New Principles of Differential Equations III

Hong Lai Zhu *

School of Physics and Electronic Information,Huaibei Normal University, Anhui 235000, China

Abstract

Using the new method proposed in this paper, in theory, it is possible to obtain general or exact solutions of an infinite number of ordinary differential equations and partial differential equations. These equations can be linear or nonlinear. We enumerate some typical cases and use the new method to prove that some equations do not have certain forms of solutions.

Keywords: general solutions; exact solutions; Riccati equation; Abel equation.

Introduction

Since the establishment of the theory of differential equations, although the analytical methods have been progressing [1-5], the types of differential equations that can be solved are still very limited, so various approximate numerical methods [6-10] and qualitative theories have been developed [11-15]. Because analysis is the foundation and core of mathematics, constantly proposing new and effective analysis methods is the eternal theme of mathematical progress.

In ordinary differential equations, the theory of the first integral is relatively complete and mature, such as a k-order ODE

$$F\left(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(k)}\right) = 0. \tag{1}$$

If it can be integrated as

$$G_1\left(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(k-1)}\right) = C_1, \tag{2}$$

where C_1 is an arbitrary constant, then (2) is called the first integral of (1). If (2) also can be integrated as

$$G_2\left(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(k-2)}\right) = C_2. \tag{3}$$

Then we call (3) the first integral of (2) and so on. If the k first integrals of a k-order ODE can be found, we can get the general solution of (1).

Finding the first integral of an ODE in a specific problem is often very difficult, so many equations have to be solved by other methods. Based on the first integral theory, we use its inverse operation to propose Method 1. General solutions of many commonly used differential equations had been obtained through various methods. In fact, these solutions can be obtained very simply by Method 1, such as

$$y' + b(x)\, y + c(x)\, y^n = 0, \tag{4}$$

$$y'' + b(y) = 0, \tag{5}$$

*E-mail address: honglaizhu@qq.com; honglaizhu@163.com

$$u_x + f(x, y)\, u = g(x, y)\,, \tag{6}$$

and many more. Method 1 can not only solve the above simple equations, in theory, this method can solve an infinite variety of linear and nonlinear ODEs and PDEs. We will introduce new laws and typical applications respectively.

1. New principles and methods I

It is well known that using algebraic methods to solve some algebraic equations may be get extraneous root, that is, the correct algebraic operation might not get the correct result; so any conclusion from correct logic is not always the correct conclusion, conclusive verification is an indispensable link to ensure correct results, thus we first put forward a verification axiom:

Validation Axioms. *Any conclusion obtained by the correct logic, which has not been conclusively corroborated, is not always the correct conclusion.*

We will follow the validation axiom to verify any result obtained in this paper to make it correct.

Theorem 1 is proposed below.

Theorem 1. *In a continuous area D, $(D \subseteq \mathbb{R}^1)$, if $F\left(x, y, y^{(1)}, y^{(2)} \ldots y^{(k)}\right) = 0$, and $F^{(m)} = \frac{d^m F}{dx^m}$ exists $(m \geq 1)$, then $F^{(m)} = 0$.*

Proof. In a continuous area D, $(D \subseteq \mathbb{R}^1)$,

$$F\left(x, y, y^{(1)}, y^{(2)} \ldots y^{(k)}\right) = 0 \Longrightarrow dF\left(x, y, y^{(1)}, y^{(2)} \ldots y^{(k)}\right) = 0 \Longrightarrow d(dF) = d^2F = 0$$

$$\Longrightarrow d\left(d^2 F\right) = d^3 F = 0 \ldots$$

Namely

$$d^m F = 0, (m \geq 1)\,. \tag{7}$$

Since x is the independent variable, therefore

$$d^m F = \frac{d^m F}{dx^m} dx^m = 0 \Longrightarrow \frac{d^m F}{dx^m} = F^{(m)}\left(x, y, y^{(1)}, y^{(2)} \ldots y^{(k+m)}\right) = 0.$$

Then the theorem is proven.□

Below we define the source equation and sub-equations of an ODE according to Theorem 1.

Definition 1. *In a continuous region D, $(D \subseteq \mathbb{R}^1)$, if $F\left(x, y, y^{(1)}, y^{(2)} \ldots y^{(k)}\right) = 0$, and $F^{(m)}\left(x, y, y^{(1)}, y^{(2)} \ldots y^{(k+m)}\right) = 0$ exists $(m \geq 1)$, then $F = 0$ is called the* **source equation***; $F^{(m)} = 0$ is called* **the first type m-order subsidiary equation** *of $F = 0$, Various $(k+m)$-order ODEs obtained by the mathematical operations of $F = 0$ and $F^{(i)} = 0, (1 \leq i \leq m)$ are called* **the second type m-order subsidiary equations** *of $F = 0$.*

A source equation can be a differential equation or a functional equation, the sum of the first type m-order sub-equations and the second type m-order sub-equations is called the m-order sub-equations groups of the source equation. Depending on Definition 1, if a source equation is infinitely differentiable, it can construct an infinite number of sub-equations. For a specific source equation, since the mathematical operations of the sub-equations of each order can be infinite, theoretically the number of any m-order sub-equations could be infinite.

Below we propose Theorem 2 using Definition 1 and the Axiom of Verification.

Theorem 2. *The solution of any solvable source equation may be the solution of its any existing sub-equation.*

Proof. Any existing sub-equation is obtained through a series of mathematical operations of the source equation. Since the source equation can be obtained by the sub-equation and the corresponding mathematical inverse operation, the solution of the source equation may be the solution of the sub-equation. The theorem is proved. □

According to the axiom of verification, the correct logic does not necessarily get the correct result, and the result needs to be verified. Theorem 2 states that any sub-equation of any solvable source equation may be solvable, and the solution of the solvable source equation may be the solution of any one of its sub-equations.

Definition 2 is proposed below.

Definition 2. *If the structure of an equation to be solved is the same as a sub-equation of a source equation, the source equation is called the* **corresponding source equation** *of the equation to be solved, and the sub-equation is called the* **corresponding sub-equation** *of the equation to be solved. If the structure of an equation to be solved is different from all the sub-equations of a source equation, then the source equation is called a* **non-corresponding source equation** *of the equation to be solved.*

According to the currently solvable function equations and differential equations, they can be used as source equations to further obtain the solutions of their infinitely many corresponding sub-equations. According to actual cases, we will find that new restrictions sometimes appear. For an equation to be solved, the corresponding source equation may not be unique.

The new method for getting solutions of differential equations according to Theorem 2 is called Method 1, and the details are as follows.

Method 1.

1. According to the structure of an equation to be solved, construct a solvable corresponding source equation with undetermined functions and derive the corresponding sub-equation.

2. Solving undetermined functions in the source equation by comparing the coefficients of the equation to be solved and the corresponding sub-equation, that is, known functions in the equation to be solved is used to represent undetermined functions in the source equation.

3. According to Theorem 2, the solution of the equation to be solved can be obtained by using the pending function in the source equation.

4. Verify the solution according to the axiom of verification.

The essence of using Method 1 to solve differential equations is to find the solvable corresponding source equation of an equation to be solved, or first set a solvable source equation and construct its various sub-equations to investigate which differential equations can be solved. Below we will use typical cases to explain the above concepts, theorems and methods.

2. General solutions and exact solutions of ODEs

In this section, unless otherwise specified, C, C_i *and* A_i *are arbitrary constants* $(i = 1, 2, \ldots)$.

When applying Method 1 to solve ODEs, there is an important note: because coefficients of an equation to be solved do not exist arbitrary constant, when comparing coefficients of the equation to be solved and the corresponding sub-equation, if coefficients of the corresponding sub-equation exists arbitrary constants, and there are no arbitrary constants in the calculated coefficients of the equation to be solved, these arbitrary constants can be retained, otherwise they need to be ordinary constants. Below we use typical cases to illustrate.

First we examine what types of solvable sub-equations can be constructed by a solvable source equation, such as

Example 1. *Analyze the typical corresponding sub-equations of the source equation (8), and obtain the general or particular solutions of some of them.*

$$f(x)\,y + g(x) = 0. \tag{8}$$

Solution. Eq. (8) is a solvable function equation, $f(x)$ and $g(x)$ are arbitrary undetermined functions, and the solution is

$$y = \frac{-g(x)}{f(x)}. \tag{9}$$

According to Theorem 1

$$f(x)\,y + g(x) = 0 \Longrightarrow f'(x)\,y + f(x)\,y' + g'(x) = 0.$$

Namely

$$y' + \frac{f'(x)}{f(x)}y + \frac{g'(x)}{f(x)} = 0. \tag{10}$$

(10) is the first type first-order sub-equation of (8), and the corresponding equation is

$$y' + b(x)\,y + c(x) = 0, \tag{11}$$

which is the first-order linear equation. According to Method 1, comparing the coefficients in (10) and (11), we get

$$\frac{f'(x)}{f(x)} = b(x) \Longrightarrow f(x) = A_1 e^{\int b(x)dx},$$

$$\frac{g'(x)}{f(x)} = c(x) \Longrightarrow g(x) = A_2 + A_1 \int c(x)\, e^{\int b(x)dx} dx,$$

where A_1 and A_2 are arbitrary constants, note there are no arbitrary constants in $b(x)$ and $c(x)$, so the general solution of (11) is

$$y = \frac{-g(x)}{f(x)} = e^{-\int b(x)dx}\left(-\frac{A_2}{A_1} - \int c(x)\, e^{\int b(x)dx} dx\right),$$

that is

$$y = e^{-\int b(x)dx}\left(C - \int c(x)\, e^{\int b(x)dx} dx\right). \tag{12}$$

We use Method 1 to get the general solution of the first-order linear ODE very simply and beautifully.

According to Theorem 1

$$f(x)\,y + g(x) = 0 \Longrightarrow f''(x)\,y + 2f'(x)\,y' + f(x)\,y'' + g''(x) = 0.$$

Namely

$$y'' + \frac{2f'(x)}{f(x)}y' + \frac{f''(x)}{f(x)}y + \frac{g''(x)}{f(x)} = 0. \tag{13}$$

(13) is the first type second-order sub-equation of (8), and the corresponding equation is

$$y'' + b(x)\,y' + c(x)\,y + d(x) = 0, \tag{14}$$

which is the second-order linear equation. According to Method 1, comparing the coefficients in (13) and (14), we get

$$\frac{2f'(x)}{f(x)} = b(x) \Longrightarrow f(x) = A_1 e^{\frac{1}{2}\int b(x)dx} \Longrightarrow f''(x) = \frac{A_1}{2} b'(x)\, e^{\frac{1}{2}\int b(x)dx} + \frac{A_1}{4} b^2(x)\, e^{\frac{1}{2}\int b(x)dx},$$

$$\frac{f''(x)}{f(x)} = \frac{1}{2} b'(x) + \frac{1}{4} b^2(x) = c(x),$$

$$d(x) = \frac{g''(x)}{f(x)} \Longrightarrow g''(x) = A_1 d(x)\, e^{\frac{1}{2}\int b(x)dx}.$$

That is

$$f(x) = A_1 e^{\frac{1}{2}\int b(x)dx},\; g(x) = A_3 + \int \left(A_2 + A_1 \int d(x)\, e^{\frac{1}{2}\int b(x)dx} dx \right) dx, \tag{15}$$

$$\frac{1}{2} b'(x) + \frac{1}{4} b^2(x) = c(x), \tag{16}$$

where A_1, A_2 and A_3 are arbitrary constants, note there are no arbitrary constants in $b(x), c(x)$ and $d(x)$. According to (16), (14) becomes

$$y'' + b(x)\,y' + \left(\frac{1}{2} b'(x) + \frac{1}{4} b^2(x) \right) y + d(x) = 0. \tag{17}$$

According to Method 1, by (9) and (15), we get

$$y = \frac{-g(x)}{f(x)} = e^{\frac{-1}{2}\int b(x)dx} \left(-\frac{A_3}{A_1} - \int \left(\frac{A_2}{A_1} + \int d(x)\, e^{\frac{1}{2}\int b(x)dx} dx \right) dx \right).$$

So the general solution of Eq. (17) is

$$y = e^{\frac{-1}{2}\int b(x)dx} \left(C_1 + C_2 x - \iint d(x)\, e^{\frac{1}{2}\int b(x)dx} dx dx \right). \tag{18}$$

