Using Geometric Algebra: A High-School-Level Demonstration of the Constant-Angle Theorem

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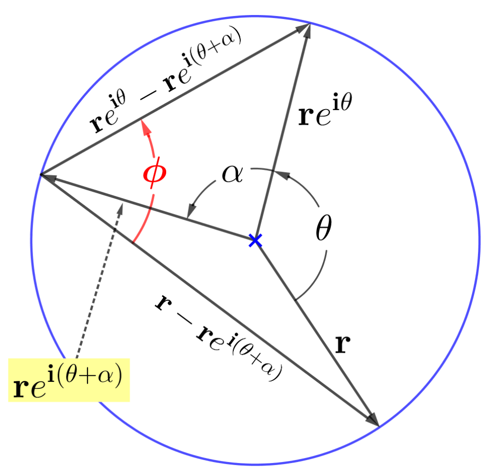
Abstract

Euclid proved (Elements, Book III, Propositions 20 and 21) proved that an angle inscribed in a circle is half as big as the central angle that subtends the same arc. We present a high-school-level version of Hestenes' GA-based proof ([1]) of that same theorem. We conclude with comments on the need for learners of GA to learn classical geometry as well.



"Demonstrate that the inscribed angle ϕ is is half as large as the central angle θ ."

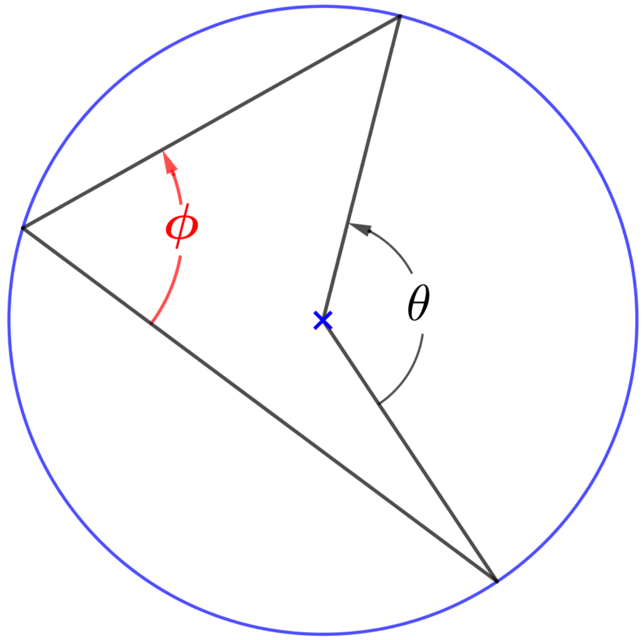


Figure 1: Demonstrate that the inscribed angle ϕ is is half as large as the central angle θ.

1 Introduction

Using the Geometric-Algebra (GA) concept of the "reverse" of a multivector, Hestenes ([1], p. 88) presents an elegant proof of the Constant-Angle Theorem. Here, we present a proof that is perhaps more accessible to high-school students. We will use basic concepts from GA, plus three simple trigonometric identities that may not be familiar to students.

2 Problem Statement

> *In Fig. 1,"Demonstrate that the inscribed angle ϕ is is half as large as the central angle θ."*

3 The Formulas and Identities that We Will Use

3.1 Trigonometric Identities

Five of the identities are familiar ones: for any two angles A and B,

- $\sin^2 A + cos^2 A = 1$;

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- $\sin(A+B) = \sin A \cos B + \cos A \sin B$;
- $\sin 2A = 2 \sin A \cos A$;
- $\cos(A+B) = \cos A \cos B - \sin A \sin B$;
- $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$.

The three less-familiar identities are

- $\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$;
- $\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$;
- $\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$.

3.2 GA Formulas and Identities

- For any real number γ, $e^{\mathbf{i}\gamma} = \cos\gamma + \mathbf{i}\sin\gamma$;
- In 2D (plane) GA, right-multiplying a vector $\mathbf{v}$ by $\lambda e^{\mathbf{i}\gamma}$ rotates $\mathbf{v}$ anticlockwise by the angle γ, and scales it by the factor λ . (Therefore, the rotation is clockwise if γ is negative.)
- Two multivectors are equal if, and only if, their respective parts of each grade are equal to each other. In 2D GA, this postulate means that the multivectors $[\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}]$ and $[\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v}]$ are equal if and only if $\mathbf{a}\cdot\mathbf{b} = \mathbf{u}\cdot\mathbf{v}$ and $\mathbf{a}\wedge\mathbf{b} = \mathbf{u}\wedge\mathbf{v}$.

