Analytical Psychology, an Information Theory approach.

Theodoros Aliferis
Contact details: t.aliferis@hotmail.com

Contents

Abstract

In the article "The introduction of the Einstein Model of a Solid, to Analytical Psychology", I documented that from an energetic perspective, the psyche as defined in Analytical Psychology (AP), can be considered an abstract Einstein Solid (ES). According to the theory of AP, and particularly the "principle of equivalence", the total number q of Values of the psyche, or equivalently the internal energy U when viewed as an ES, is a constant of the system. This allows, though, the psyche to be free to alter the number of its Structures (AP), or equivalently the number of its quantum harmonic oscillators (ES). As a consequence, the psyche's information content will vary.

In this article I study the case of small deviations of the psyche, or ES, from equilibrium and I do not do non-equilibrium statistics.

1   Introduction.

In this article the term ES and the term psyche as defined in AP, as well as all the associated terms, are considered identical and will be used interchangeably since [https://vixra.org/abs/1908.0290].
In Appendix A, I calculate the chemical potential of the ES.
In Appendix B, I introduce Information theory and particularly the concept of missing information. This concept helps us establish the second principle of statistical mechanics [3]. In the section "Discussion of the findings" I point out how the second principle can be virtually violated and the consequences this has to the psyche.
In Appendix C, I study the dependence of the missing information I, on ΔN. In the section "Discussion of the findings" I produce a graph Figure 1. of this relationship [equations (C.4) and (C.5)].
In Appendix D, I produce an estimation of the mean number of Quantum Harmonic Oscillators (QHO), $\bar{N}$. I also produce a graph of the dependence of $\bar{N}$ on T'. See Figure 2.
In Appendix E, I prove that the average energy of the interacting ES, described by the grand canonical ensemble is $U \equiv \bar{E} = qhf$. This result is accurate in the high temperature limit, i.e. $q \gg N$. This means that after the interacting ES has reached equilibrium, it can be moved to isolation (microcanonical ensemble) having the same internal energy $U = qhf$, as it started with. The number of energy quanta q, Values in AP, is practically conserved.
In the section "Discussion of the findings" I give an estimation of the energy of the system in the microcanonical ensemble E.
I give an estimation/measure of the fluctuations in the number of QHO's $\Delta N \propto \sqrt{N}$, based on [7]. These estimations are valid in the case of equilibrium statistics.

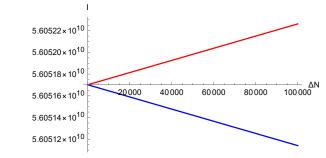# 2   Discussion of the findings.

According to [1]:
Let A be a macroscopic, system that is in thermal and chemical equilibrium with a reservoir (i.e. described by the grand canonical ensemble). It is clear that the relative fluctuations of its energy around a mean number $\bar{E}$ and of its number of particles around a mean number $\bar{N}$, are very small: [7].
As a result the system A could be moved from contact with the reservoir, to isolation. This procedure would cause no observable effects on $\bar{E}$ and $\bar{N}$. As a result we can choose, equivalently, the microcanonical ensemble to describe A:

$$U = qhf \equiv E$$

Figure 1. Equations $(C.4)$, blue line, and $(C.5)$, red line. The missing information I, as a function of $\Delta N > 0$, $\Delta N \propto \sqrt{N}$, [7]. I have set $q = 10^{12}$

and $N = 10^{10}$. In both equations (C.4) and (C.5), I is linearly dependent on ΔN.

As a result, when a patient[14] "chooses" (its actually fluctuations who "choose") to decrease N (as much or more, than equilibrium dictates) then the information that the patient posses about his/her system, increases. I.e. the missing information I decreases. This comes in contrast to the Second Principle of statistical mechanics [3]: "The probability distribution must be such that reproduces all available information, with the maximum missing information". This means that, when the missing information decreases: "the extra information must be something other than the truth" [3]. I.e. the patient loses touch with reality.

