Growth Order of Standardized Distribution Functions

Yu-Lin Chou*

Abstract

Denote by $\mathscr{D}^{0,1}(\mathbb{R})$ the class of all (cumulative) distribution functions on $\mathbb{R}$ with zero mean and unit variance; if $F \in \mathscr{D}^{0,1}(\mathbb{R})$, we are interested in the asymptotic behavior of the function sequence $(x \mapsto nF(x/\sqrt{n}))_{n\in\mathbb{N}}$. We show that $\inf_{F\in\mathscr{D}^{0,1}(\mathbb{R})} \liminf_{n\to\infty} nF(x/\sqrt{n}) \geq \Phi(x)$ for all $x \in \mathbb{R}$, which in particular would be a result obtained for the first time regarding the growth order of an arbitrary standardized distribution function on $\mathbb{R}$ near the origin.

Keywords: growth order of distribution functions; standard Gaussian distribution; standardized distribution functions; tail order of distribution functions
MSC 2020: 60F99; 60E05; 26D20; 26A12

1 Introduction

A distribution function on $\mathbb{R}$ is by definition precisely an increasing (in contrast with "strictly increasing"), right-continuous function $\mathbb{R} \to [0, 1]$ whose limit at minus infinity is 0 and whose limit at plus infinity is 1. Let $\mathscr{D}^{0,1}(\mathbb{R})$ denote the class of all distribution functions on $\mathbb{R}$ with zero mean and unit variance, i.e. the class of all standardized distribution functions on $\mathbb{R}$. If $F \in \mathscr{D}^{0,1}(\mathbb{R})$, the asymptotic behavior of the function sequence $(x \mapsto nF(x/\sqrt{n}))_{n\in\mathbb{N}}$ is unclear: For every real $x > 0$, the real sequence $(F(x/\sqrt{n}))_n$ is decreasing; but the decreasing speed seems intangible.

It turns out that the standard Gaussian distribution function $\Phi : x \mapsto \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, \mathrm{d}t$ on $\mathbb{R}$ is a pointwise bound (in x) of the limit inferior of the function sequence $(x \mapsto F(x/\sqrt{n}))_n$ for every $F \in \mathscr{D}^{0,1}(\mathbb{R})$. We then obtain a result gaining an understanding of the growth order of the standardized distribution functions on $\mathbb{R}$ near the origin.

Moreover, our main result is informative also in obtaining a further interesting, unexpected bound regarding the "tail order" of the standardized distribution functions

*Yu-Lin Chou, Institute of Statistics, National Tsing Hua University, Hsinchu 30013, Taiwan, R.O.C.; Email: y.l.chou@gapp.nthu.edu.tw.

on $\mathbb{R}$, and in checking whether a distribution function on $\mathbb{R}$ belongs to $\mathscr{D}^{0,1}(\mathbb{R})$. Here and throughout, the term "tail order" is employed in a figurative, poetic sense (and hence the presence of the quotation marks) for communicational convenience, which need not coincide with some established notion connoted by "tail order" such as in Hua and Joe [1].

Our main result is the following

Theorem 1. *If $\mathscr{D}^{0,1}(\mathbb{R})$ is the class of all distribution functions on $\mathbb{R}$ with zero mean and unit variance, then*

$$\inf_{F\in\mathscr{D}^{0,1}(\mathbb{R})} \liminf_{n\to\infty} nF(x/\sqrt{n}) \geq \Phi(x)$$

for all $x \in \mathbb{R}$. □

2 Results

We begin by giving the consequences of Theorem 1; some of them may be theoretically interesting, and others may be potentially useful regarding applications.

2.1 Consequences

There is a pointwise "equi-"bound for the elements of $\mathscr{D}^{0,1}(\mathbb{R})$:

Proposition 1. *For every $F \in \mathscr{D}^{0,1}(\mathbb{R})$ we have*

$$\lim_{x\to\infty} \liminf_{n\to\infty} nF(x/\sqrt{n}) \geq 1.$$

Proof. If $F \in \mathscr{D}^{0,1}(\mathbb{R})$, we have by Theorem 1 the inequality

$$\liminf_{n\to\infty} nF(x/\sqrt{n}) \geq \Phi(x)$$

for every $x \in \mathbb{R}$. The evident passage to the limit gives the desired inequality. □

The standardized distribution functions on $\mathbb{R}$ cannot decrease nor increase to the origin too fast:

Proposition 2. *If $F \in \mathscr{D}^{0,1}(\mathbb{R})$, then for every $x \in \mathbb{R}$ and every $\epsilon > 0$ we have*

$$F(x/\sqrt{n}) \neq o(n^{-(1+\epsilon)})$$

as $n \to \infty$.