4 Solution

4.1 Formulating the Problem in GA Terms

Fig. 2 is adapted from the proof given in [1] (p. 88).

The inscribed angle ϕ subtends the same arc as the central angle θ, which is smaller than 180°. The vertex of the angle ϕ is an arbitrary point outside that arc. Our task will be to show that $\phi = \theta/2$, regardless of the position of the vertex of the angle ϕ. That is, regardless of the angle α. We've expressed the sides of the angles θ and ϕ as differences between rotations of the vector $\mathbf{r}$. Thus, the vector $\mathbf{r}e^{\mathbf{i}\theta} - \mathbf{r}e^{\mathbf{i}(\theta+\alpha)}$ is a rotation and scaling of the vector $\mathbf{r} - \mathbf{r}e^{\mathbf{i}(\theta+\alpha)}$:

$$\mathbf{r}e^{\mathbf{i}\theta} - \mathbf{r}e^{\mathbf{i}(\theta+\alpha)} = \lambda\left\{\mathbf{r} - \mathbf{r}e^{\mathbf{i}(\theta+\alpha)}\right\}e^{\mathbf{i}\phi}.$$

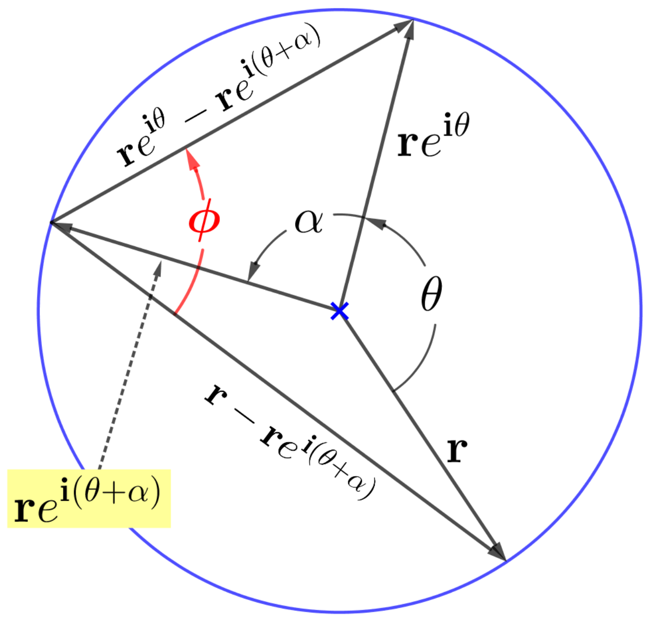

Figure 2: Formulation of the problem in GA terms. (Adapted from [1], p. 88.) The inscribed angle ϕ subtends the same arc as the central angle θ, which is smaller than 180°. The vertex of the angle ϕ is an arbitrary point outside that arc. Our task will be to show that $\phi = \theta/2$, regardless of the position of the vertex of the angle ϕ. That is, regardless of the angle α. We've expressed the sides of the two angles as differences between rotations of the vector $\mathbf{r}$. Thus, the vector $\mathbf{r}e^{\mathbf{i}\theta} - \mathbf{r}e^{\mathbf{i}(\theta+\alpha)}$ is a rotation and scaling of the vector $\mathbf{r} - \mathbf{r}e^{\mathbf{i}(\theta+\alpha)}$.