From (9, 10), we get

$$\frac{2f'(x)}{f(x)} y' = \frac{2f'(x)}{g(x)} y y' = -2\frac{(f'(x))^2}{f^2(x)} y - \frac{2f'(x)\,g'(x)}{f^2(x)} = -2y^{-1}(y')^2 - 2\frac{g'(x)}{f(x)} y^{-1} y', \tag{19}$$

$$\frac{f''(x)}{f(x)} y = \frac{-f''(x)\,g(x)}{f^2(x)} = \frac{f''(x)}{g(x)} y^2 = -\frac{f''(x)}{f'(x)} y' - \frac{f''(x)\,g'(x)}{f(x)\,f'(x)}, \tag{20}$$

$$\frac{g''(x)}{f(x)} = -\frac{g''(x)}{g(x)} y = \frac{-g''(x)\,y'}{f'(x)\,y + g'(x)}. \tag{21}$$

That is, on the basis of (13), many second type second-order sub-equations can be constructed, such as

$$y'' + \frac{2f'(x)}{f(x)} y' + \frac{f''(x)}{g(x)} y^2 + \frac{g''(x)}{f(x)} = 0, \tag{22}$$

$$y'' + \frac{2f'(x)}{f(x)} y' + \frac{f''(x)}{g(x)} y^2 - \frac{g''(x)}{g(x)} y = 0, \tag{23}$$

$$y'' + \frac{2f'(x)}{g(x)} yy' + \frac{f''(x)}{g(x)} y^2 + \frac{g''(x)}{f(x)} = 0, \tag{24}$$

$$y'' + \frac{2f'(x)}{g(x)} yy' + \frac{f''(x)}{g(x)} y^2 - \frac{g''(x)}{g(x)} y = 0, \tag{25}$$

$$y'' - 2y^{-1}(y')^2 - 2\frac{g'(x)}{f(x)} y^{-1}y' + \frac{f''(x)}{f(x)} y + \frac{g''(x)}{f(x)} = 0. \tag{26}$$

The corresponding equation of (22) is

$$y'' + b(x) y' + c(x) y^2 + d(x) = 0. \tag{27}$$

According to Method 1, comparing the coefficients in (22) and (27), we get

$$\frac{2f'(x)}{f(x)} = b(x) \Longrightarrow f(x) = A_1 e^{\frac{1}{2}\int b(x)dx} \Longrightarrow f''(x) = \frac{A_1}{2} b'(x) e^{\frac{1}{2}\int b(x)dx} + \frac{A_1}{4} b^2(x) e^{\frac{1}{2}\int b(x)dx},$$

$$d(x) = \frac{g''(x)}{f(x)} \Longrightarrow g''(x) = A_1 d(x) e^{\frac{1}{2}\int b(x)dx} \Longrightarrow g(x) = A_3 + A_2 x + A_1 \iint d(x) e^{\frac{1}{2}\int b(x)dx} dxdx,$$

$$c(x) = \frac{f''(x)}{g(x)}.$$

Namely

$$f(x) = A_1 e^{\frac{1}{2}\int b(x)dx}, g(x) = A_3 + \int \left(A_2 + A_1 \int d(x) e^{\frac{1}{2}\int b(x)dx} dx \right) dx, \tag{28}$$

$$c(x) = \frac{\frac{A_1}{2} b'(x) e^{\frac{1}{2}\int b(x)dx} + \frac{A_1}{4} b^2(x) e^{\frac{1}{2}\int b(x)dx}}{A_3 + A_2 x + A_1 \iint d(x) e^{\frac{1}{2}\int b(x)dx} dxdx}. \tag{29}$$

Because $c(x)$ does not contain arbitrary constants, A_1, A_2 and A_3 are all ordinary constants. From (9) and (28), the particular solution of (27) under (29) is

$$y = \frac{-g(x)}{f(x)} = -e^{\frac{-1}{2}\int b(x)dx} \left(\frac{A_3}{A_1} + \frac{A_2}{A_1} x + \iint d(x) e^{\frac{1}{2}\int b(x)dx} dxdx \right). \tag{30}$$

The infinitely many linear or non-linear equations corresponding to the remaining sub-equations of (8) can be similarly solved, and we will not deduct them here.

Example 2. *Analyze which solvable first-order sub-equations of the source equation (31) exist, and obtain general or particular solutions of some of the equations.*

$$f(x) y^m + g(x) = 0. \tag{31}$$

Solution. According to Theorem 1

$$f(x) y^m + g(x) = 0 \Longrightarrow mf(x) y^{m-1} y' + f'(x) y^m + g'(x) = 0,$$

that is,

$$y' + \frac{f'(x)}{mf(x)} y + \frac{g'(x)}{mf(x)} y^{1-m} = 0. \tag{32}$$

The corresponding equation of (32) is

$$y' + b(x) y + c(x) y^n = 0. \tag{33}$$

(33) is Bernoulli equation. The previous method was to set $w = y^{1-n}$, convert (33) into a first-order linear equation, and then use its general solution to obtain the general solution of (33). Using Method 1, we can directly resolve the general solution of (33). Comparing the coefficients of (32) and (33), we get

$$\frac{f'(x)}{mf(x)} = b(x) \Longrightarrow f(x) = C_1 e^{m\int b(x)dx},$$

$$\frac{g'(x)}{mf(x)} = c(x) \Longrightarrow g(x) = C_2 + C_1 \int c(x)\, e^{m\int b(x)dx} dx,$$

$$m = 1 - n.$$

Namely

$$y = \left(\frac{-g(x)}{f(x)}\right)^{\frac{1}{m}} = \left(e^{(n-1)\int b(x)dx}\left(C_3 + C_4 \int c(x)\, e^{(1-n)\int b(x)dx} dx\right)\right)^{\frac{1}{1-n}}. \tag{34}$$

Note that there are two arbitrary constants in (34). After substituting (34) into (33), we can get $C_4 = n - 1$, so the general solution of (33) is

$$y = \left(e^{(n-1)\int b(x)dx}\left(C + (n-1)\int c(x)\, e^{(1-n)\int b(x)dx} dx\right)\right)^{\frac{1}{1-n}}. \tag{35}$$

By (31) and (32), we get

$$\frac{f'(x)}{mf(x)} y = \frac{-f'(x)}{mg(x)} y^{m+1},$$

$$\frac{f'(x)}{mf(x)} y = \frac{f'(x)}{mf(x)} \frac{y^k}{\left(\frac{-g(x)}{f(x)}\right)^{\frac{k-1}{m}}} = \frac{f'(x)}{m} (f(x))^{\frac{k-m-1}{m}} (-g(x))^{\frac{1-k}{m}} y^k.$$

For $n = 1 - m$, Eq. (31) can also construct

$$y' - \frac{f'(x)}{(1-n)\, g(x)} y^{2-n} + \frac{g'(x)}{(1-n)\, f(x)} y^n = 0, \tag{36}$$

$$y' + \frac{f'(x)}{1-n} (f(x))^{\frac{k+n-2}{1-n}} (-g(x))^{\frac{1-k}{1-n}} y^k + \frac{g'(x)}{(1-n)\, f(x)} y^n = 0, \tag{37}$$

Etc. the second type first-order sub-equations of (31). Eq. (37) is the generalized Abel equation [16, 17], according to (31), the particular solutions of (36, 37) are all

$$y = \left(\frac{-g(x)}{f(x)}\right)^{\frac{1}{1-n}}. \tag{38}$$

When $n = 0$, (36) is transformed into an Riccati equation

$$y' - \frac{f'(x)}{g(x)} y^2 + \frac{g'(x)}{f(x)} = 0. \tag{39}$$

According to (38), a particular solution of (39) is

$$y = \frac{-g(x)}{f(x)}. \tag{40}$$

When $n = 0, k = 3$, (37) is transformed into an Abel equation

$$y' + \frac{f'(x)\, f(x)}{g^2(x)} y^3 + \frac{g'(x)}{f(x)} = 0. \tag{41}$$

According to (38), a particular solution of (41) is (40) too.

Example 3. *Using Method 1 analyze the particular solutions of Riccati equation.*

Solution. The form of Riccati equation is

$$y' + p(x)\, y^2 + q(x)\, y + r(x) = 0. \tag{42}$$

Set the source equation is

$$f(x)\, y + g(x) = 0. \tag{8}$$

then

$$y' + \frac{f'(x)}{f(x)} y + \frac{g'(x)}{f(x)} = 0. \tag{10}$$

A second type first-order sub-equation of Eq. (8) is

$$y' + f(x)\, y^2 + g(x)\, y + \frac{f'(x)}{f(x)} y + \frac{g'(x)}{f(x)} = 0,$$

that is

$$y' + f(x)\, y^2 + \left(g(x) + \frac{f'(x)}{f(x)}\right) y + \frac{g'(x)}{f(x)} = 0. \tag{43}$$

Comparing the coefficients in (42) and (43), we have

$$p(x) = f(x), \frac{g'(x)}{f(x)} = r(x) \Longrightarrow g(x) = \int r(x)\, p(x)\, dx,$$

$$q(x) = g(x) + \frac{f'(x)}{f(x)} = \int r(x)\, p(x)\, dx + \frac{p'(x)}{p(x)}.$$

So Eq. (42) becomes

$$y' + p(x)\, y^2 + \left(\int r(x)\, p(x)\, dx + \frac{p'(x)}{p(x)}\right) y + r(x) = 0, \tag{44}$$

and

$$y = \frac{-g(x)}{f(x)} = \frac{-\int r(x)\, p(x)\, dx}{p(x)}.$$

Thus the particular solution of Eq. (44) is

$$y_0 = \frac{-\int r(x)\, p(x)\, dx}{p(x)}. \tag{45}$$

If the source equation of Eq. (42) is

$$f(x)\, y^{-2} + g(x) = 0. \tag{46}$$

Then

$$f(x)\, y^{-2} + g(x) = 0 \Longrightarrow f'(x)\, y^{-2} - 2f(x)\, y^{-3} y' + g'(x) = 0 \Longrightarrow y' - \frac{f'(x)}{2f(x)} y - \frac{g'(x)}{2f(x)} y^3 = 0,$$

$$f(x)\,y^{-2}+g(x)=0\Longrightarrow g(x)\,y^{2}+f(x)=0.$$

A second type first-order sub-equation of Eq. (46) is

$$y'-\frac{g'(x)}{2f(x)}y^{3}+g(x)\,y^{2}-\frac{f'(x)}{2f(x)}y+f(x)=0.$$

Set

$$g(x)=p(x)=k.$$

So

$$y'+ky^{2}-\frac{f'(x)}{2f(x)}y+f(x)=0. \tag{47}$$

Comparing the coefficients in (42) and (47), we obtain

$$r(x)=f(x)\,,q(x)=-\frac{f'(x)}{2f(x)}=-\frac{r'(x)}{2r(x)}.$$

Then Eq. (42) becomes

$$y'+ky^{2}-\frac{r'(x)}{2r(x)}y+r(x)=0. \tag{48}$$

Using Eq. (46), the particular solution of Eq. (48) is

$$y^{2}=-\frac{f(x)}{g(x)}\Longrightarrow y=\pm\sqrt{-\frac{r(x)}{k}}.$$

That is

$$y=\pm\sqrt{-\frac{r(x)}{k}}. \tag{49}$$

If the source equation of Eq. (42) is

$$f(x)\,y^{m}+g(x)=0, \tag{31}$$

$$f(x)\,y^{m}+g(x)=0\Longrightarrow mf(x)\,y^{m-1}y'+f'(x)\,y^{m}+g'(x)=0.$$

Namely

$$y'+\frac{f'(x)}{mf(x)}y+\frac{g'(x)}{mf(x)}y^{1-m}=0. \tag{32}$$

Set $m=-1$, then

$$f(x)\,y^{-1}+g(x)=0\Longrightarrow f(x)+g(x)y=0,$$

$$y'-\frac{f'(x)}{f(x)}y-\frac{g'(x)}{f(x)}y^{2}+f(x)+g(x)\,y=0.$$

A second type first-order sub-equation of Eq. (31) is

$$y'-\frac{g'(x)}{f(x)}y^{2}+\left(g(x)-\frac{f'(x)}{f(x)}\right)y+f(x)=0. \tag{50}$$

The particular solution of Eq. (50) is

$$y=\frac{-f(x)}{g(x)}. \tag{51}$$

Example 3 shows that there is sometimes more than one corresponding source equation for an equation to be solved. Using different source equations can often obtain different solutions.

