MOTZKIN ISLANDS: A 3-DIMENSIONAL EMBEDDING OF MOTZKIN PATHS

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Abstract. A Motzkin Path is a walk left-to-right starting at the horizontal axis, consisting of up, down or horizontal steps, never descending below the horizontal axis, and finishing at the horizontal axis. Interpret Motzkin Paths as vertical geologic cuts through mountain ranges with limited slopes. The natural embedding of these paths defines Motzkin Islands as sets of graphs labeled on vertices by non-negative integers (altitudes), a graph cycle defining a shoreline at zero altitude, and altitude differences along edges never larger than one. We address some of these islands with simple shapes on triangular and quadratic meshes.

1. Motzkin Islands

Let a Motzkin Island be defined as a 3-dimensional extension of Motzkin Paths [3][5, A001006]: Given a connected finite graph with single, undirected edges take one cycle and call it shoreline. Vertices along a path along this cycle are labeled with zeros; labels represent geographic altitudes; the shoreline is at sea level. All other vertices are labeled by non-negative integers such that the labels of adjacent vertices differ at most by one.

In consequence, each path from shore to shore along graph edges is a Motzkin Path of up, horizontal and down steps.

The concept is likely restricted to *planar* graphs, because otherwise crossing edges are mapped to "Motzkin bridges and tunnels" in the landscape of the islands.

2. Regular Triangular Shoreline

Motzkin Islands can be defined on any type of finite graphs. One particular class are Motzkin Regular Triangular Islands: a triangular grid with a triangular shoreline. They consist of a graph of an isoceless triangle, each side split into n edges of unit length, the $3n$ edges defining the shoreline, and the interior edges defined by connecting these vertices of the shoreline by straight lines parallel to the 3 edges. Each vertex not on the shoreline has 6 adjacent vertices. For $n = 5$ this looks like

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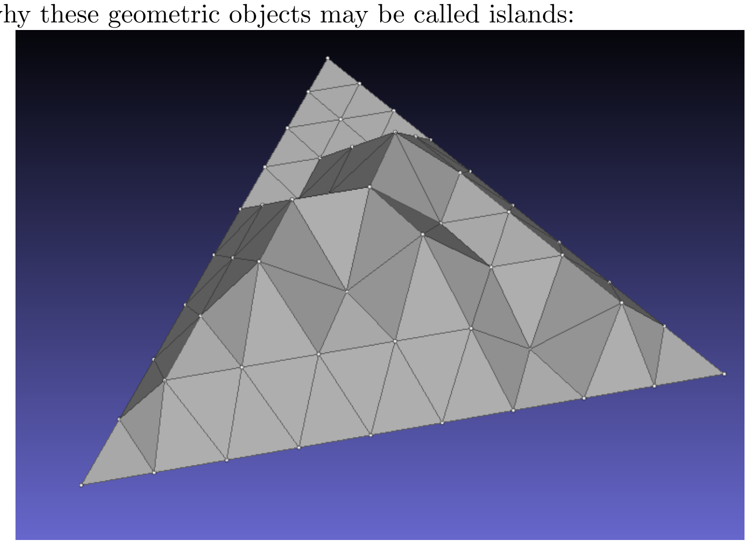
```
          .
         / \
        .---.
       / \ / \
      .---.---.
     / \ / \ / \
    .---.---.---.
   / \ / \ / \ / \
  .---.---.---.---.
 / \ / \ / \ / \ / \
.---.---.---.---.---.
```

The edges are not shown from here on, only the integer labels at the vertices. An example of such an island with $n = 9$ and highest peak at 2 is:

```
         0
        0 0
       0 0 0
      0 0 0 0
     0 0 1 1 0
    0 1 1 2 1 0
   0 1 2 2 1 1 0
  0 1 2 1 2 1 1 0
 0 1 1 1 1 1 0 1 0
0 0 0 0 0 0 0 0 0 0
```

The same island converted to a STL file and rendered with `meshlab` illustrates why these geometric objects may be called islands:



The full set of islands with edge lengths $n = 0, 1, 2$ and 3 is

```
0    0       0         0        0
    0 0     0 0       0 0      0 0
           0 0 0     0 0 0    0 1 0
                    0 0 0 0  0 0 0 0
```

Let $M^{\triangleleft}(n)$ count "fixed" islands where rotations by multiples of 120 degrees around a vertical axis or mirrors along one of the three vertical planes through

$n\backslash h$	0	1	2	3	4
1	1				
2	1				
3	1	1			
4	1	7			
5	1	63			
6	1	1023	8		
7	1	32767	872		
8	1	2097151	124800		
9	1	268435455	29455120	1000	
10	1	68719476735	12452162784	1563400	
11	1	35184372088831	9819194839552	2907994176	
12	1	36028797018963967	14784079506575424	8396820025824	1815848

Table 1. Reg-Triangular Islands $M^{\triangleleft}(n)$ with maximum altitude h. The sum of the colums $h=0$ and $h=1$ is (1). Row sums are (2).

the center may generate other islands. [This is in tune with the standard count of Motzkin paths which are considered different walking left-to-right or right-to-left. By analogy with the "free" polyominoes we could also count "free" islands where symmetry-related copies are counted only once.]

