On the Euler Integral for the positive and negative Factorial

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Abstract

We reviewed the Euler integral for the factorial, Gauss' Pi function, Legendre's gamma function and beta function, and found that gamma function is defective in $\Gamma(0)$ and $\Gamma(-x)$ because they are undefined or indefinable. And we came to a conclusion that the definition of a negative factorial, that covers the domain of the negative space, is needed to the Euler integral for the factorial, as well as the Euler Y function and the Euler Z function, that supersede Legendre's gamma function and beta function.

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A. The positive factorial and the Euler Y function

Leonhard Euler (1707–1783) developed a transcendental progression in 1730[1][1], which is read

$$\int x^e dx(1-x)^n. \tag{1}$$

From this, Euler transformed the above by changing e to $\frac{f}{g}$ for generalization into

$$\int x^{\frac{f}{g}} dx(1-x)^n. \tag{2}$$

Whence, Euler set $f = 1$ and $g = 0$, and got an integral for the factorial (!)[2],

$$\int dx(-lx)^n, \tag{3}$$

where l represents *logarithm*. This is called the Euler integral of the second kind[3], and the equation (1) is called the Euler integral of the first kind.[4, 5]

Rewriting the formula (3) as follows with limitation of domain for a positive half space,

$$\int_0^1 \ln\left(\frac{1}{x}\right)^n dx, \quad n \geq 0. \tag{4}$$

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[1] "On Transcendental progressions that is, those whose general terms cannot be given algebraically" by Leonhard Euler p.3

[2] ibid. p. 8

[3] In the article "Leonhard Euler's Integral", p. 855, Philip J. Davis

[4] ibid.p.855, someone insists $\int_0^1 t^{x-1}(1-t)^{y-1}dt$ to be the first kind.

[5] Legendre stated this came from Euler's integral $\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$ "Exercices de Calcul Intégral, Seconde Partie", Adrien-Marie Legendre, p. 221

Euler uses the above formula even in the domain for the negative half integers[6][4]. However, considering that the above formula represents the positive factorial and there is no definition of a negative factorial, it is not proper to use the above function in the negative domain of the negative space. For instance, if the above equation (4) is continuous at the negative domain, we may replace $n = -1$ and $\ln(\frac{1}{x}) = -u$, and we get

$$\begin{aligned} \int_0^1 \ln\left(\frac{1}{x}\right)^{-1} dx &= \int_{-\infty}^0 \frac{e^u}{u} du \\ &= \text{undefined}. \end{aligned} \tag{5}$$

where e is the base of the natural logarithm. The presumption is not continuous at the negative domain, because the result remains undefined. Therefore, for now, we limit the domain of the equation (4) in the positive area for $n \geq 0$. Negative factorial shall be further explored below.

Anyway, the formula (4) is well applicable in all positive integers including zero. We can see the below from one of Euler's work [7]

$$\begin{aligned} \int \partial x \left(l\frac{1}{x}\right)^0 &= 1, \\ \int \partial x \left(l\frac{1}{x}\right)^1 &= 1 \\ \int \partial x \left(l\frac{1}{x}\right)^2 &= 1 \cdot 2, \\ \int \partial x \left(l\frac{1}{x}\right)^3 &= 1 \cdot 2 \cdot 3 \\ \int \partial x \left(l\frac{1}{x}\right)^4 &= 1 \cdot 2 \cdot 3 \cdot 4, \\ \text{and go to the origin} \\ \int \partial x \left(l\frac{1}{x}\right)^n &= 1 \cdot 2 \cdot 3 \cdot 4 \cdots n. \end{aligned} \tag{6}$$

As Euler presented the above without any notation, I'd like to denote this function as a notation of Y so that we can easily distinguish the Euler integral for the factorial from others such as Gauss'

[6] Some samples of using negative half integers are quoted on page 21, "On the true value of the integral formula $\int \partial x \cdot (log(1/x))^n$ with the term extended from $x = 0$ all the way to $x = 1$", L. Eulero