We propose Theorems 3-6 based on (39, 40) and the conclusions in Example 3.

Theorem 3. *The general solution of Riccati Equation*

$$y' - \frac{f'(x)}{g(x)} y^2 + \frac{g'(x)}{f(x)} = 0, \tag{39}$$

is

$$y = \frac{-g(x)}{f(x)} + \frac{f^{-2}(x)}{C - \int f^{-2}(x)\, g^{-1}(x)\, f'(x)\, dx}. \tag{52}$$

Theorem 4. *The general solution of Riccati Equation*

$$y' + p(x)\, y^2 + \left(\int r(x)\, p(x)\, dx + \frac{p'(x)}{p(x)} \right) y + r(x) = 0, \tag{44}$$

is

$$y = \frac{-\int r(x)\, p(x)\, dx}{p(x)} + \frac{e^{\iint r(x) p(x) dx dx}}{p(x) \left(C + \int e^{\iint r(x) p(x) dx dx} dx \right)}. \tag{53}$$

Theorem 5. *The general solution of Riccati Equation*

$$y' + ky^2 - \frac{f'(x)}{2f(x)} y + f(x) = 0, \tag{47}$$

is

$$y_{\pm} = \pm \sqrt{-\frac{r(x)}{k}} + \frac{\sqrt{r(x)} e^{\mp \int \sqrt{-kr(x)} dx}}{C + k \int \sqrt{r(x)} e^{\mp \int \sqrt{-kr(x)} dx} dx}. \tag{54}$$

Theorem 6. *The general solution of Riccati Equation*

$$y' - \frac{g'(x)}{f(x)} y^2 + \left(g(x) - \frac{f'(x)}{f(x)} \right) y + f(x) = 0, \tag{50}$$

is

$$y = \frac{-f(x)}{g(x)} + \frac{f(x)\, e^{-\int g(x) dx}}{g(x) \left(C - \int \frac{g'(x)}{g(x)} e^{-\int g(x) dx} dx \right)}. \tag{55}$$

Riccati equation has always been one of the important ODEs [18-20]. At present, it is mainly solved by analytical methods [21, 22] or numerical methods [23], and the existence of the solution [24, 25] is also the focus of research.

Below we use Theorem 1 and Method 1 to propose Theorems 7 and 8.

Theorem 7. *The general solution of*

$$y' + a(x)\, b(y) + c(x)\, b(y)\, e^{-\int \frac{1}{b(y)} dy} = 0 \tag{56}$$

is

$$\int \frac{1}{b(y)} dy = \ln \left(C - \int c(x)\, e^{\int a(x) dx} dx \right) - \int a(x)\, dx. \tag{57}$$

Prove. According to Theorem 1, we set the source equation is

$$f(x)h(y) + g(x) = 0. \tag{58}$$

Then

$$f(x)h(y) + g(x) = 0 \Longrightarrow f\left(x\right)h'\left(y\right)y' + f'\left(x\right)h\left(y\right) + g'\left(x\right) = 0,$$

namely

$$y' + \frac{f'\left(x\right)h\left(y\right)}{f\left(x\right)h'\left(y\right)} + \frac{g'\left(x\right)}{f\left(x\right)h'\left(y\right)} = 0, \tag{59}$$

the corresponding equation of Eq. (59) is

$$y' + a\left(x\right)b\left(y\right) + c\left(x\right)k\left(y\right) = 0. \tag{60}$$

Comparing the coefficients in (59) and (60), we get

$$\frac{f'\left(x\right)}{f\left(x\right)} = a\left(x\right) \Longrightarrow f\left(x\right) = C_1 e^{\int a(x)dx},$$

$$\frac{h\left(y\right)}{h'\left(y\right)} = b\left(y\right) \Longrightarrow h\left(y\right) = C_2 e^{\int \frac{1}{b(y)}dy},$$

$$c\left(x\right) = \frac{g'\left(x\right)}{f\left(x\right)} \Longrightarrow g\left(x\right) = \int c\left(x\right)f\left(x\right)dx = C_3 + C_1\int c\left(x\right)e^{\int a(x)dx}dx,$$

$$k\left(y\right) = \frac{1}{h'\left(y\right)} = \frac{b\left(y\right)}{C_2}e^{-\int \frac{1}{b(y)}dy}.$$

That is

$$f\left(x\right) = C_1 e^{\int a(x)dx}, h\left(y\right) = C_2 e^{\int \frac{1}{b(y)}dy}, \tag{61}$$

$$g\left(x\right) = C_3 + C_1\int c\left(x\right)e^{\int a(x)dx}dx, k\left(y\right) = \frac{b\left(y\right)}{C_2}e^{-\int \frac{1}{b(y)}dy}. \tag{62}$$

According to (62), we set $C_2 = 1$, (60) becomes

$$y' + a\left(x\right)b\left(y\right) + c\left(x\right)b\left(y\right)e^{-\int \frac{1}{b(y)}dy} = 0. \tag{56}$$

By (58), we get

$$f(x)h(y) + g(x) = 0 \Longrightarrow C_1 e^{\int a(x)dx}e^{\int \frac{1}{b(y)}dy} + C_3 + C_1\int c\left(x\right)e^{\int a(x)dx}dx = 0$$

$$\Longrightarrow \int a\left(x\right)dx + \int \frac{1}{b\left(y\right)}dy = ln\left(-\frac{C_3}{C_1} - \int c\left(x\right)e^{\int a(x)dx}dx\right).$$

So the general solution of Eq. (56) is

$$\int \frac{1}{b\left(y\right)}dy = \ln\left(C - \int c\left(x\right)e^{\int a(x)dx}dx\right) - \int a\left(x\right)dx. \tag{57}$$

The theorem is proved. □

Theorem 8. *The general solution of*

$$y' + a\left(x\right)b\left(y\right) + c\left(x\right)b\left(y\right)\int \frac{1}{b\left(y\right)}dy = 0 \tag{63}$$

is

$$\int \frac{1}{b(y)}dy = e^{-\int c(x)dx}\left(C - \int a(x)\, e^{\int c(x)dx}dx\right). \tag{64}$$

Prove. According to Theorem 1, we set the source equation is

$$f(x)h(y) + g(x) = 0. \tag{58}$$

Using part of the formulas to prove Theorem 7, according to (58) and (59), we can get

$$\frac{f'(x)\,h(y)}{f(x)\,h'(y)} = -\frac{f'(x)\,h^2(y)}{g(x)\,h'(y)} = -\frac{f'(x)\,g(x)}{f^2(x)\,h'(y)},$$

$$\frac{g'(x)}{f(x)\,h'(y)} = \frac{-g'(x)\,h(y)}{g(x)\,h'(y)}.$$

That is to say (58) can construct many the second type first-order sub-equations, such as

$$y' + \frac{f'(x)\,h(y)}{f(x)\,h'(y)} - \frac{g'(x)\,h(y)}{g(x)\,h'(y)} = 0, \tag{65}$$

$$y' - \frac{f'(x)\,h^2(y)}{g(x)\,h'(y)} + \frac{g'(x)}{f(x)\,h'(y)} = 0, \tag{66}$$

$$y' + \frac{f'(x)\,g(x)}{f^2(x)\,h'(y)} + \frac{g'(x)}{f(x)\,h'(y)} = 0, \tag{67}$$

$$y' - \frac{f'(x)\,h^2(y)}{g(x)\,h'(y)} - \frac{g'(x)\,h(y)}{g(x)\,h'(y)} = 0, \tag{68}$$

$$y' - \frac{f'(x)\,g(x)}{f^2(x)\,h'(y)} - \frac{g'(x)\,h(y)}{g(x)\,h'(y)} = 0. \tag{69}$$

The corresponding equation of above equations is

$$y' + a(x)\,b(y) + c(x)\,k(y) = 0. \tag{60}$$

For Eq. (68), comparing (68) and (60), we get

$$\frac{g'(x)}{g(x)} = c(x) \Longrightarrow g(x) = A_1 e^{\int c(x)dx},$$

$$\frac{h^2(y)}{h'(y)} = b(y) \Longrightarrow h(y) = \frac{-1}{A_2 + \int \frac{1}{b(y)}dy}$$

$$\Longrightarrow k(y) = -\frac{h(y)}{h'(y)} = b(y)\left(A_2 + \int \frac{1}{b(y)}dy\right),$$

$$a(x) = -\frac{f'(x)}{g(x)} \Longrightarrow f(x) = -\int a(x)\,g(x)\,dx = A_4 - A_1 \int a(x)\, e^{\int c(x)dx}dx.$$

Namely

$$g(x) = A_1 e^{\int c(x)dx}, h(y) = \frac{-1}{A_2 + \int \frac{1}{b(y)}dy}, \tag{70}$$

$$f(x) = A_4 - A_1 \int a(x)\, e^{\int c(x)dx}dx, k(y) = b(y)\left(A_2 + \int \frac{1}{b(y)}dy\right). \tag{71}$$

By (71), we set $A_2 = 0$, (60) becomes

$$y' + a(x)\, b(y) + c(x)\, b(y) \int \frac{1}{b(y)} dy = 0. \tag{63}$$

By (58), we have

$$f(x)h(y) + g(x) = 0 \Longrightarrow \frac{-A_4 + A_1 \int a(x)\, e^{\int c(x)dx} dx}{\int \frac{1}{b(y)} dy} + A_1 e^{\int c(x)dx} = 0.$$

So the general solution of Eq. (63) is

$$\int \frac{1}{b(y)} dy = e^{-\int c(x)dx} \left(C - \int a(x)\, e^{\int c(x)dx} dx \right). \tag{64}$$

The theorem is proved. □

Example 4. *Using Method 1 analyzes particular solutions of the first type of Abel equation.*

Solution. The first type of Abel equation is

$$y' + p(x)\, y^3 + q(x)\, y^2 + r(x)\, y + s(x) = 0. \tag{72}$$

Let the source equation of (72) be

$$f(x)\, y + g(x) = 0. \tag{8}$$

Then

$$y' + \frac{f'(x)}{f(x)} y + \frac{g'(x)}{f(x)} = 0. \tag{10}$$

A second type first-order sub-equation of (8) is

$$y' + f(x)\, y^3 + g(x)\, y^2 + \frac{f'(x)}{f(x)} y + \frac{g'(x)}{f(x)} = 0. \tag{73}$$

Comparing (72) and (73), we get

$$p(x) = f(x), g(x) = q(x), \frac{f'(x)}{f(x)} = r(x) \Longrightarrow f(x) = p(x) = e^{\int r(x)dx},$$

$$\frac{g'(x)}{f(x)} = s(x) \Longrightarrow g(x) = q(x) = \int s(x)\, f(x)\, dx = \int s(x)\, e^{\int r(x)dx} dx.$$

Namely

$$p(x) = e^{\int r(x)dx}, q(x) = \int s(x)\, e^{\int r(x)dx} dx. \tag{74}$$

So Eq. (72) becomes

$$y' + y^3 e^{\int r(x)dx} + y^2 \int s(x)\, e^{\int r(x)dx} dx + r(x)\, y + s(x) = 0. \tag{75}$$

So

$$y = \frac{-g(x)}{f(x)} = \frac{-q(x)}{p(x)}.$$

That is, the special solution of (75) is

$$y = \frac{-q(x)}{p(x)}. \tag{75}$$

If the source equation of (72) is

$$f(x)\,y^{-2} + g(x) = 0. \tag{46}$$

So

$$f(x)\,y^{-2} + g(x) = 0 \Longrightarrow f'(x)\,y^{-2} - 2f(x)\,y^{-3}y' + g'(x) = 0 \Longrightarrow y' - \frac{f'(x)}{2f(x)}y - \frac{g'(x)}{2f(x)}y^3 = 0,$$

$$f(x)\,y^{-2} + g(x) = 0 \Longrightarrow g(x)\,y^2 + f(x) = 0.$$

Then a second type first-order sub-equation of (46) is

$$y' - \frac{g'(x)}{2f(x)}y^3 + g(x)\,y^2 - \frac{f'(x)}{2f(x)}y + f(x) = 0. \tag{46}$$

Comparing (72) and (77), we get

$$q(x) = g(x), s(x) = f(x),$$

$$p(x) = -\frac{g'(x)}{2f(x)} = -\frac{q'(x)}{2s(x)}, r(x) = -\frac{f'(x)}{2f(x)} = -\frac{s'(x)}{2s(x)}.$$

So (72) becomes

$$y' - \frac{q'(x)}{2s(x)}y^3 + q(x)\,y^2 - \frac{s'(x)}{2s(x)}y + s(x) = 0. \tag{78}$$

From (46), the particular solution of (78) is

$$y^2 = -\frac{f(x)}{g(x)} \Longrightarrow y = \pm\sqrt{-\frac{f(x)}{g(x)}} = \pm\sqrt{-\frac{s(x)}{q(x)}}.$$

That is

$$y = \pm\sqrt{-\frac{s(x)}{q(x)}}. \tag{67}$$

Next we propose Theorem 9.

