COMON'S CONJECTURE OVER THE REALS

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Abstract. We construct a symmetric tensor in

$$\mathbb{R}^{208} \otimes \mathbb{R}^{208} \otimes \mathbb{R}^{208} \otimes \mathbb{R}^{208}$$

with real rank 761 and real symmetric rank 762.

1. Introduction

Let V be a finite dimensional vector space over a field $\mathbb{F}$. A tensor $t \in V \otimes \ldots \otimes V$ is called *decomposable* if one can write $t = v_1 \otimes \ldots \otimes v_d$ with $v_1, \ldots, v_d \in V$, and t is called *symmetric* if it is invariant under the braiding isomorphisms corresponding to the permutations of $\{1, \ldots, d\}$. The *rank* of a tensor t is the smallest number r for which t can be written as the sum of r tensors decomposable over $\mathbb{F}$, and, similarly, the *symmetric rank* of a symmetric tensor σ is the smallest number s such that σ is the sum of s symmetric tensors decomposable over $\mathbb{F}$.

The earliest appearance of tensor rank is attributed to the work of Hitchcock [36] from 1927, and subsequent studies displayed growing interest to this concept in pure and applied mathematics. Rank decompositions are reported to bear practical importance in machine learning, biomedical engineering, signal processing, psychometrics, and chemometrics [17, 18, 22, 21, 28, 39, 44, 46, 62]. Being a natural measure of algebraic complexity, tensor rank appears in the study of fast matrix multiplication and other algorithmic problems involving arithmetic circuits [7, 26, 37, 50, 53, 56, 63]. The symmetric counterpart of this technique is equivalent to the so-called *Waring decompositions* of homogeneous polynomials [6], and it finds additional applications in matrix multiplication [15], parametrized algorithms [51], and independent component analysis [11, 18, 21]. A significant progress has been made with the approach of algebraic geometry, in which both the symmetric and non-symmetric ranks are natural objects of study [1, 4, 10, 40, 41, 43, 49].

2. Comon's conjecture

Do there exist symmetric tensors with different rank and symmetric rank? Many researchers recognize this as a famous question and a central and guiding problem in the field [14, 31, 33, 48, 64, 65]. It is usually termed *Comon's conjecture* as Pierre Comon posed it in 2004 at the Workshop on Tensor Decompositions in Palo Alto [19, 38] and reiterated it in several further notable publications [20, 21, 45].

Conjecture 2.1. *The rank of a symmetric tensor over $\mathbb{R}$ or $\mathbb{C}$ equals its symmetric rank.*

This paper gives a counterexample to the real case of Conjecture 2.1. Together with the counterexample in [58], which is valid over the complex numbers, this

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gives a full negative solution to Conjecture 2.1. The proof of [58] is quite complicated although restricted to three-way tensors in a certain special family, and its generalizations to tensors outside that family or to tensors of higher order remain unclear. Following the publication of [58], there has been a considerable interest to the real case of Comon's conjecture, which was discussed in subsequent studies [14, 27, 32, 45, 47, 54, 55, 65, 66, 69] but remained open. As we will see later, our current approach is different from [58], and, in fact, we build a framework that potentially allows one to construct a family of counterexamples to the real version of Comon's conjecture for tensors of any even order $d \geqslant 4$. We complete this task for $d = 4$ and prove the following theorem.

Theorem 2.2. *There exists a symmetric tensor*

$$\tau \in \mathbb{R}^{208} \otimes \mathbb{R}^{208} \otimes \mathbb{R}^{208} \otimes \mathbb{R}^{208}$$

such that τ *has real rank* 761 *and real symmetric rank* 762*.*

We proceed with a short survey of related work. First of all, we note that the rank or symmetric rank can be different over $\mathbb{R}$ and $\mathbb{C}$ even if the initial tensor is real [5, 21], so there are no a priori relation between the validity of the real and complex versions of Comon's conjecture. In fact, the complex version was invalidated with a particular example of a real symmetric tensor with different complex rank and complex symmetric rank, but that example does not seem to allow any obvious transformation to disprove the real version as well [58]. The setting of complex numbers is preferable for those researchers who work in algebraic geometry [3, 8, 33, 40], but several authors discuss Comon's conjecture from the point of view of applied mathematics and choose $\mathbb{R}$ as the ground field [45, 47, 52, 66, 68]. It is well recognized that the tensor decomposition problem is more complicated to handle over the reals [21, 55, 57], but the general perspective allows one to consider the real and complex versions together [16, 29, 30, 35, 55, 67, 69].

An early progress on Comon's conjecture came from the foundational paper by Comon, Golub, Lim, Mourrain [21], who proved the equality of the rank and symmetric rank for tensors of symmetric rank at most two. Also, the authors of [21] confirmed Conjecture 2.1 for generic elements in the family of the symmetric tensors of a fixed symmetric rank whose order and dimension are sufficiently large. Chiantini, Ottaviani, Vannieuwenhoven [16] improved their result and gave a weaker condition on the rank, order, and dimension of a generic tensor which necessarily satisfies Comon's conjecture. One instance of the result in [16] is that, for a generic element τ of the family of the symmetric tensors in $\mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ satisfying

$$\operatorname{srk} \tau \leqslant 0.5n^2 - 0.5n, \tag{2.1}$$

we have $\operatorname{rk} \tau = \operatorname{srk} \tau$. Further, Ballico and Bernardi [3] proved Comon's conjecture for tensors of border rank at most two; they used complex numbers but the corresponding result is valid over $\mathbb{R}$ as well. More generally, Zhang, Huang, Qi [67] checked the validity of Conjecture 2.1 for tensors whose rank does not exceed the order. A partial generalization of this result to the case of arbitrary characteristic was obtained by Zheng, Huang, Song, Xu [69], and also they provided another sufficient condition for the validity of Comon's conjecture in terms of the dimension of the fiber space of a tensor. Friedland [29] proved Conjecture 2.1 for tensors whose rank does not exceed the flattening rank plus one. We note that the results of [29] and [67] invalidate a solution attempt of Loperfido [47], who claimed to construct a symmetric tensor with real rank three and real symmetric rank four. Seigal [55] used the result of [29] to show that any tensor of order three and rank at most six satisfies Conjecture 2.1, and she generalized this result to tensors of rank seven with respect to $\mathbb{C}$. Several particular examples of tensors of rank seven were also considered, and, in particular, the symmetric tensors realizing the ternary forms

$$x_1(x_1x_2 + x_3^2 + x_4^2) \quad \text{and} \quad x_1(x_1^2 - x_2^2 - x_3^2 - x_4^2) \tag{2.2}$$

were shown to have real rank and real symmetric rank seven [55]. The tensors in (2.2) satisfy the complex version of Comon's conjecture as well, but, curiously, the second of these tensors has complex rank and complex symmetric rank six [13, 55]. Further sporadic families recently shown to satisfy Conjecture 2.1 include the two Coppersmith–Winograd tensors [42], which correspond to the polynomials

$$x(y_1^2 + \ldots + y_q^2) \quad \text{and} \quad x(xz + y_1^2 + \ldots + y_q^2)$$

of symmetric ranks $2q + 1$ and $2q + 3$, respectively. Landsberg and Michałek [41] showed that the ranks of these tensors equal their symmetric ranks. We mention one further example of the tensor of the 3×3 permanent

$$x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 + x_2y_3z_1 + x_3y_1z_2 + x_3y_2z_1,$$

which has rank 16 [25] and symmetric rank 16 [60]. Li, Usevich, Comon [45] proved the so-called orthogonal analogue of Comon's conjecture, and Friedland and Lim [30] established the continuous analogue of Comon's conjecture by showing that the nuclear norm of a symmetric tensor equals its symmetric nuclear norm. Zhang, Ling, Qi [68] proved that the best symmetric rank-one approximation of a symmetric tensor is its best rank-one approximation, which was regarded as a step towards Conjecture 2.1. Other related results include the works [4] and [33], but their sufficient conditions on the validity of Conjecture 2.1 are restricted to the complex case and do not seem to admit an easy generalization to $\mathbb{R}$.

We recall that the *border rank* of a tensor τ is the smallest r for which τ is a limit of a sequence of tensors with rank r. The border rank analogue of Comon's conjecture is the question of the equality between the border rank and symmetric border rank. This question is open both in the real and complex cases [8, 42, 55], but the positive answer is known for tensors of border rank at most two [3, 29], complex cubic surfaces and real cubic surfaces of subgeneric rank [55], the Coppersmith–Winograd tensors [41], the 3×3 determinant [23, 24], and the 3×3 permanent [25].

The rest of this paper is devoted to the proof of Theorem 2.2. Although the tensor τ in Theorem 2.2 belongs to the range (2.1), the results in [16] do not invalidate our counterexample because they relate to the setting of *generic* tensors. We begin our consideration in Sections 3 and 4, which specify our notation and survey the relevant definitions and basic techniques, which include the standard *substitution method* of proving lower bounds on ranks of tensors. In Section 5, we introduce a particular family of $2 \times \ldots \times 2$ tensors that we use later in our construction, and we discuss their relevant properties. In Section 6, we define the concept of a *monomial emulator* family of tensors, and we give a construction of a potential counterexample assuming the existence of such families. Section 7 presents several basic properties of linear substitutions of monomial emulator families. Sections 8–10 give a detailed proof that the presented construction is indeed a counterexample, again assuming the existence of monomial emulators. Finally, Section 11 gives an explicit construction of a monomial emulator and completes the argument.

3. Tensors and decompositions

In what follows, we restrict ourselves to work over the real field $\mathbb{R}$. We do not use complex numbers anymore, so every scalar value is supposed to be in $\mathbb{R}$, every tensor product is taken over $\mathbb{R}$, every linear space is $\mathbb{R}$-linear and finite dimensional. A family φ of vectors in a linear space is linearly dependent if φ is linearly dependent over $\mathbb{R}$, and the notation $\operatorname{span}\varphi$ stands for the $\mathbb{R}$-linear span of φ.

Remark 3.1. In fact, the only property of the ground field required by the construction of our counterexample and our proof of its correctness is that $\mathbb{R}(\sqrt{-3}) \neq \mathbb{R}$.

Let $I, I_1, \ldots, I_d$ be a family of finite non-empty indexing sets. We define $\mathbb{R}^I$ as the linear space of vectors whose coordinates are labeled with elements in I. The tensors of the format $I_1 \times \ldots \times I_d$ or, simply, the $I_1 \times \ldots \times I_d$ tensors

$$\mathbb{R}^{I_1} \otimes \ldots \otimes \mathbb{R}^{I_d} \tag{3.1}$$

can be thought of as *d-way* arrays of numbers labeled with d-tuples of indexes in $I_1 \times \ldots \times I_d$. These tensors are said to have *order* d and *size* $|I_1| \times \ldots \times |I_d|$. A tensor in (3.1) is called *decomposable* if it can be represented as $v_1 \otimes \ldots \otimes v_d$.

Definition 3.2. The *rank* $\operatorname{rk}\tau$ is the smallest number r such that a given tensor τ can be written as the sum of r decomposable tensors.

If the indexing sets $I_1, \ldots, I_d$ are all equal, then every permutation π of $\{1, \ldots, d\}$ defines the braiding isomorphism of (3.1) as the mapping $T \to T'$ with the formula

$$T'(i_1|\ldots|i_d) = T\left(i_{\pi_1}|\ldots|i_{\pi_d}\right).$$

A tensor σ in (3.1) is *symmetric* if it is invariant under any such isomorphism.

Remark 3.3. Every symmetric decomposable tensor is a scalar multiple of $v\otimes\ldots\otimes v$ for some vector v. In the case of the reals, this description further reduces to either

$$-v \otimes \ldots \otimes v \ \text{ or } \ v \otimes \ldots \otimes v$$

because every positive real number admits a d-th root.

Definition 3.4. The *symmetric rank* $\operatorname{srk}\sigma$ is the smallest number s such that a given symmetric tensor σ is the sum of s symmetric decomposable tensors.

4. Substitutions and adjoined slices

We proceed with a discussion of several basic techniques related to the substitution method of tensor rank computation [37, 41, 58, 59].

Definition 4.1. In this section, the letter $\mathcal{T}$ denotes (3.1). For any $\delta \in \{1, \ldots, d\}$, we write $\mathcal{T}_\delta$ to denote the same tensor product but with the δ-th factor removed.