Now, we left-multiply both sides by $\mathbf{r}$'s multiplicative inverse $\mathbf{r}^{-1}$ to obtain

$$e^{\mathbf{i}\theta} - e^{\mathbf{i}(\theta+\alpha)} = \lambda \left\{1 - e^{\mathbf{i}(\theta+\alpha)}\right\} e^{\mathbf{i}\phi}.$$

Next, we expand both sides of that equation by using the identity $e^{\mathbf{i}\gamma} = \cos\gamma + \mathbf{i}\sin\gamma$:

$$\begin{aligned}\cos\theta - \cos(\theta+\alpha) + \mathbf{i}\left[\sin\theta - \sin(\theta+\alpha)\right] = \lambda\left[\cos\phi - \cos\phi\cos(\theta+\alpha) + \sin\phi\sin(\theta+\alpha)\right] \\ + \lambda\mathbf{i}\left[\sin\phi - \sin\phi\cos(\theta+\alpha) - \cos\phi\sin(\theta+\alpha)\right]\end{aligned}$$

By the definition of the equality of multivectors,

$$\begin{aligned}\cos\theta - \cos(\theta+\alpha) &= \lambda\left[\cos\phi - \cos\phi\cos(\theta+\alpha) + \sin\phi\sin(\theta+\alpha)\right], \text{ and} \\ \sin\theta - \sin(\theta+\alpha) &= \lambda\left[\sin\phi - \sin\phi\cos(\theta+\alpha) - \cos\phi\sin(\theta+\alpha)\right].\end{aligned}$$

Note that for any two multivectors M and N, if $\mathbf{i}M = \mathbf{i}N$, then $M = N$.
We are now finished with GA. The rest is conventional algebra and trig identities.

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From the previous equation, we can see that both sides of the following are equal to λ, and therefore equal to each other:

$$\frac{\cos\theta - \cos(\theta+\alpha)}{\cos\phi - \cos\phi\cos(\theta+\alpha) + \sin\phi\sin(\theta+\alpha)} = \frac{\sin\theta - \sin(\theta+\alpha)}{\sin\phi - \sin\phi\cos(\theta+\alpha) - \cos\phi\sin(\theta+\alpha)}.$$

Cross-multiplying, simplifying, and then rearranging, we find that

$$\begin{aligned}&\sin\phi\left[\sin\theta\sin(\theta+\alpha) + \cos\theta\cos(\theta+\alpha) + \cos(\theta+\alpha) - \cos\theta - 1\right] \\ &\quad = \cos\left[\sin(\theta+\alpha) + \sin\theta\sin(\theta+\alpha) - \cos\theta\sin(\theta+\alpha) - \sin\theta\right].\end{aligned}$$

When simplifying, don't expand $\sin(\theta+\alpha)$ and $\cos(\theta+\alpha)$, but do watch for the terms $\sin\phi\sin^2(\theta+\alpha)$ and $\sin\phi\cos^2(\theta+\alpha)$ —which sum to $\sin\phi$ —and the cancellation of the two $\cos\phi\sin(\theta+\alpha)\cos(\theta+\alpha)$ terms.

Next, we expand the terms in blue, using the identities $\sin(A+B) = \sin A\cos B + \cos A\sin B$ and $\cos(A+B) = \cos A\cos B - \sin A\sin B$, then simplify. The result is

$$\sin\phi\left[\cos(\theta + \cos\alpha - \cos\theta)\right] = \cos\phi\left[\sin(\theta+\alpha) - \sin\theta - \sin\alpha\right]. \tag{4.1}$$

At this point, we might try to solve for $\sin\phi$ by squaring both sides, then using $\cos^2\phi = 1 - \sin^2\phi$ to obtain a quadratic in $\sin\phi$. We would hope that α would somehow be eliminated in the process. However, that route becomes extremely tedious (I tried it!), so we will instead see what we might accomplish by using the identities

1. $\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$;
2. $\cos A - \cos B = -2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$; and
3. $\sin A - \sin B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$.