Theorem 9. *If*

$$p(x) = e^{-2\int q(x)dx}\left(k + \int r(x)\,e^{2\int q(x)dx}dx\right), \tag{80}$$

then the general solution of second type of Abel equation

$$(y + p(x))y' + q(x)\,y^2 + r(x)\,y + s(x) = 0 \tag{81}$$

is

$$\begin{aligned} y_\pm = {} & e^{-2\int q(x)dx}(-k - \int r(x)\,e^{2\int q(x)dx}dx \\ & \pm \sqrt{\left(k + \int r(x)\,e^{2\int q(x)dx}dx\right)^2 - e^{2\int q(x)dx}\left(C + 2\int s(x)\,e^{2\int q(x)dx}dx\right)}). \end{aligned} \tag{82}$$

where k is a parameter and C is an arbitrary constant.

Prove. Let the source equation of (81) be

$$\Theta(x)\, y^2 + \Lambda(x)\, y + \Omega(x) = 0. \tag{83}$$

Then

$$\Theta(x)\, y^2 + \Lambda(x)\, y + \Omega(x) = 0 \Longrightarrow \Theta' y^2 + 2\Theta y y' + \Lambda' y + \Lambda y' + \Omega' = 0.$$

Namely

$$\left(y + \frac{\Lambda}{2\Theta}\right) y' + \frac{\Theta'}{2\Theta} y^2 + \frac{\Lambda'}{2\Theta} y + \frac{\Omega'}{2\Theta} = 0. \tag{84}$$

Comparing (81) and (84), we get

$$\frac{\Theta'}{2\Theta} = q(x)\,, \frac{\Lambda}{2\Theta} = p(x)\,, \frac{\Lambda'}{2\Theta} = r(x)\,, \frac{\Omega'}{2\Theta} = s(x)\,.$$

So

$$\frac{\Theta'}{2\Theta} = q(x) \Longrightarrow \Theta = C_1 e^{2\int q(x)dx},$$

$$\frac{\Lambda'}{2\Theta} = r(x) \Longrightarrow \Lambda' = 2C_1 r(x)\, e^{2\int q(x)dx} \Longrightarrow \Lambda = C_2 + 2C_1 \int r(x)\, e^{2\int q(x)dx} dx,$$

$$\frac{\Lambda}{2\Theta} = p(x) = e^{-2\int q(x)dx}\left(\frac{C_2}{2C_1} + \int r(x)\, e^{2\int q(x)dx} dx\right),$$

$$\frac{\Omega'}{2\Theta} = s(x) \Longrightarrow \Omega' = 2C_1 s(x)\, e^{2\int q(x)dx} \Longrightarrow \Omega = C_3 + 2C_1 \int s(x)\, e^{2\int q(x)dx} dx.$$

Set $\frac{C_2}{2C_1} = k$ we have

$$\Theta = C_1 e^{2\int q(x)dx}, \Lambda = 2C_1 k + 2C_1 \int r(x)\, e^{2\int q(x)dx} dx, \tag{85}$$

$$\Omega = C_3 + 2C_1 \int s(x)\, e^{2\int q(x)dx} dx, p(x) = e^{-2\int q(x)dx}\left(k + \int r(x)\, e^{2\int q(x)dx} dx\right). \tag{86}$$

If $C_1 > 0$, the general solution of (81) under (80) is

$$\begin{aligned} y_\pm &= \tfrac{-\Lambda \pm \sqrt{\Lambda^2 - 4\Theta\Omega}}{2\Theta} \\ &= \tfrac{-2C_1 k - 2C_1 \int r(x) e^{2\int q(x)dx} dx}{2C_1 e^{2\int q(x)dx}} \\ &\pm \tfrac{\sqrt{\left(2C_1 k + 2C_1 \int r(x) e^{2\int q(x)dx} dx\right)^2 - 4C_1 e^{2\int q(x)dx}\left(C_3 + 2C_1 \int s(x) e^{2\int q(x)dx} dx\right)}}{2C_1 e^{2\int q(x)dx}}, \end{aligned}$$

that is

$$\begin{aligned} y_\pm = e^{-2\int q(x)dx}(&-k - \int r(x)\, e^{2\int q(x)dx} dx \\ &\pm \sqrt{\left(k + \int r(x)\, e^{2\int q(x)dx} dx\right)^2 - e^{2\int q(x)dx}\left(C + 2\int s(x)\, e^{2\int q(x)dx} dx\right)}). \end{aligned} \tag{82}$$

So the theorem is proved. □

Abel equation is a very important ordinary differential equation [26-30], and the current research mainly adopts analytical methods [31-34]. General solutions of Abel equation in other special cases can be referred to [35-39].

3. New principles and methods II

First we propose Theorem 10 and Definition 3.

Theorem 10. *In a continuous region* D, $(D \subseteq \mathbb{R}^n)$, *if* $F(x_1, x_2 \ldots x_n, u, u_{x_1}, u_{x_2} \ldots u_{x_n}, u_{x_1x_2}, u_{x_1x_3} \ldots) = 0$, *and* $F_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)}$ *exists,* $i_1, i_2, \ldots i_n$ *are non-negative integers* $(i_1 + i_2 + \cdots + i_n = m \geq 1)$, *then* $F_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)} = 0$, *and*

$$F_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)} \triangleq \frac{\partial^m F}{\partial x_1^{i_1} \partial x_2^{i_n} \ldots \partial x_n^{i_n}}. \tag{87}$$

Proof. In a continuous area D, $(D \subseteq \mathbb{R}^n)$

$$\begin{aligned} & F(x_1, x_2 \ldots x_n, u, u_{x_1}, u_{x_2} \ldots u_{x_n}, u_{x_1x_2}, u_{x_1x_3} \ldots) = 0 \\ & \Longrightarrow dF\left(x_1, x_2 \ldots x_n, u, u_{x_1}, u_{x_2} \ldots u_{x_n}, u_{x_1x_2}, u_{x_1x_3} \ldots\right) = 0 \\ & \Longrightarrow d\left(dF\right) = d^2F = 0 \\ & \Longrightarrow d\left(d^2F\right) = d^3F = 0 \ldots \end{aligned}$$

Namely

$$d^mF = 0, (m \geq 1).$$

According to Leibniz's rule

$$d^mF = \sum_{i_1+i_2+\ldots+i_n=m} C_m^{i_1i_2\ldots i_n} F_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)} dx_1^{i_1} dx_2^{i_2} \ldots dx_n^{i_n} = 0 \Longrightarrow F_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)} = 0.$$

So the theorem is proven. □

Definition 3. *In a continuous region* D, $(D \subseteq \mathbb{R}^n)$, *if* $F(x_1, x_2 \ldots x_n, u, u_{x_1}, u_{x_2} \ldots u_{x_n}, u_{x_1x_2}, u_{x_1x_3} \ldots) = 0$, *and* $F_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)} = 0$, $(i_1 + i_2 + \cdots + i_n = m \geq 1)$, *then* $F = 0$ *is called the* **source equation**; $F_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)} = 0$ *is called* **the first type m-order subsidiary equation** *of* $F = 0$, *Various* $(k + m)$*-order PDEs obtained by the mathematical operations of* $F = 0$ *and* $F_{x_1x_2\ldots x_n}^{(j_1j_2\ldots j_n)} = 0$, $(1 \leq j_1 + j_2 + \ldots + j_n \leq m)$ *are called* **the second type m-order subsidiary equations** *of* $F = 0$, $i_1, i_2, \ldots i_n$ *and* $j_1, j_2, \ldots j_n$ *are all non-negative integers.*

Theorem 2 and Method 1 previously proposed are applicable not only to ODEs, but also to PDEs. There is a similar note in the specific calculation: Since there is no arbitrary function in the coefficients of the equation to be solved, when comparing coefficients of the equation to be solved with the corresponding sub-equation, after calculation, if there are arbitrary functions in coefficients of the corresponding sub-equation, and there is no arbitrary function in coefficients of the equation to be solved, the arbitrary functions can be retained, otherwise they need to be constants or parameters. Below we use typical cases to illustrate.

4. General solutions and exact solutions of PDEs

First we propose Theorem 11.

Theorem 11. *In* $\mathbb{R}^2$, *the general solution of*

$$u_x + b\left(x, y\right) p\left(u\right) = 0, \tag{88}$$

is

$$\varphi(y)+\int\frac{1}{p(u)}du+\int b(x,y)\,dx=0, \tag{89}$$

where $b(x,y)$ and $p(u)$ are arbitrary known functions, $\varphi(y)$ is an arbitrary functions.

Prove. According to Method 1, set the source equation of (88) be

$$f(x,y)\,h(u(x,y))=C, \tag{90}$$

where C is an arbitrary constant, according to Theorem 10

$$f(x,y)\,h(u(x,y))=C\Longrightarrow fh_u'u_x+f_xh(u)=0.$$

Namely

$$u_x+\frac{f_xh(u)}{fh_u'}=0. \tag{91}$$

(88) is the corresponding equation of (91), so

$$\frac{h(u)}{h_u'}=p(u)\Longrightarrow h(u)=C_1e^{\int\frac{1}{p(u)}du},$$

$$\frac{f_x}{f}=b(x,y)\Longrightarrow f(x,y)=\varphi_1(y)\,e^{\int b(x,y)dx},$$

where C_1 is an arbitrary constant, and $\varphi_1(y)$ is an arbitrary unary function, according to Method 1

$$\begin{aligned}&f(x,y)h(u(x,y))=C_1\varphi_1(y)\,e^{\int b(x,y)dx+\int\frac{1}{p(u)}du}=C\\&\Longrightarrow\int\tfrac{1}{p(u)}du+\int b(x,y)\,dx=ln\tfrac{C}{C_1\varphi_1(y)}.\end{aligned}$$

That is, the general solution of (88) is

$$\varphi(y)+\int\frac{1}{p(u)}du+\int b(x,y)\,dx=0.$$

The theorem is proven. □

Theorem 12. *In $\mathbb{R}^2$, the general solution of*

$$u_{xy}+b(x,y)\,u_y+b_y(x,y)\,u+d(x,y)=0 \tag{92}$$

is

$$u(x,y)=e^{-\int b(x,y)dx}\left(\varphi(y)+\int\left(\psi(x)-\int d(x,y)\,dy\right)e^{\int b(x,y)dx}dx\right), \tag{93}$$

where $b(x,y)$ and $d(x,y)$ are arbitrary known functions, $\psi(x)$ and $\varphi(y)$ are arbitrary functions.