Proof. If there are some $x \in \mathbb{R}$ and some $\epsilon > 0$ such that $F(x/\sqrt{n}) = o(n^{-(1+\epsilon)})$ as $n \to \infty$, then

$$nF(x/\sqrt{n}) = o(n^{-\epsilon})$$

as $n \to \infty$; and so

$$nF(x/\sqrt{n}) \to 0$$

as $n \to \infty$. But then

$$nF(x/\sqrt{n}) < \Phi(x)/2$$

for all sufficiently large n, and hence $F \notin \mathscr{D}^{0,1}(\mathbb{R})$ by Theorem 1. $\square$

Further, Theorem 1 may be applied to check the $\mathscr{D}^{0,1}(\mathbb{R})$-membership of any given distribution function on $\mathbb{R}$:

Proposition 3. *If $F : \mathbb{R} \to [0, 1]$ is a distribution function, and if there is some $x \in \mathbb{R}$ such that*

$$\liminf_{n\to\infty} nF(x/\sqrt{n}) < \Phi(x),$$

then $F \notin \mathscr{D}^{0,1}(\mathbb{R})$.

Proof. If $F : \mathbb{R} \to [0, 1]$ is a distribution function, and if $F \in \mathscr{D}^{0,1}(\mathbb{R})$, then $\Phi(x) \leq \liminf_{n\to\infty} nF(x/\sqrt{n})$ for all $x \in \mathbb{R}$ by Theorem 1, contradicting the other assumption. $\square$

The following is a possible theoretical situation where Proposition 3 may be helpful:

Example 1. Let $f : \mathbb{R} \to [0, +\infty[$ be an L^1 function with respect to Lebesgue measure; suppose $F : x \mapsto \int_{-\infty}^{x} f$ is a distribution function on $\mathbb{R}$ with mean zero. If it is cumbersome to compute the variance of F, but if, for instance, it is relatively easy to find that there is some real $x \geq 0$ such that

$$F(x/\sqrt{n}) < \frac{1}{3n}$$

for all sufficiently large n, then

$$\liminf_{n\to\infty} nF(x/\sqrt{n}) \leq \frac{1}{3} < \frac{1}{2} \leq \Phi(x).$$

By Proposition 3 we may proceed to conclude that

$$F \notin \mathscr{D}^{0,1}(\mathbb{R}). \quad \square$$

2.2 Proof of Main Result

We now give

Proof (*of Theorem 1*). For every distribution function F on $\mathbb{R}$, there is some sequence of independent identically distributed (i.i.d.) random variables $X_1, X_2, \ldots$ such that F is the common distribution function of each X_i. This, as well-known, follows by considering the natural projections on $\mathbb{R}^{\mathbb{N}}$ and the product probability measure, of the F-identified probability measure over $\mathbb{R}$, on the Borel sigma-algebra of $\mathbb{R}^{\mathbb{N}}$. (For a short, simple, and elegant proof of the existence of product probability measure given arbitrary probability measures, we refer the reader to Saeki [2].)

Given any $F \in \mathscr{D}^{0,1}(\mathbb{R})$, choose a sequence of i.i.d. random variables $X_1, X_2, \ldots$ having F as their common distribution function. Denote by $\mathbb{P}$ the product probability measure over $\mathbb{R}^{\mathbb{N}}$ obtained from F.

Since, letting $n^{-1/2}$ be the principal square root $1/\sqrt{n}$ of $1/n$ for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathbb{P}\left(n^{-1/2}\sum_{i=1}^{n} X_i \leq x\right) &= \mathbb{P}\left(\sum_{i=1}^{n} X_i \leq x\sqrt{n}\right)\\ &\leq \sum_{i=1}^{n} \mathbb{P}\left(X_i \leq x/\sqrt{n}\right)\\ &= n\mathbb{P}\left(X_1 \leq x/\sqrt{n}\right)\\ &= nF(x/\sqrt{n})\end{aligned}$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$, the classical Lindeberg-Lévy central limit result (e.g. Theorem 3, Section 3, Chapter III, Shiryaev [3]) implies that

$$\begin{aligned}&\liminf_{n\to\infty} \mathbb{P}\left(n^{-1/2}\sum_{i=1}^{n} X_i \leq x\right)\\ &= \Phi(x)\\ &\leq \liminf_{n\to\infty} nF(x/\sqrt{n})\end{aligned}$$

for all $x \in \mathbb{R}$.

Given any $x \in \mathbb{R}$, the argument above holds for all $F \in \mathscr{D}^{0,1}(\mathbb{R})$; and therefore we obtain

$$\inf_{F\in\mathscr{D}^{0,1}(\mathbb{R})} \liminf_{n\to\infty} nF(x/\sqrt{n}) \geq \Phi(x)$$

for all $x \in \mathbb{R}$. This completes the proof. □

References

[1] Hua, L. and Joe, H. (2011). Tail order and intermediate tail dependence of multivariate copulas. *Journal of Multivariate Analysis* **102** 1454–1471.

[2] Saeki, S. (1996). A proof of the existence of infinite product probability measures. *The American Mathematical Monthly* **103** 682–683.

[3] Shiryaev, A. N. (1996). *Probability*, second edition, translated by R. P. Boas. Springer.