In the opposite case when I increases by $\Delta N > 0$, again due to fluctuations, then the patient observes/comes closer to reality and his/her wishful scenarios, "something other than the truth", are disproved. The patient collapses emotionally.

We must strain here that although fluctuations can make I increase, they do not make $N = \bar{N} \Leftrightarrow \Delta N = 0$. This means that the statistical ensemble and the statistical element f [3], are not accurately followed. More importantly this increase in I, may go against to the second principle of statistical mechanics: The missing information I may be too large to "reproduce all available information" [3].

The above analysis can be summarized as: The patient either loses touch with reality so he/she can make wishful scenarios, or observes only a part of reality which is only enough to disprove these scenarios.

Finally in Appendix D I introduce the concept of average number of QHO's/Structures, $\bar{N}$. I also introduce the dimensionless temperature T' which could also be described as the "psychological" version of temperature. That is because T' is related to the convergence/deviation to/from equilibrium. T' is connected to the average number of Structures of the psyche by (D.5). The plot of (D.5) can be found in Figure 2.

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Appendices.

A   Calculation of the Chemical Potential of a large ES.

We first start from the definition of the multiplicity of an ES. We have from [10]:

$$\Omega = \binom{N+q-1}{q} = \frac{(N+q-1)!}{N!(q-1)!} = \frac{(N+q)!}{N!q!} \cdot \frac{q}{N+q} \qquad (A.1)$$

From Stirling's approximation (2.14) we have:

$$N! \approx N^N e^{-N} \sqrt{2\pi N} \qquad (A.2)$$

Applying (A.2) we get:

$$(N+q)! \approx (N+q)^{(N+q)} e^{-(N+q)} \sqrt{2\pi(N+q)} \qquad (A.3)$$

$$q! \approx q^q e^{-q} \sqrt{2\pi q} \qquad (A.4)$$

From (A.3) and (A.4), equation (A.1) becomes:

$$\Omega \approx \frac{(N+q)^{(N+q)} e^{-(N+q)} \sqrt{2\pi(N+q)}}{N^N e^{-N} \sqrt{2\pi N} \cdot q^q e^{-q} \sqrt{2\pi q}} \cdot \frac{N}{N+q} \Rightarrow$$

$$\Rightarrow \Omega = \left(\frac{q+N}{N}\right)^N \left(\frac{q+N}{q}\right)^q \sqrt{\frac{N}{2\pi q(q+N)}} \qquad (A.5)$$

We now have for entropy S:

$$S = k \ln \Omega \qquad (A.6)$$

Substituting $(A.5)$ to $(A.6)$ and since the logarithm is additive, we have:

$$S = k \ln \left(\frac{q+N}{N}\right)^N + k \ln \left(\frac{q+N}{q}\right)^q + k \ln \sqrt{\frac{N}{2\pi q(q+N)}} =$$

$$= kN \ln \left(\frac{q+N}{N}\right) + kq \ln \left(\frac{q+N}{q}\right) + k \ln \sqrt{\frac{N}{2\pi q(q+N)}} \qquad (A.7)$$

We can clearly notice, that the last logarithmic term in (A.7) is negligible compared to the first two, multiples of N and q, respectively. As such we can omit it. As a result $(A.7)$ becomes:

$$S \approx k \ln \left(\frac{q+N}{N}\right)^N + k \ln \left(\frac{q+N}{q}\right)^q \qquad (A.8)$$

By exponentiating $(A.8)$ we get for the multiplicity $\Omega = S/k$:

$$\Omega \approx \left(\frac{q+N}{N}\right)^N \cdot \left(\frac{q+N}{q}\right)^q \qquad (A.9)$$