Prove. According to Method 1, set the source equation of (92) be

$$u_x+f(x,y)\,u=g(x,y). \tag{94}$$

The general solution of (94) is [40]

$$u(x,y)=e^{-\int f(x,y)dx}\left(\varphi(y)+\int g(x,y)\,e^{\int f(x,y)dx}dx\right). \tag{95}$$

According to Theorem 10

$$u_x + f(x,y)\,u = g(x,y) \Longrightarrow u_{xy} + f(x,y)\,u_y + f_y(x,y)\,u = g_y(x,y)\,.$$

Namely

$$u_{xy} + f(x,y)\,u_y + f_y(x,y)\,u - g_y(x,y) = 0. \tag{96}$$

(92) is the corresponding equation of (96), so

$$b(x,y) = f(x,y), c(x,y) = f_y(x,y) = b_y(x,y)$$

$$d(x,y) = -g_y(x,y) \Longrightarrow g(x,y) = \psi(x) - \int d(x,y)\,dy.$$

According to Method 1 and (95), the general solution of (92) is

$$u(x,y) = e^{-\int b(x,y)dx}\left(\varphi(y) + \int\left(\psi(x) - \int d(x,y)\,dy\right)e^{\int b(x,y)dx}dx\right).$$

The theorem is proven. □

According to (94)

$$u = \frac{g(x,y) - u_x}{f(x,y)}. \tag{97}$$

Combining (96), the second type first-order sub-equations of (94) can also be constructed as

$$u_{xy} + f(x,y)\,u_y - \frac{f_y(x,y)}{f(x,y)}u_x + f_y(x,y)\,\frac{g(x,y)}{f(x,y)} - g_y(x,y) = 0. \tag{98}$$

The corresponding equation is

$$u_{xy} + b(x,y)\,u_y + c(x,y)\,u_x + d(x,y) = 0. \tag{99}$$

Comparing (98) and (99), we get

$$f(x,y) = b(x,y),$$

$$-\frac{f_y(x,y)}{f(x,y)} = c(x,y) \Longrightarrow f(x,y) = b(x,y) = \varphi_1(x)\,e^{-\int c(x,y)dy},$$

$$f_y(x,y)\,\frac{g(x,y)}{f(x,y)} - g_y(x,y) = d(x,y) \Longrightarrow g_y(x,y) + c(x,y)\,g(x,y) + d(x,y) = 0$$
$$\Longrightarrow g(x,y) = e^{-\int c(x,y)dy}\left(\varphi_2(x) - \int d(x,y)\,e^{\int c(x,y)dy}dy\right).$$

Because $b(x,y)$ and $d(x,y)$ do not contain arbitrary functions, set $\varphi_1(x) = 1$ and $\varphi_2(x) = 0$, that is

$$b(x,y) = f(x,y) = e^{-\int c(x,y)dy}, \tag{100}$$

$$g(x,y) = -e^{-\int c(x,y)dy}\int d(x,y)\,e^{\int c(x,y)dy}dy. \tag{101}$$

By (100), (99) becomes

$$u_{xy} + e^{-\int c(x,y)dy}u_y + c(x,y)\,u_x + d(x,y) = 0. \tag{102}$$

According to (95, 100, 101), we can obtain that an analytical solution of (102) is

$$u\left(x,y\right)=e^{-\int e^{-\int c(x,y)dy}dx}\left(\varphi\left(y\right)-\int\left(e^{\int e^{-\int c(x,y)dy}dx-\int c(x,y)dy}\int d\left(x,y\right)e^{\int c(x,y)dy}dy\right)dx\right),\tag{103}$$

where $\varphi(y)$ is an arbitrary unary function.

Using (94, 96), other solvable equations can also be constructed, and the readers can try it by themself.

Example 5. *In $\mathbb{R}^2$ space, if a source equation is*

$$f\left(x,y\right)u_x^m+g\left(x,y\right)u^n=0,(m\neq 0).\tag{104}$$

Using Method 1 obtains the analytical solution of the nonlinear PDE corresponding to it's the first type first-order sub-equation.

Solution. According to (104)

$$f\left(x,y\right)u_x^m+g\left(x,y\right)u^n=0\Longrightarrow u^{\frac{-n}{m}}u_x=\left(-\tfrac{g}{f}\right)^{\frac{1}{m}}\Longrightarrow\tfrac{m}{m-n}u^{1-\frac{n}{m}}=\phi_1\left(y\right)+\int\left(-\tfrac{g}{f}\right)^{\frac{1}{m}}dx.$$

So the general solution of (104) is

$$u=\left(\frac{m-n}{m}\left(\phi\left(y\right)+\int\left(-\frac{g}{f}\right)^{\frac{1}{m}}dx\right)\right)^{\frac{m}{m-n}}.\tag{105}$$

The first type first-order sub-equation of (104) is

$$mfu_x^{m-1}u_{xx}+f_xu_x^m+ngu^{n-1}u_x+g_xu^n=0.$$

Namely

$$u_{xx}+\frac{f_x}{mf}u_x+\frac{ng}{mf}u^{n-1}u_x^{2-m}+\frac{g_x}{mf}u^nu_x^{1-m}=0.\tag{106}$$

The corresponding equation of (106) is

$$u_{xx}+a\left(x,y\right)u_x+b\left(x,y\right)u^{n-1}u_x^{2-m}+c\left(x,y\right)u^nu_x^{1-m}=0.\tag{107}$$

Comparing the coefficients in (106) and (107), we get

$$a\left(x,y\right)=\frac{f_x}{mf}\Longrightarrow f=\phi_1\left(y\right)e^{m\int a(x,y)dx},$$

$$\frac{g_x}{mf}=c\left(x,y\right)\Longrightarrow g=\phi_2\left(y\right)+m\phi_1\left(y\right)\int c\left(x,y\right)e^{m\int a(x,y)dx}dx,$$

$$b\left(x,y\right)=\frac{ng}{mf}=e^{-m\int a(x,y)dx}\left(\frac{n\phi_2\left(y\right)}{m\phi_1\left(y\right)}+n\int c\left(x,y\right)e^{m\int a(x,y)dx}dx\right).$$

Since $b(x,y)$ does not contain any arbitrary function, set $\phi_1(y)=1$ and $\phi_2(y)=0$, incorporate n into the integral constant of $e^{-m\int a(x,y)dx}$, then

$$b\left(x,y\right)=e^{-m\int a(x,y)dx}\int c\left(x,y\right)e^{m\int a(x,y)dx}dx,\tag{108}$$

$$f=e^{m\int a(x,y)dx},g=m\int c\left(x,y\right)e^{m\int a(x,y)dx}dx.\tag{109}$$

From (105) and (109), the analytical solution of (107) under the condition of (108) is

$$u=\left(\frac{m-n}{m}\left(\phi\left(y\right)+\int e^{-\int a(x,y)dx}\left(-m\int c\left(x,y\right)e^{m\int a(x,y)dx}dx\right)^{\frac{1}{m}}dx\right)\right)^{\frac{m}{m-n}},\tag{110}$$

where $\phi(y)$ is an arbitrary function. After validation we get $m=1$, that is

$$u_{xx}+a\left(x,y\right)u_x+b\left(x,y\right)u^{n-1}u_x+c\left(x,y\right)u^n=0,\tag{111}$$

$$b\left(x,y\right)=e^{-\int a(x,y)dx}\int c\left(x,y\right)e^{\int a(x,y)dx}dx.\tag{112}$$

The analytical solution of (111) under the condition of (112) is

$$u=\left(\left(1-n\right)\left(\phi\left(y\right)-\int e^{-\int a(x,y)dx}\left(\int c\left(x,y\right)e^{\int a(x,y)dx}dx\right)dx\right)\right)^{\frac{1}{1-n}}.\tag{113}$$

If we set $\phi_1(y)$ and $\phi_2(y)$ are determined functions and n is not incorporated into the integral constant of $e^{-m\int a(x,y)dx}$, the result of the verification is $m=n$, which indicates that different settings have a different result.

Next we propose Theorem 13.

Theorem 13. *In $\mathbb{R}^2$, the general solution of*

$$u_{xx}-nu^{-1}u_x^2+a\left(x,y\right)u_x+b\left(x,y\right)u^n=0\tag{114}$$

is

$$u=\left(\left(1-n\right)\left(\phi\left(y\right)-\int e^{-\int a(x,y)dx}\left(\psi\left(y\right)+\int b\left(x,y\right)e^{\int a(x,y)dx}dx\right)dx\right)\right)^{\frac{1}{1-n}}.\tag{115}$$

where $a(x,y)$ and $b(x,y)$ are arbitrary known functions, $\phi(y)$ and $\psi(y)$ are arbitrary functions.

Prove. According to Method 1, set the source equation of (114) be

$$f\left(x,y\right)u_x+g\left(x,y\right)u^n=0.\tag{116}$$

Then

$$u=\left(\left(1-n\right)\left(\phi\left(y\right)-\int\frac{g}{f}dx\right)\right)^{\frac{1}{1-n}},\tag{117}$$

the first type first-order sub-equation of (116) is

$$u_{xx}+\frac{f_x}{f}u_x+\frac{ng}{f}u^{n-1}u_x+\frac{g_x}{f}u^n=0.\tag{118}$$

Using (116), we gain

$$f\left(x,y\right)=-\frac{gu^n}{u_x},g\left(x,y\right)=-\frac{fu_x}{u^n},u=\left(-\frac{fu_x}{g}\right)^{\frac{1}{n}},u_x=-\frac{gu^n}{f}.\tag{119}$$

By (119), a second type first-order sub-equation of (116) is

$$u_{xx}-nu^{-1}u_x^2+\frac{f_x}{f}u_x+\frac{g_x}{f}u^n=0.\tag{120}$$

The corresponding equation of (120) is

$$u_{xx} - nu^{-1}u_x^2 + a\left(x,y\right)u_x + b\left(x,y\right)u^n = 0. \tag{114}$$

Comparing the coefficients in (114) and (120), we get

$$a\left(x,y\right) = \frac{f_x}{f} \Longrightarrow f = \phi_1\left(y\right)e^{\int a(x,y)dx},$$

$$\frac{g_x}{f} = b\left(x,y\right) \Longrightarrow g = \phi_2\left(y\right) + \phi_1\left(y\right)\int b\left(x,y\right)e^{\int a(x,y)dx}dx.$$

According to (117), the general solution of (114) is

$$\begin{aligned} u &= \left((1-n)\left(\phi\left(y\right) - \int \tfrac{g}{f}dx\right)\right)^{\frac{1}{1-n}} \\ &= \left((1-n)\left(\phi\left(y\right) - \int \tfrac{\phi_2(y)+\phi_1(y)\int b(x,y)e^{\int a(x,y)dx}dx}{\phi_1(y)e^{\int a(x,y)dx}}dx\right)\right)^{\frac{1}{1-n}}. \end{aligned}$$

That is

$$u = \left((1-n)\left(\phi\left(y\right) - \int e^{-\int a(x,y)dx}\left(\psi\left(y\right) + \int b\left(x,y\right)e^{\int a(x,y)dx}dx\right)dx\right)\right)^{\frac{1}{1-n}}. \tag{115}$$

The theorem is proved. □

5. Special application of Method 1

First we propose Definition 4.

Definition 4. *If a differential equation includes arbitrary known functions, and a continuous solution could be derived from them, then it is called* **a known functional solution**.

A known functional solution may be a general solution, an analytic solution or a particular solution. Using Method 1, theoretically an infinite number of ODEs and PDEs can be solved. These equations have a universal feature that they include arbitrary known functions and have known functional solutions, such as an ODE

$$y' + b\left(x\right)y + c\left(x\right) = 0. \tag{11}$$

The general solution of (11) is

$$y = e^{-\int b(x)dx}\left(C - \int c\left(x\right)e^{\int b(x)dx}dx\right). \tag{12}$$

$b(x)$ and $c(x)$ in (11) are all arbitrary known functions. The solution formula is obtained by using arbitrary known $b(x)$ and $c(x)$, that is, (12) is a known functional solution of (11). For the partial differential equation

$$u_x + b\left(x,y\right)p\left(u\right) = 0, \tag{88}$$

which general solution is

$$\varphi\left(y\right) + \int \frac{1}{p\left(u\right)}du + \int b\left(x,y\right)dx = 0, \tag{89}$$

$b(x, y)$ and $p(u)$ in (89) are all arbitrary known functions, and (89) is the known functional solution of (88).

Now we ask two new questions. Does a differential equation with arbitrary known functions necessarily have known functional solutions? If not, how to prove it? These two questions are quite meaningful, just like the root formulas do not exist for general algebraic equations more than 5 times. If we can prove that known functional solutions of some equations do not exist, people do not need to waste time and energy in finding them. And this will promote people to further understand rules of solutions of differential equations. Here we use Method 1 to research this problem.

5.1. Typical cases for ODEs.

Next we propose Theorem 14.

