Filter exhaustiveness and filter limit theorems for k-triangular lattice group-valued set functions

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Abstract

We give some limit theorems for sequences of lattice group-valued k-triangular set functions, in the setting of filter convergence, and some results about their equivalence. We use the tool of filter exhaustiveness to get uniform (s)-boundedness, uniform continuity and uniform regularity of a suitable subsequence of the given sequence, whose indexes belong to the involved filter. Furthermore we pose some open problems.

1 Introduction

Recently there have been many studies about limit theorems for lattice group-valued set functions. A comprehensive treatment, together with a historical survey, can be found in [4, 5] (see also the references therein). In [18] it is dealt with limit theorems for k-triangular real-valued set functions, which have been extended to the lattice-group setting in [6, 7]. Some examples of k-triangular functions are the Saeki measuroids (see also [19]), which are not necessarily monotone, the aggregation functions (see also [16, 17]) and the M-measures, which are monotone continuous set functions, compatible with respect to finite suprema and infima (see also [1]).

Here we prove some Brooks-Jewett, Vitali-Hahn-Saks, Nikodým and Dieudonné-type theorems in the context of filter convergence and their equivalence, extending to k-triangular set functions earlier results proved in [3, 5, 7, 9, 10, 11, 13, 14, 16, 17] in the setting of finitely and countably additive measures. We use the tool of filter exhaustiveness, showing that it is essential to prove our main results. We consider a P-filter $\mathcal{F}$ of $\mathbb{N}$ and a sequence of lattice group-valued measures m_n, $n \in \mathbb{N}$, defined on a σ-algebra Σ, separable with respect to a suitable Fréchet-Nikodým topology, and filter convergent pointwise on a countable dense subset of Σ with respect to a single order sequence. Using filter exhaustiveness, we obtain the existence of the order filter limit set function on Σ and uniform

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(s)-boundedness, uniform τ-continuity, uniform continuity from above at $\emptyset$ and uniform regularity of a subsequence of the type m_n, $n \in M_0$, where M_0 is a suitable element of the involved filter. Finally we pose some open problems.

2 Preliminaries

Let R be a Dedekind complete lattice group. We begin with recalling the following basic concepts (see also [4, 5]).

Definitions 2.1. (a) A sequence $(\sigma_p)_p$ of positive elements of R is said to be an (O)-*sequence* iff it is decreasing and $\bigwedge_p \sigma_p = 0$.

(b) A bounded double sequence $(a_{t,l})_{t,l}$ in R is a (D)-*sequence* or a *regulator* iff $(a_{t,l})_l$ is an (O)-sequence for any $t \in \mathbb{N}$.

(c) A Dedekind complete lattice group R is said to be *super Dedekind complete* iff for every nonempty set $A \subset R$, bounded from above, there is a finite or countable subset with the same supremum as A.

(d) A lattice group R is *weakly σ-distributive* iff $\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big) = 0$ for every (D)-sequence $(a_{t,l})_{t,l}$ in R.

(e) A sequence $(x_n)_n$ in R is said to be *order convergent* (or (O)-*convergent*) to an element $x \in R$ iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that for every $p \in \mathbb{N}$ there is a positive integer n_0 with $|x_n - x| \le \sigma_p$ for each $n \ge n_0$, and in this case we write $(O) \lim_n x_n = x$.

(f) We say that a sequence $(x_n)_n$ in R is (O)-*Cauchy* iff there exists an (O)-sequence $(\tau_p)_p$ such that for every $p \in \mathbb{N}$ there is $n^* \in \mathbb{N}$ with $|x_n - x_q| \le \tau_p$ for every n, $q \ge n^*$.

(g) We call *sum* of a series $\sum_{n=1}^{\infty} x_n$ in R the limit $(O) \lim_n \sum_{l=1}^{n} x_l$, if it exists in R.

Some examples of super Dedekind complete and weakly σ-distributive lattice groups are the space $\mathbb{N}^{\mathbb{N}}$ of all permutations of $\mathbb{N}$ endowed with the usual componentwise order and the space $L^0(X, \mathcal{B}, \nu)$ of all ν-measurable functions defined on a measure space $(X, \mathcal{B}, \nu)$ with the identification up to ν-null sets, where ν is a positive, σ-additive and σ-finite extended real-valued measure, endowed with almost everywhere convergence (see also [4]).

We now recall the following basic properties of filters and filter order convergence in the lattice group context (see also [2, 4, 8]).

Definitions 2.2. (a) A *filter* $\mathcal{F}$ of $\mathbb{N}$ is a nonempty collection of subsets of $\mathbb{N}$ with $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever A, $B \in \mathcal{F}$, and such that for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$.