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By applying the First Law of Thermodynamics to the ES, we get:

$$dU = TdS - PdV + \mu dN$$

For Entropy:

$$dS = \left(\frac{\partial S}{\partial U}\right)_{N,V} dU + \left(\frac{\partial S}{\partial V}\right)_{U,N} dV + \left(\frac{\partial S}{\partial N}\right)_{U,V} dN \qquad (A.10)$$

For the temperature T:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,V} \qquad (A.11)$$

Substituting $U = qhf$ and $(A.8)$ in $(A.11)$ we have:

$$\frac{hf}{kT} = \ln\left(1 + \frac{N}{q}\right) \Rightarrow$$

$$\Rightarrow \frac{q}{N} = \frac{1}{e^{hf/kT} - 1} \qquad (A.12)$$

The definition of chemical potential μ:

$$\mu \equiv -kT\left(\frac{\partial S}{\partial N}\right)_{U,V} =$$

$$= -kT\ln\left(1 + q/N\right) \qquad (A.13)$$

From $(A.13)$ and $N \gg q$, we have that:

$$\mu = -kT\frac{q}{N} \qquad (A.14)$$

From $(A.14)$ and $(A.12)$, μ becomes:

$$\mu = \frac{kT}{1 - e^{hf/kT}} \qquad (A.15)$$

B   Information content.

Based on [6].

Let us assume that we are faced with a number n of possibilities of the outcome of an experiment. Let us further assume that these possibilities are mutually exclusive, equally probable and complete in the sense that all together imply certainty about the outcome of the experiment. Since there is a set of possibilities, there is missing information about the outcome. We denote this amount of missing information as:

$$I(n) \qquad (B.1)$$

$I(n)$ is a real number and n is an non-negative integer. We naturally expect that the greater the number of possibilities, the greater the amount of missing information. So $I(n)$ is a strictly increasing function:

$$I(n) > I(m), \quad n > m \qquad (B.2)$$

In the special case of one scenario, we expect that there is no information missing, so:

$$I(1) = 0 \qquad (B.3)$$

Let us now assume that we have two independent experiments, one with n possibilities about its outcome and another with m possibilities. Since the experiments are independent the total number of possibilities about their outcome is their Cartesian product:

$$n \cdot m \qquad (B.4)$$

We denote the total missing information as:

$$I(n \cdot m) = I(nm) \qquad (B.5)$$

In this case, the total missing information is the sum of the ignorance about the outcome of the first experiment, plus the ignorance about the second:

$$I(nm) = I(n) + I(m) \qquad (B.6)$$

So far we have chosen the number of possibilities to be a non-negative integer. We will generalize this statement to include the positive rational numbers. We define:

$$I(\frac{n}{m}) \equiv I(n) - I(m) \qquad (B.7)$$

From (B.2) and (B.7) we can extend $I(x)$ to all positive real numbers, x. We can also prove that $I(x)$, is continuous. Finally we can prove, from the above, that:

$$I(x) = k \log(x), \quad k > 0 \qquad (B.9)$$

In the case of a probability distribution, such that the probabilities P_α are non-negative numbers:

$$P_\alpha \geq 0, \quad \alpha = 1, \ldots, N \qquad (B.10)$$

and the probabilities sum to one:

$$\sum_{\alpha=1}^{N} P_\alpha = 1 \qquad (B.11)$$

then

$$I = -k \sum_{\alpha=1}^{N} P_\alpha \log P_\alpha \qquad (B.12)$$

In the case of an isolated system:

$$P_\alpha = P = constant, \quad \forall \alpha = 1, \ldots, N \qquad (B.13)$$

In the case of an isolated Einstein Solid, with multiplicity Ω:

$$I = -k \sum_{\alpha=1}^{\Omega} P_\alpha \log P_\alpha =$$

$$= -k \sum_{\alpha=1}^{\Omega} \frac{1}{\Omega} \log \frac{1}{\Omega} \Rightarrow$$

$$\Rightarrow I = k \log \Omega \qquad (B.14)$$

We will be using by convention $\ln \Omega$, instead of $\log \Omega$. As a result $(B.14)$ becomes:

$$I = \ln \Omega \qquad (B.15)$$

The entropy (divided by Boltzmann's constant (k)) is identical to the missing information, at equilibrium. [11].