Definition 4.2. Let $\tau \in \mathcal{T}$ and $\delta \in \{1, \ldots, \delta\}$. Assume j is an index in the set I_δ as in (3.1). The j-th *δ-slice* of τ is the tensor $\tau' \in \mathcal{T}_\delta$ defined as

$$\tau'(i_1|i_2|\ldots|i_{\delta-1}|i_{\delta+1}|\ldots|i_{d-1}|i_d) = \tau(i_1|i_2|\ldots|i_{\delta-1}|\,j\,|i_{\delta+1}|\ldots|i_{d-1}|i_d).$$

This notion allows the following straightforward characterization of tensor ranks.

Observation 4.3. *The rank of a tensor $\tau \in \mathcal{T}$ is the smallest integer r such that there exist r decomposable tensors whose linear span contains every 1-slice of τ.*

Proof. According to Definition 3.2, we have $\operatorname{rk}\tau \leqslant r$ if and only if there exist vectors $u_1, \ldots, u_r$ and decomposable tensors $\sigma_1 \ldots, \sigma_r$ such that

$$\tau = u_1 \otimes \sigma_1 + \ldots + u_r \otimes \sigma_r, \tag{4.1}$$

which is equivalent to the condition that, for every index $j \in I_1$, the j-th 1-slice of τ is a linear combination

$$u_{1j}\sigma_1 + \ldots + u_{rj}\sigma_r$$

in which the scalar u_{ij} corresponds to the j-th entry of the vector u_i in (4.1). □

Of course, the symmetry allows one to replace the mention of the 1-slices in Observation 4.3 by the δ-slices with any $\delta \in \{1, \ldots, d\}$. We proceed with a multilinear analogue of elementary transformations of matrices in conventional linear algebra.

Definition 4.4. Let w be a tensor in $\mathcal{T}$. For any $\delta \in \{1, \ldots, d\}$, we take a set $W_\delta \subset \mathcal{T}_\delta$ which is a linear subspace. We define

$$w \bmod (W_1, \ldots, W_\delta)$$

as the set of all tensors that can be obtained from w by the following sequence of transformations:

(Mod-1) for every 1-slice ω_1 of w, add some element of W_1 to ω_1,

(Mod-2) for every 2-slice ω_2 of what obtained, add some element of W_2 to ω_2,

...

(Mod-d) for every d-slice ω_d of what obtained, add some element of W_d to ω_d.

We note in passing that the operations (Mod-1), ..., (Mod-d) commute.

Definition 4.5. If the subsets W_δ are as in Definition 4.4 but do not necessarily form linear spaces, then we define

$$w \bmod (W_1, \ldots, W_\delta) \quad \text{as} \quad w \bmod (\operatorname{span} W_1, \ldots, \operatorname{span} W_\delta).$$

The substitution method is based on the following statement, which is easy to deduce from Observation 4.3. We omit the proof but refer the reader to Lemma B.1 in [2], Lemma 2 in [37], Proposition 3.1 in [41], Theorem 4.4 in [55] for related results. We recall that $\operatorname{span} \varnothing$ is the zero subspace.

Lemma 4.6. *Let τ be a tensor in $\mathcal{T}$, and assume that the* 1*-slices of τ are indexed with the labels $1, 2, \ldots, a,\ 1', 2', \ldots, b'$, which means that*

$$I_1 = \{1, 2, \ldots, a,\ 1', 2', \ldots, b'\}$$

in (3.1)*. Let W' be the linear span of the* 1*-slices of τ with indexes $1', \ldots, b'$. Then*

$$\operatorname{rk}\tau \geqslant \dim W' + \min \operatorname{rk} \tau \bmod (W', \varnothing, \ldots, \varnothing),$$

and the equality holds if W' admits a basis consisting of decomposable tensors.

Now we are ready to present a multidimensional generalization of the slice adjoining technique, which was used in [58, 59] for three-way tensors.

Definition 4.7. Let τ be a tensor in $\mathcal{T}$. For any $\delta \in \{1, \ldots, d\}$, we consider a finite set $W_\delta \subset \mathcal{T}_\delta$. We recall that the format of the tensors in W_δ is

$$I_1 \times I_2 \times \ldots \times I_{\delta-1} \times I_{\delta+1} \times \ldots \times I_{d-1} \times I_d, \tag{4.2}$$

where $I_{\delta'}$ is the indexing set of the δ'-slices of τ. We define

$$\mathcal{A} = \operatorname{Adjoin}(\tau, W_1, \ldots, W_d)$$

as the tensor of the format

$$(I_1 \cup W_1) \times \ldots \times (I_d \cup W_d),$$

assuming that the label of any tensor in W_δ does not repeat any label in I_δ. The entries of $\mathcal{A}$ are defined as follows:

(Adj-0) $\mathcal{A}(i_1|\ldots|i_d) = \tau(i_1|\ldots|i_d)$ if $i_\delta \in I_\delta$ for all $\delta \in \{1,\ldots,d\}$;

(Adj-δ) for any $\delta \in \{1,\ldots,d\}$ and any $w \in W_\delta$, the w-th δ-slice of $\mathcal{A}$ equals w when restricted to the entries in (4.2) and has zeros at all the positions outside (4.2).

We say that $\mathcal{A}$ is obtained by *adjoining* the slices $(W_1,\ldots,W_d)$ to τ.

Definition 4.8. Assume that

$$I_1 = \ldots = I_d \text{ and } W_1 = \ldots = W_d$$

in the setting of Definition 4.7, and, additionally, assume that the tensor τ and every tensor in W_1 are symmetric. Then the *symmetrical adjoining* of W_1 to τ is

$$\operatorname{SAdj}(\tau, W_1) := \operatorname{Adjoin}(\tau, W_1, \ldots, W_1). \tag{4.3}$$

We note that the tensor (4.3) is symmetric.

Remark 4.9. If a tensor τ represents a homogeneous polynomial f of degree d, and tensors in W correspond to homogeneous polynomials $g_1,\ldots,g_m$ of degree $d-1$, then the tensor $\operatorname{SAdj}(\tau, W)$ represents the polynomial

$$f + y_1 g_1 + \ldots + y_m g_m,$$

where $y_1,\ldots,y_m$ are variables different from any of those in $f, g_1,\ldots,g_m$.

The following is a corollary of Lemma 4.6.

Lemma 4.10. *We have*

$$\operatorname{rk}\operatorname{Adjoin}(\tau, W_1,\ldots,W_d) \;\geqslant\; \min\operatorname{rk}\tau \bmod (W_1,\ldots,W_d) \;+\; \sum_{\delta=1}^{d} \dim\operatorname{span} W_\delta,$$

and the equality holds if every tensor in $W_1 \cup \ldots \cup W_d$ is decomposable.

5. A relevant family of $2 \times \ldots \times 2$ tensors

In order to proceed with a counterexample, we need to specify some further notation that we use in the rest of the paper. We recall that, although our main result is stated for tensors of order four, the presented framework is more general, and, as said above, it potentially allows one to construct a counterexample to the real version of Comon's conjecture for tensors of any even order different from two.

Remark 5.1. We use the symbol d to denote a fixed even number $\geqslant 4$.

An explicit counterexample follows from the $d = 4$ version of our results, and this case is valid over a general family of fields as explained in Remark 3.1. We proceed with some further notation, and, since our construction requires a particular family of symmetric $2 \times \ldots \times 2$ tensors, we need to specify the corresponding indexing set.

Definition 5.2. We use the symbols e and ε to denote the elements of

$$\mathcal{B} = \{e, \varepsilon\},$$

which is an indexing set of cardinality two.

We proceed with one general observation on $2 \times \ldots \times 2$ tensors.

Remark 5.3. In Observation 5.4 below, we use the fact that

$$a^{d-1} = b^{d-1} \quad \text{implies} \quad a = b$$

for real numbers a and b. We note that the version with $d = 4$ is true over fields as in Remark 3.1, because, otherwise, the conditions $a^3 = b^3$ and $a \neq b$ imply

$$\left(\frac{a+b}{a-b}\right)^2 = \frac{(a-b)(a+b)^2}{(a-b)^3} = \frac{a^3 + a^2b - ab^2 - b^3}{a^3 - 3a^2b + 3ab^2 - b^3} = \frac{a^2b - ab^2}{-3a^2b + 3ab^2} = -\frac{1}{3},$$

and hence -3 should be a square.

Observation 5.4. *Let B and C be two $(d-1)$-way $\mathcal{B} \times \ldots \times \mathcal{B}$ tensors such that*

$$B(e|\ldots|e) = C(e|\ldots|e) \neq 0 \quad \text{and} \quad B(\varepsilon|\ldots|\varepsilon) = C(\varepsilon|\ldots|\varepsilon) \neq 0.$$

If B and C are decomposable and $B - C$ is symmetric, then $B = C$.

Proof. The multiplication of B and C by the same non-zero scalar does not change the validity of the formulation, so we can assume that

$$B = \begin{pmatrix} 1 \\ b_1 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 \\ b_{d-1} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 \\ c_1 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 \\ c_{d-1} \end{pmatrix}$$

with

$$b_1 \cdot b_2 \cdot \ldots \cdot b_{d-1} = c_1 \cdot c_2 \cdot \ldots \cdot c_{d-1} \neq 0. \tag{5.1}$$

Now we define α_i as the $(d-1)$-tuple with the ε at the i-th place and the e's everywhere else, and, by the symmetry of $B - C$, we have

$$B(\alpha_i) - C(\alpha_i) = B(\alpha_j) - C(\alpha_j)$$

for all $i, j \in \{1, \ldots, d-1\}$. This means that

$$b_i - c_i = b_j - c_j,$$

so there exists a number a such that $c_i = b_i + a$ for all i. If $a = 0$, then the conclusion of the lemma is immediate, so we can assume that $a \neq 0$. Now we consider, for any distinct i and j, the $(d-1)$-tuple β_{ij} with the ε's at the i-th and j-th places and the e's everywhere else. The symmetry of $B - C$ implies

$$B(\beta_{ij}) - C(\beta_{ij}) = B(\beta_{ik}) - C(\beta_{ik})$$

for all pairwise distinct $i, j, k \in \{1, \ldots, d-1\}$. This shows that

$$b_i b_j - (b_i + a)(b_j + a) = b_i b_k - (b_i + a)(b_k + a)$$

and hence $b_j = b_k$. So we see that $b_1 = \ldots = b_{d-1}$ and $c_1 = \ldots = c_{d-1}$, and the equality in (5.1) implies $b_1 = c_1$ in view of Remark 5.3. □

The $2 \times \ldots \times 2$ tensors used in our construction are as follows.

Definition 5.5. We use the symbol $\mathcal{U}$ to denote the d-way $\mathcal{B} \times \ldots \times \mathcal{B}$ tensor

$$\mathcal{U}(\alpha_1|\ldots|\alpha_d) = \begin{cases} 1, & \text{if } \alpha_1 = \ldots = \alpha_d = e, \\ -3, & \text{if } \alpha_1 = \ldots = \alpha_d = \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

In other words, one can define $\mathcal{U}$ as the tensor representing the mapping of the d-form $x^d - 3y^d$. The following tensors represent binary monomials.

Definition 5.6. Let a and b be positive integers. The *monomial* tensor $\mu_{a,b}$ is the $\mathcal{B} \times \ldots \times \mathcal{B}$ tensor of order $a+b$ such that

$$\mu_{a,b}(\alpha_1|\ldots|\alpha_{a+b}) = \begin{cases} 1, & \text{if } e \text{ appears exactly } a \text{ times in } (\alpha_1, \ldots, \alpha_{a+b}), \\ 0, & \text{otherwise.} \end{cases}$$

Definition 5.7. We use the symbol $\mathcal{M}$ to denote the set of tensors

$$\{\mu_{1,d-2},\ \mu_{2,d-3},\ \ldots,\ \mu_{d-3,2},\ \mu_{d-2,1}\},$$

where d is a fixed positive integer as in Remark 5.1.

Now we can explain the main idea of our construction. We look at the set

$$\mathcal{U} \bmod (\mathcal{M}, \ldots, \mathcal{M}), \tag{5.2}$$

and, since the corner entries of any tensor in $\mathcal{M}$ are zero, we note that the mod operation in (5.2) does not change the corner entries of $\mathcal{U}$. In other words, the $(e, \ldots, e)$ and $(\varepsilon, \ldots, \varepsilon)$ entries of any tensor in (5.2) remain equal to 1 and -3, respectively, and this implies that the set (5.2) contains no tensor of symmetric rank one. To see this, we recall that a such a tensor is a scalar multiple of

$$\omega \otimes \ldots \otimes \omega \quad \text{with} \quad \omega \in \mathbb{R}^{\mathcal{B}}$$

and hence the ratio of its corner entries should be a d-th power in $\mathbb{R}$. As said above, a tensor in (5.2) has this ratio equal to $-1 : 3$, so it cannot be an even power in $\mathbb{R}$. We proceed with a proof that, nevertheless, the set (5.2) admits a rank-one tensor if we do not impose the symmetry assumption.