(b) A filter of $\mathbb{N}$ is said to be *free* iff it contains the filter $\mathcal{F}_{\text{cofin}}$ of all cofinite subsets of $\mathbb{N}$.

(c) A free filter $\mathcal{F}$ of $\mathbb{N}$ is a *P-filter* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ there is a sequence $(B_n)_n$ in $\mathcal{F}$, such that the symmetric difference $A_n \triangle B_n$ is finite for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} B_n \in \mathcal{F}$.

(d) Let R be a Dedekind complete lattice group and $\mathcal{F}$ be any free filter of $\mathbb{N}$. A sequence $(x_n)_n$ in R *$(O\mathcal{F})$-converges to* $x \in R$ iff there exists an (O)-sequence $(\sigma_p)_p$ such that $\{n \in \mathbb{N} : |x_n - x| \leq \sigma_p\} \in \mathcal{F}$ for any $p \in \mathbb{N}$.

(e) We say that the sequence $(x_n)_n$ in R is *$(O\mathcal{F})$-Cauchy* iff there exists an (O)-sequence $(\tau_p)_p$ such that for all $p \in \mathbb{N}$ there is a set $V_p \in \mathcal{F}$ with $|x_n - x_q| \leq \tau_p$ whenever $n, q \in A_p$.

Remarks 2.3. (a) Observe that, when $R = \mathbb{R}$, the $(O\mathcal{F})$-convergence coincide with the usual filter convergence. Moreover, when $\mathcal{F} = \mathcal{F}_{\text{cofin}}$, $(O\mathcal{F})$-convergence is equivalent to (O)-convergence.

(b) Note that, in any Dedekind complete lattice group, a sequence is $(O\mathcal{F})$-convergent if and only if it is $(O\mathcal{F})$-Cauchy.

(c) Observe that both the filter $\mathcal{F}_{\text{cofin}}$ and the filter of all subsets of $\mathbb{N}$ having asymptotic density one are P-filters (see also [4]).

We now recall some notions and properties of submeasures and Fréchet-Nikodým topologies. Let G be any infinite set and Σ be any σ-algebra of subsets of G.

Definitions 2.4. (a) A *submeasure* $\eta : \Sigma \to [0, +\infty]$ is a set function with $\eta(\emptyset) = 0$, $\eta(A) \leq \eta(B)$ whenever $A, B \in \Sigma$, $A \subset B$, and $\eta(A \cup B) \leq \eta(A) + \eta(B)$ whenever $A, B \in \Sigma$ and $A \cap B = \emptyset$. Note that, if η is a submeasure, then $\eta\Big(\bigcup_{i=1}^{n} A_i\Big) \leq \sum_{i=1}^{n} \eta(A_i)$ for all $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \Sigma$ (see also [13, §2]). It is well-known that, given a submeasure η on Σ, the function $d(A, B) = \eta(A \Delta B)$, $A, B \in \Sigma$, is a pseudometric (see also [21, §1]).

(b) A topology τ on Σ is said to be a *Fréchet-Nikodým topology* iff the functions $(A, B) \mapsto A \Delta B$ (symmetric difference) and $(A, B) \mapsto A \cap B$ from $\Sigma \times \Sigma$ (endowed with the product topology) to Σ are continuous, and for each τ-neighborhood W of $\emptyset$ in Σ there is a τ-neighborhood U of $\emptyset$ in Σ such that, if $B \in \Sigma$ is contained in some suitable element of U, then $B \in W$ (see also [13, 21]).

Remark 2.5. Observe that a topology τ on Σ is a Fréchet-Nikodým topology if and only if there exists a family of submeasures $\mathcal{Z} := \{\eta_i : i \in \Lambda\}$, with the property that a base of τ-neighborhoods of $\emptyset$ in Σ is given by the sets of the type $U_{\varepsilon,J} := \{A \in \Sigma : \eta_i(A) < \varepsilon \text{ for all } i \in J\}$, where $\varepsilon \in \mathbb{R}^+$ and J varies in the class of all finite subsets of Λ (see also [13, Proposition 2.6 and Theorem 2.7]).

We now deal with some fundamental properties of lattice group-valued set functions (see also [5, 15, 18, 20]). Let $m : \Sigma \to R$ be a positive bounded set function and k be a fixed positive integer.

