On the minimal uncompletable word problem for unambiguous automata

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Abstract

This paper deals with finite (possibly not complete) unambiguous automata, not necessarily deterministic. In this setting, we investigate the problem of the minimal length of the uncompletable word. This problem is associated with the well-known conjecture formulated by A. Restivo. We introduce the concept of relatively maximal row for a suitable set of matrices, and show the existence of a relatively maximal row of length of quadratic order with respect to the number of the states of the treated automaton. We give some estimates of the maximal length of the minimal uncompletable word in connection with the number the states of the involved automaton and the length of a suitable relatively maximal but not maximal word, provided that it exists. In the general case, we establish an estimate of the length of the minimal uncompletable word in terms of the number of states of the studied automaton, the length of a suitable relatively maximal word and the minimal length of the uncompletable word of the automaton formed by all maximal rows associated with $\mathcal{A}$.

Keywords: Unambiguous automaton, complete automaton, uncompletable word, relatively maximal row.

1 Introduction

Automata are the subject of several studies existing in the recent literature. Automata theory is strongly related with different structures investigated in different branches of Mathematics and Computer Sciences, like for instance monoids, semirings, codes, graphs, languages, matrices, rational series, matrices, probability measures and densities. A comprehensive overview about

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2010 A. M. S. Subject Classifications: 68Q70, 68Q45.

these topics and its fundamental properties and relations can be found, for example, in [1, 12, 13, 22].

In automata theory and its literature, in connection with the main properties of languages and codes and several related applications (see also [14, 15]), a very important role is played by the problem of finding an uncompletable word of minimal length for an uncomplete automaton. Let X be a subset of an alphabet A and let X^* be its Kleene closure. We say that X is *complete* iff any word of A is a factor of some word belonging to X^*. If X is not complete, then any word which is factor of no word of X^* is said to be *uncompletable*. This problem is strictly related to the mortality problem for matrix monoids (see also [9, 16]). In [20], A. Restivo conjectured that a finite and not complete set X has always an uncompletable word whose length is quadratically bounded by the maximal length of the words of X (see also [2, 19, 21]). Some results related to this problem have been obtained in [8, 11, 18]. Note that, in suitable contexts, a word of an alphabet can be identified with a suitable matrix and with a suitable relation (see also [1, 3, 22]).

The problem we study in this article is strongly related to that of finding a synchronizing word of minimal length, which have several applications in different branches of sciences, for instance when it is dealt with a sequential transducer. Indeed, in decoding messages, the existence of errors in an input sequence can make impossible the transmission of a whole information, so that it is fundamental to have a synchronizing word of suitably small length, in order that - in the affected input sequence - the errors concerning the past do not have negative influence on the successive part of the corresponding output sequence. In this direction, there have been several studies in the literature, related to the famous Černý conjecture introduced in [7] (see also [4, 5, 6, 10, 17, 18]). In [6] several results are proved, concerning some relations between the length of the minimal uncompletable word and that of the minimal synchronizing word.

In this paper we deal with the problem of finding uncompletable words of minimal length in the context of unambiguous automata, not necessarily in the deterministic case. We identify the sets of the words A^* of the input alphabet A of the considered automaton $\mathcal{A}$ with a suitable subset of matrices whose entries are 0 or 1, which we indicate with $\phi_{\mathcal{A}}(A^*)$, where ϕ denotes the identifying morphism. We introduce the concept of relatively maximal row (and column) of a matrix, and we show that this notion is strictly weaker than that of maximality. We show the existence of a relatively maximal row of length of quadratic order with respect to the number of the states of the involved automaton. Successively we prove that, if n is the number of states of the studied automaton and $w \in A^*$ is such that $\phi_{\mathcal{A}}(A^*)$ contains a row, which is relatively maximal but not maximal, then there is an uncompletable word of length less than or equal to $2(n-1)(|w|+n-1)$, where $|w|$ denotes the length of $|w|$. This result includes, as a particular case, the setting of deterministic automata. In the general case, the minimal uncompletable word problem is still an open problem, however we relate the investigated automaton $\mathcal{A}$ with the associated automaton $\mathcal{B}$ of all maximal lines, we show that the uncompleteness of $\mathcal{B}$ is equivalent to that of $\mathcal{A}$, and establish a connection between the length of the minimal uncompletable word of $\mathcal{A}$ and that of $\mathcal{B}$, taking into account the existence of a relatively maximal row in $\mathcal{A}$.