Remark 5.8. Before we proceed, let us consider the $d \times d$ matrix S defined as

$$S_{ij} = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{if } i \neq j. \end{cases}$$

We note that the matrix $S + I$ has rank one, where I is the $d \times d$ identity matrix, and hence -1 is an eigenvalue of S with multiplicity at least $d-1$. Since the trace of S is zero, the remaining eigenvalue is $d-1$, which means that S is non-singular over the reals. In the relevant case of $d = 4$, it is non-singular over any field of characteristic different from three, and, in particular, over a field as in Remark 3.1.

Lemma 5.9. *Let V be a d-way $\mathcal{B} \times \ldots \times \mathcal{B}$ tensor which can have a non-zero at a position $\alpha = (\alpha_1|\ldots|\alpha_d)$ only if the index e appears exactly once in α. Then*

$$V \in O \bmod (\mathcal{M}, \ldots, \mathcal{M}), \tag{5.3}$$

where O is the d-way $\mathcal{B} \times \ldots \times \mathcal{B}$ zero tensor.

Proof. We are going to show a stronger result that

$$V \in O \bmod (\mu_{1,d-2}, \ldots, \mu_{1,d-2}),$$

where $\mu_{1,d-2}$ is the tensor as in Definition 5.6. In order to do this, we add, for every $\delta \in \{1, \ldots, d\}$, a copy of $\mu_{1,d-2}$ multiplied by a scalar s_δ to the ε-th δ-slice of O. Denoting $\sigma = s_1 + \ldots + s_d$, we express the entries of the resulting tensor U as

$$\begin{cases} U(e|\varepsilon|\varepsilon|\ldots|\varepsilon) = \sigma - s_1, \\ U(\varepsilon|e|\varepsilon|\ldots|\varepsilon) = \sigma - s_2, \\ \ldots \\ U(\varepsilon|\varepsilon|\ldots|\varepsilon|e) = \sigma - s_d, \end{cases}$$

and all the other entries of U are zero. The resulting mapping $(s_1, \ldots, s_d) \to U$ is injective because the corresponding $d \times d$ matrix is non-singular by Remark 5.8. This shows that the image of $(s_1, \ldots, s_d) \to U$ is precisely the space of all possible choices of the tensor V as in the formulation of the lemma. □

Lemma 5.10. *Let V be a d-way $\mathcal{B} \times \ldots \times \mathcal{B}$ tensor which can have a non-zero number at a position $\alpha = (\alpha_1 | \ldots | \alpha_d)$ only if either*
1. *the index e appears exactly once in α, or*
2. *the index ε appears exactly once in α.*

Then the condition (5.3) *holds.*

Proof. The property of being an element of the set in (5.3) is additive, so it suffices to prove it separately for tensors V_1 and V_2, each of which can have a non-zero entry at a position $\alpha = (\alpha_1 | \ldots | \alpha_d)$ only if one corresponding condition (1) or (2) is satisfied. Lemma 5.9 implies $V_1 \in O \bmod (\mathcal{M}, \ldots, \mathcal{M})$ immediately, and the corresponding inclusion for V_2 follows by switching the roles of e and ε. □

Lemma 5.11. *The set* (5.2) *contains a rank-one tensor.*

Proof. We note that the d-way $\mathcal{B} \times \ldots \times \mathcal{B}$ tensor $\mathcal{U}'$ defined as

$$\mathcal{U}'(\alpha_1 | \ldots | \alpha_d) = \begin{cases} 1, & \text{if } \alpha_1 = e, \\ -3, & \text{if } \alpha_1 = \varepsilon \end{cases}$$

is rank-one, so it suffices to check that $\mathcal{U}' - \mathcal{U}$ belongs to the set in (5.3). In order to do this, we subtract all the elements of $\mathcal{M}$ from the e-th 1-slice of $\mathcal{U}' - \mathcal{U}$, and we add the same elements multiplied by three to its ε-th 1-slice. The only non-zero entries of the resulting tensor are at $(e, \varepsilon, \ldots, \varepsilon)$ and $(\varepsilon, e, \ldots, e)$, so it satisfies the assumptions of Lemma 5.10 and hence belongs to the set in (5.3). □

6. A family of counterexamples

So we see that the set

$$\mathcal{U} \bmod (\mathcal{M}, \ldots, \mathcal{M})$$

contains a tensor of rank one but does not contain a symmetric tensor of rank one. In view of Lemma 4.10, one can try constructing a counterexample immediately as $\mathrm{SAdj}\,(\mathcal{U}, \mathcal{M})$, but this approach may not work because the tensors in $\mathcal{M}$ are not decomposable, and hence the conclusion of Lemma 4.10 may not apply with equality. Our further strategy is to replace $\mathcal{M}$ by a family of decomposable tensors that, on the one hand, contain $\mathcal{M}$ in their linear span and, on the other hand, are sufficiently ill behaved to disallow any other linear combination to be helpful for constructing alternative rank decompositions. Since the original family $\mathcal{M}$ consists of monomial tensors, we decided to refer to its possible replacements as *monomial emulators*. We work on this idea with the concept of the *clone* of a tensor introduced in [58, 59] for three-way tensors, and the current discussion requires the development of the approach of [58, 59] to the case of higher orders. First of all, we need to introduce a new indexing set related to the one in Definition 5.2.

Definition 6.1. We use the letter c to denote a fixed integer $\geqslant 2$.

Definition 6.2. We define the new indexing sets

$$E = \{e_1, \ldots, e_c\}, \quad \mathcal{E} = \{\varepsilon_1, \ldots, \varepsilon_c\}, \quad \mathbb{B} = E \cup \mathcal{E}.$$

Definition 6.3. Let q be a positive integer, and let T be a q-way $\mathcal{B}\times\ldots\times\mathcal{B}$ tensor. The *clone* of T is the q-way $\mathbb{B}\times\ldots\times\mathbb{B}$ tensor T_c whose entries are defined as

$$T_c\left({}^1\alpha_{i_1}\left|{}^2\alpha_{i_2}\right|\ldots\left|{}^q\alpha_{i_q}\right.\right)=T\left({}^1\alpha|{}^2\alpha|\ldots|{}^q\alpha\right)$$

for all ${}^1\alpha,\ \ldots,\ {}^q\alpha\in\mathcal{B}$ and $i_1,\ldots,i_q\in\{1,\ldots,c\}$.

Definition 6.4. Let q be a positive integer, and let A be a family of q-way $\mathcal{B}\times\ldots\times\mathcal{B}$ tensors. Then the notation A_c stands for the set of clones of all tensors in A.

We proceed with one general property of the class of clone tensors.

Lemma 6.5. *Let P be a q-way decomposable $\mathbb{B}\times\ldots\times\mathbb{B}$ tensor in which*

(1) *all entries of the $E\times\ldots\times E$ block are equal to the same non-zero number,*
(2) *all entries of the $\mathcal{E}\times\ldots\times\mathcal{E}$ block are equal to the same non-zero number.*

Then P is a clone.

Proof. We have

$$B=\begin{pmatrix}v_{1E}\\ v_{1\mathcal{E}}\end{pmatrix}\otimes\ldots\otimes\begin{pmatrix}v_{qE}\\ v_{q\mathcal{E}}\end{pmatrix},$$

and we can note that, for any fixed $i,j\in\{1,\ldots,q\}$, the entries of v_{iE} are all equal because of the item (1), and the entries of $v_{j\mathcal{E}}$ are equal because of the item (2). □

Now we can present a list of properties that we need to impose on a monomial emulator family to get a counterexample in the fashion described above. The existence of such families is a non-trivial question, and we postpone its consideration to Section 11, where we give an explicit example of a monomial emulator for $d=4$.

Remark 6.6. In the following Definition 6.7, the notations T_E and $T_{\mathcal{E}}$ stand for, respectively, the $E\times\ldots\times E$ and $\mathcal{E}\times\ldots\times\mathcal{E}$ blocks of a q-way $\mathbb{B}\times\ldots\times\mathbb{B}$ tensor T. If F is a family of such tensors, then F_E and $F_{\mathcal{E}}$ denote the set of all corresponding blocks of the tensors in F. Also, we write $\mathbb{I}(E,q)$ and $\mathbb{I}(\mathcal{E},q)$ for, respectively, the q-way $E\times\ldots\times E$ and $\mathcal{E}\times\ldots\times\mathcal{E}$ tensors with all entries equal to one.

Definition 6.7. A finite set $\mathcal{W}$ of symmetric $\mathbb{B}\times\ldots\times\mathbb{B}$ tensors (of order $d-1$) is called a *monomial emulator* if the following conditions are satisfied:

(1) $\mathcal{W}$ is linearly independent,
(2) every tensor in $\mathcal{W}$ is decomposable,
(3) $\operatorname{span}\mathcal{W}$ contains the clone of every tensor in $\mathcal{M}$,
(4e) $\mathbb{I}(E,d)$ is the only rank-one tensor in $\mathbb{I}(E,d)\bmod(\mathcal{W}_E,\ldots,\mathcal{W}_E)$,
(4ε) $\mathbb{I}(\mathcal{E},d)$ is the only rank-one tensor in $\mathbb{I}(\mathcal{E},d)\bmod(\mathcal{W}_{\mathcal{E}},\ldots,\mathcal{W}_{\mathcal{E}})$,
(5e) $\mathbb{I}(E,d-1)$ is the only rank-one tensor in $\operatorname{span}\mathcal{W}_E+\mathbb{I}(E,d-1)$,
(5ε) $\mathbb{I}(\mathcal{E},d-1)$ is the only rank-one tensor in $\operatorname{span}\mathcal{W}_{\mathcal{E}}+\mathbb{I}(\mathcal{E},d-1)$,
(6) if a tensor $u\otimes\ldots\otimes u$ belongs to $\operatorname{span}\mathcal{W}$, then the following sets are disjoint:

$$u_E\otimes\left(\underbrace{\mathbb{R}^E\otimes\ldots\otimes\mathbb{R}^E}_{d-1\text{ times}}\right)\quad\text{and}\quad\mathbb{I}(E,d)\bmod(\mathcal{W}_E,\ldots,\mathcal{W}_E).$$

Now we are ready to construct a counterexample to Comon's conjecture assuming that we are given a monomial emulator family as a black-box. The following notation for such families is to be used in the rest of our paper.

Definition 6.8. We use the symbol $\mathcal{W}$ to denote a monomial emulator family.

Definition 6.9. We use the symbol $\mathcal{S}$ to denote the tensor

$$\mathcal{S} = \operatorname{SAdj}(\mathcal{U}_c, \mathcal{W}), \tag{6.1}$$

where $\mathcal{U}_c$ is the clone of the tensor $\mathcal{U}$ as in Definition 5.5.

Remark 6.10. Following the conventions of Definitions 4.7 and 4.8, we assume that no tensor in $\mathcal{W}$ is indexed with a label in the set $\mathbb{B}$. So the d-way product

$$\mathbb{R}^{\mathbb{B}\cup\mathcal{W}} \otimes \ldots \otimes \mathbb{R}^{\mathbb{B}\cup\mathcal{W}}$$

is the tensor space containing $\mathcal{S}$.

The rank of $\mathcal{S}$ can be computed immediately.

Lemma 6.11. *We have* $\operatorname{rk}\mathcal{S} = d\,|\mathcal{W}| + 1$.

Proof. Since the family $\mathcal{W}$ is linearly independent by the item (1) of Definition 6.7, we have $|\mathcal{W}| = \dim \mathcal{W}$. Also, the tensors in $\mathcal{W}$ are decomposable by the item (2) of Definition 6.7, so we can apply the equality case of Lemma 4.10 and get

$$\operatorname{rk}\mathcal{S} = \min\operatorname{rk}\mathcal{U}_c \bmod (\mathcal{W}, \ldots, \mathcal{W}) + d\,|\mathcal{W}|. \tag{6.2}$$

Now we use the definition of the clone to see that

$$\left(\mathcal{U} \bmod (\mathcal{M}, \ldots, \mathcal{M})\right)_c \subseteq \mathcal{U}_c \bmod (\mathcal{M}_c, \ldots, \mathcal{M}_c), \tag{6.3}$$

and the item (3) of Definition 6.7 shows a further inclusion

$$\mathcal{U}_c \bmod (\mathcal{M}_c, \ldots, \mathcal{M}_c) \subseteq \mathcal{U}_c \bmod (\mathcal{W}, \ldots, \mathcal{W}). \tag{6.4}$$

A comparison of (6.3) and (6.4) gives the inequality

$$\min\operatorname{rk}\mathcal{U}_c \bmod (\mathcal{W}, \ldots, \mathcal{W}) \leqslant \min\operatorname{rk}\mathcal{U} \bmod (\mathcal{M}, \ldots, \mathcal{M}),$$

in which the right-hand side is at most one by Lemma 5.11. This implies

$$\min\operatorname{rk}\mathcal{U}_c \bmod (\mathcal{W}, \ldots, \mathcal{W}) \leqslant 1, \tag{6.5}$$

and we also have the opposite inequality

$$\min\operatorname{rk}\mathcal{U}_c \bmod (\mathcal{W}, \ldots, \mathcal{W}) \geqslant 1 \tag{6.6}$$

by the item (4e) of Definition 6.7. Now the proof is complete, because the conditions (6.2), (6.5), (6.6) imply the desired conclusion. □

Now let us check the inequality $\operatorname{srk}\mathcal{S} \leqslant \operatorname{rk}\mathcal{S} + 1$.