A lower bound is

$$M^{\triangleleft}(n) \geq 2^{T(n-2)} \tag{1}$$

where $T(n) \equiv n(n+1)/2$ are the triangular numbers.

Proof. $T(n-2)$ is the number of vertices that are not on the shoreline [7]. Any set of labels of ones and zeros at these internal vertices creates a Motzkin Island, and there are $2^{T(n-2)}$ possible multisets of zeros and ones. So there are $2^{T(n-2)}$ islands with maximum peak height 1, and more if peak heights ≥ 2 are achievable, i.e., at sufficiently large islands such that the distance to the shore is larger. □

The number of Motzkin Reg-Triangular Islands is

$$M^{\triangleleft}(n) = 1, 1, 1, 2, 8, 64, 1032, 33640, 2221952, 297891576, \\ 81173202920, 45006474922560, 50821273347381064, \ldots \quad (n \geq 0). \tag{2}$$

One can refine these enumerations by counting Reg-Triangular Islands with maximum altitude h: Table 1.

The first case where the lower bound (1) is surpassed is $M^{\triangleleft}(6) = 1032 = 8 + 2^{10} = 8 + 2^{T(4)}$ where a peak height of 2 may be reached in the center of the triangle:

```
      0
     0 0
    0 a 0
   0 1 1 0
  0 1 2 1 0
 0 b 1 1 c 0
0 0 0 0 0 0 0
```

The $8 = 2^3$ islands of that shape are those where any combination of zeros and ones appears at the three vertices a, b and c: Entry $n = 6, h = 2$ in Table 1.

3. Regular Rectangular Shoreline

3.1. **Overview.** On the simple $m \times n$ square grid with 4 neighbors adjacent to each internal vertex and an area of mn unit squares, the prototypical shore line is a rectangle.

```
.--.--.--.--.--.--.--.
|  |  |  |  |  |  |  |
.--.--.--.--.--.--.--.
|  |  |  |  |  |  |  |
.--.--.--.--.--.--.--.
|  |  |  |  |  |  |  |
.--.--.--.--.--.--.--.
```

The 3×4 Motzkin Islands have the labels arranged like this,

```
0  0  0  0  0
0  a  b  c  0
0  d  e  f  0
0  0  0  0  0
```

where $|b-a| \le 1$, $|b-c| \le 1$, $|b-e| \le 1$, $|c-0| \le 1$, $|c-f| \le 1$, $|d-a| \le 1$ and so on are the 4 requirements at each of the vertices $a, \ldots, f$.

Definition 1. *(Reg-Rect Islands) $M^{\square}_{m\times n}$ is the number of "fixed" Motzkin islands of this class with paths on a (projected) grid of squares and a rectangular shoreline of length $2(m+n)$.*

Because there are $(m-1)(n-1)$ internal vertices, the number of islands with maximum peak height 1 is given by distributing ones and zeros on all nodes in all possible ways, and, similar to (1), the associated lower bound is

$$M^{\square}_{m\times n} \ge 2^{(m-1)(n-1)}. \tag{3}$$

For small widths of the islands, all points are close to shores, so no peaks of height > 1 are possible, and

$$M^{\square}_{0\times n} = 1; \tag{4}$$

$$M^{\square}_{m\times n} = 2^{(m-1)(n-1)}, \quad 1 \le m \le 3, n \ge 1. \tag{5}$$

Table 2 shows numeric results for $M^{\square}_{m\times n}$.

Glueing a Motzkin Path of length m at the right of a Motzkin Island to get an array of $n+1$ columns with a finite set of paths that are compatible with the altitude constraint shows that the Transfer Matrix method [6] is applicable to relate $M^{\square}_{m\times n}$ for fixed m, so linear recurrences with constant coefficients arise along each row and each column of Table 2.