C The dependence of the missing information I, on ΔN.

For an Einstein Solid with $q \gg N$, we have from [4]:

$$\Omega \cong \left(\frac{eq}{N}\right)^N \qquad (C.1)$$

So, for N decreasing by $\Delta N > 0$ and $N \gg \Delta N$, since $\Delta N \propto \sqrt{N}$ and from $(B.15)$, we have for the missing information I:

$$I \cong \ln \left(\frac{eq}{N - \Delta N}\right)^{(N-\Delta N)} = (N - \Delta N) \ln \left(\frac{eq}{N - \Delta N}\right) \Rightarrow$$

$$\Rightarrow I \cong N \ln \left(\frac{eq}{N - \Delta N}\right) - (\Delta N) \ln \left(\frac{eq}{N - \Delta N}\right) \qquad (C.2)$$

We have:

$$\ln (N - \Delta N) = \ln N \left(1 - \frac{\Delta N}{N}\right) =$$

$$= \ln \left(1 - \frac{\Delta N}{N}\right) + \ln N \qquad (C.3)$$

Since $\Delta N \propto \sqrt{N}$ and $ln(1 + x) = x$, when $x \to 0$, $(C.3)$ becomes:

$$\ln (N - \Delta N) \cong -\frac{\Delta N}{N} + \ln N \cong$$

$$\cong \ln N$$

As a result $(C.2)$ becomes:

$$I \cong (N - \Delta N)\ln\left(\frac{eq}{N}\right) \Rightarrow$$

$$I \cong N\ln\left(\frac{eq}{N}\right) - \Delta N\ln\left(\frac{eq}{N}\right) \qquad (C.4)$$

In the opposite case, i.e. when N increases by $\Delta N > 0$, equation $(C.4)$ becomes:

$$I \cong N\ln\left(\frac{eq}{N}\right) + \Delta N\ln\left(\frac{eq}{N}\right) \qquad (C.5)$$

Since the entropy S for an Einstein solid is symmetrical under the interchange of N and q, for $N \gg q$, we have from $(C.1)$:

$$\Omega \cong \left(\frac{eN}{q}\right)^q \qquad (C.6)$$

As a result when N increases/decreases by $\Delta N > 0$, I becomes:

$$I \cong q\ln\left(\frac{e(N \pm \Delta N)}{q}\right) =$$

$$= q\left(\ln\frac{e}{q} + \ln N \pm \frac{\Delta N}{N}\right) \Rightarrow$$

$$I \cong \left(q\ln\frac{eN}{q} \pm \frac{q}{N}\Delta N\right) \qquad (C.7)$$

D   Calculation of the average number of QHO's.

We will only examine the case of $N \gg q$. From reference [8], we have:

$$Q_{(\alpha,\beta)} = \sum_{N=0}^{\infty} e^{\alpha N} Z_N(\alpha, \beta) \qquad (3.127)$$

$$\beta = \frac{1}{kT} \qquad \alpha = \frac{\mu}{kT} \qquad (3.128)$$

$$\bar{N} = \left(\frac{\partial \ln Q}{\partial \alpha}\right)_\beta \qquad (3.129)$$

Setting: $T_E = hf/k$ $(D.1)$ we define: $\mu' \equiv \mu/hf$ $(D.2)$ and
$T' \equiv T/T_E$ $(D.3)$. The chemical potential of an ES in the case of $N \gg q$, is given by $(A.14)$ and $(A.15)$.
Z_1 is the partition function of 1 quantum harmonic oscillator, neglecting the ground state energy, and is given by:

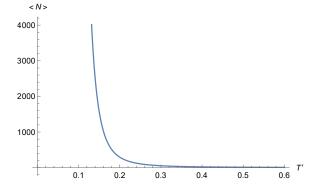$$Z_1 = 1/(1 - e^{-\frac{1}{T'}}) \qquad (D.4)$$

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Considering the oscillators as distinguishable: $Z_N = Z_1^N$. From $(D.2)$, $(D.3)$ and $(D.4)$, (3.127) becomes:

$$Q_{(\alpha,\beta)} = \sum_{N=0}^{\infty} e^{((n/(1-e^{(1/T')}))}/(1 - e^{(-1/T')})^n =$$

$$= 1/(1 - (e^{(1/(1-e^{1/T'}))}/(1 - e^{-1/T'})))$$

And (3.129) becomes:

$$\bar{N} = \frac{\partial \ln Q}{\partial T'} \bigg/ \frac{\partial \alpha}{\partial T'} \qquad (D.5)$$

Figure 2. Equation $(D.5)$. The average number of oscillators $< N >\equiv \bar{N}$, as a function of T'. In the case of $T' = 0.6$, we find $\bar{N} \approx 8$. In the case of $T' = 0.06$, we find $\bar{N} \approx 3.46942 \cdot 10^7$.

Where in the calculations and figures, Wolfram Mathematica has been used.

E   Calculation of the average energy in the grand canonical ensemble.

$$\mu = -T \left(\frac{\partial S}{\partial N} \right)_{U,V} = -kT \ln \left(1 + \frac{q}{N} \right) \qquad (A.13)$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U} \right)_{V,N} = \ln \left(1 + \frac{N}{q} \right) \qquad (A.12)$$

We only consider the case of high temperatures, i.e. $q \gg N$. $\qquad (E.1)$

From $(A.13)$ and $(E.1)$ we have:

$$\mu = kT \ln \left(\frac{N}{q} \right) \qquad (E.2)$$

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From $(A.12)$ and $(E.1)$ we have:

$$T = \frac{q}{N} \qquad (E.3)$$

The canonical partition function for one oscillator, neglecting the ground state energy is:

$$Z_1 = \frac{1}{1 - e^{\frac{-1}{T}}} \qquad (D.4)$$

$$Z_N = Z_1^N \qquad (E.4)$$

$$\alpha = \frac{\mu}{kT} \Rightarrow e^\alpha = \frac{N}{q} = \frac{1}{T} \qquad (3.128)$$
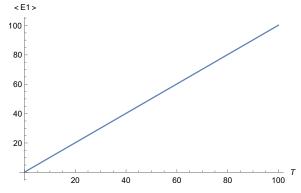
The grand partition function Q is:

$$Q_{(\alpha,\beta)} = \sum_{N=0}^{\infty} e^{\alpha N} Z_N(\alpha, \beta) \qquad (3.127)$$

$$\Rightarrow Q = \frac{1}{1 - (\frac{1}{T})(\frac{1}{1 - e^{-\frac{1}{T}}})} \qquad (E.5)$$

From (3.129), we have for the average energy $< E_1 >$ of one oscillator:

$$< E_1 >= -\frac{\partial \ln Q}{\partial T} / \frac{\partial \beta}{\partial T} \qquad (E.6)$$

Figure 3. Equation $(E.6)$. The average energy for of one oscillator $< E_1 > \equiv \bar{E}_1$, as a function of T.

We see from the plot in the Mathematica file, that the average energy for one oscillator is linearly depending on T and particularly that it is the angle bisector. For N oscillators we multiply by N. It is:

$$\bar{E} = NThf = N\frac{q}{N}hf = qhf. \qquad (E.7)$$

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This agrees with result for the internal energy of the ES in the microcanonical ensemble, neglecting the ground state energy of each oscillator. This means that we can transfer the ES after it has interacted (grand canonical ensemble) to isolation (microcanonical ensemble). The number of energy quanta (Values in AP) remains practically the same in these two ensembles, in the high temperature limit.

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