Lemma 6.12. *We have* $\operatorname{srk}\mathcal{S} \leqslant d\,|\mathcal{W}| + 2$.

Proof. The $\mathbb{B}\times\ldots\times\mathbb{B}$ block of $\mathcal{S}$ is the clone of the rank-two tensor $\mathcal{U}$. Subtracting this block from $\mathcal{S}$, we get the tensor $\operatorname{SAdj}(O, \mathcal{W})$, and hence the inequality

$$\operatorname{srk}\mathcal{S} \leqslant 2 + \operatorname{srk}\operatorname{SAdj}(O, \mathcal{W}) \leqslant 2 + s|\mathcal{W}|$$

follows from Remark 4.9, where s is the symmetric rank of xy^{d-1} considered as a tensor. It remains to observe an easy fact that $s = d$, see also [3, 12, 21, 67]. □

7. Substitutions of monomial emulators

In order to proceed with the proof that $\operatorname{srk}\mathcal{S} = \operatorname{rk}\mathcal{S} + 1$, we need some further information on the monomial emulator families introduced in Definition 6.7. This section collects several basic properties of monomial emulators that can be obtained by linear transformations from families similar to those in Definition 6.7.

Observation 7.1. *Let $\mathcal{W}'$ be a family that satisfies all the assumptions in Definition 6.7 except possibly the condition (1). If $\operatorname{span}\mathcal{W}'$ admits a basis $\mathcal{V}$ consisting of decomposable tensors, then every such $\mathcal{V}$ is a monomial emulator.*

Proof. The validity of the conditions (1) and (2) in Definition 6.7 for $\mathcal{V}$ is immediate. The remaining conditions appeal to $\mathcal{V}$ only via $\operatorname{span}\mathcal{V}$, so they are true since they hold for $\mathcal{W}'$ and $\operatorname{span}\mathcal{V} = \operatorname{span}\mathcal{W}'$ by the assumptions of the lemma. □

In what follows, a family as in Definition 6.7 is called a monomial emulator *on* its span. If $\mathcal{V}$ and $\mathcal{W}$ are monomial emulators with $\operatorname{span}\mathcal{V} = \operatorname{span}\mathcal{W}$, then

$$\mathcal{S} = \operatorname{SAdj}(\mathcal{U}_c, \mathcal{W}) \quad \text{and} \quad \mathcal{S}' = \operatorname{SAdj}(\mathcal{U}_c, \mathcal{V}), \tag{7.1}$$

which are the tensors constructed as in Definition 6.9, differ by an invertible linear transformation of the variables of the corresponding homogeneous polynomials. Let us give a formal description of the tensorial version of this mapping.

Definition 7.2. Let $\mathcal{V}, \mathcal{W}$ be monomial emulators on the same space. Assume

$$v = \sum_{w \in \mathcal{W}} s_v^w w$$

is an expression of any $v \in \mathcal{V}$ as a linear combination of $\mathcal{W}$. We set

$$\Lambda : \left(\underbrace{\mathbb{R}^{\mathcal{B}\cup\mathcal{W}} \otimes \ldots \otimes \mathbb{R}^{\mathcal{B}\cup\mathcal{W}}}_{d \text{ times}} \right) \to \left(\underbrace{\mathbb{R}^{\mathcal{B}\cup\mathcal{V}} \otimes \ldots \otimes \mathbb{R}^{\mathcal{B}\cup\mathcal{V}}}_{d \text{ times}} \right)$$

as the mapping that acts in the d steps below. Here, the notation $\varphi_{\delta w}$ stands for the w-th δ-slice of the tensor obtained at the completion of the $\delta - 1$ of these steps:

(Step 1) Adjoin, as the 1-slice, for any $v \in \mathcal{V}$, the linear combination

$$\sum_{w \in \mathcal{W}} s_v^w \varphi_{1w}$$

and remove the w-th 1-slices of the resulting tensor for all $w \in \mathcal{W}$,

...

(Step d) adjoin, as the d-slice, for any $v \in \mathcal{V}$, the linear combination

$$\sum_{w \in \mathcal{W}} s_v^w \varphi_{dw}$$

and remove the w-th d-slices of the resulting tensor for all $w \in \mathcal{W}$.

Remark 7.3. The mapping Λ is linear and acts on decomposable tensors as

$$v_1 \otimes \ldots \otimes v_d \to (I \oplus S)v_1 \otimes \ldots \otimes (I \oplus S)v_d,$$

where I is the $\mathbb{B} \times \mathbb{B}$ unity matrix, and S is the $\mathcal{V} \times \mathcal{W}$ matrix (s_v^w).

Remark 7.4. We have $\Lambda(\mathcal{S}) = \mathcal{S}'$ in the notation of (7.1) and Definition 7.2.

Observation 7.5. *The mapping Λ in Definition 7.2 preserves the rank and symmetric rank, so we have* $\operatorname{rk}\mathcal{S} = \operatorname{rk}\mathcal{S}'$ *and* $\operatorname{srk}\mathcal{S} = \operatorname{srk}\mathcal{S}'$ *for the tensors as in* (7.1).

We proceed with one useful notational convention.

Definition 7.6. Let I_1, $J_1, \ldots, I_q$, J_q be finite indexing sets such that $I_t \subseteq J_t$ for all $t \in \{1, \ldots, q\}$. Let A_J be a $J_1 \times \ldots \times J_q$ tensor with

- the $I_1 \times \ldots \times I_q$ block equal to a tensor A,
- all the entries outside the $I_1 \times \ldots \times I_q$ block zero.

We say that A_J is obtained by *padding* A to the format $J_1 \times \ldots \times J_q$. If the formats of A and A_J are clear, we say that A_J is the *appropriate padding* of the tensor A.

Remark 7.7. The w-th δ-slice of the tensor $\mathcal{A}$ in Definition 4.7 is a padding of w.

We finalize the section with one more property of the mapping Λ.

Observation 7.8. *Let Λ be the mapping as in Definition 7.2 and $\delta \in \{1, \ldots, d\}$. Assume that the δ-slices of a d-way tensor*

$$T \in \mathbb{R}^{\mathcal{B}\cup\mathcal{W}} \otimes \ldots \otimes \mathbb{R}^{\mathcal{B}\cup\mathcal{W}}$$

are collinear to the padded version of an element w of a monomial emulator $\mathcal{W}$. Then the δ-slices of $\Lambda(T)$ are also collinear to the appropriately padded w.

Proof. The δ'-th step of Definition 7.2 does not affect T unless $\delta' = \delta$, and the δ-th step preserves the δ-slices of T up to collinearity because T is decomposable. □

8. Symmetric decompositions of $\mathcal{S}$

The three forthcoming sections are intended to complete the proof that $\mathcal{S}$ is a desired counterexample. We recall that, according to Lemmas 6.11 and 6.12,

$$\operatorname{rk}\mathcal{S} = d|\mathcal{W}| + 1 \text{ and } \operatorname{srk}\mathcal{S} \in \{\operatorname{rk}\mathcal{S}, \operatorname{rk}\mathcal{S} + 1\},$$

and we are going to show that the correct value of $\operatorname{srk}\mathcal{S}$ is $\operatorname{rk}\mathcal{S} + 1$. We argue by contradiction, that is, we are going to show that the equality $\operatorname{srk}\mathcal{S} = d|\mathcal{W}| + 1$ is impossible. In Sections 8–10, we assume that there exists a decomposition

$$\mathcal{S} = \Psi_1^0 + \Psi_2^0 + \ldots + \Psi_{md}^0 + \Psi_{md+1}^0, \tag{8.1}$$

where the summands of the right-hand side are symmetric decomposable tensors as in Remark 3.3, and we also write $m = |\mathcal{W}|$. Our argument requires one more definition, which allows us to develop a technique in [58] for higher order tensors.

Definition 8.1. Let $\delta \leqslant q$ be positive integers. Assume that $t = (t_1, \ldots, t_r)$ and $t' = (t'_1, \ldots, t'_r)$ are two *rank decompositions* of a q-way tensor τ, which means that $\operatorname{rk}\tau = r$, the tensors in both t and t' are decomposable, and

$$\sum_{i=1}^{r} t_i = \sum_{i=1}^{r} t'_i = \tau.$$

We say that t' is obtained from t by a *δ-transformation* if the δ-slices of every tensor in t' belong to the linear span of the δ-slices of the tensors in t.

Observation 8.2. *For any i, let σ_i be a non-zero δ-slice of a tensor t_i in a rank decomposition $(t_1, \ldots, t_r)$. Then the tensors $(\sigma_1, \ldots, \sigma_r)$ are linearly independent.*

Proof. Otherwise, we would be able to cover the δ-slices of $t_1 + \ldots + t_r$ with a subspace spanned by less than r decomposable tensors, which implies, according to Observation 4.3, that the rank of $t_1 + \ldots + t_r$ is less than r. □

Observation 8.3. *If t, t' are rank decompositions of one tensor, then the conditions*

- *t' is obtained from t by a δ-transformation and*
- *t is obtained from t' by a δ-transformation*

are equivalent.

Proof. Each of these conditions, taken separately from the other one, implies that the linear span of the δ-slices of the tensors in either t or t' is a subspace of the corresponding linear span of the other decomposition. According to Observation 8.2, these linear spans should have the same dimension and hence coincide. □

Observation 8.4. *Let $\mathcal{V} = V \otimes \ldots \otimes V$ be a q-fold tensor product of a linear space V, and let π be the braiding isomorphism of $\mathcal{V}$ induced by a permutation that fixes a number $\delta \in \{1, \ldots, q\}$. Let t and t' be rank decompositions of one tensor in $\mathcal{V}$. If t' can be obtained from t by a δ-transformation, and if also $\pi(t) = t$, then $\pi(t') = t'$.*

Proof. Since δ is fixed by the permutation, its braiding isomorphism acts separately on every δ-slice, and the δ-slices of the tensors in t are invariant under π by the condition $\pi(t) = t$. Since t' is obtained from t by a δ-transformation, the δ-slices in t' are spanned by those in t, and hence they are invariant under π as well. □

The δ-transformations are quite powerful for tensors with many rank-one slices, and they lead to the following improvement of Lemma 4.6.

Lemma 8.5. *Let τ be a tensor. Assume that the linear span of the δ-slices of τ contains a linearly independent set $\mathcal{L}$ consisting of rank-one tensors. Then, for any rank decomposition $t = (t_1, \ldots, t_r)$ of τ, there is a δ-transformation of t which contains, for any $\ell \in \mathcal{L}$, a tensor whose δ-slices are collinear to ℓ.*

Proof. For any i, we pick a non-zero δ-slice of a tensor t_i and denote it by σ_i. Since t is a decomposition of τ, we have

$$\mathcal{L} \subset \operatorname{span}\{\sigma_1, \ldots, \sigma_r\}.$$

Since $\mathcal{L}$ is linearly independent, it can be extended by $r - |\mathcal{L}|$ appropriately chosen tensors in $\{\sigma_1, \ldots, \sigma_r\}$ so that the resulting family forms a basis of span $\{\sigma_1, \ldots, \sigma_r\}$. This basis consists of rank-one tensors, and it can be lifted to a desired decomposition as in the proof of Observation 4.3. □

Now we want to apply Lemma 8.5 repeatedly to the 1-slices of the symmetric decompostion (8.1), then to the 2-slices of the resulting expression and so forth. The following procedure goes by the induction on $\delta \in \{1, \ldots, d\}$, and we think of the decomposition (8.1) as the starting point corresponding to $\delta = 1$.