2 Preliminaries

Let A be a finite set, and A^* be the free monoid generated by A. The set A is called *alphabet*. The elements of A and A^* are called *letters* and *words*, respectively. We denote by ε the *empty word*, that is the neutral element of A^*. Given a word w, the symbol $|w|$ indicates the *length* of w, defined inductively by $|\varepsilon| = 0$, $|wa| = |w| + 1$, for every $w \in A^*$, $a \in A$.

An *automaton* is a triple (A, Q, δ), where $A = \{\alpha_1, \alpha_2, \ldots, \alpha_h\}$ is the *input alphabet*, $Q = \{q_1, q_2, \ldots, q_n\}$ is the *set of the states*, $\delta : Q \times A \to \mathcal{P}(Q)$ is the *transition function*, that is the action of A on Q. An automaton $\mathcal{A}$ is said to be *deterministic* iff $\mathrm{card}(\delta(q, \alpha)) \leq 1$ for each $\alpha \in A$ and $q \in Q$. We associate to the automaton $\mathcal{A}$ the *monoid morphism* $\phi_{\mathcal{A}} : (A^*, \circ) \to (\mathbb{N}^{Q\times Q}, \cdot)$, defined by

$$(\phi_{\mathcal{A}}(\alpha))_{i,j} = \begin{cases} 1, & \text{if } q_j \in \delta(q_i, \alpha), \\ 0, & \text{otherwise.} \end{cases}$$

Here, the symbols $\circ$ and $\cdot$ denote the product on the monoid A^* and the row-column product of matrices, respectively. From now on, as no confusion will arise, we will write ab instead of $a \circ b$. Note that $\phi_{\mathcal{A}}(\varepsilon)$ is equal to the identity matrix I_n.

A word $w \in A^*$ is said to be *uncompletable* in A iff $\phi_{\mathcal{A}}(w) = \underline{0}$, where $\underline{0}$ is the identically null matrix.

We denote by $\{0,1\}^{Q\times Q}$ the set of all matrices whose entries are all 0 or 1. An automaton is said to be *unambiguous* iff $\phi_{\mathcal{A}}(A^*) \subset \{0,1\}^{Q\times Q}$.

We say that an unambiguous automaton is *complete* iff no word w of A^* is uncompletable.

For every $m \in \{0,1\}^{Q\times Q}$ and $i, j = 1, \ldots, n$, the symbols m_{i*} and m_{*j} indicate the i-th row and the j-th column of m, respectively. A nonempty subset $M \subset \{0,1\}^{Q\times Q}$ is said to be *transitive* iff for each $i, j \in Q$ there is an element $m \in M$ with $m_{i,j} = 1$. From now on, we assume that the set $\phi_{\mathcal{A}}(A^*)$ is transitive.

Given any two $n \times n$ matrices $A = (a_{i,j})_{i,j}$ and $B = (b_{i,j})_{i,j}$ with non-negative entries, we say that $A \leq B$ iff $a_{i,j} \leq b_{i,j}$ for every i, $j = 1, 2, \ldots, n$. If $A \leq B$ and $A \neq B$, then we write $A < B$. It is not difficult to see that, if $A \leq B$ and $C \geq 0$, then $AC \leq BC$ and $CA \leq CB$.

We say that a row $\underline{a} = (a_1\, a_2 \ldots a_n)$ of a matrix of $\phi_{\mathcal{A}}(A^*)$ is *maximal* iff it is a maximal element in the set of the rows of the matrices of $\phi_{\mathcal{A}}(A^*)$.

A row $\underline{a} = (a_1\, a_2 \ldots a_n) = (\phi_{\mathcal{A}}(u))_{q*}$ of a matrix of $\phi_{\mathcal{A}}(A^*)$ is said to be *relatively maximal* iff for every $m \in \phi_{\mathcal{A}}(A^*)$ and for each $p \in Q$ with $m_{p,q} = 1$, it is $(m\, \phi_{\mathcal{A}}(u))_{p*} = (\phi_{\mathcal{A}}(u))_{q*} = \underline{a}$.