[7] Euler Leonhard, ibid. p. 15

pi function (Π) and the gamma function (Γ). i.e.,

$$\begin{aligned} Y(x) &= \int_0^1 \ln\left(\frac{1}{t}\right)^x dt, \quad x \geq 0, \\ &= x!. \end{aligned} \tag{7}$$

By replacing $\ln\left(\frac{1}{t}\right)$ by u, we get

$$\begin{aligned} Y(x) &= \int_0^1 \ln\left(\frac{1}{t}\right)^x dt \\ &= \int_0^\infty e^{-u} u^x du, \quad x \geq 0. \end{aligned} \tag{8}$$

This is exactly the same as Gauss' pi function $\Pi(x)$, but Gauss$(1777-1855)$, as a later mathematician than Euler, developed pi function much later. Therefore the Gauss' pi function[6] shall be discarded, because Y function was developed in 1730 much earlier than Gauss'.

From here and after, we shall call the function (8) as Euler-Gauss Y function or simply Euler Y function.

Meanwhile, Adrien-Marie Legendre$(1752-1833)$ re-presented a general formula of the above Euler integral (3) in his book [8][8],

$$\int_0^1 x^{m-1} dx \left(log\frac{1}{x}\right)^n. \tag{9}$$

This integral is integrated by parts for n times as

$$\begin{aligned} \int_0^1 x^{m-1} dx \left(log\frac{1}{x}\right)^n &= \frac{n}{m}\int_0^1 x^{m-1} dx \left(log\frac{1}{x}\right)^{n-1} \\ &= \frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdots 1}{m^{n+1}}. \end{aligned} \tag{10}$$

If $m = 1$, this is identical with the Euler integral (4).

B. The Euler Z function at the positive space

Consider the integral (1), which we denote this integral as the Euler Z function, which would supersede the beta function,

$$Z(x,y) = \int_0^1 t^x (1-t)^y dt, \quad x \geq 0, y \geq 0. \tag{11}$$

[8] Exercices de Calcul Intégral p. 276 Des intégrales Eulériennes de la seconde espèce

To derive this relation, we may apply the Euler Y function by multiplying $Y(x)$ by $Y(y)$[6] of the equation (8), then we have

$$\begin{aligned} Y(x)Y(y) &= \int_{u=0}^{\infty} e^{-u}u^x du \int_{v=0}^{\infty} e^{-v}v^y dv \\ &= \int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-u-v}u^x v^y du dv. \end{aligned} \tag{12}$$

Substituting variables by $u = mn$ and $v = m(1-n)$, this provides (Jacobian Matrix[10] is applied.)

$$\begin{aligned} Y(x)Y(y) &= \int_{m=0}^{\infty}\int_{n=0}^{1} e^{-m}(mn)^x(m(1-n))^y m dn dm \\ &= \int_{0}^{\infty} e^{-m}m^{x+y+1}dm \cdot \int_0^1 n^x(1-n)^y dn \\ &= Y(x+y+1)\cdot Z(x,y). \end{aligned} \tag{13}$$

Dividing both sides by $Y(x+y+1)$,[9] this results the equation (11),

$$\frac{Y(x)Y(y)}{Y(x+y+1)} = Z(x,y) = \int_0^1 t^x(1-t)^y dt. \tag{14}$$

Another way to prove the Z function (11) is integrating directly from beginning to end, i.e.,

$$\begin{aligned} Z(x,y) &= \int_0^1 t^x(1-t)^y dt \\ &= \frac{y(y-1)(y-2)\cdots(3)(2)(1)}{(x+1)(x+2)\cdots(x+y-1)(x+y)}\int_0^1 t^{x+y}dt \\ &= \frac{y(y-1)(y-2)\cdots(3)(2)(1)}{(x+1)(x+2)\cdots(x+y-1)(x+y)(x+y+1)} \\ &\quad \text{multiplying both sides by } (1)(2)(3)\cdots(x-1)(x), \\ &= \frac{y(y-1)(y-2)\cdots(3)(2)(1)\cdot(1)(2)(3)\cdots(x-1)(x)}{(1)(2)(3)\cdots(x-1)(x)\cdot(x+1)(x+2)\cdots(x+y-1)(x+y)(x+y+1)} \\ &= \frac{Y(x)Y(y)}{Y(x+y+1)}. \qquad \text{Q.E.D.} \end{aligned} \tag{15}$$