Procedure 8.6. *We assume $\mathcal{S}^0 = \mathcal{S}$ and, for any $\delta \in \{1, \ldots, d\}$, we have*

$$\mathcal{S}^{\delta-1} = \Psi_1^{\delta-1} + \Psi_2^{\delta-1} + \ldots + \Psi_{(d-\delta+1)m}^{\delta-1} + \Psi_{(d-\delta+1)m+1}^{\delta-1} \tag{8.2}$$

as a rank decomposition. As justified by Remark 8.8 below, we can apply Lemma 8.5 to (8.2) *with $\mathcal{W}$ in the role of $\mathcal{L}$, and the corresponding δ-transformation results in*

$$\mathcal{S}^{\delta-1} = \sum_{w \in \mathcal{W}} \Psi_w^\delta + \left(\Psi_1^\delta + \Psi_2^\delta + \ldots + \Psi_{(d-\delta)m}^\delta + \Psi_{(d-\delta)m+1}^\delta\right), \tag{8.3}$$

where all the summands are decomposable, and, moreover, we know that the δ-slices of Ψ_w^δ are collinear to the appropriately padded w. We define the tensor $\mathcal{S}^\delta$ inductively as the sum of the terms in the brackets in (8.3)*, which gives*

$$\mathcal{S}^\delta = \Psi_1^\delta + \Psi_2^\delta + \ldots + \Psi_{(d-\delta)m}^\delta + \Psi_{(d-\delta)m+1}^\delta \tag{8.4}$$

and completes the inductive definition of the tensors $\mathcal{S}^0, \ldots, \mathcal{S}^d$ and the summands in the corresponding decompositions of the forms (8.3) *and* (8.4)*.*

Three remarks on Procedure 8.6 are in order.

Remark 8.7. The decomposition (8.4) may not be determined uniquely by (8.3), and hence the outcome of Procedure 8.6 is uniquely identified neither by a given tensor $\mathcal{S}^0$ nor by its decomposition (8.1). Every possible outcome of Procedure 8.6 is said to be a *realization* of this procedure on an input of the form (8.1).

Remark 8.8. The tensor $\mathcal{S}^\delta$ differs from $\mathcal{S}^{\delta-1}$ by the sum of tensors whose δ-slices are collinear to padded w's, and hence the δ-slices of $\mathcal{S}^\delta - \mathcal{S}^{\delta-1}$ have zero entries at all the positions outside $\mathbb{B} \times \ldots \times \mathbb{B}$. Therefore,

$$\text{the } w\text{-th } \delta'\text{-slices are the same in } \mathcal{S}^{\delta-1} \text{ and } \mathcal{S}^\delta \text{ if } \delta' \neq \delta \text{ and } w \in \mathcal{W}. \tag{8.5}$$

In particular, the w-th δ-slice of $\mathcal{S}^{\delta-1}$ is the padded w, which justifies the inductive application of Lemma 8.5 when we pass from (8.2) to (8.3) in Procedure 8.6.

Remark 8.9. Let $\delta \in \{1, \ldots, d\}$, and let $\mathcal{V}$ be a monomial emulator on $\operatorname{span} \mathcal{W}$. Then, for all $v \in \mathcal{V}$, there exist tensors Ξ_v^δ such that

$$\sum_{w \in \mathcal{W}} \Psi_w^\delta = \sum_{v \in \mathcal{V}} \Xi_v^\delta$$

and every δ-slice of Ξ_v^δ is collinear to the padded v. Now we replace the tensors Ψ_w^δ by the tensors Ξ_v^δ in the decomposition (8.3), and we apply the mapping Λ as in Definition 7.2 to the decompositions obtained from (8.3) and (8.4) after this replacement. In view of Observation 7.8, the resulting decompositions can be obtained from $\mathcal{S}' = \operatorname{SAdj}(\mathcal{U}_c, \mathcal{V})$ as a realization of Procedure 8.6.

Lemma 8.10. *For any $w \in \mathcal{W}$ and $\delta \in \{1, \ldots, d\}$, the tensor Ψ_w^δ is a unique summand in* (8.3) *which has a non-zero w-th δ-slice.*

Proof. We note that both

(1) the total of the w-th δ-slices of the summands in (8.3), and
(2) a non-zero δ-slice of Ψ_w^δ

are collinear to the padded version of w. Therefore, the sum of (1) with an appropriate multiple of (2) is zero, and the assertion follows from Observation 8.2. $\square$

Lemma 8.11. *For $w \in \mathcal{W}$ and integers j, δ, δ'' satisfying*

$$1 \leqslant \delta'' \leqslant \delta \leqslant d \quad \textit{and} \quad 1 \leqslant j \leqslant (d-\delta)m + 1,$$

the w-th δ''-slice of Ψ_j^δ is zero.

Proof. If $\delta'' = \delta$, the statement follows by Lemma 8.10, so the w-th δ''-slice of $\mathcal{S}^{\delta''}$ is zero for all $w \in \mathcal{W}$. For $\delta'' < \delta$, we conclude, as in the assertion (8.5) in Remark 8.8, that the w-th δ''-slices of $\mathcal{S}^\delta$ are zero, and hence the corresponding slices should be zero in the tensors in the rank decompositions of $\mathcal{S}^\delta$ by Observation 8.2. $\square$

Lemma 8.12. *For any $\delta' \in \{0, \ldots, d-2\}$, the tensors*

$$\Psi_1^{\delta'}, \quad \ldots, \quad \Psi_{(d-\delta')m+1}^{\delta'}$$

are invariant under the braiding isomorphisms of $\mathbb{R}^{\mathbb{B}\cup\mathcal{W}} \otimes \ldots \otimes \mathbb{R}^{\mathbb{B}\cup\mathcal{W}}$ corresponding to the permutations of $\{\delta'+1, \delta'+2, \ldots, d\}$.

Proof. Follows from Observation 8.4 by the induction on δ'. □

We are going to finalize the section with a description of the tensor $\mathcal{S}^d = \Psi_1^d$, which appears as the $\delta = d$ case of (8.4). One auxiliary lemma is needed.

Lemma 8.13. *If Z is the restriction of the tensor*

$$\sum_{\delta=1}^{d} \sum_{w\in\mathcal{W}} \Psi_w^{\delta}$$

to its $\mathbb{B} \times \ldots \times \mathbb{B}$ block, then $Z \in O \bmod (\mathcal{W}, \ldots, \mathcal{W})$.

Proof. For any fixed δ, the $\mathbb{B} \times \ldots \times \mathbb{B}$ block of

$$\sum_{w\in\mathcal{W}} \Psi_w^{\delta}$$

belongs to

$$O \bmod (\varnothing, \ldots, \varnothing, \mathcal{W}, \varnothing, \ldots, \varnothing),$$

where the appearance of $\mathcal{W}$ corresponds to the δ-th place. □

Lemma 8.14. *The tensor Ψ_1^d is the padding of the clone of a $\mathcal{B} \times \ldots \times \mathcal{B}$ tensor P' satisfying $P'(e|\ldots|e) = 1$ and $P'(\varepsilon|\ldots|\varepsilon) = -3$.*

Proof. According to Lemma 8.11, the w-th δ''-slices of Ψ_1^d are zero for all $w \in \mathcal{W}$ and $\delta'' \in \{1, \ldots, d\}$, which means that Ψ_1^d is the padding of a $\mathbb{B} \times \ldots \times \mathbb{B}$ tensor P. Since the padding operation preserves decomposability, the tensor P is decomposable. In view of the equations (8.3) and (8.4), we see that

$$\Psi_1^d = \mathcal{S} - \sum_{\delta=1}^{d} \sum_{w\in\mathcal{W}} \Psi_w^{\delta} \tag{8.6}$$

and the restriction of (8.6) to the $\mathbb{B} \times \ldots \times \mathbb{B}$ blocks gives the equality $P = \mathcal{U}_c - Z$, where Z is the tensor as in Lemma 8.13. We get

$$P \in \mathcal{U}_c \bmod (\mathcal{W}, \ldots, \mathcal{W}),$$

and since P is decomposable, we can apply the items (4e) and (4ε) of Definition 6.7. We get that the $E \times \ldots \times E$ and $\mathcal{E} \times \ldots \times \mathcal{E}$ blocks of P are equal to the corresponding blocks of $\mathcal{U}_c$, which means that these blocks are the clones of the $1 \times \ldots \times 1$ tensors equal to 1 and -3, respectively. Now we see that P is a clone by Lemma 6.5. □

9. Backtrack analysis of Procedure 8.6

As explained above, Lemma 8.14 gives a description of the last step of the decomposition (8.4), that is, a description of the $\delta = d$ case. As we can see in the lemma below, this result allows us to figure out which decomposition appeared at the previous step, that is, to reveal the $\delta = d - 1$ case in (8.4).

Lemma 9.1. *We consider non-zero tensors*

$$\psi_1^{d-1}, \ldots, \psi_{m+1}^{d-1} \tag{9.1}$$

each of which is collinear to the d-slices of the corresponding tensor Ψ_j^{d-1} in the decomposition (8.4)*. Then there exists one padded clone among the tensors* (9.1)*, and the unpadded versions of the m others form a monomial emulator on* span $\mathcal{W}$.

Proof. The decomposition (8.4) with $\delta = d-1$ is related to

$$\mathcal{S}^{d-1} = \sum_{w\in\mathcal{W}} \Psi_w^d + \Psi_1^d \tag{9.2}$$

by a d-transformation. Those summands in (9.2) which correspond to some $w \in \mathcal{W}$ have their d-slices collinear to the paddings of the corresponding w's; the d-slices of the remaining tensor are collinear to a padded clone σ as in Lemma 8.14. The possible candidates for the d-slices of the tensors in the decompositions obtained from (9.2) by the d-transformations are rank-one tensors of the form

$$\psi = \sum_{w\in\mathcal{W}} \lambda_w w + \lambda_\sigma \sigma$$

with scalar λ_w and λ_σ. Let us consider separately two possible cases.

Case 1. If $\lambda_\sigma \neq 0$, then the conditions (5e) and (5ε) of Definition 6.7 show that each of the $E \times \ldots \times E$ and $\mathcal{E} \times \ldots \times \mathcal{E}$ blocks of ψ consists of equal numbers, which are the same as those located at the corresponding positions in the tensor $\lambda_\sigma\sigma$. Using Lemma 6.5, we conclude that ψ is a clone itself, and by the symmetry of w the tensor $\psi - \lambda_\sigma\sigma$ is symmetric. Therefore, we have $\psi = \lambda_\sigma\sigma$ from Observation 5.4.

Case 2. If $\lambda_\sigma = 0$, then ψ belongs to span $\mathcal{W}$. Using the result of Step 1 with Observation 8.2, we conclude that at most one tensor in (9.1) can satisfy $\lambda_\sigma \neq 0$, and hence there are at least m tensors that fall into Case 2. Since $\dim \operatorname{span} \mathcal{W} = m$, there are exactly m tensors that remain for Step 2, and, according to Observation 8.2, they form a basis of span $\mathcal{W}$. In other words, those m tensors in (9.1) which satisfy $\lambda_\sigma = 0$ are a monomial emulator on span $\mathcal{W}$ as in Observation 7.1. As explained in Step 1, the remaining tensor in (9.1) is a padded clone. □

Our further strategy is to use Lemma 9.1 to describe the $\delta = d-2$ case of (8.3), which in turn can be used to characterize the $\delta = d-3$ case, and eventually we are going to climb back to the decomposition (8.1) with this type of argument. In order to make this intuition precise, we need one more auxiliary definition.

Definition 9.2. Let I be a finite indexing set and $w \in I$. The *w-skeleton* of a q-way $I \times \ldots \times I$ tensor T is the set of all q-way $I \times \ldots \times I$ tensors T' such that

$$T(i_1|\ldots|i_q) = T'(i_1|\ldots|i_q) \quad \text{whenever} \quad w \in \{i_1, \ldots, i_q\}.$$

In other words, the w-skeleton of a given tensor T is obtained if we void all the entries of T in which the index w does not appear in any coordinate. Now we can point out a property relevant for our method of the backward δ-transformations.

Claim 9.3. *Let $\delta \in \{0, \ldots, d-1\}$. For any $w \in \mathcal{W}$, there exists an indexing family*

$$\tau(\delta, w) \subset \{1, \ldots, (d-\delta)m + 1\}$$

of cardinality $d - \delta$ such that

1. *for distinct w and ω, the sets $\tau(\delta, w)$ and $\tau(\delta, \omega)$ are disjoint,*
2. *if $t \notin \tau(\delta, w)$, then the w-skeleton of Ψ_t^δ is zero,*

(3) *if* $t \in \tau(\delta, w)$ *and* $\delta \geqslant 1$, *then*

$$\Psi_t^\delta \in \underbrace{u \otimes \ldots \otimes u}_{\delta\ times} \otimes \underbrace{\mathbb{R}^{\mathbb{B}\cup\mathcal{W}} \otimes \ldots \otimes \mathbb{R}^{\mathbb{B}\cup\mathcal{W}}}_{d-\delta\ times}$$

with $u \neq 0$ *being a vector such that* $u \otimes \ldots \otimes u$ *is collinear to the padded* w.

