22 Thomson's lamp revisited

Adapted from a chapter of the book Infinity put to the test by Antonio León (next publication).

Abstract.-The argument of Thomson lamp and Benacerraf's critique are reexamined from the perspective of the w-order legitimated by the hypothesis of the actual infinity subsumed into the Axiom of Infinite. The conclusions point to the inconsistency of that hypothesis.

Keywords: Thomson lamp, actual infinity, supertask, P. Benacerraf, counting machine.

Introduction

P1 To perform an ω-supertask (supertask hereafter) means to perform an ω-ordered sequence of actions (tasks) in a finite interval of time. Supertasks are useful theoretical devices for the philosophy of mathematics, particularly for the discussions on certain problems related to infinity [39, 8, 11, 33, 6, 41, 33]. Although their physical possibilities and implications have also been discussed [28, 29, 33, 35, 17, 19, 18, 29, 30, 31, 15, 32, 27, 3, 4, 34, 41, 21, 13, 14, 27, 12, 36]. In this book all supertasks will be conceptual.

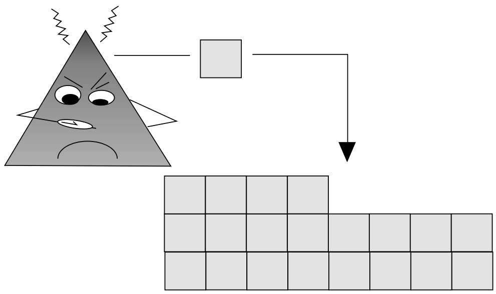

Figura 22.1 – God performing Gregory's supertask.

P2 Probably Gregory of Rimini was the first to propose how a supertask could be accomplished ([26], p. 53):

> If God can endlessly add a cubic foot to a stone -which He can- then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on *ad infinitum*. He would then have before Him an infinite stone at the end of the hour.

But the term "*supertask*" was introduced by J. F. Thomson in his seminal paper of 1954 [39]. Thomson's paper was motivated by Black's argument [7] on the impossibility to perform infinitely many successive actions and by the discussions of Black's argument by R. Taylor [38] and J. Watling [40]. In his paper Thomson tried to prove the impossibility of supertasks. Thomson argument was, in turn, criticized in another seminal paper, in this case by P. Benacerraf [5]. Benacerraf's successful criticism finally motivated the foundation of a new infinitist theory independent of set theory: supertask theory.

P3 The basic idea of Benacerraf's criticism of Thomson's argument is the impossibility to derive formal conclusions on the final state of the supermachine that performs the supertask from the sequence of states the machine traverses as a consequence of performing the supertask. Although Benacerraf's criticism of Thomson's lamp argument is well founded (see below), it is far from being complete. As we will see here, it is possible to consider a new line of argument, which Benacerraf only incidentally considered, based on the formal definition of the lamp. That line of argument leads to a contradiction that put into question the formal consistency of the ω-order involved in supertasks.

P4 In fact, if the world continues to be the same world it was before the execution of a supertask, and one is still allowed to think in rational terms in the same framework of the laws of logic, then Thomson's argument can be reoriented towards the formal definition of the machine that performs the supertask. A definition that is assumed to be independent of the number of performed tasks with that machine, and then a definition that holds before, during and after performing the supertask, whenever the execution, as such an execution, of a supertask does not arbitrarily change a legitimate definition previously established (Principles of Invariance and of Autonomy).

P5 The possibilities to perform an uncountable infinitude of actions were examined, and ruled out, by P. Clark and S. Read [11]. Supertasks have also been considered from the perspective of nonstandard analysis [25, 24, 2, 23], although the possibilities to perform a *hypertask* along a hyperreal interval of time have not been discussed, despite the fact that finite hyperreal intervals can be divided into hypercountably many successive infinitesimal intervals (hyperfinite partitions)[37, 16, 22, 20], etc. But most of the supertasks are ω-supertasks, i.e. ω-ordered sequences of actions performed in a finite (or perceived as finite) interval of time.