Particularly, in case $x = y$, we may obtain the following by substituting t with $\sin^2(\theta)$

$$Z(x,x) = \frac{1}{2^{2x}}\int_0^{\frac{\pi}{2}} (\sin(2\theta))^{2x+1}d\theta. \tag{16}$$

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[9] Legendre mentioned this equation as an integral of a second member

$$Z = \int x^\alpha(1-x)^\beta dx = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)}.$$

Exercices de Calcul Intégral Troisième partie, Exemple II. p. 348-351

If the Euler Y function is continuous in negative space beyond the positive domain, one may substitute y with $-x$ of the above equation (14), and get

$$Z(x,-x) = \int_0^1 t^x (1-t)^{-x} dt. \tag{17}$$

In case $x = 1$, this provides

$$\begin{aligned} Z(1,-1) &= \int_0^1 \frac{t}{1-t} dt \\ &= \Big[-t - \ln(1-t) \Big]_0^1 \\ &= -1 + \infty. \end{aligned} \tag{18}$$

This tells us that the above presumption is false because it is not continuous but divergent at the negative domain. The Euler Y function, as it is now, cannot be applied beyond the positive to the negative domain. Here, we already see at the equation (5) that $Y(-1)$ is undefined, and this is because we have defined the use of the above equation to the positive domain. And we do not know what value $Y(-1)$ has without knowing the negative factorial that covers properly for the negative domain. In the below section, we would like to further draw a definition of a negative factorial.

C. The Definition of a negative factorial

Now let's explore a negative factorial, (-n)! [11] [10]. The Euler Y function is apparently located at the positive half space. As we don't know about a negative factorial, we need to define a negative factorial that covers the negative half space.

Therefore we hereby present a negative factorial at the negative space as follows. The factorial of a positive integer n, expressed as $n!$, is a product of all positive integers from 1 to n, i.e.,

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \cdots \cdot 3 \cdot 2 \cdot 1. \tag{19}$$

We can consider about $(-n)!$. If we put $-n$ in the place of n, we may obtain $(-n)!$.

Lemma 1

$$Y(-n) = (-1)^n Y(n). \tag{20}$$

[10] Refer to "Factorials of real negative and imaginary numbers - A new perspective", Ashwani K. Thukral, 2014 Nov 6.

Proof:

$$\begin{aligned} Y(-n) &= (-n)! \\ &= (-n)\cdot(-n+1)\cdot(-n+2)\cdot(-n+3)\cdots(-3)\cdot(-2)\cdot(-1) \\ &= (-1)^n\cdot n! \\ &= (-1)^n Y(n), \quad n>0. \end{aligned} \tag{21}$$

By the use of the Euler Y function, it can be defined as

$$\begin{aligned} Y(-s) &= (-1)^{|-s|}\int_{-\infty}^{0} t^{|-s|}e^{t}dt \qquad (22) \\ &= (-1)^s\int_0^\infty t^s e^{-t}dt \\ &= (-1)^s Y(s), \quad s>0. \end{aligned}$$

We adopt an absolute value symbol || for a negative variable so that it may follow the rule of the equation (21), and then we omit it for further use since it can be distinguished easily from the accompanied minus symbol as per $Y(-s)$.