A detailed discussion of Claim 9.3 is given separately in Section 10. Now let us confirm that this claim is sufficient to compute the symmetric rank of $\mathcal{S}$.

Lemma 9.4. *The* $\delta = 0$ *case of Claim 9.3 implies that* $\operatorname{srk} \mathcal{S} = \operatorname{rk} \mathcal{S} + 1$.

Proof. For the fixed value $\delta = 0$, the family $\tau_w := \tau(0, w)$ in Claim 9.3 has cardinality d and depends only on w. For any $w \in \mathcal{W}$, we define

$$\Phi_w = \sum_{t \in \tau_w} \Psi_t^0 \tag{9.3}$$

to rewrite the equality (8.1) as

$$\mathcal{S} = \Psi_\pi^0 + \sum_{w \in \mathcal{W}} \Phi_w, \tag{9.4}$$

where π is the unique element of the set $\{1, \ldots, md + 1\}$ that does not belong to τ_w with any w. By the conclusion (2) of Claim 9.3, the tensor Φ_w has the same w-skeleton as $\mathcal{S}$, and hence the padded w's are adjoined to Φ_w as the w-slices as in the construction of Definition 4.7. The equality (9.3) implies $\operatorname{rk} \Phi_w \leqslant d$, and this is possible, according to Lemma 4.10, only if the $\mathbb{B} \times \ldots \times \mathbb{B}$ block of Φ_w belongs to $O \bmod (w, \ldots, w)$ and hence to $O \bmod (\mathcal{W}, \ldots, \mathcal{W})$. Since the $\mathbb{B} \times \ldots \times \mathbb{B}$ block of $\mathcal{S}$ is the tensor $\mathcal{U}_c$ as in Definition 6.9, we use the equality (9.4) to get that

$$\Psi_\pi^0 \in \mathcal{U}_c \bmod (\mathcal{W}, \ldots, \mathcal{W}).$$

According to the items (4e) and (4ε) of Definition 6.7, this means that each of the $E \times \ldots \times E$ and $\mathcal{E} \times \ldots \times \mathcal{E}$ blocks of Ψ_π^0 consists of equal numbers, which are in turn equal to those located at the corresponding positions in $\mathcal{U}_c$. So we see that Ψ_π^0

- is a symmetric d-way tensor,
- has 1 at the $(e_1, \ldots, e_1)$ position,
- has -3 at the $(\varepsilon_1, \ldots, \varepsilon_1)$ position,

so we get a contradiction because d is even and -3 is not a square. □

Therefore, the two statements are now sufficient to disprove the real case of Comon's conjecture: Claim 9.3 and the existence of monomial emulators. In the forthcoming Section 10, we deal with Claim 9.3, and the subsequent Section 11 completes the argument with an explicit example of a monomial emulator.

10. On the validity of Claim 9.3

Our proof of Claim 9.3 requires an additional assumption.

Assumption 10.1. *The monomial emulator obtained in Lemma 9.1 coincides with the monomial emulator* $\mathcal{W}$ *that was used in Definition 6.9.*

It is not immediately clear whether Assumption 10.1 forces any loss of generality or not, so we decided to formulate several results of this section in the *ad hoc* way. We return to the unconditional versions of the relevant results in the end of the section, which concludes with the proof of the validity of our counterexample.

Lemma 10.2. *Assumption 10.1 implies Claim 9.3 with $\delta = d-1$.*

Proof. Let $w \in \mathcal{W}$. Using the notation of Lemma 9.1, we define

$$\tau(d-1, w) = \{j\},$$

where j is such that ψ_j^{d-1} is collinear to the padded w. The conditions (1) and (3) in Claim 9.3 are now immediate, and the condition (2) follows by Observation 8.2. □

Before we consider the remaining cases of Claim 9.3, we describe the summands of the decomposition (8.3) assuming the correctness of this claim for some δ.

Lemma 10.3. *If Claim 9.3 is true for $\delta = \delta' \in \{1, \ldots, d-1\}$, then the w-skeleta of the δ'-slices of the tensors $\Psi_j^{\delta'}$ with $j \in \tau(\delta', w)$ are linearly independent.*

Proof. The decompositions (8.3) and (8.4), taken with $\delta \in \{\delta', \ldots, d\}$, allow one to construct two further rank decompositions of $\mathcal{S}^{\delta'}$ as

$$\Psi_1^{\delta'} + \ldots + \Psi_{(d-\delta')m+1}^{\delta'} = \Psi_1^d + \left(\sum_{i=\delta'+1}^{d} \sum_{w \in \mathcal{W}} \Psi_w^i \right). \tag{10.1}$$

We fix an arbitrary $w \in \mathcal{W}$ and proceed by restricting (10.1) to the w-skeleta. By the assertion (2) of Claim 9.3, the non-zero w-skeleta at the left-hand side of (10.1) do only appear at the summands with the index in $\tau(\delta', w)$. Therefore,

$$\text{the } w\text{-skeleton of the sum of } \Psi_j^{\delta'} \text{ over all } j \in \tau(\delta', w) \tag{10.2}$$

equals the w-skeleton of the right-hand side of (10.1). Further, the tensor Ψ_w^i has

- the w-th i-slice equal to the padded w, by Lemma 8.10,
- zero w-th j-slices whenever $j \neq i$, by the decomposability,

and since Ψ_1^d is a padding of a $\mathbb{B} \times \ldots \times \mathbb{B}$ tensor by Lemma 8.14, we have that

$$\text{the } w\text{-skeleton of } \sum_{i=\delta'+1}^{d} \Bigg(\underbrace{b_w \otimes \ldots \otimes b_w}_{i-1 \text{ times}} \Bigg) \otimes \overline{e_w} \otimes \Bigg(\underbrace{b_w \otimes \ldots \otimes b_w}_{d-i \text{ times}} \Bigg), \tag{10.3}$$

equals the w-skeleton as in (10.2), where b_w is a vector for which the padding of w is collinear to $b_w \otimes \ldots \otimes b_w$, and $\overline{e_w}$ is the vector with a one at the w-th position and zeros everywhere else. In the notation of Definition 4.7, the tensors in (10.3) have the w-th i-slices adjoined with any $i \in \{\delta'+1, \ldots, d\}$, and hence the tensors with the w-skeleton as in (10.3) should have rank at least $d - \delta'$ by Lemma 4.10. Since the sum in (10.2) contains exactly $d - \delta'$ terms, this sum is a rank decomposition, and if the w-skeleta of the δ'-slices of these terms were linearly dependent, we would get a contradiction to Observation 8.2. □

Now we are ready to prove Claim 9.3 for δ between 1 and $\delta - 2$. In view of Lemma 10.2, the only other case that remains uncovered is $\delta = 0$.

Lemma 10.4. *Assumption 10.1 implies Claim 9.3 with $\delta \in \{1, \ldots, d-2\}$.*

Proof. Lemma 10.2 and the induction allow us to assume that Claim 9.3 is true with $\delta + 1$ instead of δ. The formulas (8.3) and (8.4) give the two rank decompositions

$$\Psi_1^{\delta} + \ldots + \Psi_{(d-\delta)m+1}^{\delta} = \sum_{w \in \mathcal{W}} \Psi_w^{\delta+1} + \Psi_1^{\delta+1} + \ldots + \Psi_{(d-\delta-1)m+1}^{\delta+1} \tag{10.4}$$

of the tensor $\mathcal{S}^\delta$, and, according to Procedure 8.6, these decompositions are related by a $(\delta+1)$-transformation. In particular, the $(\delta+1)$-slices of the tensors on the left-hand side of (10.4) are linear combinations of the corresponding $(\delta+1)$-slices on the right-hand side of (10.4). In the rest of this proof, we write

$$\psi_j^k \quad \text{for some arbitrarily fixed non-zero } (\delta+1)\text{-slice of} \quad \Psi_j^k,$$

and we recall that the $(\delta+1)$-slices of $\Psi_w^{\delta+1}$ are collinear to $\overline{w}$, where $\overline{w}$ denotes the padding of w to the format $(\mathbb{B}\cup\mathcal{W})\times\ldots\times(\mathbb{B}\cup\mathcal{W})$. Therefore, we have

$$\psi_j^\delta = \sum_{w\in\mathcal{W}} \lambda_j^w \overline{w} + \sum_{\zeta=1}^{M_\delta} \lambda_j^\zeta \psi_\zeta^{\delta+1} \tag{10.5}$$

with $M_\delta = (d-\delta-1)m+1$ and the families

$$\left(\lambda_j^w\right)_{w\in\mathcal{W}} \quad \text{and} \quad \left(\lambda_j^\zeta\right)_{\zeta\in\{1,\ldots,M_\delta\}}$$

consisting of scalars and depending on $j\in\{1,\ldots,(d-\delta)m+1\}$. Further, we note that, according to the assertion (2) of Claim 9.3 for $\delta+1$, the condition

$$\lambda_j^q \neq 0 \ \text{ for some } \ q\in\tau(\delta+1,w) \tag{10.6}$$

is necessary for the w-skeleton of (10.5) to be non-zero, and, using Lemma 10.3, we see that the condition (10.6) is also sufficient for this skeleton being non-zero. Now we use the assertion (3) of the $\delta+1$ version of Claim 9.3, and we see that the w-skeleton of (10.5) equals the w-skeleton of some tensor in

$$\Big(\underbrace{b_w\otimes\ldots\otimes b_w}_{\delta \text{ times}}\Big)\otimes\Big(\underbrace{\mathbb{R}^{\mathbb{B}\cup\mathcal{W}}\otimes\ldots\otimes\mathbb{R}^{\mathbb{B}\cup\mathcal{W}}}_{d-\delta-1 \text{ times}}\Big), \tag{10.7}$$

where b_w is a vector for which $\overline{w}$ is collinear to $b_w\otimes\ldots\otimes b_w$. In fact, every tensor (10.5) satisfying the condition (10.6) should itself be contained in (10.7) by the decomposability. According to the item (1) of Definition 6.7, the vectors b_w and b_ω cannot be collinear for distinct $w,\omega\in\mathcal{W}$, and hence the decomposability of (10.5) implies that either the w-skeleton or the ω-skeleton is zero. In other words, for every fixed j, the condition (10.6) can be valid with at most one w.

In order to prove Claim 9.3 for the current value of δ, we take $\tau(\delta,w)$ as the set of all j for which the tensor (10.5) has a non-zero w-skeleton. Equivalently,

$$\tau(\delta,w) \ \text{ is the set of all } j \text{ for which the condition (10.6) is valid.}$$

In this notation, the tensor (10.5) with $j\notin\tau(\delta,w)$ has a zero w-skeleton, and hence the corresponding tensors Ψ_j^δ should have zero w-skeleta by Lemma 8.12; this gives the conclusion (2) in Claim 9.3. Also, the conclusion (1) in Claim 9.3 is immediate from the last sentence of the previous paragraph. Finally, the form (10.7) of the tensor (10.5) with $j\in\tau(\delta,w)$ gives the conclusion (3) of Claim 9.3.