Thomson's lamp

P6 As Thomson did in 1954 ([39], p. 5), in the following discussion we will make use of one of those:

> ... reading-lamps that have a button in the base. If the lamp is *off* and you press the button the lamp goes *on*, and if the lamp is *on* and you press the button the lamp goes *off*.

Let us complete Thomson's definition by explicitly declaring the following conditions regarding the (theoretical) functioning of the lamp:

a) Thomson's lamp has two, and only two, states: *on* and *off*.
b) The state of the lamp (*on*/*off*) changes if, and only if, its button is pressed down.
c) Each change of state takes place at a precise and definite instant.
d) The pressing down (clicking) of the button and the corresponding lamp change of state are both instantaneous and simultaneous events.
e) Thomson's lamp remains unaltered after performing any finite or infinite number of clickings.

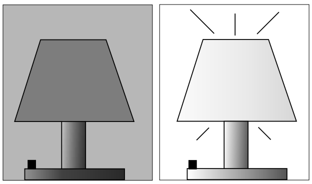

Figura 22.2 – Thomson's lamp has two, and only two, states: *off* and *on*. The state of Thomson's lamp changes if, and only if, its button is pressed.

P7 Assume now the button of Thomson's lamp is clicked at each of the infinitely many successive instants t_i, and only at them, of a strictly increasing ω-ordered sequence of instants $\langle t_n \rangle$ defined within a finite interval of time (t_a, t_b), being t_b the limit of the sequence $\langle t_n \rangle$. In these conditions, at instant t_b the button of the lamp will have undergone an ω-ordered sequence $\langle c_n \rangle$ of clicks (each click c_i performed at the precise instant t_i) and, consequently, the state of the lamp will have changed an ω-ordered infinitude of times. Or in other words, at t_b Thomson's supertask will have been completed. Don't forget this is a purely conceptual argument, so that we are not concerned here with the physical details.

P8 Thomson tried to derive a contradiction from his supertask by speculating on the final state of the lamp at instant t_b in terms of the sequence of switchings completed along the supertask ([39], p. 5):

> [The lamp] cannot be *on*, because I did not ever turn it *on* without at once turning it *off*. It cannot be *off*, because I did in the first place turn it *on*, and thereafter I never turned *off* without at once turning it *on*. But the lamp must be either *on* or *off*. This is a contradiction.

P9 It is worth noting, as we have just seen, that Thomson based his argument on the sequence of actions carried out on the lamp: it was never turned on without turning it off after, and viceversa. What Thomson tried to do is to derive the final state of the lamp, the state of the lamp at t_b, from the successive changes of state the lamp underwent during the supertask: The reason why the lamp cannot be *on* is because it was always turned *off* after turning it *on*. And for the same reason it cannot be *off* either. This way of arguing was severely criticized by Benacerraf

P10 Benacerraf argued against Thomson's argument as follows ([5], p. 768):

> The only reasons Thomson gives for supposing that his lamp will not be *off* at t_b are ones which hold only for times *before* t_b. The explanation is quite simply that Thomson's instructions do not cover the state of the lamp at t_b, although they do tell us what will be its state at every instant *between* t_a and t_b (including t_a). Certainly, the lamp must be *on* or *off* (provided that it hasn't gone up in a metaphysical puff of smoke in the interval), but nothing we are told implies which it is to be. The arguments to the effect that it can't be either just have no bearing on the case. To suppose that they *do* is to suppose that a description of the physical state of the lamp at t_b (with respect to the property of being *on* or *off*) is a *logical* consequence of a description of its state (with respect to the same property) at times prior to t_b. [t_a and t_b appears respectively as t_0 and t_1 in Benacerraf's paper].

