Some New Type Laurent Expansions and Division by Zero Calculus; Spectral Theory

Hiroshi Okumura
Maebashi-shi, 371-0123, Japan
hokmr@yandex.com
and
Saburou Saitoh
Institute of Reproducing Kernels
Kawauchi-cho, 5-1648-16, Kiryu 376-0041, Japan
saburou.saitoh@gmail.com

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Abstract: In this paper we introduce a very interesting property of the Laurent expansion in connection with the division by zero calculus and Euclid geometry by H. Okumura ([14]). The content may be related to analytic motion of figures. We will refer to some similar problems in the spectral theory of closed operators.

Key Words: Laurent expansion, division by zero, division by zero calculus, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0, [(z^n)/n]_{n=0} = \log z$, $[e^{(1/z)}]_{z=0} = 1$, spectral theory, closed operator, spectrum decomposition.

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1 Okumura's results

First, we recall Okumura's results from ([14]). Let C be a point on the segment AB such that $|BC| = 2a$ and $|CA| = 2b$. We consider the three circles α, β and γ with diameters CB, AC and AB, respectively. We use a rectangular coordinates system with origin C such that the point B has coordinates $(2a, 0)$. We call the line AB the baseline. Let $c = a + b$ and $d = \sqrt{ab}/c$. Then, we have:

A circle γ_z touches the circles α and β if and only if it has radius r_z^γ and center of coordinates (x_z^γ, y_z^γ) given by

$$r_z^\gamma = |q_z^\gamma| \quad \textit{and} \quad (x_z^\gamma, y_z^\gamma) = \left(\frac{b-a}{c} q_z^\gamma, 2zq_z^\gamma\right), \ \textit{where} \ q_z^\gamma = \frac{abc}{c^2 z^2 - ab}$$

for a real number $z \neq \pm d$.

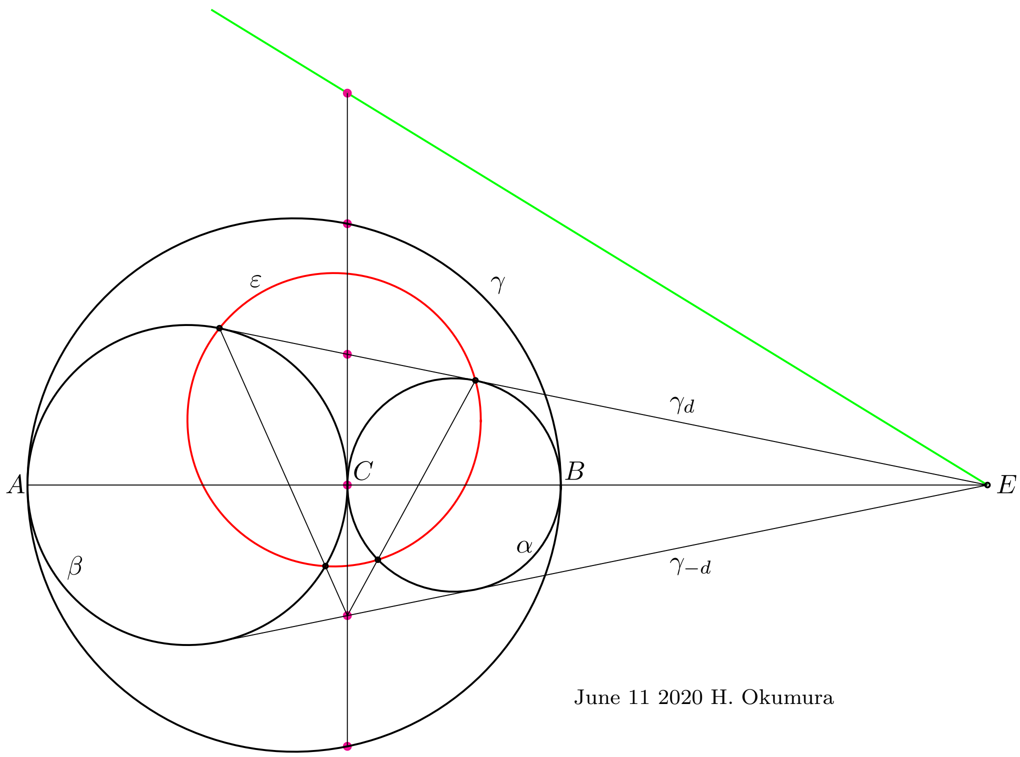
The circle γ_z touches α and β internally (resp. externally) if and only if $|z| < d$ (resp. $|z| > d$). The external common tangents of α and β have following equations:

$$(a - b)x \mp 2\sqrt{ab}y + 2ab = 0,$$

which are denoted by $\gamma_{\pm d}$.