Theorem 14. *In a continuous region D, $(D \subseteq \mathbb{R})$, for an arbitrary known function $b(x)$, the equation $y'' + b(x)\, y = 0$ does not have known functional solutions in the form of $y = y(x)$.*

Proof. Assuming $y'' + b(x)\, y = 0$ has a known functional solution in the form of $y = y(x)$ in the continuous area D, the solution must be expressed as

$$f(x)y + k = 0. \tag{121}$$

So

$$f(x)y + k = 0 \Longrightarrow f(x)\, y' + f'(x)\, y = 0 \Longrightarrow f(x)\, y'' + 2f'(x)\, y' + f''(x)\, y = 0.$$

Namely

$$y'' + \frac{2f'(x)}{f(x)} y' + \frac{f''(x)}{f(x)} y = 0. \tag{122}$$

Comparisons with

$$y'' + b(x)\, y = 0. \tag{123}$$

In D, we get

$$2\frac{f'(x)}{f(x)} \equiv 0, \tag{124}$$

$$\frac{f''(x)}{f(x)} = b(x). \tag{125}$$

By (124), $f'(x) \equiv 0$ can be obtained in D, and then $f''(x) \equiv 0$. According to (125), $b(x) \equiv 0$ can be get. Since there is a contradiction that $b(x)$ is an arbitrary known function, so (123) cannot have a solution in the form of $y = y(x)$, Theorem 14 is proved. □

(123) is an equation that has been extensively and deeply studied [41-44], Hill differential equation and Mathieu differential equation [45] are its special cases.

$y'' + b(x)y = 0$ is an equation with a very simple structure. According to the equation, which has no known functional solutions, it can be initially judged that related more complex equations also have no known functional solutions, such as

$$y'' + b(x)\, y + c(x) = 0, \tag{126}$$

$$y'' + a(x)\, y' + b(x)\, y = 0, \tag{127}$$

$$y'' + a(x)\, y' + b(x)\, y + c(x) = 0, \tag{128}$$

where $a(x), b(x)$ and $c(x)$ are arbitrary known functions. In fact, according to Method 1, it can be proved that (126-128) have no known functional solutions, namely

Theorem 15. *In a continuous area D, $(D \subseteq \mathbb{R})$, for arbitrary known functions $a(x), b(x)$ and $c(x)$, the equations*

$$y'' + b(x)y + c(x) = 0, \tag{126}$$

$$y'' + a(x)y' + b(x)y = 0, \tag{127}$$

$$y'' + a(x)y' + b(x)y + c(x) = 0, \tag{128}$$

do not have known functional solutions in the form of $y = y(x)$.

Proof. Assuming $y'' + a(x)y' + b(x)y + c(x) = 0$ has a known functional solution in the form of $y = y(x)$ in D, the solution must be expressed as

$$y = -\frac{g(x)}{f(x)}.$$

So

$$y = -\frac{g(x)}{f(x)} \Longrightarrow f(x)y + g(x) = 0 \Longrightarrow f(x)y'' + 2f'(x)y' + f''(x)y + g''(x) = 0.$$

Namely

$$y'' + 2\frac{f'(x)}{f(x)}y' + \frac{f''(x)}{f(x)}y + \frac{g''(x)}{f(x)} = 0. \tag{129}$$

Comparisons with

$$y'' + a(x)y' + b(x)y + c(x) = 0. \tag{128}$$

In D, we get

$$2\frac{f'(x)}{f(x)} = a(x), \tag{130}$$

$$\frac{f''(x)}{f(x)} = b(x), \tag{131}$$

$$2\frac{f'(x)}{f(x)} = a(x) \Longrightarrow f = C_1 e^{\frac{1}{2}\int a(x)dx} \Longrightarrow f'' = \frac{1}{2}C_1 a'(x) e^{\frac{1}{2}\int a(x)dx} + \frac{1}{4}C_1 a^2(x) e^{\frac{1}{2}\int a(x)dx}.$$

So

$$b(x) = \frac{f''(x)}{f(x)} = \frac{\frac{1}{2}C_1 a'(x) e^{\frac{1}{2}\int a(x)dx} + \frac{1}{4}C_1 a^2(x) e^{\frac{1}{2}\int a(x)dx}}{C_1 e^{\frac{1}{2}\int a(x)dx}} = \frac{1}{2}a'(x) + \frac{1}{4}a^2(x),$$

there is a contradiction that $a(x)$ and $b(x)$ are arbitrary known functions, so (128) cannot have a solution in the form of $y = y(x)$. Similarly, it can be proved that (126, 127) cannot have a known function solution in the form $y = y(x)$, Theorem 15 is proved. □

(127) is an ordinary differential equation that has been studied emphatically [46, 47]. If $a(x)$ and $b(x)$ are special functions, obviously (127) has a solution. If set $y = sinx, a(x) = x$, then

$$\begin{aligned} &y'' + a(x)y' + b(x)y = sinx + xcosx + b(x)sinx = 0 \\ &\Longrightarrow b(x) = \tfrac{-sinx - xcosx}{sinx} = -1 - x\cot x. \end{aligned}$$

Namely

$$y'' + xy' - (1 + x\cot x)y = 0. \tag{132}$$

The particular solution of (132) is $y = sinx$. This case is not contradictory to Theorem 15, because the solution of (132) does not have a unified functional relationship with different $a(x)$

and $b(x)$.

Set $y = e^{\int zdx}$, the linear equation $y'' + a(x)y' + b(x)y = 0$ can be transformed into Riccati Equation

$$z' + z^2 + a(x)z + b(x) = 0.$$

So Theorem 15 indirectly proves that this type of Riccati Equation has not a known function solution! So we can propose Theorem 16.

Theorem 16. *In a continuous area D, $(D \subseteq \mathbb{R})$, for arbitrary known functions $a(x)$ and $b(x)$, the Riccati Equation*

$$z' + z^2 + a(x)z + b(x) = 0 \tag{133}$$

does not have known functional solutions in the form of $z = z(x)$.

According to Theorem 16, it can be preliminarily judged that Riccati Equation and Abel Equation in the general form have not a known function solution.

Next we propose Theorem 17.

Theorem 17. *In a continuous area D, $(D \subseteq \mathbb{R})$, for an arbitrary known functions $a(x)$, $b(y)$, the equation $y' + a(x) + b(y) = 0$ does not have known functional solutions in the form of $\varphi(y) = \phi(x)$.*

Proof. Assuming $y' + a(x) + b(y) = 0$ has a known functional solution in the form of $\varphi(y) = \phi(x)$ in the continuous area D, note that φ and ϕ cannot be arbitrary functions, then

$$\varphi(y) = \phi(x) \Longrightarrow \varphi'(y)\,y' - \phi'(x) = 0.$$

That is

$$y' - \frac{\phi'(x)}{\varphi'(y)} = 0. \tag{134}$$

Comparisons with

$$y' + a(x) + b(y) = 0. \tag{135}$$

So

$$-\frac{\phi'(x)}{\varphi'(y)} = a(x) + b(y).$$

Namely

$$-\phi'(x) = a(x)\,\varphi'(y) + b(y)\,\varphi'(y). \tag{136}$$

Since $\varphi(y)$ is not an arbitrary function, and $b(y)$ is an arbitrary known function, $a(x)\varphi'(y) + b(y)\varphi'(y)$ must include y, so (136) cannot set up, then Theorem 17 is proved. □

On the basis of Theorem 17, it can be preliminarily judged that $y' + a(x)b(y) + c(x) = 0$ and $y' + a(x)b(y) + d(y) = 0$ also do not have a known function solution in the form of $\varphi(y) = \phi(x)$.

5.2. Typical cases for PDEs.

The idea and method to prove that there are no known functional solutions for PDEs are almost completely similar to ODEs, and it will be explained through specific cases.

Theorem 18. *In a continuous area D, $(D \subseteq \mathbb{R}^n, n \geq 2)$, for an arbitrary known functions*

$b(x_1, \ldots x_n)$, *the equations* $u_{x_ix_i} + b(x_1, \ldots x_n)u = 0$ *and* $u_{x_ix_j} + b(x_1, \ldots x_n)u = 0, (i, j = 1, 2, \ldots n)$ *do not have a known functional solution in the form of* $u = \varphi(x_1, \ldots x_n)$.

Proof. Assuming $u_{x_ix_i} + b(x_1, \ldots x_n)u = 0$ has a known functional solution in the form of $u = \varphi(x_1, \ldots x_n)$ in the continuous area D, the solution must be expressed as

$$f(x_1, \ldots x_n)u + k = 0. \tag{137}$$

So

$$f(x_1, \ldots x_n)u + k = 0 \Longrightarrow fu_{x_i} + f_{x_i}u = 0 \Longrightarrow fu_{x_ix_i} + 2f_{x_i}u_{x_i} + f_{x_ix_i}u = 0.$$

Namely

$$u_{x_ix_i} + \frac{2f_{x_i}}{f}u_{x_i} + \frac{f_{x_ix_i}}{f}u = 0. \tag{138}$$

Comparisons with

$$u_{x_ix_i} + b(x_1, \ldots x_n)u = 0. \tag{139}$$

In D, we get

$$\frac{2f_{x_i}}{f} \equiv 0, \tag{140}$$

$$\frac{f_{x_ix_i}}{f} = b(x_1, \ldots x_n). \tag{141}$$

By (140), $f_{x_i} \equiv 0$ can be obtained in D, and then $f_{x_ix_i} \equiv 0$. According to (141), $b(x_1, \ldots x_n) \equiv 0$ can be obtained, this is inconsistent with $b(x_1, \ldots x_n)$ being any known function, so (139) cannot have a known functional solution in the form of $u = (x_1, \ldots x_n)$. Similar methods can be used to prove that $u_{x_ix_j} + b(x_1, \ldots x_n)u = 0, (i, j = 1, 2, \ldots n)$ does not have a continuous known functional solution in the form of $u = \varphi(x_1, \ldots x_n)$, so the theorem is proved. □

Theorem 19. *In a continuous area* D, $(D \subseteq \mathbb{R}^n, n \geq 2)$, *for arbitrary known functions* $b(x_1, \ldots x_n)$ *and* $c(x_1, \ldots x_n)$, $u_{x_ix_i} + b(x_1, \ldots x_n)u + c(x_1, \ldots x_n) = 0$ *and* $u_{x_ix_j} + b(x_1, \ldots x_n)u + c(x_1, \ldots x_n) = 0$ *do not have a known functional solution in the form of* $u = \varphi(x_1, \ldots x_n)$.

Proof. Assuming $u_{x_ix_i} + b(x_1, \ldots x_n)u + c(x_1, \ldots x_n) = 0$ has a known functional solution in the form of $u = \varphi(x_1, \ldots x_n)$ in D, the solution must be expressed as

$$u = -\frac{g(x_1, \ldots x_n)}{f(x_1, \ldots x_n)}, (f(x_1, \ldots x_n) \neq 0). \tag{142}$$

So

$$\begin{aligned} u = -\tfrac{g(x_1, \ldots x_n)}{f(x_1, \ldots x_n)} &\Longrightarrow f(x_1, \ldots x_n)u + g(x_1, \ldots x_n) = 0 \\ &\Longrightarrow fu_{x_ix_i} + 2f_{x_i}u_{x_i} + f_{x_ix_i}u + g_{x_ix_i} = 0. \end{aligned}$$

Namely

$$u_{x_ix_i} + 2\frac{f_{x_i}}{f}u_{x_i} + \frac{f_{x_ix_i}}{f}u + \frac{g_{x_ix_i}}{f} = 0. \tag{143}$$

Comparisons with

$$u_{x_ix_i} + b(x_1, \ldots x_n)u + c(x_1, \ldots x_n) = 0. \tag{144}$$

In D, we get

$$2\frac{f_{x_i}}{f} \equiv 0, \tag{145}$$

$$\frac{f_{x_i x_i}}{f} = b\left(x_1, \ldots x_n\right), \tag{146}$$

$$\frac{g_{x_i x_i}}{f} = c\left(x_1, \ldots x_n\right). \tag{147}$$

By (145),$f_{x_i} \equiv 0$ can be obtained in D, and then $f_{x_i x_i} \equiv 0$. According to (146), $b\left(x_1, \ldots x_n\right) \equiv 0$ can be obtained, which is inconsistent with $b\left(x_1, \ldots x_n\right)$ being any known function, so (144) cannot have a known functional solution in the form of $u = \varphi\left(x_1, \ldots x_n\right)$. Similar methods can be used to prove that $u_{x_i x_j} + b\left(x_1, \ldots x_n\right) u + c(x_1, \ldots x_n) = 0$ does not have a known functional solution in the form of $u = \varphi\left(x_1, \ldots x_n\right)$, so the theorem is proved. □

Theorem 20. *In a continuous area D, $(D \subseteq \mathbb{R}^n, n \geq 2)$, for arbitrary known functions $a(x_1, \ldots x_n)$, $b(x_1, \ldots x_n)$ and $c(x_1, \ldots x_n)$, $u_{x_i x_i} + au_{x_i} + bu + c = 0$ and $u_{x_i x_j} + au_{x_i} + bu + c = 0$ do not have a known functional solution in the form of $u = \varphi\left(x_1, \ldots x_n\right)$.*

The proving method of Theorem 20 is the same as Theorem 19, and readers can try it by themselves.