P11 In short, according to Benacerraf, the problem posed by Thomson is not sufficiently described since no constraint have been placed on what happens at t_b [1]. But the only constraint on what happens at t_b is that Thomson's lamp continue to be Thomson's lamp. Or in other words, that the execution of a supertask does not change the formal definitions of the involved theoretical artifacts (Principle of Invariance). As we will see, the state of Thomson's lamp at t_b is not "a *logical consequence of a description of its state (with respect to the same property) at times prior to* t_b", it is a logical consequence of remaining a Thomson lamp after performing Thomson supertask (Principles of Invariance). And this is pertinent to the case. It will be the key of the next argumentation.

P12 Consider the instant t_b, the limit of the sequence $\langle t_n \rangle$ of instants at which the successive clicks $\langle c_n \rangle$ have been performed. That instant is, therefore, the first instant *after completing* the sequence of switchings. The first instant at which the button of the lamp is no longer clicked. Let now S_b be the state of the lamp at instant t_b. Being the state of a Thomson lamp, it can only be either *on* or *off*. And this conclusion has nothing to do with the number of previously performed switchings. The lamp will be either *on* or *off* because, being a Thomson's lamp, it has only two states: *on* and *off*, and it is not affected by the number of times it has been turned on and off (Principle of Invariance). Therefore the sate S_b of the lamp at the instant t_b can only be either on or off, regardless of the number of times it has been turned on or off.

P13 Some infinitist claim, however, that at t_b, after performing Thomson's supertask, the lamp could be in any unknown state, even in an exotic one. But a lamp that can be in an unknown state is not a Thomson's lamp: the only possible states of a Thomson's lamp are *on* and *off*. No other alternative is possible *without arbitrarily violating* the formal legitimate definition of Thomson's lamp. And we presume no formal theory is authorized to violate arbitrarily a formal definition, nor, obviously to change, in the same arbitrary terms, the nature of the world (Principle of invariance). It goes without saying that if that were the case any thing could be expected from that theory, because the case could be applied to any other argument.

P14 Others claim the state S_b is the consequence of completing the ω-ordered sequence of clicks $\langle c_n \rangle$, since that sequence, and only that sequence, has been carried out. But if to complete the sequence of clicks $\langle c_n \rangle$ means to perform each and every of the infinitely many clicks $c_{i=1,2,3,\ldots}$ of $\langle c_n \rangle$, and only them, then we have a problem. The problem that no click c_i of $\langle c_n \rangle$ originates S_b. None. Indeed, if c_v is any element of $\langle c_n \rangle$ it cannot originates S_b because in such a case the button would have been clicked only a finite number v of times. That is to say, if we remove from $\langle c_n \rangle$ all clicks that do not originate S_b, then all of them would be removed. Or in other, set theoretical, words, if from the set of performed clicks $\langle c_i \rangle$ we remove all clicks that do not originate S_b, all clicks would be removed and we would get the empty set. It is not, therefore, a question of indeterminacy but of impossibility: no click of the sequence $\langle c_i \rangle$ originates S_b. None.

P15 In those conditions, how can it be claimed that the completion of the sequence of clicks $\langle c_n \rangle$, *none* of whose elements originates S_b, originates just S_b? Is the completion of the sequence an additional click different from all elements of $\langle c_n \rangle$? If that were the case the sequence of performed clicks would be $(\omega+1)$-ordered in the place of ω-ordered, but ω-supertasks are ω-ordered not $(\omega+1)$-ordered.

P16 At this point some infinitists claim the lamp could be at S_b by reasons unknown. But, once again, that claim violates the definition of the lamp: the state of a Thomson's lamp changes exclusively by pressing down its button, by clicking its button. So a lamp that changes its state by reasons unknown is not, by definition, a Thomson's lamp (Principles of Invariance and of Autonomy).