The 4th column obeys a 3rd order linear recurrence obtained by inverting a 9×9 transfer matrix:

$$M^{\square}_{4\times n} = 9M^{\square}_{4\times(n-1)} - 4M^{\square}_{4\times(n-2)} - 16M^{\square}_{4\times(n-3)}, \quad n \ge 4, \tag{6}$$

with generating function

$$\sum_{n\ge 0} M^{\square}_{4\times n} x^n = 1 + x\frac{1-x-4x^2}{1-9x+4x^2+16x^3}. \tag{7}$$

$n\backslash m$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	4	8	16	32
3	1	4	16	64	256	1024
4	1	8	64	528	4368	36176
5	1	16	256	4368	75536	1312656
6	1	32	1024	36176	1312656	48185392
7	1	64	4096	299664	22844432	1776652720
8	1	128	16384	2482384	397748880	65621158928
9	1	256	65536	20563984	6926263568	2425367471888
10	1	512	262144	170351696	120616891280	89665360360880
11	1	1024	1048576	1411191184	2100502259216	3315246641703216
12	1			11690290128	36579688651920	122581250828532112
13	1			96842219280	637026388506384	4532510993637490576
14	1			802239754064	11093664235296144	
15	1			6645744267408	193193569061312528	

$n\backslash m$	7	8	9
1	1	1	1
2	64	128	256
3	4096	16384	65546
4	299664	2482384	20563984
5	22844432	397748880	6926263568
6	1776652720	65621158928	2425367471888
7	139414770480	10984586881360	867077331528016
8	10984586881360	1852311458443344	313509786289629136
9	867077331528016	313509786289629136	
10	68500322235665264		
11	5413646938769199472		

TABLE 2. The number of Motzkin Rectangular Islands $M^{\square}_{m\times n}$. The array is symmetric, $M^{\square}_{m\times n} = M^{\square}_{n\times m}$, so only one triangular part of it needs to be shown. The columns $m = 2, 3$ are entirely characterized by (5) and cut short to save space.

The order of the transfer matrix for the $m\times n$ shape is the m-th Motzkin number [5, A001006]. Because the counts are symmetric, $M^{\square}_{m\times n} = M^{\square}_{n\times m}$, column generating functions are also row generating functions.

Definition 2. $M^{\square}_{m\times}(x) = \sum_{n\geq 0} M^{\square}_{m\times n} x^n$ *are the (rational) generating functions for Reg-Rect Islands of fixed width m.*

The Motzkin path of n steps through the centre axis of the $4\times n$ island is a word of length $n-1$ of the alphabet $\{0,1,2\}$ avoiding the patterns 02 and 20.

The growth of the sequences can commonly be estimated with the Binet formulas from the smallest roots (in absolute value) of the denominator polynomials of the generating functions:[8, §5.2]

$$1 + x\frac{1-x-4x^2}{1-9x+4x^2+16x^3} \approx \frac{3}{4} + \frac{0.04303}{0.55768-x} + \frac{0.0565}{x+0.928} + \frac{0.0135}{0.12071-x} \tag{8}$$

$$\therefore M^{\square}_{4\times n} \propto \frac{0.0135}{0.12071^{1+n}} \approx 0.0135 \times 8.284^{n+1}. \tag{9}$$

The 5th column obeys a 5th order linear recurrence obtained by inverting a 21×21 transfer matrix:

$$M^{\square}_{5\times n} = 21M^{\square}_{5\times(n-1)} - 52M^{\square}_{5\times(n-2)} - 184M^{\square}_{5\times(n-3)} + 32M^{\square}_{5\times(n-4)} + 128M^{\square}_{5\times(n-5)}, \quad n \ge 6, \tag{10}$$

with generating function

$$\sum_{n\ge 0} M^{\square}_{5\times n} x^n = 1 + x\frac{1-5x-28x^2+8x^3+32x^4}{1-21x+52x^2+184x^3-32x^4-128x^5}. \tag{11}$$

$$M^{\square}_{5\times n} \propto \frac{0.0027}{0.0574^{1+n}} \approx 0.0027 \times 17.415^{n+1}. \tag{12}$$

The 6th column obeys a 13th order linear recurrence obtained by inverting a 51×51 transfer matrix The generating function is

$$\sum_{n\ge 0} M^{\square}_{6\times n} x^n = 1+x\frac{p_{6,3}(x)}{q_{6,3}(x)} = 1+x+32x^2+1024x^3+36176x^4+1312656x^5+48185392x^6+\cdots \tag{13}$$

where

$$p_{6,3} \equiv 1-17x-185x^2+1339x^3+7130x^4-32536x^5-61584x^6+186080x^7+97536x^8 - 298496x^9 + 43008x^{10} + 98304x^{11} - 32768x^{12};$$

$$q_{6,3} \equiv 1-49x+359x^2+3851x^3-23750x^4-68392x^5+321168x^6+352480x^7-1284352x^8- 401408x^9 + 1615872x^{10} - 319488x^{11} - 393216x^{12} + 131072x^{13}.$$