It remains to prove that the cardinality of $\tau(\delta,w)$ is $d-\delta$. As we noted in the proof of Lemma 10.3, the w-skeleton of (10.4) cannot correspond to a tensor of rank less than $d-\delta$, so we have $|\tau(\delta,w)|\geqslant d-\delta$. Therefore, we can complete the proof by showing that the decomposable tensors of the form (10.5) that have non-zero w-skeleta as in (10.7) cannot span a subspace of dimension greater than

$d-\delta$. To this end, we are going to check that their span is contained in the subspace $H_w + Q_w$ with $H_w = \operatorname{span}\{\overline{w}\}$ and

$$Q_w \text{ is the linear span of all } \psi_\zeta^{\delta+1} \text{ with } \zeta \in \tau(\delta+1, w),$$

and this is sufficient because $\dim(H_w + Q_w) \leqslant 1 + (d-\delta-1) = d-\delta$. More precisely, it is enough to show that any decomposable tensor φ which has both the forms (10.5) and (10.7) should belong to H_w+Q_w. As said above, the formula (10.6) cannot hold with any ω different from w, and this implies $\varphi = q_w+\varphi'$ with $q_w \in Q_w$ and φ' being the padding of a $\mathbb{B} \times \ldots \times \mathbb{B}$ tensor φ''. Since both φ and q_w belong to (10.7), their difference φ' belongs to (10.7) as well, and hence

$$\varphi' = \left(\underbrace{b_w \otimes \ldots \otimes b_w}_{\delta \text{ times}}\right) \otimes T_\varphi \tag{10.8}$$

with some tensor T_φ of the order $(d-\delta-1)$ such that T_φ is the padding of a $\mathbb{B} \times \ldots \times \mathbb{B}$ tensor to the format $(\mathbb{B} \cup \mathcal{W}) \times \ldots \times (\mathbb{B} \cup \mathcal{W})$. If T_φ was collinear to $b_w \otimes \ldots \otimes b_w$, then φ' would be collinear to $\overline{w}$, and hence it would belong to H_w, which would imply $\varphi \in H_w + Q_w$ and conclude the argument. Therefore, it suffices to reach the contradiction starting at the assumption that

$$T_\varphi \text{ is not collinear to } b_w \otimes \ldots \otimes b_w, \tag{10.9}$$

which means, in view of the formula (10.8), that the tensor φ' is not symmetric, and hence its non-padded version φ'' cannot belong to $\operatorname{span}\mathcal{W}$ in the case of (10.9). However, the adjoining of φ' as the $(\delta+1)$-slice to $\mathcal{S}$ does not increase the rank because φ' is a linear combination of the $(\delta+1)$-slices of the tensors in a rank decomposition of $\mathcal{S}$. Using Lemma 4.10, we get

$$\operatorname{rk}\mathcal{S} = \min \operatorname{rk} \mathcal{U}_c \operatorname{mod}(\mathcal{W}, \ldots, \mathcal{W}, \mathcal{W} \cup \{\varphi''\}) + dm + 1,$$

and, since $\operatorname{rk}\mathcal{S} = dm+1$ by Lemma 6.11, we get the property

$$O \in \mathcal{U}_c \operatorname{mod}(\mathcal{W}, \ldots, \mathcal{W}, \mathcal{W} \cup \{\varphi''\}),$$

which shows that the sets

$$O \operatorname{mod}(\varnothing, \ldots, \varnothing, \varphi'') \quad \text{and} \quad \mathcal{U}_c \operatorname{mod}(\mathcal{W}, \ldots, \mathcal{W})$$

have common elements, but this is false by the condition (6) in Definition 6.7. $\square$

The case of $\delta = 0$ in Claim 9.3 does not immediately follow from the argument in Lemma 10.4. In fact, the left expression in the brackets in (10.7) gets void with $\delta = 0$, and we cannot use the condition (6) of Definition 6.7 to complete the proof as in Lemma 10.4. However, we can argue as in Lemma 10.4 to get conditions similar to (10.5), (10.6) and change the strategy. This requires two additional lemmas.

Lemma 10.5. *Let u be a vector in a linear space U. If a symmetric q-way tensor $A \in U \otimes \ldots \otimes U$ lies in $O \operatorname{mod}(w, \ldots, w)$ with $w = u \otimes \ldots \otimes u$, then A is represented as either (1) ax^q or (2) $x^{q-1}y + ax^q$ for some $a \in \mathbb{R}$ and linear forms x, y.*

Proof. When written with respect to a basis whose first vector equals u, the tensor A can have a non-zero entry at a position $(i_1|\ldots|i_q)$ only if at least $q-1$ of the values $i_1, \ldots, i_q$ are ones. In this case, we end up with the conclusion (1) if all entries except $(1|\ldots|1)$ are zero, and we get the assertion (2) otherwise. $\square$

Lemma 10.6. *If linear forms $\ell_1, \ldots, \ell_{d-1} \in \mathbb{R}[x, y, z]$ are such that*

$$x^{d-2}y + ax^{d-1} \in \operatorname{span}\left\{(\ell_1)^{d-1}, \ldots, (\ell_{d-1})^{d-1}\right\}$$

with $a \in \mathbb{R}$, then the dependence of every $\ell_1, \ldots, \ell_{d-1}$ on z is void.

Proof. Otherwise, we can substitute z with a linear form in x and y so that ℓ_1 and ℓ_2 become collinear. This is a contradiction because, according to Lemma 4.10, the rank of the tensor corresponding to $x^{d-2}y + ax^{d-1}$ cannot be less than $d-1$. $\square$

We are ready to proceed with the remaining case of Claim 9.3.

Lemma 10.7. *Assumption 10.1 implies Claim 9.3 with $\delta = 0$.*

Proof. Similarly to the proof of Lemma 10.4, we get

$$\psi_j^0 = \sum_{w \in \mathcal{W}} \lambda_j^w \overline{w} + \sum_{\zeta=1}^{M} \lambda_j^\zeta \psi_\zeta^1 \tag{10.10}$$

with $M = (d-1)m + 1$, where ψ_j^k is an arbitrarily fixed non-zero 1-slice of Ψ_j^k and

$$\left(\lambda_j^w\right)_{w \in \mathcal{W}} \text{ and } \left(\lambda_j^\zeta\right)_{\zeta \in \{1, \ldots, M\}}$$

are families of scalars that depend on $j \in \{1, \ldots, dm + 1\}$. Also, we write $\overline{w}$ to denote the appropriate padding of $w \in \mathcal{W}$. We continue to argue as in the proof of Lemma 10.4, and we conclude that the condition

$$\lambda_j^q \neq 0 \text{ for some } q \in \tau(1, w) \tag{10.11}$$

holds if and only if ψ_j^0 has a non-zero w-skeleton, and, for any fixed j, this can happen for at most one choice of $w \in \mathcal{W}$. Similarly to the proof of Lemma 10.4, we define $\tau(0, w)$ as the set of all j for which the condition (10.11) applies. Therefore, the tensor (10.10) with $j \notin \tau(0, w)$ has a zero w-skeleton, and the corresponding tensor Ψ_j^0 has a zero w-skeleton because it is symmetric; this gives the conclusion (2) in Claim 9.3. Also, the corresponding conclusion (1) is valid because the condition (10.11) cannot hold simultaneously with different w.

Since the conclusion (3) is void for $\delta = 0$, it remains to prove $|\tau(0, w)| = d$ for any $w \in \mathcal{W}$. Using the argument in the proof of Lemma 10.4, we see that $|\tau(0, w)| \geqslant d$, so we need to check $|\tau(0, w)| \leqslant d$. We do this by showing that those decomposable tensors of the form (10.10) which have non-zero w-skeleta belong to a subspace of dimension at most d. To this end, we consider a non-zero vector u such that $u \otimes \ldots \otimes u$ is collinear to $\overline{w}$, and we denote by $\pi \in \mathbb{B}$ an arbitrary position at which u is non-zero. Further, we write g to denote the π-th 1-slice of $\mathcal{S}$, and, according to Definitions 4.7 and 4.8, this tensor g has a non-zero scalar multiple of the $(d-2)$-way product $u \otimes \ldots \otimes u$ symmetrically adjoined as the w-th slice.

Since the only tensors ψ_ζ^1 with non-zero w-skeleta are those with $\zeta \in \tau(1, w)$, there exists a linear combination φ of these tensors whose w-skeleton equals that of g. Since the cardinality of $\tau(1, w)$ is $d-1$ by the $\delta = 1$ version of Claim 9.3, we have $\operatorname{rk} \varphi \leqslant d-1$. In view of Lemma 4.10, we have

$$\varphi \in O \bmod \left(u^{\otimes(d-2)}, \ldots, u^{\otimes(d-2)}\right),$$

and hence, according to Lemma 10.5, the tensor φ represents a polynomial of the form $ax^{d-1} + x^{d-2}y$ with $a \in \mathbb{R}$ and some linear forms x, y. According to Lemma 10.6, the tensors ψ_ζ^1 with $\zeta \in \tau(1, w)$ correspond to polynomials in x, y as well. Therefore, the w-th slices of a tensor φ' as in (10.10) are polynomials in x and y, and, hence, if these w-th slices are non-zero, the tensor φ' being decomposable should correspond to a polynomial in x and y itself. Since the space of all binary homogeneous polynomials of degree $(d-1)$ has dimension d, this implies the desired conclusion that the space of all possible φ' has dimension at most d. □

We are ready to complete the proof of the main result of Sections 8–10.

Theorem 10.8. *We have* $\operatorname{srk} \mathcal{S} = \operatorname{rk} \mathcal{S} + 1$.

Proof. Using the letter $\mathcal{V}$ to denote the monomial emulator obtained in Lemma 9.1, we consider the tensors $\mathcal{S}$ and $\mathcal{S}'$ as in the formulas (7.1). The decompositions introduced in Remark 8.9 arise from a realization of Procedure 8.6 on $\mathcal{S}'$, and, as we apply Lemma 9.1 to this new realization, we take the d-slices of

$$\Lambda\left(\Psi_1^{d-1}\right), \ \ldots, \ \Lambda\left(\Psi_{m+1}^{d-1}\right) \tag{10.12}$$

in the role of (9.1). According to Lemma 9.1, there exists a monomial emulator $\mathcal{V}'$ consisting of m tensors collinear to the d-slices of those in (10.12). This $\mathcal{V}'$ is to be the same as $\mathcal{V}$ because, for every fixed j, the d-slices of the tensors

$$\Psi_j^{d-1} \quad \text{and} \quad \Lambda\left(\Psi_j^{d-1}\right)$$

are all collinear by Observation 7.8. Therefore, Assumption 10.1 can be satisfied by the decompositions of $\mathcal{S}'$, and hence the conclusion of Lemma 10.7 applies. We use Lemma 9.4 to get $\operatorname{srk} \mathcal{S}' = \operatorname{rk} \mathcal{S}' + 1$, and this completes the proof because we have $\operatorname{rk} \mathcal{S} = \operatorname{rk} \mathcal{S}'$ and $\operatorname{srk} \mathcal{S} = \operatorname{srk} \mathcal{S}'$ by Remark 7.5. □

11. A monomial emulator

Since Theorem 10.8 is proved, the existence of monomial emulator families is the only remaining obstruction towards our counterexample. This section gives an example of such a family, and we decided to focus on the case $d = 4$ due to technical difficulties that arise in the general situation. In order to proceed, we go back to the setting of Section 6 and define all parameters appearing in Definition 6.7.

Definition 11.1. We take $c = 10$ for the value as in Definition 6.1. The family $\mathcal{W}$ is going to contain 190 symmetric $20 \times 20 \times 20$ tensors of the form

$$u \otimes u \otimes u \quad \text{with} \quad u = (u_E | u_{\mathcal{E}}),$$

where u_E and $u_{\mathcal{E}}$ are two vectors of length 10.

If we assume the existence of a monomial emulator $\mathcal{W}$ as in Definition 11.1, then the tensor $\mathcal{S}$ as in Definition 6.9 satisfies

$$\operatorname{rk} \mathcal{S} = 4 \cdot 190 + 1 = 761 \quad \text{and} \quad \operatorname{srk} \mathcal{S} = \operatorname{rk} \mathcal{S} + 1 = 762,$$

where the conclusion about the rank follows from Lemma 6.11, and the symmetric rank is computed in Theorem 10.8. According to Definition 6.9, the tensor $\mathcal{S}$ is obtained by the symmetrical adjoining of the slices in $\mathcal{W}$ to the symmetric tensor $\mathcal{U}_c$ with $2c$ slices, and hence $\mathcal{S}$ has $2c + |\mathcal{W}| = 210$ slices.

Remark 11.2. As we will see in Definition 11.4, the corresponding vectors u_E and $u_{\mathcal{E}}$ in Definition 11.1 have the sums of their entries at the even positions equal to the corresponding sums at the odd positions, which means that the vectors $u = (u_E | u_{\mathcal{E}})$ embed in an 18-dimensional space. This allows one to reduce the number of the slices of $\mathcal{S}$ by two, so we use the number 208 instead of 210 in Theorem 2.2.

It remains to construct an appropriate family $\mathcal{W}$, and we begin with a description of building blocks for the vectors u as in Definition 11.1.

Definition 11.3. For any $i \in \{1, \dots, 5\}$, we define $\alpha_i \in \mathbb{R}^{10}$ as the vector with two ones at the positions $2i-1$, $2i$ and eight zeros at the remaining places.

We are ready to complete the construction of $\mathcal{W}$.