Definitions 2.6. (a) We say that m is *k-subadditive* on Σ iff $m(\emptyset) = 0$ and

$$m(A \cup B) \leq m(A) + k\, m(B) \quad \text{whenever } A, B \in \Sigma,\ A \cap B = \emptyset;$$

k-triangular on Σ, iff m is k-subadditive and

$$m(A) - k\, m(B) \leq m(A \cup B) \quad \text{whenever } A, B \in \Sigma,\ A \cap B = \emptyset.$$

(b) We call *semivariation* of m, shortly $v(m)$, the set function defined by

$$v(m)(A) := \bigvee \{m(B) : B \in \Sigma,\ B \subset A\}, \quad A \in \Sigma.$$

(c) Let $\mathcal{E} \subset \Sigma$ be a lattice. A set function $m : \Sigma \to R$ is said to be *$\mathcal{E}$-(s)-bounded* iff there exists an (O)-sequence $(\sigma_p)_p$ such that, for every disjoint sequence $(C_l)_l$ in $\mathcal{E}$, $(O)\lim_l v(m)(C_l) = 0$ with respect to $(\sigma_p)_p$. We say that m is *(s)-bounded* iff it is Σ-(s)-bounded.

(d) We say that the set functions $m_n : \Sigma \to R$, $n \in \mathbb{N}$, are *$\mathcal{E}$-uniformly (s)-bounded* on Σ iff there exists an (O)-sequence $(\sigma_p)_p$ such that, for every disjoint sequence $(C_l)_l$ in $\mathcal{E}$,

$$(O)\lim_l \Big(\bigvee_n v(m_n)(C_l) \Big) = 0$$

with respect to $(\sigma_p)_p$. The m_n's are said to be *uniformly (s)-bounded* iff they are Σ-uniformly (s)-bounded.

(e) We say that a set function $m : \Sigma \to R$ is *continuous from above at $\emptyset$* iff there is an (O)-sequence $(\sigma_p)_p$ with $(O)\lim_l v(m)(H_l) = 0$ with respect to $(\sigma_p)_p$, whenever $(H_l)_l$ is a decreasing sequence in Σ with $\bigcap_{l=1}^{\infty} H_l = \emptyset$.

(f) The set functions $m_n : \Sigma \to R$, $n \in \mathbb{N}$, are said to be *uniformly continuous from above at $\emptyset$* iff there is an (O)-sequence $(\sigma_p)_p$ with

$$(O)\lim_l \Big(\bigvee_n v(m_n)(H_l) \Big) = 0$$

with respect to $(\sigma_p)_p$, for each decreasing sequence $(H_l)_l$ in Σ with $\bigcap_{l=1}^{\infty} H_l = \emptyset$.

(g) We say that the set functions $m_n : \Sigma \to R$, $n \in \mathbb{N}$, are *equibounded* on Σ iff there is an element $u \in R$ with $|m_n(A)| \le u$ for each $n \in \mathbb{N}$ and $A \subset \Sigma$.

Remark 2.7. Observe that continuity from above at $\emptyset$ of a k-triangular set function with respect to an (O)-sequence $(\sigma_p)_p$ implies its (s)-boundedness with respect to the (O)-sequence $((k+1)\sigma_p)_p$. Analogously it is possible to check that uniform continuity from above at $\emptyset$ implies uniform (s)-boundedness (see also [6, Remark 2.12]).

Definitions 2.8. (a) Let τ be a Fréchet-Nikodým topology. A set function $m : \Sigma \to R$ is said to be *τ-continuous* on Σ iff it is (s)-bounded on Σ and for each decreasing sequence $(H_l)_l$ in Σ, with τ-$\lim_l H_l = \emptyset$, we get $(O)\lim_l m(H_l) = 0$ with respect to a single (O)-sequence.

(b) The set functions $m_n : \Sigma \to R$, $n \in \mathbb{N}$, are *uniformly τ-continuous* on Σ iff they are uniformly (s)-bounded and for every decreasing sequence $(H_l)_l$ in Σ with τ-$\lim_l H_l = \emptyset$, we get $(O)\lim_l \Big(\bigvee_n m_n(H_l) \Big) = 0$ with respect to a single (O)-sequence.

(c) Let $\mathcal{G}$, $\mathcal{H}$ be two sublattices of Σ, such that $\mathcal{G}$ is closed under countable unions, and the complement of every element of $\mathcal{H}$ belongs to $\mathcal{G}$. A set function $m : \Sigma \to R$ is said to be *regular* iff there exists an (O)-sequence $(\sigma_p)_p$ such that for every $E \in \Sigma$ there are two sequences $(V_l)_l$ in $\mathcal{G}$ and $(K_l)_l$ in $\mathcal{H}$ with $V_l \supset E \supset K_l$ for every $l \in \mathbb{N}$ and such that for each $p \in \mathbb{N}$ there is $l_0 \in \mathbb{N}$ with $v(m)(V_l \setminus K_l) \leq \sigma_p$ whenever $l \geq l_0$.