Theorem 21. *In a continuous area D, $(D \subseteq \mathbb{R}^n, n \geq 2)$, for arbitrary known functions $a(x_1, \ldots x_n)$ and $b(u)$, $u_{x_i} + a\left(x_1, \ldots x_n\right) + b\left(u\right) = 0$ does not have a known functional solution in the form of $\varphi\left(u\right) = \phi\left(x_1, \ldots x_n\right)$.*

Proof. Assuming $u_{x_i} + a\left(x_1, \ldots x_n\right) + b\left(u\right) = 0$ has a known functional solution in the form of $\varphi\left(u\right) = \phi\left(x_1, \ldots x_n\right)$ in D, note that φ and ϕ cannot be arbitrary functions, then

$$\varphi\left(u\right) = \phi\left(x_1, \ldots x_n\right) \Longrightarrow \varphi'\left(u\right) u_{x_i} - \phi_{x_i} = 0,$$

That is

$$u_{x_i} - \frac{\phi_{x_i}}{\varphi'\left(u\right)} = 0, \tag{148}$$

Comparisons with

$$u_{x_i} + a\left(x_1, \ldots x_n\right) + b\left(u\right) = 0. \tag{149}$$

Then

$$-\frac{\phi_{x_i}}{\varphi'\left(u\right)} = a\left(x_1, \ldots x_n\right) + b\left(u\right),$$

namely

$$-\phi_{x_i}\left(x_1, \ldots x_n\right) = a\left(x_1, \ldots x_n\right) \varphi'\left(u\right) + b\left(u\right) \varphi'\left(u\right). \tag{150}$$

Since $\varphi(u)$ is not an arbitrary function, and $b(u)$ is an arbitrary known function, $a\left(x_1, \ldots x_n\right) \varphi'\left(u\right) + b\left(u\right) \varphi'\left(u\right)$ must include u, so (150) cannot set up, then Theorem 21 is proved. □

Theorem 22. *In a continuous area D, $(D \subseteq \mathbb{R}^n, n \geq 2)$, for an arbitrary known functions $a(x_1, \ldots x_n)$, the equations $au_{x_1} + u_{x_2} + \ldots + u_{x_k} = 0$ does not have a known functional solution in the form of $u = \varphi\left(x_1, \ldots x_n\right)$.*

Proof. Assuming $au_{x_1} + u_{x_2} + \ldots + u_{x_k} = 0$ has a known functional solution in the form of $u = \varphi\left(x_1, \ldots x_n\right)$ in the continuous area D, the solution must be expressed as

$$u = f(a). \tag{151}$$

So

$$au_{x_1} + u_{x_2} + \ldots + u_{x_k} = af'_a a_{x_1} + f'_a a_{x_2} + \ldots + f'_a a_{x_k} = 0.$$

Namely

$$aa_{x_1} + a_{x_2} + \ldots + a_{x_k} = 0. \tag{152}$$

Since a general function $a(x_1, \ldots x_n)$ does not satisfy $aa_{x_1} + a_{x_2} + \ldots + a_{x_k} = 0$, so (152) cannot have a known functional solution in the form of $u = \varphi\,(x_1, \ldots x_n)$, the theorem is proved. □

If a differential equation does not have a known function solution, it does not mean that it is unsolvable under any circumstances, and it might be solved in special circumstances. For example, theorem 15 points out that $y'' + b\,(x)\,y' + c\,(x)\,y + d\,(x) = 0$ has no known function solution, but its special case

$$y'' + b\,(x)\,y' + \left(\frac{1}{2}b'\,(x) + \frac{1}{4}b^2\,(x)\right)y + d\,(x) = 0, \tag{17}$$

the general solution of (17) is

$$y = e^{\frac{-1}{2}\int b(x)dx}\left(C_1 + C_2 x - \iint d\,(x)\,e^{\frac{1}{2}\int b(x)dx}dxdx\right). \tag{18}$$

That is (17) is solvable.

For another example, Theorem 22 states that $a(x, y)u_x + u_y = 0$ does not have a known function solution, but its special case $b(x)c(y)u_x + u_y = 0$, we can use the characteristic equation method to obtain its general solution $u = f\left(\int \frac{dx}{b(x)} - \int c\,(y)\,dy\right)$, note that the solution still cannot be expressed by the known function $a(x, y) = b(x)c(y)$.

6. New principles and methods III

Below we propose definitions 5.

Definition 5. *In a continuous area D, $(D \subseteq \mathbb{R}^n, n \geq 1)$, if a differential equation has known functional solutions, it is called an Φ equation; if a differential equation has no known functional solution, it is called a Ψ equation.*

In the preceding paper [48], we proposed the concept and laws of the dependent variable transformation equation, and now we go further into this concept. The specific choice for a dependent variable transformation can be an Φ equation or a Ψ equation, so we propose definition 6.

Definition 6. *A dependent variable transformation using an Φ equation is called an Φ transformation; a dependent variable transformation using a Ψ equation is called a Ψ transformation.*

We proved in Section 3 that

$$y'' + b\,(x)\,y + c\,(x) = 0 \tag{126}$$

is a Ψ equation, so

$$z\,(x) = y'' + b\,(x)\,y, \tag{153}$$

this dependent variable transformation is a Ψ transformation, and

$$z\,(x) = y' + b\,(x)\,y, \tag{154}$$

is an Φ transformation because the transformation uses the Φ equation $y' + b(x)y + c(x) = 0$.

Based on the above concepts, we can propose the following theorems.

Theorem 23. *After a Ψ transformation, an Φ equation becomes a Ψ equation.*

Proof. For ODEs, according to the theorem [48]:

In the domain D, $(D \subset \mathbb{R}^1)$, if the solution $w = f(x)$ of an ODE $G\left(x, w, w', w'', \dots w^{(n)}\right) = 0$ is known, set $w = h\left(x, y, y', y'', \dots y^{(m)}\right)$, then the solution of its DVTE $F\left(x, y, y', y'', \dots y^{(m+n)}\right) = 0$ is the solution of $h\left(x, y, y', y'', \dots y^{(m)}\right) = f(x)$.

A Ψ transformation $w = f(x) = h\left(x, y, y', y'', \dots y^{(m)}\right)$ does not have any known functional solution. Since the dependent variable transformation equations $F\left(x, y, y', y'', \dots y^{(m+n)}\right) = 0$ and $h\left(x, y, y', y'', \dots y^{(m)}\right) = f(x)$ have the same solutions, $F = 0$ is a Ψ equation.

For PDEs, according to the theorem [48]:

In the domain D, $(D \subset \mathbb{R}^n)$, if the solution $v = f(x_1, \dots x_n)$ of a PDE $G(x_1, \dots x_n, v, v_{x_1}, \dots v_{x_n}, v_{x_1 x_2}, \dots) = 0$ is known, set $v = h(x_1, \dots x_n, u, u_{x_1}, \dots u_{x_n}, u_{x_1 x_2}, \dots)$, then the solution of its DVTE $F(x_1, \dots x_n, u, u_{x_1}, \dots u_{x_n}, u_{x_1 x_2}, \dots) = 0$ is the solution of $h(x_1, \dots x_n, u, u_{x_1}, \dots u_{x_n}, u_{x_1 x_2}, \dots) = f(x_1, \dots x_n)$.

A Ψ transformation $v = f(x_1, \dots x_n) = h(x_1, \dots x_n, u, u_{x_1}, \dots u_{x_n}, u_{x_1 x_2}, \dots)$ has no known functional solution. Since the dependent variable transformation equations $F(x_1, \dots x_n, u, u_{x_1}, \dots u_{x_n}, u_{x_1 x_2}, \dots) = 0$ and $h(x_1, \dots x_n, u, u_{x_1}, \dots u_{x_n}, u_{x_1 x_2}, \dots) = f(x_1, \dots x_n)$ have the same solutions, $F = 0$ is a Ψ equation, so the theorem is proved. □

Using methods similar to the proof of Theorem 23, we can get Theorems 24-26, which readers can try by themselves.

Theorem 24. *After a Ψ transformation, a Ψ equation is still a Ψ equation.*

Theorem 25. *After a Φ transformation, a Ψ equation is still a Ψ equation.*

Theorem 26. *After a Φ transformation, an Φ equation is still an Φ equation.*

Theorem 26 has particularly important application value, because the essence of using the transformation of dependent variables to solve differential equations is to use theorem 26.

In [48], we put forward the concept and law of the transformation equation of the independent variable of PDEs. Below we propose the analogous concept and law of ODEs.

Definition 7. *In a continuous region D, $(D \subset \mathbb{R}^1)$, set $x = x(t), t = t(x)$ are known, and transform the n-th order ODE $F\left(x, y, y', y'', \dots y^{(n)}\right) = 0$ into the n-th order ODE $G\left(t, y, y', y'', \dots y^{(n)}\right) = 0$, then $G = 0$ is the independent variable transformation equation of $F = 0$.*

Theorem 27. *In a continuous D, $(D \subset \mathbb{R}^1)$, if the solution $y = f(x)$ of the n-th order ODE $F\left(x, y, y', y'', \dots y^{(n)}\right) = 0$ is known, then the solution of the independent variable transformation equation $G\left(t, y, y', y'', \dots y^{(n)}\right) = 0$ is $y = f(x) = g(t)$.*

Proof. Use $t = t(x)$ to transform $G\left(t, y, y', y'', \dots y^{(n)}\right) = 0$ into $F\left(x, y, y', y'', \dots y^{(n)}\right) = 0$. Since the solution $y = f(x)$ of $F = 0$ is known, it is also the solution of $G = 0$, so use $x = x(t)$ convert $y = f(x)$ to $y = g(t), y = g(t)$ is the solution of $G = 0$, the theorem is proved. □

Below we propose Theorem 28, 29.

Theorem 28. *An independent variable transformation equation of a Ψ equation is still a Ψ equation.*

Proof. Disprove method: For ODEs, suppose that the independent variable transformation equation $G\left(t, y, y', y'', \ldots y^{(n)}\right) = 0$ of the Ψ equation $F\left(x, y, y', y'', \ldots y^{(n)}\right) = 0$ is an Φ equation, and the solution of $G = 0$ is $y = g(t)$. Then use $t = t(x)$ to transform $G\left(t, y, y', y'', \ldots y^{(n)}\right) = 0$ into $F\left(x, y, y', y'', \ldots y^{(n)}\right) = 0$, so $F = 0$ has a known function solution $y = f(x)$, which contradicts that $F = 0$ is a Ψ equation, so the theorem is established for ODEs.

For PDEs, assume that the independent variable transformation equation $G\left(y_1, \ldots y_n, u, u_{y_1} \ldots u_{y_n}, u_{y_1 y_2} \ldots\right) = 0$ of the Ψ equation $F\left(x_1, \ldots x_n, u, u_{x_1} \ldots u_{x_n}, u_{x_1 x_2} \ldots\right) = 0$ is an Φ equation, and the solution of $G = 0$ is $u = g(y_1, y_2, \ldots y_n)$. Then use $y_i = y_i\left(x_1, x_2 \ldots x_n\right)$ to transform $G\left(y_1, \ldots y_n, u, u_{y_1} \ldots u_{y_n}, u_{y_1 y_2} \ldots\right) = 0$ into $F\left(x_1, \ldots x_n, u, u_{x_1} \ldots u_{x_n}, u_{x_1 x_2} \ldots\right) = 0$, so $F = 0$ has a known function solution $u = g\left(y_1, y_2, \ldots y_n\right) = f\left(x_1, x_2 \ldots x_n\right)$, which contradicts that $F = 0$ is a Ψ equation, that is, the theorem holds for PDEs, so the theorem is proved. □

Theorem 29. *An independent variable transformation equation of an Φ equation is still an Φ equation.*

The proof of Theorem 29 is similar to Theorem 28, and readers can try it by themselves.

