Gauss-Bonnet Theorem for Surfaces Embedded in a Four Dimensional Space

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Abstract

We study the Gauss-Bonnet theorem applied to a specific example.

Key Words: differential geometry, curvature.

1 Introduction

Is the Gauss-Bonnet theorem (see [1]) valid for The Real projective space? And more generally, is the Gauss-Bonnet theorem valid for a 2D manifold embedded in $\mathbb{R}^4$ as an ambient space?.

While analysing this question we have found a peculiar example that we would like to study in this paper.

Of course there is something wrong with our analysis and we do not know what. If anybody can tell where is the mistake is, please email your answer at the address below.

2 The Space Under Study

Given the following curve γ composed of two semicircles of radius $R = 1$ on the (x,y) plane:

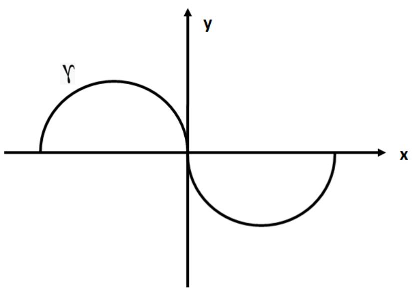

Figure 1: The curve γ

We want to get a surface of revolution in $\mathbb{R}^4$ with axis (x, y, z, w) and we want our surface to be a manifold. In order to do so, we rotate the above curve by an angle θ around the x axis in the (x, y, z) space by 360 degs so that we get a closed surfaces Σ. However, since we do not want Σ to self-intersect, we displace $\gamma(\theta)$ on the plane (z, w) for points close to the origin around a tiny circle of radius ϵ.

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We get a surface in the two parameters $x \in [-2, 2]$ and $\theta \in [0, 2\pi]$:

$$\begin{cases} S = \epsilon \cos\left(\frac{\pi}{4}x\right)\sin(\theta) \\ C = \epsilon \cos\left(\frac{\pi}{4}x\right)\cos(\theta) \\ \tilde{y} = \sqrt{1-(\bar{x}+1)^2} \ \text{ per } \ \bar{x} \leq 0 \\ \underline{\tilde{y} = -\sqrt{1-(x-1)^2} \ \text{ per } \ x \geq 0} \\ x = x \\ y = \tilde{y}\cos(\theta) \\ z = C + \tilde{y}\sin(\theta) \\ w = S \end{cases} \tag{1}$$

for $\epsilon = 0$ this is the wedge product of two spheres and for $\epsilon > 0$ is a manifold. This manifold, for $\epsilon \to 0$ and $x \neq 0$ has Gaussian curvature $K \to 1$ being $R = 1$ the radius of the two spheres.

This surface is clearly a closed surface. We need to prove that it is a manifold end in order to do so, we need to show that is does not self-intersect. If the surface self intersect in a point $P \in \mathbb{R}^4$, there must exist two value of theta, let's say θ_1 and θ_2, for which the curves $\gamma(\theta_1)$ and $\gamma(\theta_2)$ intersect in P. However, the two above curves have the same coordinate w only if:

$$w_1 = \epsilon \cos\left(\frac{\pi}{4}x\right)\sin(\theta_1) = \epsilon \cos\left(\frac{\pi}{4}x\right)\sin(\theta_2) = w_2 \tag{2}$$

and the two curves lay in the 3D slice $w = w_1 = w_2$ of $\mathbb{R}^4$. This clearly happen when:

$$\theta_2 = \pi - \theta_1 \tag{3}$$

If we plot $\gamma(\theta_1)$ and $\gamma(\theta_2)$ in the 3D slice mentioned above, we can finally convince ourself that the two curves never intersect.

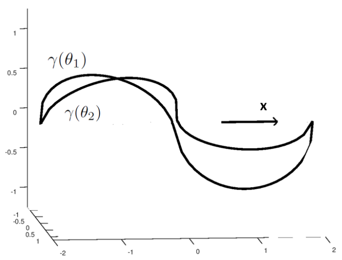

Figure 2: Curve $\gamma(\theta_1)$ vs $\gamma(\theta_2)$

This is because the two curves lay on two planes that intersect on the line where the two curves have the flex and therefore they can meet only in the two flexes. If we make the flex of $\gamma(\theta)$ to go around a small circle of radius ϵ in the (z, w) plane and as a function of θ, they will never intersect.

In particular, being $\tilde{y} = 0$ in the flex, the flexes of the two curves will have the same z coordinate when:

$$z_1 = \epsilon \cos\left(\frac{\pi}{4}x\right)\cos(\theta_1) = \epsilon \cos\left(\frac{\pi}{4}x\right)\cos(\theta_2) = z_2 \tag{4}$$

which will be verified for:

$$\theta_2 = 2\pi - \theta_1 \tag{5}$$

and the condition (3) and (5) never happen at the same time, as expected.

Moreover, this surface is orientable and simply connected and therefore it is homomorphic to a sphere S^2. We want to evaluate it's total curvature for $\epsilon \to 0$. In this limit, clearly the Gaussian curvature K of each point of the surfaces for $x \neq 0$ goes to 1, being the radii of the two spheres 1, and therefore the total curvature is equal of the total curvature of two sphere:

$$K = 4\pi\chi \tag{6}$$

in contradiction to the Gauss-Bonnet theorem.

3 Differentiability of the Surface

The reader may argue that the above surface is not differentiable for $x = 0$ and therefore it may contain a discrete (infinite) curvature (see [2]) on $\Sigma|_{x=0}$ which is a circle of radius ϵ. However, for $x = 0$ the curves $\gamma(\theta)$ has no cusps (it has first derivative continuous) and therefore does not curry discrete curvature. Moreover, this surface may be smoothed for points at $x = 0$ and the final total curvatures would not change much.

On circle $\Sigma|_{x=0}$ obviously one of the two principal curvatures goes to infinite (being equal to $\frac{1}{\epsilon}$) but the other one is on a flex and therefore vanishes constantly for each ϵ. This will give a vanishing Gaussian curvature. For this reason the curvature of Σ is equal to:

$$\begin{cases} K = R^2 = 1 & \text{for} \quad x \neq 0 \\ K = 0 & \text{for} \quad x = 0. \end{cases} \tag{7}$$

References

[1] Wikipedia. Gauss Bonnet Theorem

[2] V. Nardozza. *Product of Distributions Applied to Discrete Differential Geometry.* www.vixra.org/abs/1211.0099 (2012).