The 7th column obeys a 25 order linear recurrence obtained by inverting a 127×127 transfer matrix. The generating function is

$$\sum_{n\ge 0} M^{\square}_{7\times n} x^n = 1 + x\frac{p_{7,3}(x)}{q_{7,3}(x)} \tag{14}$$

where

$$p_{7,3} \equiv 1-57x-434x^2+30928x^3+52707x^4-5195215x^5-2295186x^6+330479424x^7-86356656x^8 -9531144448x^9+7586615232x^{10}+131598114560x^{11}-146226275584x^{12}-901632193024x^{13} + 1173476425728x^{14} + 3008236953600x^{15} - 4337494769664x^{16} - 4485232918528x^{17} + 7302221398016x^{18} + 2352974135296x^{19} - 5122050490368x^{20} - 72225914880x^{21} + 1221716869120x^{22} - 30870077440x^{23} - 90194313216x^{24};$$

$$q_{7,3} \equiv 1-121x+3214x^2+21184x^3-1052701x^4-188559x^5+111757902x^6-91039024x^7 -5161897360x^8+5669119488x^9+115099761600x^{10}-146008447232x^{11}-1294977514752x^{12} +1790831527424x^{13}+7375685763072x^{14}-10831638134784x^{15}-20396717817856x^{16} + 31750174605312x^{17} + 25819486879744x^{18} - 44558677180416x^{19} -11796274479104x^{20}+27229555785728x^{21}+110528299008x^{22}-5667209347072x^{23} + 123480309760x^{24} + 360777252864x^{25}.$$

3.2. **Reduction to Island without straight canals.** A Motzkin Path may return multiple times to the horizontal line, and the Motzking Ballot numbers [5, A091836] count how Motzking Paths can be described as left-to-right compositions of "atomic" Motzkin Paths without intermediate returns to the horizontal.

In the same style, a list of Rectangular Motzkin Islands of size $m \times n$, $m \times n'$, $m \times n''$ etc can be chained to form an island of size $m \times (n+n'+n''+\cdots)$ by mergers of eastern and western shorelines. The composite island has straight "canals" of length m at sea level at the places of these mergers.

```
0 0 0 0 0 0 0      0 0 0 0 0       0 0 0 0 0 0 0 0 0 0 0
0 a b . . . 0      0 A B . 0       0 a b . . . 0 A B . 0
0 . . . . . 0  <-> 0 . . . 0    => 0 . . . . . 0 . . . 0
0 . . . . . 0      0 . . . 0       0 . . . . . 0 . . . 0
0 0 0 0 0 0 0      0 0 0 0 0       0 0 0 0 0 0 0 0 0 0 0
                                               ^canal
```

Definition 3. *$M^{\square(i)}_{m\times n}$ is the number of Reg-Rect Islands of shape $m \times n$ which are composed of i atomic Reg-Rect Islands (i.e., islands without straight canals of length m). The associated (rational) generating funtion is $M^{\square(i)}_{m\times}(x) \equiv \sum_{n\ge 0} M^{\square(i)}_{m\times n} x^n$.*

The inverse INVERT transformation [1][4, Th. I.1] decomposes an integer sequence into the number of atomic parts, and the ith power of the generating function of the atomic parts provides the number of composite islands:

$$M^{\square(1)}_{m\times}(x) = 1 - \frac{1}{M^{\square}_{m\times}(x)}; \tag{15}$$

$$M^{\square(i)}_{m\times}(x) = [M^{\square(1)}_{m\times}(x)]^i. \tag{16}$$

A consequence of these formulas: because the $M^{\square}_{m\times n}$ have rational generating functions, the $M^{\square(i)}_{mtimesn}$ also have rational generating functions.

Example 1. *The case $m = 3$, where $M^{\square}$ are powers of 4, has an inverse INVERT transformation which are the powers of 3, so we arrive that the table of $m \times n$ Reg-Rect islands composed of i smaller islands* [5, A027465]*: Table 3.*

Example 2. *The $4 \times n$ Reg-Rect Motzkin Islands which are mergers of smaller $4 \times n'$ Islands without canals are counted in Table 4.*

3.3. **Statistics for Maximum Peak altitudes.** Refining the counts of Table 2 by the maximum altitude h in the islands is simply a matter of leaving out the Motzkin Paths from the transfer matrices which climb too high. The reduction of the order of the transfer matrix by deleting entries with a given height from the table is tabulated in the Online Encyclopedia of Integer Sequences (OEIS) [5,

$n\backslash i$	1	2	3	4	5	6	$\sum_i$
1	1						1
2	3	1					4
3	9	6	1				16
4	27	27	9	1			64
5	81	108	54	12	1		256
6	243	405	270	90	15	1	1024

TABLE 3. The number of $M^{\square(i)}_{3\times n}$ islands that consist of i atomic islands that have no straight canals of length 3.