P17 It makes no sense to argue about the last term of an ω-ordered sequence because such a last term does not exist. By contrast, it is always possible to argue about the limit of an ω-ordered sequence, whenever that limit exists, because it is a well defined object, though it is not an element of the sequence. Similarly, whilst it makes no sense to argue about the last instant at which the button of Thomson's lamp is clicked, the instant t_b is plenty of meaning: it is limit of the sequence of instants at which the successive switchings are carried out. It is the first instant after completing the sequence of switchings. It is the first instant at which the button of the lamp is no longer clicked. It is the first instant after all instant of (t_a, t_b).

P18 And the relevant question on the state S_b is: at which instant Thomson's lamp becomes S_b? It is immediate to prove that instant can only be the precise instant t_b. We know the state of the lamp is S_b at instant t_b, but assume there exist an instant t within (t_a, t_b) at which the lamp becomes S_b. Since t_b is the limit of the sequence $\langle t_n \rangle$, we will have:

$$\exists v: \ t_v \leq t < t_{v+1} \tag{1}$$

which means that at t only a finite number v of clicks have been carried out, and then that infinitely many clickings still remain to be carried out. Therefore, no instant t exists in (t_a, t_b) at which the lamp becomes S_b. None. The precise instant at which the lamp becomes S_b is not within the interval (t_a, t_b). Therefore, the state S_b can only originate at the first instant after all instants of (t_a, t_b). And that instant is just t_b.

P19 But at t_b the button of the lamp is not clicked. At t_b nothing happens that can cause a change in the state of the lamp. Consequently, the state S_b, which according to P18 can only originate at the instant t_b, cannot originate at the instant t_b. The state S_b is, therefore, an impossible state. It is the consequence of assuming that it is possible to complete an incompletable sequence of actions, incompletable because there is no final element to complete the sequence.

P20 The fact that the elements of two incompletable sequences can be paired off by a one to one correspondence, as in the case of the above sequences of clicks and of instants, does not prove both sequences exist as complete infinite totalities: they could also be potentially infinite. The possibility of pairing off the elements of two impossible totalities does not make them possible

P21 At this point, all that one can expect from infinitists is to be declared incompetent to understand the meaning of the sentence: "*the state of the lamp at t_b is the result of completing the ω-ordered sequence $\langle c_n \rangle$ of clicks, a result that manifests for the first time just at t_b*". But, wait a moment, is not S_b the result of a pressing down the button of the lamp? Do not forget that Thomson's lamp can only change its state if, and only if, its button is clicked. And that both events, the clicking and the corresponding lamp change of state, are instantaneous and simultaneous by definition. In addition, the lamp is not altered by pressing its button any finite or infinite number of times. So, if S_b appears for the first time at the precise instant t_b and at t_b the button of the lamp is not clicked, then S_b is impossible.

P22 In short, S_b must of necessity be originated just at instant t_b, otherwise only a finite number of clicks would have been performed, according to [P18-P17]. But, on the other hand, it cannot be originated at t_b because:

1.- The state of the lamp changes only by clicking its button.

2.- The clicking of the button and the corresponding lamp change of state are instantaneous and simultaneous events that takes place at a definite and precise instant.

3.- Being the clicking of the button and the corresponding lamp change of state instantaneous and simultaneous events, and being the state S_b originated at the precise instant t_b, the button must be clicked at that precise instant t_b.

4.- But at t_b the button of the lamp is not clicked.

Therefore, it has to be concluded that the state S_b originates and does not originate at the instant t_b. Or what is the same, in the instant t_b the button of the lamp is pressed and it is not pressed. And this is a contradiction.

P23 S_b could only be, therefore, the impossible last state of an ω-ordered sequence of states in which no last state exists. The imprint of an inconsistency. The consequence of assuming the hypothesis of the actual infinity from which derives the existence of ω-ordered sequences as *complete totalities*, in spite of the fact that no last element completes them. The state S_b forces the actual infinity to leave a trace of its existence and what it leaves is an inconsistency.