D. The Euler Z function at the negative space

With this, the Euler Z function of (11) provides

$$Z(x,-y) = (-1)^y\int_0^1 t^x(1-t)^y dt. \tag{23}$$

To derive this relation, we get as per $Z(x,y)$ of the above (13)

$$\begin{aligned} Y(x)Y(-y) &= \int_{u=0}^{\infty} e^{-u}u^x du\cdot(-1)^y\int_{v=0}^{\infty} e^{-v}v^y dv \qquad (24) \\ &= (-1)^y\int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-u-v}u^x v^y dudv. \end{aligned}$$

Substituting variables by $u=mn$ and $v=m(1-n)$, we have

$$\begin{aligned} Y(x)Y(-y) &= (-1)^y\int_{m=0}^{\infty}\int_{n=0}^{1} e^{-m}(mn)^x(m(1-n))^y mdndm \qquad (25) \\ &= (-1)^y\int_0^\infty e^{-m}m^{x+y+1}dm\cdot\int_0^1 n^x(1-n)^y dn \\ &= Y(x+y+1)\cdot Z(x,-y). \end{aligned}$$

Finally, we get

$$\frac{Y(x)Y(-y)}{Y(x+y+1)} = Z(x,-y) = (-1)^y \int_0^1 t^x (1-t)^y dt. \tag{26}$$

We can obtain $Y(-x)$ from the above, by replacing $t = \sin^2(\theta)$

$$Y(-x) = (-1)^x \frac{Y(2x+1)}{2^{2x}Y(x)} \int_0^{\frac{\pi}{2}} (\sin(2\theta))^{2x+1}\, dt. \tag{27}$$

Now, we have a negative Euler Z function as well as a negative factorial that works at the negative domain.

As the Euler Y function is available for a half integer, we may have $Y(\frac{1}{2})$ as follows from (16)

$$\frac{Y(\frac{1}{2})Y(\frac{1}{2})}{Y(\frac{1}{2}+\frac{1}{2}+1)} = \frac{\left(Y(\frac{1}{2})\right)^2}{2} = \frac{\pi}{8}. \tag{28}$$

Therefore, we get $Y(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.

Similar to the above, we may have $Y(-\frac{1}{2})$ for a negative half integer from (26),

$$\begin{aligned} \frac{Y(\frac{1}{2})Y(-\frac{1}{2})}{Y(\frac{1}{2}+\frac{1}{2}+1)} &= (-1)^{\frac{1}{2}} \int_0^1 (t(1-t))^{\frac{1}{2}} dt \\ &= \frac{i\pi}{8}. \end{aligned} \tag{29}$$

As $Y(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$, so we get $Y(-\frac{1}{2}) = \frac{i\sqrt{\pi}}{2}$, where i is the imaginary unit.

In general, a half integer of $Z(z-\frac{1}{2}, z-\frac{1}{2})$ may be given as follows;

$$\begin{aligned} Z(z-\frac{1}{2}, z-\frac{1}{2}) &= \int_0^1 (t(1-t))^{z-\frac{1}{2}} dt, \qquad z = 1,2,3,\cdots \\ &= \int_0^1 (t(1-t))^z \frac{dt}{(t(1-t))^{\frac{1}{2}}}. \end{aligned} \tag{30}$$

If we replace t by $\sin^2(\theta)$, we obtain

$$Z(z-\frac{1}{2}, z-\frac{1}{2}) = \frac{1}{2^{3z-1}} \int_0^{\frac{\pi}{2}} (1-\cos(4\theta))^z d\theta. \tag{31}$$

And for a negative half integer, we have

$$Z(z-\frac{1}{2}, \frac{1}{2}-z) = \frac{(-1)^{z-\frac{1}{2}}}{2^{3z-1}} \int_0^{\frac{\pi}{2}} (1-\cos(4\theta))^z\, d\theta. \tag{32}$$