$n\backslash i$	1	2	3	4	5	6	7	8	$\sum_i$
1	1								1
2	7	1							8
3	49	14	1						64
4	359	147	21	1					528
5	2641	1404	294	28	1				4368
6	19463	12709	3478	490	35	1			36176
7	143473	111082	37407	6924	735	42	1		299664
8	1057703	947127	378051	86339	12085	1029	49	1	2482384

TABLE 4. The number of $M^{\square(i)}_{4\times n}$ islands that consist of i atomic $M^{\square}_{4\times n'}$ islands.

A097862]. The generic result is that the $M^{\square}_{m\times n}$ that reach at most an altitude h have rational generating functions. Looking at a straight Motzkin Path of m steps reveals that the maximum altitude is $h \le \lfloor m/2 \rfloor$, $h \le \lfloor n/2 \rfloor$.

For maximum altitude $h = 1$ the outcome is given by Eq. (5). The generating function of these Reg-Rect Islands is

$$M^{\square}_{m\times,h\le 1}(x) = 1 + \sum_{n\ge 1} 2^{(m-1)(n-1)}x^n = 1 + 2^{m-1}\frac{1}{1-2x}, \quad m > 0. \tag{17}$$

Subtracting this from Eq. (7) gives the $4 \times n$ Reg-Rect Islands with maximum altitude 2; subtracting this from Eq. (11) gives the $5 \times n$ Reg-Rect Islands with maximum altitude 2.

The first slightly more complicated result occurs at $m = 6$ and $m = 7$, where altitudes of $h = 3$ are possible. Inverting a 50×50 transfer matrix we obtain

$$M^{\square}_{6\times,h\le 2}(x) = 1+x\frac{p_{6,2}(x)}{q_{6,2}(x)} = 1+x+32x^2+1024x^3+36176x^4+1312656x^5+48175392x^6+\cdots \tag{18}$$

where

$$p_{6,2} \equiv 1-16x-192x^2+1084x^3+7144x^4-22816x^5-64704x^6+95616x^7+139264x^8 - 114688x^9 - 49152x^{10} + 32768x^{11};$$

$$q_{6,2} \equiv 1-48x+320x^2+3820x^3-18984x^4-71232x^5+214592x^6+430720x^7-655872x^8 - 759808x^9+606208x^{10}+196608x^{11}-131072x^{12}.$$

The results for maximum altitudes exactly equal to some h are derived from these intermediate results with the inclusion-exclusion principle.

Example 3. *Eq.* (18) *determines* $M^{\Box}_{6\times,h\le 2}(x)$ *and* (13) *determines* $M^{\Box}_{6\times,h\le 3}(x)$*:*

$$M^{\Box}_{6\times,h=3}(x) = M^{\Box}_{6\times,h\le 3}(x) - M^{\Box}_{6\times,h\le 2}(x) = 10000x^6+1036400x^7+69542224x^8+3853469712x^9+\cdots \tag{19}$$

Example 4. *Inverting a* 120×120 *transfer matrix yields* $M^{\Box}_{7\times,h\le 2}(x)$ *and*

$$M^{\Box}_{7\times,h=3}(x) = M^{\Box}_{7\times,h\le 3}(x) - M^{\Box}_{7\times,h\le 2}(x) = 1036400x^6+248197680x^7+37655895568x^8+4643162738256x^9+\cdots \tag{20}$$

The order 120 of the transfer matrix is the order 127 without constraint that governs Eq. (14) *minus the 7 Motzkin paths of length 7 and height 3* [5, A097862]*.*

One may also mix the results of this chapter and the previous one, looking at the Reg-Rect Islands with some maximum h that have no straight canals of length m. The basic ansatz is that islands which have altitudes not larger than H can be decomposed into atomic islands of altitudes not larger than H:

$$M^{\Box}_{m\times n,h\le H} = \sum_i M^{\Box(i)}_{m\times n,h\le H}. \tag{21}$$

Equations (15)–(16) for the associated generating functions remain valid if the constraint $h \le H$ is added (and leads to another family of rational generating functions).