An analogous concept can be formulated also if it is dealt with columns.

It is not difficult to check that every maximal row is relatively maximal too, while the converse, in general, is not true, as the next example shows.

Example 2.1. Let $\mathcal{A} = (A, Q, \delta)$ be the automaton defined by setting $A = \{a, b\}$, $Q = \{\underline{1}, \underline{2}, \underline{3}\}$,

$$\phi_{\mathcal{A}}(a) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \phi_{\mathcal{A}}(b) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$\delta(q, a) = q \cdot \phi_{\mathcal{A}}(a)$, $\delta(q, b) = q \cdot \phi_{\mathcal{A}}(b)$, $q \in Q$. We prove that the first row of $\phi_{\mathcal{A}}(a)$, namely $(0\,1\,0)$, is relatively maximal. Observe that, to this aim, if $\mathcal{D}$ denotes the set of all words of A^* whose last letter is a, then it will be enough to show that for each $d \in \mathcal{D}$ and $k = 1, 2, 3$ it is $(\phi_{\mathcal{A}}(d))_{k*} \not> (0\,1\,0)$, namely

$$(\phi_{\mathcal{A}}(d))_{k*} \notin \{(1\,1\,0), (0\,1\,1), (1\,1\,1)\}. \tag{1}$$

Easy calculations show that

$$\phi_{\mathcal{A}}(a^2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \phi_{\mathcal{A}}(a^3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \phi_{\mathcal{A}}(a);$$

$$\phi_{\mathcal{A}}(ba) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \phi_{\mathcal{A}}(ba^2) = \phi_{\mathcal{A}}(b^2a) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix};$$

$$\phi_{\mathcal{A}}(aba) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq \phi_{\mathcal{A}}(a), \quad \phi_{\mathcal{A}}(ab^2a) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq \phi_{\mathcal{A}}(a^2); \quad \phi_{\mathcal{A}}(b^3a) = \phi_{\mathcal{A}}(ba).$$

Thus, for every $d \in \mathcal{D}$, any row of d is majorated by some row of one of the matrices above, and therefore it contains at most one non-null entry, getting (1). However, the row $(0\,1\,0)$ is not maximal, since $(0\,1\,1) > (0\,1\,0)$, and $(0\,1\,1)$ is the first row of $\phi_{\mathcal{A}}(b)$. Furthermore, note that the automaton $\mathcal{A}$ is uncomplete, since $\phi_{\mathcal{A}}(ab^2ab) = \underline{0}$.

Now we recall the next result, which will be useful in the sequel.

Proposition 2.2. (see also [3, Proposition 2.1]) *Let $\mathcal{A}$ be an unambiguouos automaton and i, j be two fixed states.*

2.2.1) *If there exists a word $u \in A^*$ with $(\phi_{\mathcal{A}}(u))_{i,j} = 1$, then there is a word $w \in A^*$ such that $(\phi_{\mathcal{A}}(w))_{i,j} = 1$ and $|w| \leq n - 1$.*

2.2.2) *If there are $u \in A^*$ and $p_0 \in Q$ with $(\phi_{\mathcal{A}}(u))_{p_0,i} = (\phi_{\mathcal{A}}(u))_{p_0,j} = 1$, then there exist $v \in A^*$ and $q_0 \in Q$ such that $(\phi_{\mathcal{A}}(v))_{q_0,i} = (\phi_{\mathcal{A}}(v))_{q_0,j} = 1$ and*

$$|v| \leq \frac{1}{2}n(n-1).$$

3 The main results

We begin with the following technical proposition, which extends [3, Lemma 4.2] to our context.

