Division by Zero Calculus and Laplace Transform

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Abstract: In this paper, we will discuss the Laplace transform from the viewpoint of the division by zero calculus with typical examples. The images of the Laplace transform are analytic functions on some half complex plane and meanwhile, the division by zero calculus gives some values for isolated singular points of analytic functions. Then, how will be the Laplace transform at the isolated singular points? For this basic question, we will be able to obtain a new concept for the Laplace integral.

Recall that David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Meanwhile,

Oliver Heaviside: *Mathematics is an experimental science, and definitions do not come first, but later on.*

Key Words: Division by zero, division by zero calculus, Laplace transform, isolated singular point, analytic function, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0, [(z^n)/n]_{n=0} = \log z,\ [e^{(1/z)}]_{z=0} = 1$.

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1 Introduction

In this paper, we will discuss the Laplace transform from the viewpoint of the division by zero calculus with typical examples. The images of the Laplace transform are analytic functions on some half complex plane and meanwhile, the division by zero calculus gives some values at isolated singular points of analytic functions. Then, how will be the Laplace transform at the isolated singular points? For this basic question, we will be able to obtain a new concept for the Laplace integral. At the isolated singular points, of course, the Laplace transform (integral) does not exist in the usual sense and so the problem is delicate and new.

2 Division by zero calculus – definition

We would like to consider some values for isolated singular points for analytic functions. The very typical problem is to consider some value of the fundamental function $W = 1/z$ at the origin. We found that its value is zero. When the result is written as

$$\frac{1}{0} = 0,$$

it will have a serious sense, because it looks like the division by zero that has a mysteriously long history ([1, 3, 20, 32, 33, 34]). However, note that $0 \times 0 \neq 1$. We showed that our result gave great impacts widely with over 1100 items. For example, look the papers cited in the reference.

The essence is stated as follows:

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{2.1}$$

we will define

$$f(a) = C_0. \tag{2.2}$$

For the correspondence (2.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (2.1), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

With this assumption, we can obtain many new results and new concepts.

Typically, we found a beautiful and important circle by this division by zero calculus, see [14] and [18].

However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problem. – In this viewpoint, **the division by zero calculus may be considered as an axiom.**

3 Examples

1. For the Laplace transform of the function

$$\frac{t^{n-1}e^{-at}}{(n-1)!}, \quad n = 1, 2, 3, ...,$$

we have

$$\frac{1}{(s+a)^n}.$$

Then, for $s = -a$, by the division by zero calculus (DBZC), we have

$$\frac{1}{(s+a)^n}(-a) = 0.$$

Then, how will be the corresponding Laplace transform

$$\int_0^\infty \frac{t^{n-1}e^{-at}}{(n-1)!}e^{at}dt = \int_0^\infty \frac{t^{n-1}}{(n-1)!}dt$$

? Note that this integral is zero, because infinity may be represented by 0. For many geometrical examples and analytical meanings, see the papers cited in the references. For example [2, 6, 19, 27].

Conversely, from this argument for the general function for any positive k

$$\frac{\Gamma(k)}{(s+a)^k}$$

that is the Laplace transform of the function

$$t^{k-1}e^{-at},$$

we can derive the result

$$\frac{\Gamma(k)}{(s+a)^k}(-a) = 0.$$

Indeed, since this result is not defined by DBZC for general positive k, this result now was derived here, by this logic.

2. For the Laplace transform of the function

$$\frac{e^{-at} - e^{-bt}}{b-a}, \quad a < b$$

we have

$$\frac{1}{(s+a)(s+b)}.$$

Then, for $s = -a$, by DBZC, we have

$$\frac{1}{(s+a)(s+b)}(-a) = -\frac{1}{(b-a)^2}.$$

Then, the corresponding Laplace transform

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{b-a} e^{at} dt = \frac{1}{b-a} \int_0^\infty \left(1 - e^{-(b-a)t}\right) dt$$

$$= -\frac{1}{(b-a)^2},$$

that is right.

3. For the Laplace transform of the function

$$\frac{ae^{-at} - be^{-bt}}{a-b}, \quad a < b$$

we have

$$\frac{s}{(s+a)(s+b)}.$$

Then, for $s = -a$, by DBZC, we have

$$\frac{s}{(s+a)(s+b)}(-a) = \frac{b}{(b-a)^2}.$$

Then, the corresponding Laplace transform

$$\int_0^\infty \frac{ae^{-at} - be^{-bt}}{a-b} e^{at} dt = \frac{1}{a-b} \int_0^\infty \left(a - be^{-(b-a)t}\right) dt$$

$$= \frac{b}{(b-a)^2},$$

that is right.