Let's begin with the right-hand side of Eq. (4.1). Using the first of the above identities, $-\sin\theta - \sin\alpha$ becomes

$$-\left[\sin\theta + \sin\alpha\right] = -2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right).$$

Now, we see what we might do with the term $\sin(\theta+\alpha)$ on the right-hand side. We'll write it as

$$\sin(\theta+\alpha) = \sin\left[2\left(\frac{\theta+\alpha}{2}\right)\right] = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta+\alpha}{2}\right).$$

Thus, the right-hand side becomes

$$\cos\phi\left[2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta+\alpha}{2}\right) - 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)\right]. \qquad (4.2)$$

Next, we'll work on the left-hand side of Eq. (4.1). To use the second of the above identities, we'll write the difference $\cos\alpha - \cos\theta$ as $-(\cos\theta - \cos\theta)$, which becomes

$$-(\cos\theta - \cos\alpha) = 2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right).$$

Now, we see how we might eliminate the factor $\sin\left(\frac{\theta+\alpha}{2}\right)$: we'll write $\cos(\theta+\alpha)$ as $\cos\left[2\left(\frac{\theta+\alpha}{2}\right)\right]$, which is

$$\cos^2\left(\frac{\theta+\alpha}{2}\right) - \sin^2\left(\frac{\theta+\alpha}{2}\right) = 1 - 2\sin^2\left(\frac{\theta+\alpha}{2}\right).$$

Making these substitutions, the left-hand side of Eq. (4.1)becomes

$$\cos\phi\left[1 - 2\sin^2\left(\frac{\theta+\alpha}{2}\right) + 2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) - 1\right], \text{or}$$

$$\cos\phi\left[-2\sin^2\left(\frac{\theta+\alpha}{2}\right) + 2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)\right]. \qquad (4.3)$$

Returning to Eq. (4.1), and using the expressions in Eqs. (4.2) and (4.3), we see that

$$\frac{\sin\phi}{\cos\phi} = \frac{2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta+\alpha}{2}\right) - 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)}{-2\sin^2\left(\frac{\theta+\alpha}{2}\right) + 2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)}, \text{ or}$$

$$\tan\phi = \frac{\cos\left(\frac{\theta+\alpha}{2}\right) - \cos\left(\frac{\theta-\alpha}{2}\right)}{\sin\left(\frac{\theta-\alpha}{2}\right) - \sin\left(\frac{\theta+\alpha}{2}\right)}. \qquad (4.4)$$

To proceed further, we'll simplify the numerator of Eq. 4.4 by using the identity $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$:

$$\cos\left(\frac{\theta+\alpha}{2}\right) - \cos\left(\frac{\theta-\alpha}{2}\right) = -2\sin\frac{\theta}{2}\sin\frac{\alpha}{2}.$$

We simplify the denominator of Eq. 4.4 by using the identity $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$:

$$\sin\left(\frac{\theta-\alpha}{2}\right) - \sin\left(\frac{\theta+\alpha}{2}\right) = 2\cos\frac{\theta}{2}\sin\frac{-\alpha}{2} = -2\cos\frac{\theta}{2}\sin\frac{\alpha}{2}.$$

Hence, Eq. 4.4 becomes

$$\tan\phi = \frac{-2\sin\frac{\theta}{2}\sin\frac{\alpha}{2}}{-2\cos\frac{\theta}{2}\sin\frac{\alpha}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}. \tag{4.5}$$

Because $\theta < 180°$, the finding that $\tan\phi = \tan\frac{\theta}{2}$ means that $\phi = \theta/2$. In addition, because this relationship is true for an arbitrary point on the circle outside the subtended arc, said relationship holds for all such points because of the Law of Universal Generalization ([2]). □

5 Discussion

I hope the present document has been a useful example of how to use GA. That having been said, those who are familiar with Euclid's proof of this same theorem (Elements, Book III. Propositions 20 and 21) recognized immediately that our GA-based proof was going to be considerably more complicated. Thus the observations of Professor Miroslav Josipović are on-target):

> [T]raditional geometry is our heritage and we must preserve it. Traditional geometry is an expression of pure human genius and its study has a very favorable effect on brain development (in the light of modern knowledge about neuroplasticity). On the other hand, geometric algebra, regardless of its power, is not a substitute for everything, and especially it is not a "magic wand" or some "royal path" into geometry. Geometric algebra simply introduces clarity into the question of vector multiplication, with far-reaching consequences. However, caution is required: (in my opinion) you can't be good at GA if you skip Euclid.

References

[1] D. Hestenes, 1999, *New Foundations for Classical Mechanics*, (Second Edition), Kluwer Academic Publishers (Dordrecht/Boston/London).

[2] Old Dominion Computer Science, "Universal Generalization" , retrieved 6 December 2020.