Definition 11.4. We define $\mathcal{W}_1$ as the set of all tensors $u \otimes u \otimes u$, where u can be one of the following vectors:

$$(\alpha_i + \alpha_j | \alpha_k), \quad (\alpha_i + \alpha_j | \, 0), \quad (3\alpha_i \,|\, 4\alpha_k), \quad (\alpha_i | \, 0), \quad (0 \,| \alpha_k), \tag{11.1}$$

where $i, j, k \in \{1, \dots, 5\}$ and $i < j$. We define $\mathcal{W}_2$ as the set of all tensors of the from $u_\pi \otimes u_\pi \otimes u_\pi$ with u as in (11.1), where the mapping π acts as

$$\pi : (v|w) \to (w_\sigma | v_\sigma), \tag{11.2}$$

and σ is the cyclic permutation

$$\sigma : (x_1 \;\, x_2 \;\, \dots \;\, x_9 \;\, x_{10}) \;\; \to \;\; (x_2 \;\, x_3 \;\, \dots \;\, x_{10} \;\, x_1).$$

Finally, we set $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$.

The rest of the paper is devoted to the confirmation of the assumptions (1)–(6) in Definition 6.7. As said above, this would complete the proof of Theorem 2.2.

Remark 11.5. Observation 7.1 allows us to omit the proof of the assumption (1).

Remark 11.6. The condition (2) is immediate from Definition 11.4.

Lemma 11.7. *The assumption (3) in Definition 6.7 is valid.*

Proof. As said in Definition 11.4, the linear mapping defined by

$$u \otimes u \otimes u \to u_\pi \otimes u_\pi \otimes u_\pi \tag{11.3}$$

with π as in (11.2) restricts to a bijection $\mathcal{W}_1 \to \mathcal{W}_2$. Also, the mapping (11.3) sends the clone of the tensor $\mu_{2,1}$ to the clone of $\mu_{1,2}$, so it suffices to check that the clone of $\mu_{2,1}$ belongs to $\operatorname{span} \mathcal{W}_1$. To this end, we check that the tensor

$$\sum_{i,k=1}^{5} \sum_{j=i+1}^{5} (\alpha_i + \alpha_j | \alpha_k)^{\otimes 3} - \frac{1}{12} \sum_{i,k=1}^{5} (3\alpha_i \,|\, 4\alpha_k)^{\otimes 3}$$

has all ones at the $E \times E \times \mathcal{E}$ block and all zeros at the $E \times \mathcal{E} \times \mathcal{E}$ block. Also, the $E \times E \times E$ and $\mathcal{E} \times \mathcal{E} \times \mathcal{E}$ blocks can be cleaned up by adding

$$-5 \sum_{i=1}^{5} \sum_{j=i+1}^{5} (\alpha_i + \alpha_j | \, 0)^{\otimes 3} + k_2 \sum_{i=1}^{5} (\alpha_i | \, 0)^{\otimes 3} + k_3 \sum_{k=1}^{5} (0 \,| \alpha_k)^{\otimes 3}$$

with $k_2 = 45/4$ and $k_3 = 50/3$. □

We proceed with the proofs of the assumptions (4e), (4ε), (5e), (5ε).

Lemma 11.8. *The assumptions (5e) and (5ε) in Definition 6.7 are valid.*

Proof. Due to the symmetry, we can only prove (5e). To this end, we take a tensor

$$w \in \operatorname{span} \mathcal{W}_E,$$

where $\mathcal{W}_E$ is the set of all $E \times \ldots \times E$ blocks of the tensors in $\mathcal{W}$. The desired statement is that we can have

$$\mathbb{I}(E, d-1) + w = u_1 \otimes u_2 \otimes u_3 \tag{11.4}$$

only if $w = O$, where $\mathbb{I}(E, q)$ is the q-way $E \times \ldots \times E$ tensor of all ones. According to Definition 11.4, the tensor w lies in the linear span of the tensors of the form

$$(\alpha_i + \alpha_j) \otimes (\alpha_i + \alpha_j) \otimes (\alpha_i + \alpha_j)$$

with possibly equal i and j, we recall Definition 11.3 and conclude that the condition

$$w(s_1|s_2|s_3) = 0 \tag{11.5}$$

holds for all tuples (s_1, s_2, s_3) in which every pair of entries differ by at least two. The application of (11.5) with (s_1, s_2, s_3) equal to

$$(1, 8, 10),\ (2, 8, 10),\ \ldots,\ (6, 8, 10) \tag{11.6}$$

shows that the tensor (11.4) has ones at the positions in (11.6), and hence the vector u_1 has equal entries at the positions $1, 2, 3, 4, 5, 6$. Similarly, we can use (11.5) with

$$(6, 3, 1),\ (7, 3, 1),\ \ldots,\ (10, 3, 1)$$

to show that the positions $6, 7, 8, 9, 10$ of u_1 are also equal to each other. Therefore, all entries of u_1 are equal, and the symmetry translates this conclusion to u_2 and u_3 as well. Therefore, the tensor (11.4) has all entries equal, and, since its $(1, 8, 10)$ entry is one, this tensor should coincide with $\mathbb{I}(E, d-1)$. $\square$

Lemma 11.9. *The assumptions (4e) and (4ε) in Definition 6.7 are valid.*

Proof. Using the symmetry again, we can focus on (4e). We take a tensor

$$w \in O \bmod (\mathcal{W}_E, \ldots, \mathcal{W}_E)$$

and proceed with a proof that the equality

$$\mathbb{I}(E, d) + w = u_1 \otimes u_2 \otimes u_3 \otimes u_4 \tag{11.7}$$

is possible only if $w = O$. The argument as in Lemma 11.8 shows that

$$w(s_1|s_2|s_3|s_4) = 0 \tag{11.8}$$

whenever every pair of entries in (s_1, s_2, s_3, s_4) differ by at least two. Similarly to the proof of Lemma 11.8, the application of (11.8) with

$$(1, 6, 8, 10),\ (2, 6, 8, 10),\ (3, 6, 8, 10),\ (4, 6, 8, 10)$$

shows that the vector u_1 has equal numbers at the positions $1, 2, 3, 4$. Further, the application of (11.8) with

$$(4, 1, 8, 10),\ (5, 1, 8, 10),\ (6, 1, 8, 10)$$

shows that the corresponding positions from 1 to 6 are equal. Finally, we can use

$$(6, 1, 3, 10),\ (7, 1, 3, 10),\ (8, 1, 3, 10)$$

to prove the same statement for all positions from 1 to 8, and then the tuples

$$(8, 1, 3, 5),\ (9, 1, 3, 5),\ (10, 1, 3, 5)$$

allow one to conclude that all entries in the vector u_1 are equal, which completes the argument similarly to Lemma 11.8. $\square$

Now we can focus on the remaining assumption (6) in Definition 6.7. We recall that the tensors in $\mathcal{W}$ have the format $(E \cup \mathcal{E}) \times (E \cup \mathcal{E}) \times (E \cup \mathcal{E})$, where

$$E = \{e_1, \ldots, e_{10}\} \text{ and } \mathcal{E} = \{\varepsilon_1, \ldots, \varepsilon_{10}\}.$$

Definition 11.10. Let w be a tensor in $\operatorname{span} \mathcal{W}$. For any $\chi \in \{1, 2, 3\}$ and $i \in \{1, \ldots, 9\}$, the differences between the χ-slices of w with indexes

- e_i and e_{i+1},
- ε_i and ε_{i+1}

are called *odd differences* of w for odd i and *even differences* of w for even i.

Observation 11.11. *Every odd difference of a tensor in* $\operatorname{span} \mathcal{W}_1$ *is zero. Every even difference of a tensor in* $\operatorname{span} \mathcal{W}_2$ *is zero.*

Proof. This is immediate from Definitions 11.3 and 11.4. $\square$

Lemma 11.12. *If every even difference of a tensor* $w_1 \in \operatorname{span} \mathcal{W}_1$ *is zero, then* w_1 *is collinear to the clone of* $\mu_{2,1}$*, which is the tensor with ones at the positions*

$$(e, e, \varepsilon),\ (e, \varepsilon, e),\ (\varepsilon, e, e)$$

and zeros everywhere else.

Proof. Using Observation 11.11 together with the assumption of the lemma, we conclude that all slices of w_1 with indexes in E are equal, and all slices of w_1 with indexes in $\mathcal{E}$ are equal. Therefore, the tensor w_1 is a clone, and also w_1 is symmetric since the tensors in $\mathcal{W}$ are symmetric. Finally, we remark that the $E \times E \times E$ and $\mathcal{E} \times \mathcal{E} \times \mathcal{E}$ blocks have to be zero by the argument in the proof of Lemma 11.8, and the $E \times \mathcal{E} \times \mathcal{E}$ block is zero because is contains a zero entry, for similar reasons, for instance, at the position $(e_1, \varepsilon_1, \varepsilon_3)$. $\square$

We need the following description of rank-one tensors in $\operatorname{span} \mathcal{W}_1$ and $\operatorname{span} \mathcal{W}_2$.

Lemma 11.13. *For any vector* u *such that the tensor* $u \otimes u \otimes u$ *belongs to the linear span of either* $\mathcal{W}_1$ *or* $\mathcal{W}_2$*, the projection* u_E *contains at least one zero entry.*

Proof. Let $w \in \operatorname{span} \mathcal{W}_1$ be such a tensor. Using Definition 11.4, we can note that the $E \times E \times E$ block of w belongs to the linear span of

$$(\alpha_i + \alpha_j) \otimes (\alpha_i + \alpha_j) \otimes (\alpha_i + \alpha_j) \tag{11.9}$$

with possibly equal $i, j \in \{1, 2, 3, 4, 5\}$. It is straightforward to note that this $E \times E \times E$ block should be collinear to a tensor in (11.9), and hence is has several zeros on the diagonal. This means that the corresponding vector u_E has zero entries and completes the proof of the $\mathcal{W}_1$ case. The situation with $\mathcal{W}_2$ is similar, because the $E \times E \times E$ blocks of the tensors in $\operatorname{span} \mathcal{W}_2$ lie in the span of the tensors

$$\alpha_i \otimes \alpha_i \otimes \alpha_i$$

up to the cyclic permutation as in Definition 11.4. $\square$

Lemma 11.14. *If tensors* $w_1 \in \operatorname{span} \mathcal{W}_1$ *and* $w_2 \in \operatorname{span} \mathcal{W}_2$ *satisfy* $\operatorname{rk}(w_1 + w_2) = 1$*, then either* $w_1 = 0$ *or* $w_2 = 0$.

Proof. We argue by contradiction and assume that $w_1 \neq 0$, $w_2 \neq 0$.

Further, we assume that every even difference of $w_1 + w_2$ is zero. In view of Observation 11.11, this means that every even difference of w_1 is zero, and hence w_1 is a tensor as in Lemma 11.12. Since w_1 is non-zero, we can assume without loss of generality that every entry in the $E \times E \times \mathcal{E}$ block of w_1 is one. Using Definitions 11.3 and 11.4, one can note that a tensor in $\operatorname{span} \mathcal{W}_2$ should have a zero at the $(e_a, e_b, \varepsilon_t)$ position whenever $|a - b| \geqslant 2$. An argument similar to Lemmas 11.8 and 11.9 shows that the $E \times E \times \mathcal{E}$ block of $w_1 + w_2$ should contain only ones, and hence $w_1 + w_2$ should be a symmetric rank-one clone. However, this is impossible because the $E \times E \times E$ and $\mathcal{E} \times \mathcal{E} \times \mathcal{E}$ blocks of $w_1 + w_2$ should be zero by Lemma 11.8. Using this contradiction and the symmetry, we conclude that

$$w_1 + w_2 \text{ admits both even and odd non-zero differences.} \tag{11.10}$$

These differences and the slices of $w_1 + w_2$ should be collinear to one non-zero matrix ω because $\operatorname{rk}(w_1 + w_2) = 1$. If all entries of ω are equal, then all entries of $w_1 + w_2$ are equal as well, which contradicts to (11.10). Otherwise, the matrix ω contains two consecutive different rows, and their non-zero difference is either an odd difference or an even difference. Using this conclusion together with (11.10), we see that there is an odd difference of the 1-slices of $w_1 + w_2$ which in turn admits a non-zero even difference of its 2-slices, and hence we get a contradiction to Observation 11.11. $\square$

Now we can confirm the assumption (6) and finalize the paper.

Lemma 11.15. *The condition (6) in Definition 6.7 is valid.*

Proof. Let u_E be a vector as in (6). In view of Lemmas 11.13 and 11.14, this vector contains a zero at some position i. In this case, every tensor in the left set as in the assumption (6) has all zeros at the i-th 1-slice, but every tensor in the corresponding right set has ones in every slice by the argument in Lemma 11.9. $\square$

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