4. Regular Parallelogram Shoreline

If the underlying grid is the triangular grid with 6 neighbors adjacent to each internal vertex, the shoreline may be a parallelogram with 2 sides of length m and 2 sides of length n. The example of a $M^{\Diamond}_{3\times 5}$ distribution of vertices looks like this:

```
      .---.---.---.---.---.
     / \ / \ / \ / \ / \ /
    .---.---.---.---.---.
   / \ / \ / \ / \ / \ /
  .---.---.---.---.---.
 / \ / \ / \ / \ / \ /
.---.---.---.---.---.
```

The number of internal vertices is $(n-1)(m-1)$ as in the case of the square lattice, and basically every aspect related to the availability of transfer matrix algorithms and canals (they are slanted now) remains intact.

The lower bound obtained for islands with maximum height of 1 is again 2 to the power of the number of internal vertices,

$$M^{\Diamond}_{m\times n} \le 2^{(m-1)(n-1)}, \quad n \ge 1. \tag{22}$$

The basic counting numbers are in Table 5. The columns $m \le 3$ are the regular powers of 2. Further generating functions are

$$\sum_{n\ge 0} M^{\diamondsuit}_{4\times n} x^n = 1 + x\frac{1-x-x^2}{1-9x+7x^2+4x^3}. \tag{23}$$

$$\sum_{n\ge 0} M^{\diamondsuit}_{5\times n} x^n = 1 + x\frac{(1-x)(1-4x-8x^2+3x^3)}{1-21x+76x^2+7x^3-99x^4-44x^5+16x^6}. \tag{24}$$

$$\sum_{n\ge 0} M^{\diamondsuit}_{6\times n} x^n = 1 + x\frac{r_{6,3}(x)}{s_{6,3}(x)}, \tag{25}$$

with

$$r_{6,3} = 1-19x+53x^2+286x^3-1266x^4-200x^5-348x^6+12112x^7-6896x^8-6752x^9 + 384x^{10} + 512x^{11};$$

$$s_{6,3} = (1-47x+481x^2-702x^3-1866x^4+1268x^5+7228x^6-3616x^7-4064x^8-64x^9+256x^{10}) \times (1-4x-8x^2).$$

$$\sum_{n\ge 0} M^{\diamondsuit}_{7\times n} x^n = 1 + x\frac{r_{7,3}(x)}{s_{7,3}(x)}, \tag{26}$$

with

$$r_{7,3} = -1+63x-1143x^2+3648x^3+91019x^4-768871x^5+960584x^6-327136x^7 + 61740672x^8 - 441671236x^9 + 1577898892x^{10} - 1632794220x^{11} - 6633776796x^{12} + 11128738368x^{13} + 17012212988x^{14} - 16432634428x^{15} - 27719932816x^{16} + 551175824x^{17} + 23451095152x^{18} + 12993764672x^{19} - 7682412560x^{20} - 7410479680x^{21} - 22526464x^{22} + 1183185920x^{23} + 245771264x^{24} - 18113536x^{25} - 7204864x^{26} + 32768x^{27} + 65536x^{28};$$

$$s_{7,3} = -1+127x-5175x^2+85916x^3-503349x^4-1111931x^5+17547656x^6+18640380x^7 - 390222952x^8 - 288723512x^9 + 5057701996x^{10} + 3601279348x^{11} - 36244480044x^{12} - 26797173296x^{13} + 116223283500x^{14} + 137289708452x^{15} - 129459816960x^{16} - 288061565440x^{17} - 64508507216x^{18} + 197636554032x^{19} + 162786560752x^{20} - 7310842880x^{21} - 49814909440x^{22} - 8841032192x^{23} + 4239736832x^{24} + 1174548480x^{25} - 50548736x^{26} - 28753920x^{27} + 131072x^{28} + 262144x^{29}.$$

Remark 1. *The heuristics $M^{\diamondsuit}_{m\times n} \le M^{\square}_{m\times n}$ is a reduction of counts caused by the additional constraint of having 2 more neighbors with restricted slope in the triangular grid than in the square grid.*

$n\backslash m$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	4	8	16	32
3	1	4	16	64	256	1024
4	1	8	64	516	4164	33608
5	1	16	256	4164	67972	1110784
6	1	32	1024	33608	1110784	36817540
7	1	64	4096	271260	18157476	1221583492
8	1	128	16384	2189428	296834852	40546108744
9	1	256	65536	17671600	4852696576	1345953965468
10	1	512	262144	142633364	79332968772	44681762461244
11	1	1024	1048576	1151241364	1296954937540	1483327227159344
12	1			9292052328	21202946992384	49243159687702764
13	1			74999247948	346631167980516	1634766151787023540
14	1			605343899780	5666814604807012	

$n\backslash m$	7	8	9
1	1	1	1
2	64	128	256
3	4096	16384	65546
4	271260	2189428	17671600
5	18157476	296834852	4852696576
6	1221583492	40546108744	1345953965468
7	82365445328	5558783791956	375309375734172
8	5558783791956	763486687369180	104956410336715568
9	375309375734172	104956410336715568	29397499665406126792
10	25343833012948964		
11	1711534424576065500		

TABLE 5. The number of Motzkin Parallelogram Islands $M^{\Diamond}_{m\times n}$. The array is symmetric, $M^{\Diamond}_{m\times n} = M^{\Diamond}_{n\times m}$, so only one triangular part of it needs to be shown. The columns $m = 2, 3$ are entirely characterized by (22) and cut short to save space.