Proposition 3.1. *Let $\mathcal{A}$ be an n-state unambiguous automaton (not necessarily complete), $u \in A^*$ be a word with $\phi_{\mathcal{A}}(A) \neq \underline{\{0\}}$, and $\underline{a}$ be a row of $\phi_{\mathcal{A}}(u)$, not relatively maximal. Then there exist a word $v \in A^*$ and a row $\underline{a'}$ of $\phi_{\mathcal{A}}(v)$ with $\underline{a'} > \underline{a}$ and*

$$|v| \leq |u| + \frac{1}{2}n(n-1). \tag{2}$$

Proof: Let $\underline{a} = (\phi_{\mathcal{A}}(u))_{q*}$. Since $\underline{a}$ is not relatively maximal, there exist a word $z \in A^*$ and a state $p \in \{1, 2, \ldots, n\}$ with $(\phi_{\mathcal{A}}(zu))_{p*} > (\phi_{\mathcal{A}}(u))_{q*}$ and $(\phi_{\mathcal{A}}(z))_{p,q} = 1$. Now we claim that there exists a state $q' \neq q$ with $(\phi_{\mathcal{A}}(z))_{p,q'} = 1$ and $(\phi_{\mathcal{A}}(u))_{q'*} \neq 0$. Otherwise, if for every $q' \neq q$ either $(\phi_{\mathcal{A}}(z))_{p,q'} = 0$ or $(\phi_{\mathcal{A}}(u))_{q'*} = 0$, then a not difficult calculation shows that $(\phi_{\mathcal{A}}(zu))_{p*} = (\phi_{\mathcal{A}}(u))_{q^*}$, a contradiction. By 2.2.2), there are a word $v' \in \phi_{\mathcal{A}}(A^*)$ and a state q_0 with $(\phi_{\mathcal{A}}(v'))_{q_0,q} = (\phi_{\mathcal{A}}(v'))_{q_0,q'} = 1$ and

$$|v'| \leq \frac{1}{2}n(n-1). \tag{3}$$

Let now $v = v'u$ and $\underline{a'} = (\phi_{\mathcal{A}}(v))_{q_0*}$. For each $r = 1, 2, \ldots, n$ it is

$$\begin{aligned}(\phi_{\mathcal{A}}(v))_{q_0,r} &= (\phi_{\mathcal{A}}(v'))_{q_0,q}(\phi_{\mathcal{A}}(u))_{q,r} + (\phi_{\mathcal{A}}(v'))_{q_0,q'}(\phi_{\mathcal{A}}(u))_{q',r} + \sum_{s\neq q, s\neq q'} (\phi_{\mathcal{A}}(v'))_{q_0 s}(\phi_{\mathcal{A}}(u))_{s,r} \geq \\ &\geq (\phi_{\mathcal{A}}(v'))_{q_0,q}(\phi_{\mathcal{A}}(u))_{q,r} + (\phi_{\mathcal{A}}(v'))_{q_0,q'}(\phi_{\mathcal{A}}(u))_{q',r} = a_r + (\phi_{\mathcal{A}}(u))_{q',r}.\end{aligned}$$

Since $(\phi_{\mathcal{A}}(u))_{q'*} \neq 0$, then for at least an index $r_0 \in \{1, \ldots, n\}$ we get $(\phi_{\mathcal{A}}(v))_{q_0,r_0} = a_{r_0} + 1 > a_{r_0}$, namely $\underline{a'} > \underline{a}$. Inequality (2) follows from (3) and the definition of length. This ends the proof. $\square$

Remark 3.2. Note that an analogous of Proposition 3.1 holds even if it is dealt with columns instead of rows.

The next result extends [3, Proposition 4.3].

Proposition 3.3. *Let $\mathcal{A}$ be as in Proposition 3.1. Then there are two words u, $v \in A^*$ such that u (resp. v) contains a relatively maximal row $\underline{a} = (a_1\, a_2 \ldots a_n)$ (resp. column $\underline{b} = (b_1\, b_2 \ldots b_n)$), $a_1 = b_1 = 1$, and*