(d) The set functions $m_n : \Sigma \to R$, $n \in \mathbb{N}$, are *uniformly regular* iff there is an (O)-sequence $(\sigma_p)_p$ such that for any $E \in \Sigma$ and $n \in \mathbb{N}$ there exist two sequences $(F_l)_l$ in $\mathcal{H}$ and $(G_l)_l$ in $\mathcal{G}$, with $V_l \supset E \supset K_l$ for each $l \in \mathbb{N}$ and such that for each $p \in \mathbb{N}$ there is $l_0 \in \mathbb{N}$ with $\bigvee_n v(m_n)(V_l \setminus K_l) \leq \sigma_p$ whenever $l \geq l_0$.

We now give the concept of (uniform) filter exhaustiveness, which plays a fundamental role in our results.

Definition 2.9. *Let τ be a Fréchet-Nikodým topology on Σ and $\mathcal{F}$ be a free filter of $\mathbb{N}$. A sequence of set functions $m_n : \Sigma \to R$, $n \in \mathbb{N}$ is said to be $\mathcal{F}$-exhaustive at $\emptyset$ (resp. uniformly $\mathcal{F}$-exhaustive)with respect to τ iff there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there are a τ-neighborhood U of $\emptyset$ in Σ and a set $V \in \mathcal{F}$ with $|m_n(F)| \leq \sigma_p$ whenever $n \in V$ and for any $F \in \Sigma$ with $F \in U$ (resp. $|m_n(E) - m_n(F)| \leq \sigma_p$ whenever E, $F \in \Sigma$ with $E \Delta F \in U$ and for each $n \in V$).*

3 The main results

From now on, let R be a super Dedekind complete and weakly σ-distributive lattice group. We begin with recalling the following results.

Proposition 3.1. (see also [4, Proposition II.2.19]) *Let $(x_{n,j})_{n,j}$ be a double sequence in R, $\mathcal{F}$ be a P-filter of $\mathbb{N}$, let $j \in \mathbb{N}$ and suppose that*

$$x_j = (O\mathcal{F}) \lim_n x_{n,j}. \tag{1}$$

Then there is a set $B_0 \in \mathcal{F}$ with

$$(O) \lim_{n \in B_0} x_{n,j} = x_j, \tag{2}$$

and in (2) it is possible to take the same (O)-sequence as in (1).

Theorem 3.2. (see also [6, Theorems 3.3, 3.5, 3.6 and 3.10]) *Let $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be a sequence of equibounded k-triangular set functions, such that the limit $m_0(E) := (O) \lim_n m_n(E)$ exists in R for every $E \in \Sigma$ with respect to a single (O)-sequence. Then, the following results are true and equivalent.*

3.2.1) *If the m_n's are (s)-bounded on Σ, then they are uniformly (s)-bounded on Σ, and m_0 is k-triangular and (s)-bounded on Σ.*

3.2.2) *If the m_n's are continuous from above at $\emptyset$, then they are uniformly continuous from above at $\emptyset$, and m_0 is k-triangular and continuous from above at $\emptyset$ on Σ.*

3.2.3) *If τ is a Fréchet-Nikodým topology on Σ, and the m_n's are τ-continuous, then they are uniformly τ-continuous, and m_0 is k-triangular and τ-continuous on Σ.*

Lemma 3.3. (see also [6, Lemma 3.4]) *Let $\mathcal{G}$ and $\mathcal{H}$ be two sublattices of Σ, such that the complement of every element of $\mathcal{H}$ belongs to $\mathcal{G}$, $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be a sequence of k-triangular and $\mathcal{G}$-uniformly (s)-bounded set functions. Fix $W \in \mathcal{H}$ and a decreasing sequence $(H_l)_l$ in $\mathcal{G}$, with $W \subset H_l$ for each $l \in \mathbb{N}$. If*

$$(O)\lim_l \Big(\bigvee_{A\in\mathcal{G}, A\subset H_l\setminus W} m_n(A)\Big) = \bigwedge_l \Big(\bigvee_{A\in\mathcal{G}, A\subset H_l\setminus W} m_n(A)\Big) = 0 \textit{ for every } n \in \mathbb{N}$$

with respect to a single (O)-sequence $(\sigma)_p$, then

$$(O)\lim_l \Big(\bigvee_n \Big(\bigvee_{A\in\mathcal{G}, A\subset H_l\setminus W} m_n(A)\Big)\Big) = \bigwedge_l \Big(\bigvee_n \Big(\bigvee_{A\in\mathcal{G}, A\subset H_l\setminus W} m_n(A)\Big)\Big) = 0$$

with respect to $(\sigma_p)_p$.

We recall the following Dieudonné-type theorem for k-triangular lattice group-valued set functions

Theorem 3.4. (see also [7, Theorem 3.3]) *Let $\mathcal{G}$, $\mathcal{H}$ be as in Lemma 3.3, $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be a sequence of equibounded, regular, k-triangular and (s)-bounded set functions. Then the m_n's are uniformly (s)-bounded and uniformly regular.*

Now we prove the next result.