According to Theorems 2, in general, sub-equations of a Ψ equation are Ψ equations, sub-equations of an Φ equation are Φ equations. For ODEs, such as

$$y'' + b\left(x\right) y + c\left(x\right) = 0. \tag{126}$$

If it is taken as a source equation, the first-order sub-equation of the first kind is

$$y''' + b\left(x\right) y' + b'\left(x\right) y + c'\left(x\right) = 0. \tag{155}$$

The second type first-order sub-equations are

$$y''' + b\left(x\right) y' + b'\left(x\right) y + c'\left(x\right) = 0, \tag{156}$$

$$y''' + b\left(x\right) y' - b'\left(x\right) \frac{y'' + c\left(x\right)}{b\left(x\right)} + c'\left(x\right) = 0.$$

Namely

$$y''' - \frac{b'\left(x\right)}{b\left(x\right)} y'' + b\left(x\right) y' - \frac{b'\left(x\right) c\left(x\right)}{b\left(x\right)} + c'\left(x\right) = 0, \tag{157}$$

and so on, it can be preliminarily judged that (155-157) are all Ψ equations.

For PDEs, such as the Ψ equation

$$u_{x_i x_i} + b\left(x_1, \ldots x_n\right) u + c\left(x_1, \ldots x_n\right) = 0. \tag{144}$$

If it is taken as a source equation, the first-order sub-equation of the first kind is

$$u_{x_i x_i x_i} + b u_{x_i} + b_{x_i} u + c_{x_i} = 0, \tag{158}$$

$$u_{x_i x_i x_j} + b u_{x_j} + b_{x_j} u + c_{x_j} = 0. \tag{159}$$

The second type first-order sub-equations are

$$u_{x_ix_ix_i} + au_{x_ix_i} + bu_{x_i} + (b_{x_i} + ab)\,u + c_{x_i} + ac = 0, \tag{160}$$

$$u_{x_ix_ix_i} + bu_{x_i} - b_{x_i}\frac{u_{x_ix_i} + c}{b} + c_{x_i} = 0.$$

That is

$$u_{x_ix_ix_i} - \frac{b_{x_i}}{b}u_{x_ix_i} + bu_{x_i} - \frac{cb_{x_i}}{b} + c_{x_i} = 0, \tag{161}$$

and so on, it can be preliminarily judged that (158-161) are all Ψ equations.

7. Conclusions and discussions

The solution of a differential equation may be a function, such as (12) and (113); it may also be a functional equation, such as (57), (89) and so on. In order to obtain solutions of a differential equation, we propose Method 1 and verification axiom for the first time. According to actual cases, it can be found that, whether ODEs or PDEs, almost all solvable differential equations or functional equations can be used as source equations, and any solvable source equation theoretically has an infinite number of solvable sub-equations. The solutions obtained by Method 1 are sometimes general solutions, sometimes exact solutions and sometimes require additional conditions.

A differential equation may be solved through multiple different source equations, and the conditions and exact solutions for different source equations are often different. If there are n arbitrary known functions in the differential equation to be solved and m pending functions in the source equation, generally, an unconditional solution can be obtained when $m \geq n$, and a conditional solution can be obtained when $m < n$. Since the order of the source equation is lower than that of the equation to be solved, generally $m \leq n$.

Method 1 can be used to judge whether there is a known functional solution to a certain differential equation. To this end, we propose the concepts of Φ equation, Ψ equation, Φ transformation, and Ψ transformation, and research the relevant laws. Through some cases, we point out that if a differential equation does not have a known function solution, it does not mean that it is unsolvable under any circumstances, and is often solvable under special conditions.

References

[1] J. Biazar, M. Eslami, Analytic solution for Telegraph equation by differential transform method, Phys. Lett. A 374 (2010) 2904-2906.

[2] O. I. Akhmetov, V. S. Mingalev, I. V. Mingalev, et al., Solution of the Cauchy Problem for the Three-Dimensional Telegraph Equation and Exact Solutions of Maxwell's Equations in a Homogeneous Isotropic Conductor with a Given Exterior Current Source, Comput. Math. Math. Phys. 58 (2018) 604-611.

[3] J. Biazar, P. Gholamin, K. Hosseini, Variational iteration method for solving Fokker-Planck equation, J. Franklin I. 347 (2010) 1137-1147.

[4] H. Triki, A.-M. Wazwaz, Traveling wave solutions for fifth-order KdV type equations with time-dependent coefficients, Commun. Nonlinear Sci. Numer. Simulat. 19 (2014) 404-408.

[5] S. N. Baranovskaya, E. N. Novikov, N. I. Yurchuk, Directional Derivative Problem for the Telegraph Equation with a Dirac Potential, Differ. Equations 54 (2018) 1147-1155.

[6] T. S. Jang, A new solution procedure for the nonlinear telegraph equation, Commun. Nonlinear Sci. Numer. Simulat. 29 (2015) 307-326.
[7] B. D. Abramov, Method of Equivalent Differences for the Neutron Transport Equation, Comput. Math. Math. Phys. 53 (9) (2013), 1336-1349.
[8] C. Kesan, Chebyshev polynomial solutions of second-order linear partial differential equations, Appl. Math. Comput. 134 (2003) 109-124.
[9] T. Ellis, L. Demkowicz, J. Chan, Locally conservative discontinuous Petrov-Galerkin finite element methods for fluid problems, Comput. Math. Appl. 68 (2014) 1530-1549.
[10] B. Blbla, M. Sezerb, A numerical approach for solving generalized Abel-type nonlinear differential equations, Appl. Math. Comput. 262 (2015) 169-177.
[11] R. Dascaliuc, E. A.Thomann, E. C.Waymire, Stochastic explosion and non-uniqueness for -Riccati equation, J. Math. Anal. Appl. 476 (2019) 53-85.
[12] J. L. Bravo, J. Torregrosa, Abel-like differential equations with no periodic solutions, J. Math. Anal. Appl. 342 (2008) 931-942.
[13] M. J. lvarez, J. L. Bravo, M. Fernndez, Existence of non-trivial limit cycles in Abel equations with symmetries, Nonlinear Anal. 84 (2013) 18-28.
[14] N. Lungu, D. Popa, Hyers-Ulam stability of a first order partial differential equation, J. Math. Anal. Appl. 385 (2012) 86-91.
[15] C. M. Guo, C. B. Zhai, R. P. Song, An existence and uniqueness result for the singular Lane-Emden-Fowler equation, Nonlinear Anal. TMA 72 (2010) 1275-1279.
[16] M. J. Álvareza, J. L. Bravob, M. Fernndezb, R. Prohensa, Alien limit cycles in Abel equations, J. Math. Anal. Appl. 482 (2020) 123525.
[17] J. Llibrea, C. Vallsb, Centers of planar generalized Abel equations, J. Differ. Equations 268 (2020) 6481-6487.
[18] M. Bohnera, A. Gasullb, C. Vallsc, Periodic solutions of linear, Riccati, and Abel dynamic equations, J. Math. Anal. Appl. 470 (2019) 733-749.
[19] L. A. Markovich, R. Grimaudo, A. Messina, H. Nakazato, An example of interplay between Physics and Mathematics: Exact resolution of a new class of Riccati Equations, Ann. Phys. 385 (2017) 522-531.
[20] G. A. Grigoryan, Some Properties of Solutions of a Scalar Riccati Equation with Complex Coefficients, Differ. Equations, 52, 10 (2016) 1366-1370.
[21] V. S. Samovol, On Convergent Series Expansions of Solutions of the Riccati Equation, Math. Notes, 105, 4 (2019) 592-603.
[22] M. K. Mak, T. Harko, New integrability case for the Riccati equation, Appl. Math. Comput. 218 (2012) 10974-10981.
[23] A. R. Vahidi, Z. Azimzadeh, M. Didgar, An efficient method for solving Riccati equation using homotopy perturbation method, Indian J. Phys. 87 (2013) 447-454.
[24] V. V. Palin, E. V. Radkevich, Behavior of stabilizing solutions of the Riccati equation, J. Math. Sci. 234 (4) (2018) 455-469.
[25] V. S. Samovol, Asymptotic Integration of the Riccati Equation by Methods of Power Geometry, Dokl. Math. 96 (1) (2017) 373-376.
[26] M. J. Álvarez, A. Gasull, H. Giacomini, A new uniqueness criterion for the number of periodic orbits of Abel equations, J. Differ. Equations 234 (2007) 161-176.
[27] M. P. Markakis, Closed-form solutions of certain Abel equations of the first kind, Appl. Math. Lett. 22 (2009) 1401-1405.
[28] O. G. Hudayberenow, N. O. Hudaywerenow, An Estimate of the Number of Periodic Solutions of the Abel Equation, Differ. Equations, 40, 8 (2004) 1209-1212.
[29] J. -M. Belley, A. Gueye, Periodic solutions of Abel equations with jumps, J. Math. Anal. Appl. 472 (2019) 1106-1131.
[30] J. Llibre, R. Ramrez, N. Sadovskaia, Differential equations with a given set of solutions,

Appl. Math. Comput. 365 (2020) 124659.
[31] A. Cima, A. Gasull, F. Mañosas, On the number of polynomial solutions of Bernoulli and Abel polynomial differential equations, J. Differ. Equations 263 (2017) 7099-7122.
[32] A. Cima, A. Gasull, F. Mañosas, Periodic orbits in complex Abel equations, J. Differ. Equations 232 (2007) 314-328.
[33] M. A. M. Alwash, The composition conjecture for Abel equation, Expo. Math. 27 (2009) 241-250.
[34] D. E. Panayotounakos, Exact analytic solutions of unsolvable classes of first and second order nonlinear ODEs (Part I: Abel's equations), Appl. Math. Lett. 18 (2005) 155-162.
[35] A.D. Polyanin and V.F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations, CRC Press, Florida (2003).
[36] J. Gin, X. Santallusia, Abel differential equations admitting a certain first integral, J. Math. Anal. Appl. 370 (2010) 187-199.
[37] Stefan C. Mancas, Haret C. Rosu, Integrable dissipative nonlinear second order differential equations via factorizations and Abel equations, Phys. Lett. A 377 (2013) 1434-1438.
[38] L. Bougoffa, New exact general solutions of Abel equation of the second kind, Appl. Math. Comput. 216 (2010) 689-691.
[39] M. K. Mak, New method for generating general solution of Abel differential Equation, Comput. Math. Appl. 43 (2002) 91-94.
[41] J. Sugie, Interval oscillation criteria for second-order linear differential equations with impulsive effects, J. Math. Anal. Appl. 479 (2019) 621-642.
[42] I. Jadlovská, J. Džurina, Kneser-type oscillation criteria for second-order half-linear delay differential equations, Appl. Math. Comput. 380 (2020) 125289.
[43] G. E. Chatzarakis, J. Džurina, I. Jadlovská, New oscillation criteria for second-order half-linear advanced differential equations, Appl. Math. Comput. 347 (2019) 404-416.
[44] J. Sugie, K. Ishibashi, Nonoscillation of Mathieu equations with two frequencies, Appl. Math. Comput. 346 (2019) 491-499.
[45] A. A. Abramov, S. V. Kurochkin, Calculation of Solutions to the Mathieu Equation and of Related Quantities, Comput. Math. Math. Phys. 47 (3) (2007) 397-406.
[46] I. Jadlovsk, Oscillation criteria of Kneser-type for second-order half-linear advanced differential equations, Appl. Math. Lett. 106 (2020) 106354.
[47] A. Ruiz, C. Muriel, On the integrability of Liénard I-type equations via λ-symmetries and solvable structures, Appl. Math. Comput. 339 (2018) 888-898.
[48] H. L. Zhu, New Principles of Differential Equations , http://vixra.org/abs/1705.0410 (2017).