$$|uv| \leq \frac{1}{2} n(n-1)^2.$$

Proof: Let us construct two finite sequences of words of A^*, $u_1, u_2, \ldots, u_t$, $v_1, v_2, \ldots, v_s$, and corresponding rows $\underline{a}^{(i)}$ of $\phi_{\mathcal{A}}(u_i)$ and columns $\underline{b}^{(j)}$ of $\phi_{\mathcal{A}}(v_j)$, as follows. At the first step, put $u_1 = \varepsilon$, $\underline{a}^{(1)} = (1\,0 \ldots 0) = (\phi_{\mathcal{A}}(u_1))_{1*}$. Now suppose to have find words $u_1, u_2, \ldots u_i$ and rows $a^{(1)} < a^{(2)} < \ldots < a^{(i)}$ of $\phi_{\mathcal{A}}(u_1), \phi_{\mathcal{A}}(u_2), \ldots, \phi_{\mathcal{A}}(u_i)$ respectively, such that $|u_i| \leq \dfrac{i-1}{2} n(n-1)$. If $\underline{a}^{(i)}$ is relatively maximal, then we take $t = i$ and stop. Otherwise, we apply Proposition 3.1 and Remark 3.2, and we find a word $u_{i+1} \in A^*$ and a row $a^{(i+1)}$ of $\phi_{\mathcal{A}}(u_{i+1})$, such that $a^{(i+1)} > a^{(i)}$ and

$$|u_{i+1}| \leq |u_i| + \frac{1}{2} n(n-1) \leq \frac{i}{2} n(n-1).$$

We observe that, after a finite number of steps, this procedure ends, because the greatest row (resp. column) which can be reached is $(1\,1 \ldots 1)$ (resp. $(1\,1 \ldots 1)^T$), by virtue of unambiguity. The words v_1, $v_2 \ldots v_s$ and the rows $b^{(1)}$, $b^{(2)}, \ldots, b^{(s)}$ are defined symmetrically. By construction, one has $a_1^{(t)} = b_1^{(s)} = 1$,

$$|u_t| \leq \frac{t-1}{2} n(n-1), \qquad |v_s| \leq \frac{s-1}{2} n(n-1),$$

and hence

$$|u_t\, v_s| = |u_t| + |v_s| \leq \frac{1}{2} n(n-1)(t+s-2).$$

Thus, in order to prove the proposition, it is enough to demonstrate that $t + s - 2 \leq n - 1$. For every state $q \neq \underline{1}$, we get either $a_q^{(t)} = 0$ or $b_q^{(s)} = 0$. Otherwise, since $b_q^{(s)} = b_q^{(s)} = 1$, the (scalar) product between $\underline{a}^{(t)}$ and $\underline{b}^{(s)}$ should be strictly greater than 1: this is impossible, since the involved automaton is unambiguous. In passing from $\underline{a}^{(i-1)}$ to $\underline{a}^{(i)}$ (resp. from $\underline{b}^{(j-1)}$ to $\underline{b}^{(j)}$), the number of null entries becomes strictly smaller. So, after t (resp. s) steps, we get that the row $\underline{a}^{(t)}$ (resp. the column and $\underline{b}^{(s)}$) contains at most $n-t$ (resp. $n-s$) null entries. Thus, the sum of the number of null entries of $\underline{a}^{(t)}$ and that of $\underline{b}^{(s)}$ is between $n-1$ and $2n-t-s$. From this it follows in particular that $n - 1 \leq 2\,n - t - s$, and hence $t + s - 2 \leq n - 1$. This ends the proof. $\square$

Proposition 3.4. *Under the same hypotheses and notations above, let us assume that there are two words $z, \gamma \in A^*$ such that $(\phi_{\mathcal{A}}(z))_{p*}$ is a relatively maximal row and $n_{q*} = (\phi_{\mathcal{A}}(\gamma))_{q*}$ is a not maximal row. Then there exists a word $y \in A^*$ with $n_{q*}\, \phi_{\mathcal{A}}(y) = \underline{0}$, and*

$$|y| \leq |z| + n - 1. \tag{4}$$

Proof: By hypothesis, there are a matrix $\mu \in \phi_{\mathcal{A}}(A^*)$ and a state $r \in \{1, 2, \dots, n\}$, with $\mu_{r*} > n_{q^*}$, namely $\mu_{r,t} \geq n_{r,t}$ for every $t = 1, 2, \dots, n$ and there exists $s \in \{1, 2, \dots, n\}$ with $\mu_{r,s} > n_{q,s}$, that is $\mu_{r,s} = 1$ and $n_{q,s} = 0$, thanks to unambiguity. Hence