Theorem 3.5. *Theorems 3.2 and 3.4 are equivalent.*

Proof. Let $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be a sequence of (s)-bounded and regular equibounded k-triangular set functions, with $(O)\lim_n m_n(E) = m(E)$ for each $E \in \Sigma$. By Theorem 3.2, the m_n's are uniformly (s)-bounded. Choose arbitrarily $E \in \Sigma$, let $(\sigma_p)_p$ be an (O)-sequence in R and $(V_l)_l$, $(K_l)_l$ be two sequences in $\mathcal{G}$ and $\mathcal{H}$, respectively, with $V_l \supset E \supset K_l$ for each $l \in \mathbb{N}$, associated with regularity of the m_n's (in [7] it is shown that $(V_l)_l$ and $(K_l)_l$ can be taken independently of n). There exists an (O)-sequence $(\sigma_p)_p$ with $(O)\lim_l v(m_n)(V_l \setminus K_l) = 0$ for every $n \in \mathbb{N}$ with respect to $(\sigma_p)_p$. By Lemma 3.3, we obtain $(O)\lim_l \Big(\bigvee_n v(m_n)(V_l \setminus K_l)\Big) = 0$ with respect to $(\sigma_p)_p$, and thus we get global regularity of the m_n's. So, Theorem 3.2 implies Theorem 3.4.

We now prove the converse implication. Let $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be a sequence of (s)-bounded and equibounded k-triangular set functions, with $(O)\lim_n m_n(E) = m(E)$ for each $E \in \Sigma$. Of course, if we take $\mathcal{G} = \mathcal{H} = \Sigma$, then the m_n's are regular on Σ. Uniform (s)-boundedness of the m_n's follows directly from Theorem 3.4. This concludes the proof. $\square$

The following result about filter exhaustive k-triangular set functions extends [12, Proposition VII.9] to lattice groups and the filter setting.

Proposition 3.6. *Let Σ be endowed with a Fréchet-Nikodým topology τ. If $m_n : \Sigma \to R$, $n \in \mathbb{N}$, is a sequence of k-triangular set functions, then $(m_n)_n$ is $\mathcal{F}$-exhaustive if and only if $(m_n)_n$ is uniformly $\mathcal{F}$-exhaustive on Σ.*

Proof. We prove only the "only if" part, since the "if" part is straightforward. Let $(\sigma_p)_p$ be an (O)-sequence and, in correspondence with $p \in \mathbb{N}$, pick U and V, associated with the $\mathcal{F}$-exhaustiveness of $(m_n)_n$ at $\emptyset$. Choose arbitrarily $n \in V$ and $E, F \in \Sigma$ with $E \triangle F \in U$. Since τ is a Fréchet-Nikodým topology on Σ, we can assume, without loss of generality, that $E \setminus F \in U$ and $F \setminus E \in U$. Taking into account k-triangularity of m_n, we get

$$\begin{aligned} |m_n(E) - m_n(F)| &\leq |m_n(E) - m_n(E \cap F)| + |m_n(F) - m_n(E \cap F)| \leq \\ &\leq k\, m_n(E \setminus F) + k\, m_n(F \setminus E) \leq 2\, k\, \sigma_p. \end{aligned}$$

□

From now on, we give the following

Assumption 3.7. Let Σ be a σ-algebra, separable with respect to a suitable Fréchet-Nikodým topology τ on Σ, and let $\mathcal{B} := \{F_j : j \in \mathbb{N}\}$ be a countable τ-dense subset of Σ.

We prove the following result on k-triangular extensions of filter limit set functions taking values in lattice groups, extending [3, Theorem 3.8].

Theorem 3.8. *Under Assumption 3.7, let $\mathcal{F}$ be any free filter of $\mathbb{N}$, $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be a sequence of k-triangular set functions, uniformly $\mathcal{F}$-exhaustive on Σ (with respect to τ), and suppose that $m(F_j) := (O\mathcal{F})\lim\limits_n m_n(F_j)$, $j \in \mathbb{N}$, exists in R with respect to a single (O)-sequence. Then there is a k-triangular extension $m_0 : \Sigma \to R$ of m, with $(O\mathcal{F})\lim\limits_n m_n(E) = m_0(E)$ for every $E \in \Sigma$ with respect to a single (O)-sequence.*