Some of positive and negative half integers are given as below

$$\begin{aligned} Y(\frac{1}{2}) &= \frac{\sqrt{\pi}}{2} & Y(-\frac{1}{2}) &= i\frac{\sqrt{\pi}}{2} \\ Y(\frac{3}{2}) &= \frac{3\sqrt{\pi}}{4} & Y(-\frac{3}{2}) &= -i\frac{3\sqrt{\pi}}{4} \\ Y(\frac{5}{2}) &= \frac{15\sqrt{\pi}}{8} & Y(-\frac{5}{2}) &= i\frac{15\sqrt{\pi}}{8} \\ Y(\frac{7}{2}) &= \frac{105\sqrt{\pi}}{16} & Y(-\frac{7}{2}) &= -i\frac{105\sqrt{\pi}}{16}. \end{aligned} \tag{33}$$

E. Defectiveness of the gamma function

Now we need to find out the relationship between the Euler Y function and the Legendre gamma function[8]. Strictly speaking, the Legendre gamma function is only a part of the Euler Y function, and cannot be stood independently as shown as below. As the equation (8) is an improper integral, we may integrate it by part as follows, and come across gamma function,

$$\begin{aligned} Y(x) &= \int_0^\infty e^{-u}u^x du \\ &= \Big[-e^{-u}u^x\Big]_0^\infty + x\int_0^\infty e^{-u}u^{x-1}du \\ &= x\int_0^\infty e^{-u}u^{x-1}du \\ &= x\Gamma(x), \quad x \geq 1. \end{aligned} \tag{34}$$

By using l'Hôpital's rule, we can identify that $\Big[-e^{-u}u^x\Big]_0^\infty$ vanishes in case $x \geq 1$. Here Γ [11] represents the gamma function denoted as $\Gamma(x) = (x-1)!$.

The above equation is not continuous at zero because, while $Y(0)$ represents 0! which is defined to be 1, the equation (34) generates zero at $x = 0$. Therefore we limit the domain $x \geq 1$. Taking a look at the above equation, we can identify that gamma function $\Gamma(x)$ itself cannot fully replace the positive factorial as a whole, but a coupling variable x is multiplied to $\Gamma(x)$ so that it may

[11] Euler denoted this formula as Δ[12] as

$$\int_0^\infty x^{n-1}\partial x \cdot e^{-x} = \Delta.$$

Euler, Leonhard, "De valoribus integralium a termino variabilis $x = 0$ usque ad $x = \infty$ extensorum" (1794) p. 340.

represent a real factorial. Now, consider $Y(x) = x\Gamma(x)$ in cases, $x = 1$ and $x = 0$,

$$Y(1) = 1 \cdot \Gamma(1) = 1 \cdot 0! = 1, \tag{35}$$
$$\Gamma(0) = (0-1)! = (-1)!.$$

The second term $\Gamma(0) = (-1)!$ is not defined, and negative gamma $\Gamma(-x)$, neither. Lack of definition of $\Gamma(0)$ is a deadly obstacle to the path of the negative factorial.

F. Conclusion

The Euler integral of the second kind (3), derived from the Euler transcendental progression, is denoted as the Euler Y function in this paper, which works well in positive domain and at zero. The Euler Z function (1) can be derived from the product of $Y(x)Y(y)$, and vice versa.

The Euler integral of the second kind is integrable in negative domain only if the variable x is a half of an integer. Considering in other part of the negative domain, the Euler integral of the second kind is not complete and this is proved by $Y(-1)$ which is undefined. But Euler's half integer is still selectively used in specific cases without a clear definition of what a negative factorial is.

In order to solve this problem, it was necessary to clearly define a negative factorial. Using the Euler Y function, we defined the negative factorial as $Y(-x) = (-1)^x Y(x)$ for $x > 0$. We have proved above that this newly defined $Y(-x)$ works well in all parts of the negative domain, also including the negative half integers.

As above, without using Legendre's gamma function, Euler Y function, Euler Z function and the definition of the negative factorial work well to each other, whereas Legendre's gamma function can be derived from Euler Y function, and $\Gamma(0) = (-1)!$ is undefined or indefinable, so $\Gamma(-x)$ cannot be defined either. The same goes for the beta function.