$$\mu_{r*} \geq n_{q*} + e_s, \tag{5}$$

where e_s denotes the vector of $\mathbb{R}^n$ whose s-th component is 1 and whose other components are 0. Note that, by (left or right) multiplying both members of the inequality in (5) by any matrix with non-negative entries, the sign of the inequality remains the same. By transitivity and 2.2.1), there is a word $x \in A^*$ with $(\phi_{\mathcal{A}}(x))_{s,p} = 1$ and $|x| \leq n - 1$. Observe that

$$e_s \phi_{\mathcal{A}}(x) = (\phi_{\mathcal{A}}(x))_{s*} \geq e_p, \tag{6}$$

because $(\phi_{\mathcal{A}}(x))_{s,p} = 1$. Furthermore note that, since $\mu_{r,s} = (\phi_{\mathcal{A}}(x))_{s,p} = 1$, then, thanks to unambiguity, it is not difficult to deduce that $(\mu\, \phi_{\mathcal{A}}(x))_{r,p} = 1$. From this and relative maximality of $(\phi_{\mathcal{A}}(z))_{p*}$ it follows that

$$(\phi_{\mathcal{A}}(z))_{p*} = (\mu\, \phi_{\mathcal{A}}(x)\, \phi_{\mathcal{A}}(z))_{r*}. \tag{7}$$

From (5), (6) and (7) we obtain

$$\begin{aligned}(\phi_{\mathcal{A}}(z))_{p*} &= (\mu\, \phi_{\mathcal{A}}(x)\, \phi_{\mathcal{A}}(z))_{r*} = (\mu\, \phi_{\mathcal{A}}(x))_{r*}\, \phi_{\mathcal{A}}(z) \geq n_{q*}\, \phi_{\mathcal{A}}(x)\, \phi_{\mathcal{A}}(z) + e_s\, \phi_{\mathcal{A}}(x)\, \phi_{\mathcal{A}}(z) \\ &\geq n_{q*}\, \phi_{\mathcal{A}}(x)\, \phi_{\mathcal{A}}(z) + e_p \geq \phi_{\mathcal{A}}(z) = n_{q*}\, \phi_{\mathcal{A}}(x)\, \phi_{\mathcal{A}}(z) + (\phi_{\mathcal{A}}(z))_{p*}.\end{aligned} \tag{8}$$

So, all inequalities in (8) are actually equalities, and hence, setting $y = x\, z$, we get that

$$n_{q*}\, \phi_{\mathcal{A}}(y) = n_{q*}\, \phi_{\mathcal{A}}(x)\, \phi_{\mathcal{A}}(z) = \underline{0}$$

and

$$|y| = |z| + |x| \leq |z| + n - 1.$$

This concludes the proof. $\square$

The next proposition will be useful in the sequel.

Proposition 3.5. *Let $\underline{a} = (\phi_{\mathcal{A}}(\alpha))_{s*}$ be a not identically zero row, and $(\phi_{\mathcal{A}}(w))_{i*}$ be a relatively maximal row. Then there is a word $v \in A^*$ with $|v| \leq n - 1$ and $\underline{a}\, \phi_{\mathcal{A}}(vw) = (\phi_{\mathcal{A}}(w))_{i*}$.*

Proof: Let $\underline{a} = (a_1\, a_2 \ldots a_n)$. By hypothesis, there is $j \in \{1, \ldots, n\}$ with $a_j = 1$. By 2.2.1) there exists a word $v \in A^*$ with $|v| \leq n-1$ and $(\phi_{\mathcal{A}}(v))_{j,i} = 1$. Since $a_j = 1$, $(\phi_{\mathcal{A}}(v))_{j,i} = 1$ and $(\phi_{\mathcal{A}}(w))_{i*}$ is relatively maximal, it follows that

$$\underline{a}\, \phi_{\mathcal{A}}(v\, w) = \underline{a}\, \phi_{\mathcal{A}}(v)\, \phi_{\mathcal{A}}(w) = (\phi_{\mathcal{A}}(w))_{i*},$$

getting the assertion. $\square$

A consequence of Propositions 3.4 and 3.5 is the following result.