Proof. Let $(\sigma_p)_p$ be an (O)-sequence associated with the uniform $\mathcal{F}$-exhaustiveness of the sequence $(m_n)_n$, and choose arbitrarily $E \in \Sigma$. For each $p \in \mathbb{N}$ there exist a τ-neighborhood U of $\emptyset$ and a set $V \in \mathcal{F}$, with $|m_n(E) - m_n(F)| \leq \sigma_p$ for every $F \in \Sigma$ with $E \Delta F \in U$ and whenever $n \in V$. By separability of Σ, there is $\overline{j} \in \mathbb{N}$ with $E \Delta F_{\overline{j}} \in U$. By the Cauchy criterion, there is an (O)-sequence $(\zeta_p)_p$ such that for each $j, p \in \mathbb{N}$ there is a set $Z_p^{(j)} \in \mathcal{F}$ with $|m_k(F_j) - m_n(F_j)| \leq \zeta_p$ whenever $k, n \in Z_p^{(j)}$. In particular we get

$$\begin{aligned} &|m_k(E) - m_n(E)| \leq |m_k(E) - m_k(F_{\overline{j}})| + |m_k(F_{\overline{j}}) - m_n(F_{\overline{j}})| + \\ + \quad &|m_n(F_{\overline{j}}) - m_n(E)| \leq 2\sigma_p + \zeta_p \end{aligned}$$

for any $k, n \in V \cap Z_p^{(\overline{j})}$. Again by the Cauchy criterion, there exists a set function $m_0 : \Sigma \to R$, extending m, with $(O\mathcal{F}) \lim\limits_n m_n(E) = m_0(E)$ with respect to a single (O)-sequence. It is not difficult to see that m_0 is k-triangular on Σ. □

Now we give our main limit theorems for lattice group-valued k-triangular set functions with respect to filter convergence, extending [3, Lemma 3.9], [6, Theorems 3.3, 3.5, 3.6 and 3.10] to k-triangular lattice group-valued set functions. Assume that $\mathcal{F}$ is a P-filter.

Theorem 3.9.

3.9.1) *Let $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be a uniformly $\mathcal{F}$-exhaustive sequence of k-triangular set functions and assume that the limit $(O\mathcal{F})\lim\limits_n m_n(F_j) =: m(F_j)$ exists in R for every $j \in \mathbb{N}$ with respect to a single (O)-sequence. Then there are a set $M_0 \in \mathcal{F}$ and a k-triangular extension m_0 of m, defined on Σ, with $(O) \lim\limits_{n \in M_0} m_n(E) = m_0(E)$ for each $E \in \Sigma$ with respect to a single (O)-sequence.*

3.9.2) (Brooks-Jewett (BJ)) *If every m_n, $n \in \mathbb{N}$, is (s)-bounded, then there exists a set $M_0 \in \mathcal{F}$ such that the set functions m_n, $n \in M_0$, are uniformly (s)-bounded on Σ.*

3.9.3) (Vitali-Hahn-Saks (VHS)) *Under Assumption* 3.7, *if each m_n is τ-continuous, then there exists $M_0 \in \mathcal{F}$, such that the set functions m_n, $n \in M_0$, are uniformly τ-continuous on Σ.*

3.9.4) (Nikodým (N)) *If the m_n's, $n \in \mathbb{N}$, are continuous from above at $\emptyset$, then there is $M_0 \in \mathcal{F}$, such that the set functions m_n, $n \in M_0$, are uniformly continuous from above at $\emptyset$.*

3.9.5) (Dieudonné (D)) *If each m_n is (s)-bounded and regular, then there is a set $M_0 \in \mathcal{F}$ such that the set functions m_n, $n \in M_0$, are uniformly (s)-bounded and uniformly regular on Σ.*