5. Regular Rhomboid Shoreline

5.1. **Subclassing Parallelograms.** The shoreline with the lozenge/rhomboid shape of n edges on each of the 4 sides defines Reg-Rhomb Motzkin Islands $M^{\Diamond}_{n\times n}$. They are a special case of the Parallelogram Islands for equilateral sides: There are $(n+1)^2$ vertices, of which $4n$ are on the shoreline and $(n-1)^2$ inside the island. The islands for $n = 0, 1, 2$ and 4 of the 16 islands for $n = 3$ have altitudes like this:

```
0    0 0      0 0 0     0 0 0      0 0 0 0    0 0 0 0   0 0 0 0   0 0 0 0
       0 0      0 0 0     0 1 0      0 0 0 0    0 0 0 0   0 0 0 0   0 1 1 0
                  0 0 0     0 0 0      0 1 0 0    0 0 1 0   0 1 1 0   0 1 0 0
                                         0 0 0 0    0 0 0 0   0 0 0 0   0 0 0 0
```

From the diagonal of Table 5 we extract

$$M^{\Diamond}_{n\times n} = 1, 1, 2, 16, 516, 67972, 36817540, \ldots \quad (n \geq 0). \tag{27}$$

Glueing two Reg-Triangular islands such that the two shorelines overlap creates a sub-class of Reg-Rhomb islands with a canal across the short diagonal of the rhomboid, so these yield the lower bound

$$M^{\Diamond}_{n\times n} \ge [M^{\triangleleft}(n)]^2. \tag{28}$$

5.2. **Hand-counted Examples.** The first case where the lower bound (22) is surpassed are the 4 islands with $n = 4$ and maximum height 2, $M^{\Diamond}_{4\times 4} = 2^{(4-1)^2} + 4 = 516$, which have the altitude map

```
    0 0 0 0 0
   0 1 1 b 0
  0 1 2 1 0
 0 a 1 1 0
0 0 0 0 0
```

where a and b are one of the $2^2 = 4$ combinations of zeros and ones.

For $n = 5$ the islands with maximum height 2 have one of the following maps:

- A single 2 at one of the 4 inner vertices:

```
     0 0 0 0 0 0   0 0 0 0 0 0   0 0 0 0 0 0   0 0 0 0 0 0
    0 1 1 a b 0   0 a 1 1 b 0   0 a b c d 0   0 a b c d 0
   0 1 2 1 c 0   0 c 1 2 1 0   0 1 1 e f 0   0 e 1 1 f 0
  0 d 1 1 e 0   0 d e 1 1 0   0 1 2 1 g 0   0 g 1 2 1 0
 0 f g h i 0   0 f h h i 0   0 h 1 1 i 0   0 h i 1 1 0
0 0 0 0 0 0   0 0 0 0 0 0   0 0 0 0 0 0   0 0 0 0 0 0
```

  where a–i are 9 vertices supporting $2^9 = 512$ combinations of zeros and ones. The 4 places of the peak altitude give $4 \times 512 = 2048$ islands.

- Two 2's parallel to sides:

```
     0 0 0 0 0 0   0 0 0 0 0 0   0 0 0 0 0 0   0 0 0 0 0 0
    0 1 1 a b 0   0 1 1 1 a 0   0 a b c d 0   0 a 1 1 b 0
   0 1 2 1 c 0   0 1 2 2 1 0   0 1 1 1 e 0   0 c 1 2 1 0
  0 1 2 1 d 0   0 b 1 1 1 0   0 1 2 2 1 0   0 d 1 2 1 0
 0 e 1 1 f 0   0 c d e f 0   0 f 1 1 1 0   0 e f 1 1 0
0 0 0 0 0 0   0 0 0 0 0 0   0 0 0 0 0 0   0 0 0 0 0 0
```

  where a–f are 6 vertices supporting $2^6 = 64$ combinations of zeros and ones. The 4 places of the peak altitude specify $4 \times 64 = 256$ islands.