Theorem 3.6. *Under the same notations and hypotheses as above, let there exist a word $w \in A^*$ which contains a relatively maximal but not maximal row $(\phi_{\mathcal{A}}(w))_{q*}$. Then $\mathcal{A}$ has an uncompletable word ω of length at most $2(n-1)(|w|+n-1)$.*

We construct an uncompletable word of $\mathcal{A}$ as follows, arguing by induction. First of all, thanks to Proposition 3.4, we find a word $x_0 \in A^*$ with $|x_0| \leq |w| + n - 1$ and

$$(\phi_{\mathcal{A}}(w))_q\, \phi_{\mathcal{A}}(x_0) = (\phi_{\mathcal{A}}(w\, x_0))_{q*} = \underline{0}. \tag{9}$$

At the first step, let $y_1 = \varepsilon$ be the empty word and choose arbitrarily $q_1 \in \{1, 2, \ldots, n\}$. If $(\phi_{\mathcal{A}}(y_1))_{q_1*}$ is a not maximal row of $\phi_{\mathcal{A}}(y_1)$, then, by virtue of Proposition 3.4, there is a word $x_1 \in A^*$ with

$$(\phi_{\mathcal{A}}(y_1))_{q_1*}\, \phi_{\mathcal{A}}(x_1) = (\phi_{\mathcal{A}}(y_1\, x_1))_{q_1*} = \underline{0}$$

and $|x_1| \leq |w| + n - 1$. If $(\phi_{\mathcal{A}}(y_1))_{q_1*} = \underline{a}$ is a maximal line of $\phi_{\mathcal{A}}(y_1)$, then, by applying Proposition 3.5 to $\underline{a}$ and $(\phi_{\mathcal{A}}(w))_{q_1*}$ and by (9) we find a word $v_1 \in A^*$, with $|v_1| \leq n-1$ and $(\phi_{\mathcal{A}}(y_1\, v_1\, w\, x_0))_{q_1*} = \underline{0}$. Put

$$y_2 = \begin{cases} x_1, & \text{if } \underline{a} \text{ is not maximal;} \\ v_1\, w\, x_0, & \text{if } \underline{a} \text{ is maximal.} \end{cases}$$

We get $|y_2| \leq 2(|w| + n - 1)$ and $(\phi_{\mathcal{A}}(y_1\, y_2))_{q_1*} = \underline{0}$.

At the second step, proceeding analogously as above, we pick an integer $q_2 \in \{1, 2, \ldots, n\}$, $q_2 \neq q_1$, distinguish the two cases $(\phi_{\mathcal{A}}(y_2))_{q_2*}$ not maximal and $(\phi_{\mathcal{A}}(y_2))_{q_2*}$ maximal and find a word $y_3 \in A^*$ such that $|y_3| \leq 2(|w| + n - 1)$ and $(\phi_{\mathcal{A}}(y_2\, y_3))_{q_2*} = \underline{0}$.

Proceeding by induction, at the $n-1$-th step we find a word $y_n \in A^*$ and a $k_n \in \{1, 2, \ldots, n\}$ with $(\phi_{\mathcal{A}}(y_{n-1}\, y_n))_{k_n*} = \underline{0}$, and $|y_n| \leq 2(|w| + n - 1)$. If

$$\omega = y_1\, y_2\, \ldots y_{n-1}\, y_n = y_2\, \ldots y_{n-1}\, y_n,$$

then ω is an uncompletable word of $\mathcal{A}$ and $|\omega| \leq 2(n-1)(|w| + n - 1)$. This ends the proof. $\square$

Remark 3.7. Observe that, when $\mathcal{A}$ is deterministic, since each row of every element of $\phi_{\mathcal{A}}(A)$ contains at most one 1 and has all other entries equal to 0, it is not difficult to show that every non-zero row is maximal and every identically null row is relatively maximal but not maximal.

In the general case, given an automaton (A, Q, δ) with an associated transition function $\phi_{\mathcal{A}}$, it is advisable to consider the automaton $\mathcal{B} = (A, Q', \delta')$, where Q' is the set of all maximal rows, $\delta'(q, \alpha) = q \cdot \phi_{\mathcal{A}}(\alpha)$, $\alpha \in A$, $q \in Q$. Note that the uncompleteness of $\mathcal{B}$ follows from that of $\mathcal{A}$, since the product of every maximal line of each element of $\phi_{\mathcal{A}}(A^*)$ with the null matrix is equal to the null line.