Furthermore, the statements 3.9.j), $j = 2, \dots, 5$ *are equivalent.*

Proof. We first prove 3.9.1). By the uniform $\mathcal{F}$-exhaustiveness of $(m_n)_n$, there is an (O)-sequence $(\sigma_p)_p$ such that for every $p \in \mathbb{N}$ there are a τ-neighborhood U_p of $\emptyset$ in Σ and a set $M'_p \in \mathcal{F}$ with $|m_n(E) - m_n(F)| \le \sigma_p$ whenever E, $F \in \Sigma$ with $E \Delta F \in U_p$ and $n \in M'_p$. As $\mathcal{F}$ is a P-filter, in correspondence with the sequence $(M'_p)_p$ there is a sequence $(M_p)_p$ in $\mathcal{F}$ such that $M_p \Delta M'_p$ is finite for any $p \in \mathbb{N}$ and $\bigcap\limits_{p=1}^{\infty} M_p \in \mathcal{F}$. Let $M := \bigcap\limits_{p=1}^{\infty} M_p$ and $Z_p := M \setminus M'_p$ for all $p \in \mathbb{N}$. Note that Z_p is finite for every $p \in \mathbb{N}$, and so we get $|m_n(E) - m_n(F)| \le \sigma_p$ whenever E, $F \in \Sigma$ with $E \Delta F \in U_p$ and $n \in M \setminus Z_p$. Moreover, thanks to Proposition 3.1, we find a set $B_0 \in \mathcal{F}$ such that for every j, $p \in \mathbb{N}$ there is $\overline{n} \in B_0$ with $|m_n(F_j) - m(F_j)| \le \sigma_p$ whenever $n \ge \overline{n}$, $n \in B_0$. Without loss of generality, we can take $\overline{n} \in B_0 \cap M$. Set $M_0 := B_0 \cap M$: we get $M_0 \in \mathcal{F}$. Moreover the sequence m_n, $n \in M_0$, is uniformly $\mathcal{F}_{\text{cofin}}$-exhaustive and $(O) \lim\limits_{n \in M_0} m_n(F_j) = m(F_j)$ with respect to a single (O)-sequence. From this and Theorem 3.8 applied to m_n, $n \in M_0$ and $\mathcal{F}_{\text{cofin}}$ we get the existence of a k-triangular extension m_0 of m, defined on Σ, with $(O) \lim\limits_{n \in M_0} m_n(E) = m_0(E)$ for each $E \in \Sigma$ and with respect to a single (O)-sequence. Thus, M_0 is the requested set, and 3.9.1) is proved.

The assertions 3.9.j), $j = 2, \dots, 5$ and their equivalence follow by observing that there exists a set $M_0 \in \mathcal{F}$, satisfying 3.9.1), and by applying 3.2.1), 3.2.2), 3.2.3) and Theorems 3.4, 3.5 to the sequence m_n, $n \in M_0$, respectively. $\square$

The following example shows that in general the hypothesis of uniform $\mathcal{F}$-exhaustiveness cannot be dropped.

Example 3.10. (see also [3, Example 3.11]) Let $\Sigma = \mathcal{P}(\mathbb{N})$, $R = \mathbb{R}$, $\mathcal{F}$ be the filter of all subsets of $\mathbb{N}$ of asymptotic density one, and define $\lambda : \Sigma \to \mathbb{R}$ by $\lambda(A) = \sum_{n \in A} \frac{1}{2^n}$, $A \in \Sigma$. It is easy to see that Σ is separable with respect to the Fréchet-Nikodým topology τ generated by λ (indeed, the set $\mathcal{I}_{\text{fin}}$ of all finite subsets of $\mathbb{N}$ is countable and τ-dense in Σ).

For any $A \subset \mathbb{N}$ and $n \in \mathbb{N}$ set $\delta_n(A) = 1$ if $n \in A$, and $\delta_n(A) = 0$ if $n \in \mathbb{N} \setminus A$. It is not difficult to check that δ_n is a set function, continuous from above at $\emptyset$ for each $n \in \mathbb{N}$, and that for every $W \in \mathcal{I}_{\text{fin}}$ we have $\lim_n \delta_n(W) = 0$ and hence $(O\mathcal{F}) \lim_n \delta_n(W) = 0$.

However, observe that for every $\delta > 0$ there is a cofinite set $Z_\delta \subset \mathbb{N}$ with $\lambda(Z_\delta) < \delta$, and hence $\lambda(E \triangle F) \leq \lambda(Z_\delta) < \delta$ whenever $E \cup F \subset Z_\delta$. Note that for every infinite subset $M \subset N$, and even a fortiori for each $M \in \mathcal{F}$, it is possible to find an integer $\overline{n} \in M$ large enough and two sets E, $F \in \Sigma$, with $E \cup F \subset Z_\delta$ and $\overline{n} \in E \setminus F$, so that $|\delta_{\overline{n}}(E) - \delta_{\overline{n}}(F)| = \delta_{\overline{n}}(E) = 1$. Thus the set functions δ_n, $n \in \mathbb{N}$, are not uniformly $\mathcal{F}$-exhaustive on Σ. Moreover, it is not true that the limit $(O\mathcal{F})\lim_n \delta_n(A)$ exists in $\mathbb{N}$ for every $A \subset \mathbb{N}$. Indeed, let $C \subset \mathbb{N}$ be with $C \notin \mathcal{F}$ and $\mathbb{N} \setminus C \notin \mathcal{F}$: note that such a set C does exist (see also [4]). We have $\delta_n(C) = 1$ if and only if $n \in C$ and $\delta_n(C) = 0$ if and only if $n \notin C$. Let now $l \neq 0$ and $\varepsilon_0 := \frac{|l|}{2} > 0$. Then for each $n \in \mathbb{N} \setminus C$ we get $|\delta_n(C) - l| = |l| > \varepsilon_0$, and thus $\{n \in \mathbb{N} : |\delta_n(C) - l| \leq \varepsilon_0\} \notin \mathcal{F}$, because it is contained in C and $C \notin \mathcal{F}$. When $l = 0$, take $\varepsilon_0 = \frac{1}{2}$. For every $n \in C$ we have $|\delta_n(C)| = 1 > \varepsilon_0$. So, $\{n \in \mathbb{N} : |\delta_n(C)| \leq \varepsilon_0\} \notin \mathcal{F}$, since it is contained in $\mathbb{N} \setminus C$ and $\mathbb{N} \setminus C \notin \mathcal{F}$. Hence, the limit $(O\mathcal{F})\lim_n \delta_n(C)$ does not exist in $\mathbb{R}$. Furthermore, given any infinite subset $M \subset \mathbb{N}$ and $k \in M$, we get $\sup_{n \in M} \delta_n(\{k\}) = 1$, and so the set functions δ_n, $n \in M$, are not uniformly (s)-bounded on Σ. $\square$