The distance between the center of the circle γ_z and the baseline equals $2|z|r_z^\gamma$.

The ratio of the distance from the center of γ_z to the perpendicular to the baseline at C to the radius of γ_z is constant and equals to $|a - b|/c$ for $z \neq \pm d$.



Let $g_z(x, y) = (x - x_z^\gamma)^2 + (y - y_z^\gamma)^2 - (r_z^\gamma)^2$. Then $g_z(x, y) = 0$ is an equation of the circle γ_z for $z \neq \pm d$. Let

$$g_z(x, y) = \cdots + C_{-2}(z-d)^{-2} + C_{-1}(z-d)^{-1} + C_0 + C_1(z-d) + \cdots$$

be the Laurent expansion of $g_z(x, y)$ around $z = d$, then we have

$$\cdots = C_{-4} = C_{-3} = C_{-2} = 0,$$

$$C_{-1} = d((a-b)x - 2\sqrt{ab}y + 2ab),$$

$$C_0 = \left(x - \frac{a-b}{4}\right)^2 + \left(y - \frac{\sqrt{ab}}{2}\right)^2 - \left(\frac{\sqrt{a^2+18ab+b^2}}{4}\right)^2,$$

$$C_n = -\frac{1}{2}\left(\frac{-1}{2d}\right)^n ((a-b)x + 2\sqrt{ab}y + 2ab),\ \ for\ n = 1, 2, 3, \cdots.$$

Therefore $C_{-1} = 0$ gives an equation of the line γ_d. Also $C_n = 0$ gives an equation of the line γ_{-d} for $n = 1, 2, 3, \cdots$.

Let ε be the circle given by the equation $C_0 = 0$. Then, it has the following beautiful properties that were given in [15]:

(i) The points, where γ_d touches α and β, lie on ε.
(ii) The radical center of the three circles α, β and ε has coordinates $(0, -\sqrt{ab})$, and lies on the line γ_{-d}.
(iii) The radical axis of the circles ε and γ passes though the points of coordinates $(0, 3\sqrt{ab})$ and $(2ab/(b-a), 0)$, where the latter coincides with the point of intersection of γ_d and γ_{-d}.
The y-axis meets γ and $\gamma_{\pm d}$ in the points of coordinates $(0, \pm 2\sqrt{ab})$ and $(0, \pm\sqrt{ab})$, respectively. Hence the six points, where the y-axis meets γ, $\gamma_{\pm d}$, the baseline, the radical axis of γ and ε, are evenly spaced. Reflecting the figure in the baseline, we also get similar results for the Laurent expansion of $g_z(x, y)$ around $z = -d$.

Meanwhile, for the Laurent expansion of $g_z(x, y)$ around $z = 0$, we have:

$$\cdots = C_{-3} = C_{-2} = C_{-1} = 0,$$

$$C_0 = (x - 2a)(x + 2b) + y^2 = g_0(x, y),$$

$$C_n = \frac{4(a+b)^n}{(ab)^{(n-1)/2}} y; \quad n = 1, 3, 5, \cdots,$$

and

$$C_n = -\frac{2(a+b)^n}{(ab)^{(n/2)}}(a-b)\left(x - \frac{2ab}{b-a}\right); \quad n = 2, 4, 6, \cdots.$$

Therefore $C_0 = 0$ is an equation of the circle γ_0. $C_n = 0$ is an equation of the x-axis for $n = 1, 3, 5, \cdots$, and $C_n = 0$ is an equation of the line $x = 2ab/(b-a)$ for $n = 2, 4, 6, \cdots$. This line passes through the point of coordinates

$$\left(\frac{2ab}{b-a}, 0\right),$$

which is denoted by E. Notice that if a circle touches α and β at two points P and Q, respectively, then the line PQ passes through E.

For more beautiful properties and Figures, see [15].

In particular, we would like to recall that:

David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Look 4 points on the red circle and 6 points on the y axis; they are very beautiful. The circle was discovered by the division by zero calculus.

2 From the division by zero calculus

In the equation $g_z(x, y) = 0$, when we apply the division by zero calculus at $z = d$ we obtain the equation of ε. Meanwhile, in the equation $(z - d)g_z(x, y) = 0$, when we apply the division by zero calculus at $z = d$ we obtain the equation of γ_d ([10]).

Meanwhile, in the equation $g_z(x, y) = 0$ by letting $z \to \pm\infty$, we obtain the point of C.

Here, in particular note the very interesting fact that around $\pm d$ the equation $g_z(x, y) = 0$ represents the circle taching both circles α, β even near $\pm d$, however, the function $g_z(x, y)$ has poles of order one at $\pm d$, it looks like ∞ that is a contradiction with $g_z(x, y) = 0$. This fact will show some naturality of the division by zero calculus at $\pm d$.