4. For the Laplace transform of the function

$$\frac{1}{a} \sinh at$$

we have

$$\frac{1}{(s-a)(s+a)}.$$

Then, for $s = a$, by DBZC, we have

$$\frac{1}{(s-a)(s+a)}(a) = -\frac{1}{4a^2}.$$

Then, the corresponding Laplace transform

$$\frac{1}{2a} \int_0^\infty \left(1 - e^{-2at}\right) dt = -\frac{1}{4a^2},$$

that is right.

5. For the Laplace transform of the function

$$\cosh at$$

we have

$$\frac{s}{(s-a)(s+a)}.$$

Then, for $s = a$, by DBZC, we have

$$\frac{s}{(s-a)(s+a)}(a) = \frac{1}{4a}.$$

Then, the corresponding Laplace transform

$$\frac{1}{2}\int_0^\infty \left(1 + e^{-2at}\right) dt = \frac{1}{4a},$$

that is right.

6. For the Laplace transform of the function

$$\frac{1}{a^3}\left(at - \sin at\right)$$

we have

$$\frac{1}{s^2(s^2+a^2)}.$$

Then, for $s = 0$, by DBZC, we have

$$\frac{1}{s^2(s^2+a^2)}(0) = -\frac{1}{a^4}.$$

Then, the corresponding Laplace transform

$$\frac{1}{a^3}\int_0^\infty \left(at - \sin at\right) e^{-0t} dt = -\frac{1}{a^4},$$

that is right. However, here note that

$$\int_0^\infty \sin at dt = \frac{1}{a},$$

in the sense of distribution theory.

7. For the Laplace transform of the function

$$\frac{1}{a^2}(1 - \cos at),$$

we have

$$\frac{1}{s(s^2+a^2)}.$$

Then, for $s = 0$, by DBZC, we have

$$\frac{1}{s(s^2+a^2)}(0) = 0.$$

Then, the corresponding Laplace transform

$$\frac{1}{a^2}\int_0^{\infty}(1-\cos at)e^{-0t}dt = 0,$$

that is right.

8. For the step function $u(t)$, the Laplace transform of the function $u(t-k)$ is given by

$$\frac{1}{s}e^{-ks}.$$

Then, by DBZC, we have

$$\left(\frac{1}{s}e^{-ks}\right)(0) = -k.$$

Then, its Laplace transform is

$$\int_k^{\infty} e^{0t}dt = [t]_k^{\infty} = -k,$$

that is right. Note that $\infty = 0$.

9. The Laplace transform of the function $(t-k)u(t-k)$ is given by

$$\frac{1}{s^2}e^{-ks}.$$

Then, by DBZC, we have

$$\left(\frac{1}{s^2}e^{-ks}\right)(0) = \frac{k^2}{2}.$$

Then, its Laplace transform is

$$\int_k^{\infty}(t-k) = \left[\frac{t^2}{2} - kt\right]_k^{\infty} = \frac{k^2}{2},$$

that is right.

10. For the Laplace transform of the function

$$1 - 3e^{-t} + 3e^{-2t}$$

we have

$$\frac{s^2+2}{s(s+1)(s+2)}.$$

Then, for $s = 0$, by DBZC, we have

$$\frac{s^2+2}{s(s+1)(s+2)}(0) = -\frac{3}{2}.$$

Note that by the theory of Oliver Heaviside we can calculate the inverse Laplace transform of the form

$$\frac{p(s)}{q(s)}$$

that is for polynomials $p(s), q(s)$.

Then, the corresponding Laplace transform

$$\int_0^\infty \left(1 - 3e^{-t} + 3e^{-2t}\right) dt = -\frac{3}{2},$$

that is right.

11. For the Laplace transform of the function

$$-\gamma - \log t,$$

we have

$$\frac{1}{s}\log s.$$

Then, for $s = 0$, we have

$$\left(\frac{1}{s}\log s\right)(0) = 0.$$

Note that by the general definition of the division by zero calculus for differentiable functions, this result is derived also in this way

$$\left(\frac{1}{s}\log s\right)(0) = (\log s)'(0) = \left(\frac{1}{s}\right)(0) = 0$$

([30]).

In general, we obtain

$$\left(\frac{1}{s^k}\log s\right)(0) = 0, \quad k > 0.$$

Of course, we can derive many and many examples.

4 Remark

For the Dirac delta distribution δ, we have

$$\delta(\omega) = \frac{1}{\pi}\int_0^\infty \cos\omega t dt.$$

Then we see that

$$\delta(0) = 0.$$

By taking derivative, we have

$$\delta'(\omega) = \frac{1}{\pi}\int_0^\infty -t\sin\omega t dt.$$

Hence,

$$\delta'(0) = 0.$$

In general, we obtain that

$$\delta^{(n)}(0) = 0, \quad n = 0, 1, 2, 3, ...$$

5 Conclusion

We are interesting in some definite statement for the relation of the division by zero calculus and Laplace integrals.

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