Further developments. It is still an open problem, whether limit theorems for k-triangular set functions hold also for other kinds of non-additive lattice group-valued set functions and whether it is possible to find weaker conditions on the involved filter $\mathcal{F}$ in Theorem 3.9.

References

[1] K. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications,* Physica-Verlag, Heidelberg, 1999.

[2] A. Boccuto, D. Candeloro, A. R. Sambucini, L^p spaces in vector lattices and applications, *Math. Slovaca* **67** (6) (2017), 1409–1426. DOI: 10.1515/ms-2017-0060.

[3] A. Boccuto and X. Dimitriou, Some new results on limit theorems with respect to ideal convergence in (ℓ)-groups, *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia* **58** (2011), 163-174.

[4] A. Boccuto and X. Dimitriou, *Convergence Theorems for Lattice Group-Valued Measures,* Bentham Science Publ., U. A. E., 2015. ISBN 9781681080109

[5] A. Boccuto and X. Dimitriou, *Non-additive lattice group-valued set functions and limit theorems,* Lambert Acad. Publ., 2017. ISBN 9786134913355.

[6] A. Boccuto and X. Dimitriou, Equivalence between limit theorems for lattice group-valued k-triangular set functions, *Mediterranean J. Math.* **15** (4) (2018), 1–20.

[7] A. Boccuto and X. Dimitriou, Dieudonné-type theorems for lattice group-valued k-triangular set functions (2018), viXra:1811.0496.

[8] D. Candeloro, A. R. Sambucini, Filter convergence and decompositions for vector lattice-valued measures, *Mediterranean J. Math.* **12** (3) (2015), 621–637. DOI: 10.1007/s00009-014-0431-0

[9] P. Cavaliere, P. de Lucia, On Brooks-Jewett, Vitali-Hahn-Saks and Nikodým convergence theorems for quasi-triangular functions, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **20** (4) (2009), 387–396.

[10] P. Cavaliere, P. de Lucia, F. Ventriglia, On Drewnowski lemma for non-additive functions and its consequences, *Positivity* **14** (1) (2010), 1–16.

[11] P. Cavaliere, P. de Lucia, A. De Simone, F. Ventriglia, On the Lebesgue decomposition for non-additive functions, *Positivity* **18** (3) (2014), 489–503.

[12] J. Diestel, *Sequences and series in Banach spaces,* Springer - Verlag, Berlin, Heidelberg, New York, 1984.

[13] L. Drewnowski, Topological rings of sets, Continuous set functions, Integration. I, II, III, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **20** (1972), 269-276, 277-286, 439-445.

[14] L. Drewnowski, Equivalence of Brooks - Jewett, Vitali - Hahn - Saks and Nikodym theorems, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **20** (1972), 725-731.

[15] M. Grabisch, k-order additive discrete fuzzy measures and their representation, *Fuzzy Sets Systems,* **92** (1997) 167–189.

[16] A. Kolesárová, J. Li, R. Mesiar, k-additive aggregation function and their characterization, *European J. of Operational Research,* **265** (2018) 985–992.

[17] A. Kolesárová, J. R. Mesiar, k-maxitive aggregation functions, *Fuzzy Sets Systems* (2018). DOI: 10.1016/j.fss.2017.12.016

[18] E. Pap, *Null-Additive Set Functions,* Kluwer Acad. Publishers/Ister Science, Bratislava, 1995.

[19] S. Saeki, The Vitali-Hahn-Saks theorem and measuroids, *Proc. Amer. Math. Soc.* **114** (3) (1992), 775-782.

[20] Z. Wang and G. J. Klir, *Generalized Measure theory,* Springer, Berlin, Heidelberg, New York, 2009.

[21] H. Weber, FN-topologies and group-valued measures, in: *E. Pap (Ed.), Handbook on Measure Theory, vol. I,* Elsevier, Amsterdam, 2002, 703-743.