- Two 2's on the short or long diagonal:

```
     0 0 0 0 0 0   0 0 0 0 0 0
    0 a 1 1 b 0   0 1 1 a b 0
   0 1 1 2 1 0   0 1 2 1 c 0
  0 1 2 1 1 0   0 d 1 2 1 0
 0 c 1 1 d 0   0 e f 1 1 0
0 0 0 0 0 0   0 0 0 0 0 0
```

  where a–f are vertices with combinations of zeros and ones. This specifies $2^4 + 2^6 = 80$ islands.

- Three 2's:

$n\backslash h$	0	1	2	3	$\Sigma_{h\geq 0}$
1	1				1
2	1	1			2
3	1	15			16
4	1	511	4		516
5	1	65535	2436		67972
6	1	33554431	3263008	100	36817540

Table 6. Reg-Rhomb Islands $M^{\Diamond}_{n\times n}$ with maximum altitude h. The sum of the columns $h = 0$ and $h = 1$ is (22). Row sums are (27).

```
     0 0 0 0 0 0    0 0 0 0 0 0    0 0 0 0 0 0    0 0 0 0 0 0
    0 1 1 1 a 0    0 1 1 1 a 0    0 a 1 1 b 0    0 1 1 a b 0
   0 1 2 2 1 0    0 1 2 2 1 0    0 1 1 2 1 0    0 1 2 1 c 0
  0 1 2 1 1 0    0 b 1 2 1 0    0 1 2 2 1 0    0 1 2 2 1 0
 0 b 1 1 c 0    0 c d 1 1 0    0 c 1 1 1 0    0 d 1 1 1 0
0 0 0 0 0 0    0 0 0 0 0 0    0 0 0 0 0 0    0 0 0 0 0 0
```

where a–d are vertices with combinations of zeros and ones. This specifies $2^3 + 2^4 + 2^3 + 2^4 = 48$ islands.

- Four 2's

```
     0 0 0 0 0 0
    0 1 1 1 a 0
   0 1 2 2 1 0
  0 1 2 2 1 0
 0 b 1 1 1 0
0 0 0 0 0 0
```

where a–b are vertices with combinations of zeros and ones. This specifies $2^2 = 4$ islands.

The total $M^{\Diamond}_{5\times 5}$ with maximum height 2 is $2048 + 256 + 80 + 48 + 4 = 2436$, entry $(n = 5, h = 2)$ in Table 6.

The $M^{\Diamond}_{6\times 6}$ with maximum height 3 have the altitude maps

```
      0 0 0 0 0 0 0
     0 1 1 1 . . 0
    0 1 2 2 a . 0
   0 1 2 3 2 1 0
  0 . b 2 2 1 0
 0 . . 1 1 1 0
0 0 0 0 0 0 0
```

where a–b can be any combination of ones and twos.

- If $a = b = 2$, the 4 out of 6 dotted vertices must be 1, and only two places (4 islands) remain.
- If $a = b = 1$, the 6 dotted vertices can be any combinations of ones and zeros, a total of $2^6 = 64$ islands.
- If $a = 2$ and $b = 1$ or vice versa, two of the 6 dotted vertices are fixed, leaving $2^4 = 16$ islands.

This leads to $4 + 64 + 2 \times 16 = 100$ islands, entry $(n = 6, h = 3)$ in Table 6.

6. Summary

We have introduced Motzkin Islands as finite vertex-labeled graphs defined on meshes with a perimeter describing a coastline, considered in particular the triangular and simple-square meshes and pointed out that the generating functions for simple shapes of the coast can be calculated with the transfer matrix method.

References

1. Mira Bernstein and Neil J. A. Sloane, *Some canonical sequences of integers*, Lin. Alg. Applic. **226–228** (1995), 57–72, (E:) [2]. MR 1344554
2. Richard A. Brualdi, *From the editor-in-chief*, Lin. Alg. Applic. **320** (2000), no. 1–3, 209–216. MR 1796542
3. Robert Donaghey and Louis W. Shapiro, *Motzkin numbers*, J. Comb. Theory A **23** (1977), 291–301. MR 0505544
4. Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009. MR 2483235
5. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2020), https://oeis.org/. MR 3822822
6. Richard J. Mathar, *Tilings of rectangular regions by rectangular tiles: counts derived from transfer matrices*, arXiv:1406.7788 [math.CO] (2014).
7. ———, *Lozenge tilings of the equilateral triangle*, arXiv:1909.06336 (2019).
8. Herbert S. Wilf, *Generatingfunctionology*, Academic Press, 2004. MR 2172781

URL: `https://www.mpia.de/~mathar`

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