P24 Thomson's lamp is a theoretical device intentionally invented to facilitate a formal discussion on the actual infinity hypothesis that legitimizes the existence of ω-ordered sequences as complete totalities [9], [10, Theorem 15-A]. Supertasks are an example of such sequences, and contradiction [P22] clearly indicates the hypothesis on which they are founded is inconsistent.

The counting machine

P25 The Counting Machine (CM) we will examine in this section poses a problem similar to the one posed by Thomson's lamp we have just examined. As its name suggests, CM counts natural numbers, and it does it by counting the successive numbers 1, 2, 3... at each of the successive instants t_1, t_2, t_3... of the above sequence $\langle t_n \rangle$. CM counts each number n at the precise instants t_n. In addition, the machine has a red LED **L** that turns *on* if, and only if, the machine counts an even number; and turns *off* if, and only if, the machine counts an odd number, and so that the counting of the number and the change of state of **L** are simultaneous and instantaneous events. Obviously, **L** is a perfect LED that never fails.

P26 The one to one correspondence f between $\langle i \rangle$ and $\langle t_i \rangle$

$$f : \langle t_i \rangle \mapsto \mathbb{N} \tag{2}$$

$$f(t_n) = n, \ \forall t_n \in \langle t_i \rangle \tag{3}$$

proves that at t_b our machine will have counted all natural numbers. All. The conclusions on the state of **L** at t_b will not be deduced from its successive states while performing the supertask of counting all natural numbers, as Thomson did with his lamp, otherwise Benacerraf's criticism would be inevitable. They will deduced from the fact that the LED of CM has two, and only two, states, *on* and *off*, so that no other alternative exist. Thus, if after performing the supertask, CM continues to be the same counting machine it was before beginning the supertask, i.e. if performing a supertask does not arbitrarily violate a legitimate formal definition, as that of CM, then its LED **L** can only be either *on* or *off*, simply because, according to its legitimate definition, **L** can only be either *on* or *off*, and it will always be either *on* or *off*, independently of the number of times it has been turned *on* and *off*.

P27 Assume then that at t_b the red LED is *on* (a similar argument would apply if it were *off*). One of the following two exhaustive and mutually exclusive alternatives must be true:

1.- The red LED **L** is *on* because CM counted a last even number that left it *on*.

2.- The red LED **L** is *on* because of any other reason.

The first alternative is impossible if *all* natural numbers have been in fact counted: each even number has an immediate odd successor and then there is not a last natural number, neither even nor odd. The second alternative would imply the formal definition of CM has been arbitrarily violated: its red LED **L** turns *on* if, and only if, the machine counts an even number, which excludes the possibility of being turned *on* by any other reason (Principle of Invariance).

P28 Since the same argument applies if **L** is *off* at t_b, we must conclude that if the ω-ordered list of natural numbers exists as a complete infinite totality, then, once completed the supertask of counting all of them, **L** can be neither *on* nor *off*; though, by definition, it will be either *on* or *off*. The alternative to this contradiction is the arbitrary violation of a legitimate definition with the only purpose to justify that **L** can change its state by reasons different from the reason defined as the unique reason by which **L** can change its state: if, and only if, CM counts a natural number, being both events simultaneous and instantaneous. But assuming the arbitrary violation of a definition when convenient means any thing can be proved. So this alternative is formally unacceptable.

P29 Notice again that, as in the case of Thomson lamp, the above contradiction on the state of **L** at t_b has not been drawn from its successive states while performing the supertask, but from the fact of being a LED with two definite, precise and unique states: *on* and *off*, and so that it turns *on* if, and only if, CM counts an even number; and it turns *off* if, and only if, CM counts an odd number. Thus, as in the case of Thomson lamp, CM definition forces the actual infinity to leave a track of its existence through the state of **L** at t_b, and what it leaves is an inconsistency. By contrast, from the hypothesis of the potential infinity, only finite totalities of numbers can be counted, as large as wished but always finite, and depending of the parity of the last counted number, **L** will be either *on* or *off*, in agreement with the definition of CM.

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