So, although Legendre's gamma function and beta function are derived from the functions used by Euler, they are incomplete and should be discarded[12], as the Euler Y function, Euler Z function

[12] Philip J. Davis[5] wrote " Legendre's notation shifts the argument. Gauss introduced a notation $\pi(x)$ free of this defect. Legendre's notation won out, but continues to plague many people. The notations Γ, π, and ! can all be found today". Leonhard Euler's Integral: A historical Profile of the Gamma Function in memoriam: Milton Abramowitz, Philip J. Davis 1959 p.855

and negative factorial can take their places entirely.

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[1] https://scholarlycommons.pacific.edu/euler-works/19/E019en.pdf "On transcendental progressions, that is, those whose general terms cannot be given algebraically" by Leonhard Euler Article 19 in Eneström's index Commentarii academiae scientiarum Petropolitanae 5 (1730/1), 1738, p.36-57 Translated from the Latin by Stacy G. Langton, University of San Diego, 1999

[2] https://scholarlycommons.pacific.edu/euler-works/19/E019de.pdf "1738 De progressionibus transcendentibus seu quarum termini generales dari nequeunt, Leonhard Euler"

[3] https://scholarlycommons.pacific.edu/euler-works/122/E122de.pdf "1750 De productis ex infinitis factoribus ortis, Leonhard Euler"

[4] https://scholarlycommons.pacific.edu/euler-works/662/ "De vero valore formulae integralis $\int \partial x \cdot (log(1/x))^n$ a termino $x = 0$ usque ad terminum $x = 1$ extensae (1776) L. Eulero."

[5] https://www.maa.org/sites/default/files/pdf/upload_library/22/Chauvenet/Davis.pdf " Leonhard Euler's Integral: A historic Profile of the Gamma Function in Memoriam: Milton Abramowitz, Philip J. Davis, 1959, National Bureau of Standards, Washington, D. C."

[6] https://en.wikipedia.org/wiki/Gamma function and Π function

[7] https://math.stackexchange.com/questions/234209/derivative-of-gauss-pi-function

[8] https://ia800208.us.archive.org/5/items/bub_gb_$riIOAAAAQAAJ$/bu_gb_$riIOAAAAQAAJ$.pdf "Exercices de Calcul Intégral sur Divers Ordres de Transcendantes et sur les Quadratures Par A. M. LE GENDRE, TOME PREMIER, Paris, 1811"

[9] https://scholarlycommons.pacific.edu/euler-works/663/E663de.pdf "Euler, Leonhard, Plenior expositio serierum illarum memorabilium, quae ex unciis potestatum binomii formantur (1794)."

[10] https://en.wikipedia.org/wiki/Jacobian_matrix_and_determinant

[11] https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4247832/"Factorials of real negative and imaginary numbers - A new perspective, Ashwani K Thukral Nov. 6, 2014"

[12] https://scholarlycommons.pacific.edu/euler-works/675 "De valoribus integralium a termino variabilis $x = 0$ usque ad $x = \infty$ extensorum (1794). Leonhard Euler."

[13] https://scholarlycommons.pacific.edu/euler-works/640/E640-paper.pdf "Comparatio valorum formulae integralis $\int \frac{x^{p-1}\partial x}{\sqrt{(1-x^n)^{(n-q)/n}}}$ a termino $x = 0$ usque ad $x = 1$ extensae L. Eulero. 10 Oct. 1776"

[14] https://scholarlycommons.pacific.edu/euler-works/640/E640-add.pdf "Comparatio valorum formulae integralis $\int \frac{x^{p-1}\partial x}{\sqrt{(1-x^n)^{(n-q)/n}}}$ a termino $x = 0$ usque ad $x = 1$ extensae L. Eulero. 17 Oct. 1776"

[15] https://scholarlycommons.pacific.edu/euler-works/321 "Observationes circa integralia formularum $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ posito post integrationem $x = 1$ (1766). Euler, Leonhard"

[16] https://web.archive.org/web/20161112081854/http://www.plouffe.fr/simon/math/Artin%20E.%20 The%20Gamma%20Function%20(1931)(23s).pdf