Fractional Dynamics and Hamiltonian Time Crystals

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Abstract

Hamiltonian time crystals (HTC) are time-dependent solutions of the equation of motion that develop in a minimum energy configuration. A subset of HTC's includes periodic solutions with minimal energy that spontaneously break time-translation invariance. Recent studies suggest that HTC's may be realized as novel topological structures formed by knotted molecules. In this brief note we show that fractional dynamics of harmonic oscillators may replicate the properties of HTC's outside the realm of traditional condensed matter applications.

Key words: Hamiltonian time crystals, knotted molecules, fractional dynamics, fractional oscillator, out of equilibrium dynamics.

Time crystals are temporal analogs of ordinary crystals which explicitly break space translation symmetry in solid-state physics [1-2]. At first sight, HTC's can be discarded on straightforward theoretical grounds. One invokes the argument that, on any compact closed manifold, the minimum of the Hamiltonian $H(p^a, q^a)$, ($a = 1, 2, ..., N$) is a critical point where Hamilton's equation reads

$$\frac{dq^a}{dt} = \frac{\partial H}{\partial p^a} \tag{1a}$$

$$\frac{dp^a}{dt} = -\frac{\partial H}{\partial q^a} \tag{1b}$$

Because critical points are defined by vanishing derivatives of H, (1) implies that these points are necessarily time independent. However, there are legitimate counterarguments against this point of view. In particular, HTC's can be shown to exist under these conditions [1-2]:

a) Hamilton's equation has symmetries that lead to conserved Noether charges,

b) HTC's spontaneously break time-translation invariance,

c) HTC's are both a minimum of the energy *and* a time-periodic trajectory generated by linear combination of conserved charges.

Moreover, [2] argues that HTC's may be topologically implemented using closed molecule strings dubbed "*knotty molecular motors*".

The purpose of this (exceedingly) brief note is to indicate that fractional dynamics of harmonic oscillators has the potential of complying with conditions a)-c) *outside* the traditional framework of condensed matter theory. To this end, consider the *fractional Lagrangian* describing the behavior of harmonic oscillators endowed with memory attributes [3]

$$L = \frac{1}{2}({}_0D_t^{\alpha}x)^2 - \frac{1}{2}\omega^2 x^2 \tag{2}$$

subject to the initial conditions $x(0) = 0$ and $x'(0) = D_t^1 x(0) = 1$. The equation of motion derived from (2) is given by

$${}_tD_1^{\alpha}({}_0D_t^{\alpha}x) = -\omega^2 x \tag{3}$$

Here, D_t^{α} is the fractional time operator defined in section 5.1 of [3], $\alpha \in (0,1]$ stands for the order of differentiation and ω for the oscillator frequency. It can be shown that (3)

generates solutions that generalize the exponential function of ordinary differential equations and may be interpreted as defining *quasi-periodic trajectories* [5-6]. It can be also shown that the fractional Noether theorem detailed in [3] extrapolates the standard conservation law to

$$E^{\alpha} = \frac{1}{2}({}_0D_t^{\alpha}x)^2 + \frac{1}{2}\omega^2x^2 + \int_0^t(-{}_0D_s^{\alpha}x' \cdot {}_0D_s^{\alpha}x + x' \cdot {}_sD_1^{\alpha}({}_0D_s^{\alpha}x))\,ds = \text{const.} \qquad (4)$$

If we limit the discussion to the *minimal fractal manifold* set by $\alpha = 1-\varepsilon,\ \varepsilon = O\,(m^2/\Lambda_{UV}^2) << 1$ [4], the integral in (4) nearly cancels out and one recovers the expression of the standard oscillator energy as in

$$E^1 \Rightarrow E_c = \frac{1}{2}(x')^2 + \frac{1}{2}\omega^2x^2 \qquad (5)$$

It is apparent that turning off the non-negative contribution of the integral in (4) automatically implies that (5) sets the *lowest bound* of (4).

These observations point out that (2) – (5) are likely to meet all conditions a) – c) outside the realm of conventional condensed matter applications, in general, and molecular knots, in particular.

References

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