Now we show that the converse is true too, and in particular we establish a connection between the length of the minimal uncompletable word of $\mathcal{A}$ and that of $\mathcal{B}$.

Theorem 3.8. *Let $\mathcal{A}$ be as in Proposition 2.2, n_0 be the length of the minimal uncompletable word of $\mathcal{B}$ and $w \in A^*$ be a word such that $\phi_{\mathcal{A}}(w)$ contains a relatively maximal row. Then $\mathcal{A}$ has an uncompletable word χ with $|\chi| = (n-1)\,(\max\{|w| + n - 1,\, n_0\})$.*

Proof: Analogously as in Theorem 3.6, we construct inductively an uncompletable word of $\mathcal{A}$. At the first step, let $y_1 = \varepsilon$ be the empty word and choose arbitrarily $q_1 \in \{1, 2, \ldots, n\}$. If $(\phi_{\mathcal{A}}(y_1))_{q_1 *}$ is a not maximal row of $\phi_{\mathcal{A}}(y_1)$, then, by virtue of Proposition 3.4, there is a word $x_1 \in A^*$ with

$$(\phi_{\mathcal{A}}(y_1))_{q_1 *}\, \phi_{\mathcal{A}}(x_1) = (\phi_{\mathcal{A}}(y_1\, x_1))_{q_1 *} = \underline{0}$$

and $|x_1| \leq |w| + n - 1$. If $(\phi_{\mathcal{A}}(y_1))_{q_1 *}$ is a maximal line of $\phi_{\mathcal{A}}(y_1)$, then there is a word $v_1 \in A^*$, with $|v_1| \leq n_0$ and $(\phi_{\mathcal{A}}(y_1))_{q_1 *}\, \phi_{\mathcal{A}}(v_1) = (\phi_{\mathcal{A}}(y_1\, v_1))_{q_1 *} = \underline{0}$. Put

$$y_2 = \begin{cases} x_1, & \text{if } (\phi_{\mathcal{A}}(y_1))_{q_1 *} \text{ is not maximal;} \\ v_1, & \text{if } (\phi_{\mathcal{A}}(y_1))_{q_1 *} \text{ is maximal.} \end{cases}$$

We get $|y_2| \leq \max\{|w| + n - 1, n_0\}$ and $(\phi_{\mathcal{A}}(y_1\, y_2))_{q_1 *} = \underline{0}$.

At the second step, arguing analogously as above, we take an integer $q_2 \neq q_1$, consider the two cases $(\phi_{\mathcal{A}}(y_2))_{q_2 *}$ not maximal and $(\phi_{\mathcal{A}}(y_2))_{q_2 *}$ maximal and construct a word $y_3 \in A^*$ with $|y_3| \leq \max\{|w| + n - 1, n_0\}$ and $(\phi_{\mathcal{A}}(y_2\, y_3))_{q_2 *} = \underline{0}$.

By induction, we find a word $y_n \in A^*$ and an index $k_n \in \{1, 2, \ldots, n\}$ with $(\phi_{\mathcal{A}}(y_{n-1}\, y_n))_{k_n *} = \underline{0}$, and $|y_n| \leq \max\{|w| + n - 1, n_0\}$. Let

$$\chi = y_1\, y_2\, \ldots y_{n-1}\, y_n = y_2\, \ldots y_{n-1}\, y_n.$$

Then, χ is an uncompletable word of $\mathcal{A}$, and $|\chi| \leq (n-1)\,(\max\{|w| + n - 1,\, n_0\})$. This completes the proof. $\square$

4 Conclusions

The problem of finding an uncompletable word of minimal length is still an open problem, in particular in the non-deterministic case, in which it seems that there are not many results in the literature. A conjecture which can arise is to find better estimates, possibly sharp, of the length of the minimal uncompletable word, possibly by dropping the hypothesis of existence of a relatively maximal but not maximal row, and/or which are independent of the length of maximal uncompletable word of the automaton of all maximal words.

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