For many differential equations with analytical and isolated singularities, this property is similar and we have interesting and general problems.

For an analytic function $W = f(z)$ on a domain D, of course, we have

$$W - f(z) = 0, \quad z \in D.$$

Then, note that for any $a \in D$ and for any integer n,

$$\frac{W - f(z)}{(z - a)^n} = 0, \quad z \in D.$$

For the division by zero and division by zero calculus, see the references.

3 Analytic motion of figures

The equation $g_z(x, y) = 0$ may be understood as an analytic motion of the circles γ_z with parameter z. Then, the problem may be considered as a general concept in mathematics.

As a simple and typical case, we will recall that for a general ordinary differential equation, we have a general solution with an arbitrary constant C; that is the general solution may be, in general, represented by using an analytic parameter.

For example, recall Clairau's differential equation

$$y = y'x + \frac{1}{y\prime}$$

and its general solutions containing any real number ξ are

$$y = x\xi + \frac{1}{\xi}.$$

Here note that as the singular solution of the differential equation and as the envelop of the general solutions, we have the parabolic curve

$$y^2 = 4x.$$

Firstly, note that for $\xi = 0$, we have $y = 0$ by the division by zero and it is a very natural solution of the Clairau's equation.

In connection with the Okumura's Laurent expansion, we have already the expansion at $\xi = 0$. Then, we obtain the results that

$$C_{-1} = 1, C_0 = -y, C_1 = x$$

and other Laurent expansion coefficients are all zero.

We see certainly that the coefficients C_0 and C_1 have their meanings for the general solutions. However, the Clairau's equation is given by the very general way

$$y = xy\prime + f(y'),$$

and so it certainly seems that the Okumura's Laurent expansion is mysterious.

Meanwhile, we recall another example by Okumura ([7]):

On the sector

$$\Delta_\alpha = \left\{ |\arg z| < \alpha; 0 < \alpha < \frac{\pi}{2} \right\},$$

we shall change the angle and we consider a circle $C_a, a > 0$ with a fixed radius a inscribed in the sectors. We see that when the circle tends to $+\infty$, the angles α tend to zero. How will be the case $\alpha = 0$? Then, we will not be able to see the position of the circle. Surprisingly enough, then C_a is the circle with its center at the origin 0. This result is derived from the division by zero calculus for the formula

$$k = \frac{a}{\sin \alpha}.$$

The two lines $\arg z = \alpha$ and $\arg z = -\alpha$ were tangential lines of the circle C_a and now they are the positive real line. The gradient of the positive real line is of course zero. Note here that the gradient of the positive y axis is zero by the division by zero calculus that means $\tan \frac{\pi}{2} = 0$. Therefore, we can understand that the positive real line is still a tangential line of the circle C_a.

Here note the expansion:

$$k = \frac{a}{\sin \alpha} = \frac{a}{\alpha} + \frac{a\alpha}{6} + \frac{7a\alpha^3}{369} + \frac{31a\alpha^5}{15120} + \cdots.$$

The Okumura's Laurent expansion is very curious, because all the Laurent expansion coefficients have their graphic meanings.

When we consider the Laurent expansion of the function $g_z(x, y)$ we consider, of course, that x, y are fixed. However, when we consider the expansion with the equation $g_z(x, y) = 0$, x, y are depending on z.

Why the Okumura's Laurent expansions are so beautiful?

4 Spectral theory for closed operators

In the Okumura Laurent expansion, the coefficients C_{-1} and C_0 are, in particular, important and we can see some beautiful relations among the Laurent expansion coefficients. We will be able to see the similar interesting properties in the spectral theory for closed operators.

If μ is isolated in the spectrum σ of a closed operator $A : D(A) \subset X \to X$, the resolvent $R(\lambda, A)$ of A at the point λ can be expanded as a Laurent series

$$R(\lambda, A) = \sum_{n=-\infty}^{\infty} (\lambda - \mu)^n U_n$$

on a ring domain $0 < |\lambda - \mu| < \delta$ with some small $\delta > 0$. The coefficients U_n of this expansion are bounded operators. In particular, U_{-1} is the spectral projection P corresponding to the decomposition $\sigma(A) = \{\mu\} \cup (\sigma(A) \setminus \mu)$ of the spectrum of A.

In general, we have, for any $n, m > 0$

$$U_{(n+1)} = (A - \mu)^n P$$

and

$$U_{(n+1)} \cdot U_{(m+1)} = U_{(n+m+1)}.$$

See [2], Chapter 4 for the details.

Then, by the division by zero calculus, we can consider the operator U_0 that is corresponding to the Laurent coefficient for $n = 0$.

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