.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ is h-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is h-continuous. Let U be any open in Z. Then U is h-open set by Theorem.2.1. Since, g is h-continuous, then $g^{-1}(U)$ is h-open set in Y. Since, f is h-irresolute, then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is h-open in X. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is h-irresolute. $\square$

Definition 3.4. *A bijective function $f : (X, \tau) \to (Y, \sigma)$ is said to be h-homeomorphism if f is h-continuous and h-open function.*

Theorem 3.9. *If $f : (X, \tau) \to (Y, \sigma)$ is homomorphism, then f is h-homomorphism.*

Proof. Since, every continuous function is h-continuous by Theorem.3.1. Also, since every open function is h-open by Theorem.3.3. Further, since f is bijective. Therefore, f is h-homomorphism. $\square$

Remark 3.5. *The converse of the Theorem.3.9, need not be true as shown in the following example.*

Example 3.8. *Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a, c\}\}, \tau^h = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}, \sigma = \{\emptyset, Y, \{b, c\}\}$ and $\sigma^h = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Clearly, the identity function $f : (X, \tau) \to (Y, \sigma)$ is h-homomorphism, but it is not homomorphism.*

Definition 3.5. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be h-totally continuous, if $f^{-1}(U)$ is clopen set in X for every h-open set U in Y.*

Example 3.9. *Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b, c\}\}, \sigma = \{\emptyset, Y, \{a\}\}$ and $\sigma^h = \{\emptyset, Y, \{a\}, \{b, c\}\}$. Clearly, the identity function $f : (X, \tau) \to (Y, \sigma)$ is h-totally continuous function.*

Theorem 3.10. *Every h-totally continuous function is totally continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be h-totally continuous and U be any open set in Y. Since, every open set is h-open set by Theorem.2.1, then U is h-open set in Y. Since, f is h-totally continuous function, then $f^{-1}(U)$ is clopen set in X. Therefore, f is totally continuous. $\square$

Remark 3.6. *The converse of the Theorem.3.10, need not be true as shown in the following example.*

Example 3.10. *Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b, c\}\}, \sigma = \{\emptyset, Y, \{b, c\}\}$ and $\sigma^h = \{\emptyset, Y, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Clearly, the identity function $f : (X, \tau) \to (Y, \sigma)$ is totally continuous function but it is not h-totally continuous.*

Theorem 3.11. *Every h-totally continuous function is h-irresolute.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be h-totally continuous and U be h-open set in Y. Since, f is h-totally continuous function, then $f^{-1}(U)$ is clopen set in X, which implies $f^{-1}(U)$ open, it follow $f^{-1}(U)$ is h-open set in X. Therefore, f is h-irresolute. $\square$

Remark 3.7. *The converse of the Theorem.3.11, need not be true as shown in the following example.*

Example 3.11. *In Example.3.5, the identity function $f : (X, \tau) \to (Y, \sigma)$ is h-irresolute but not h-totally continuous.*

Theorem 3.12. *The composition of two h-totally continuous function is also h-totally continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two h-totally continuous. Let U be any h-open in Z. Since, g is h-totally continuous, then $g^{-1}(U)$ is clopen set in Y, which implies $f^{-1}(U)$ open set, it follow $f^{-1}(U)$ is h-open set. Since, f is h-totally continuous, then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is clopen in X. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is h-totally continuous. $\square$

Theorem 3.13. *If $f : (X, \tau) \to (Y, \sigma)$ be h-totally continuous and $g : (Y, \sigma) \to (Z, \eta)$ be h-irresolute, then $g \circ f : (X, \tau) \to (Z, \eta)$ is h-totally continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be h-totally continuous and $g : (Y, \sigma) \to (Z, \eta)$ be h-irresolute. Let U be h-open set in Z. Since, g is h-irresolute, then $g^{-1}(U)$ is h-open set in Y. Since, f is h-totally continuous, then $f^{-1}((g^{-1}(U)) = (g \circ f)^{-1}(U)$ is clopen set in X. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is h-totally continuous. $\square$

Theorem 3.14. *If $f : (X, \tau) \to (Y, \sigma)$ is h-totally continuous and $g : (Y, \sigma) \to (Z, \eta)$ is h-continuous, then $g \circ f : (X, \tau) \to (Z, \eta)$ is totally continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be h-totally continuous and $g : (Y, \sigma) \to (Z, \eta)$ is h-continuous. Let U be open set in Z. Since, g is h-continuous, then $g^{-1}(U)$ is h-open set in Y. Since, f is h-totally continuous, then $f^{-1}((g^{-1}(U)) = (g \circ f)^{-1}(U)$ is clopen set in X. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is totally continuous. $\square$

Definition 3.6. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be h-contra-continuous if $f^{-1}(U)$ is h-closed set in X for every open set U in Y.*

Example 3.12. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\sigma = \{\emptyset, Y, \{a\}\}$ *and* $\tau^h = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. *Clearly, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is a h-contra-continuous.*

Theorem 3.15. *Every contra-continuous function is h-contra-continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be contra-continuous function and U any open set in Y. Since, f is contra-continuous, then $f^{-1}(U)$ is closed sets in X. Since, every closed set is h-closed set, then $f^{-1}(U)$ is h-closed set in X. Therefore, f is h-contra-continuous. □

Remark 3.8. *The converse of the Theorem.3.15, need not be true as shown in the following example.*

Example 3.13. *In Example.3.12, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is h-contra-continuous but not contra-continuous.*

Theorem 3.16. *Every totally continuous function is h-contra-continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be totally continuous and U be any open set in Y. Since, f is totally continuous function, then $f^{-1}(U)$ is clopen set in X, and hence closed, it follows h-closed set. Therefore, f is h-contra-continuous. □

Remark 3.9. *The converse of the Theorem.3.16, need not be true as shown in the following example.*

Example 3.14. *In Example.3.12, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is h-contra-continuous but not totally continuous.*

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