QUANTUM PERMUTATIONS AND QUANTUM REFLECTIONS

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ABSTRACT. The permutation group S_N has a free analogue S_N^+, which is non-classical and infinite at $N \geq 4$. We review here the known basic facts on S_N^+, with emphasis on algebraic and probabilistic aspects. We discuss as well the structure of the closed subgroups $G \subset S_N^+$, with particular attention to the quantum reflection groups.

CONTENTS

INTRODUCTION

The compact quantum groups were introduced by Woronowicz in [99], [100]. The axioms are very simple, obtained by looking at the compact Lie groups $G \subset U_N$, and removing the commutativity assumption on the algebras $C(G)$. The basic examples are the compact Lie groups, $G \subset U_N$, as well as the abstract duals $G = \widehat{\Gamma}$ of the finitely generated discrete groups $F_N \to \Gamma$. The general theory is efficient as well, with an existence result for the Haar measure, a Peter-Weyl theory, and a Tannakian duality.

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Woronowicz's axioms and theory allow the construction of many interesting examples of compact quantum groups. A quite standard procedure, called "liberation and twisting", is that of starting with a compact Lie group $G \subset U_N$, and replacing the commutation relations $ab = ba$ between the standard coordinates $u_{ij} : G \to \mathbb{C}$ with other relations, such as $ab = \pm ba$, $abc = cba$, $abc = \pm cba$ and so on, or with nothing at all.

The liberation and twisting procedure applies well to O_N, U_N and other classical Lie groups, and allows the construction of liberations O_N^+, U_N^+, twists $\bar{O}_N, \bar{U}_N$, half-liberations O_N^*, U_N^*, twisted half-liberations $\bar{O}_N^*, \bar{U}_N^*$, and so on. All these quantum groups are quite interesting, from a general noncommutative geometry perspective.

In the discrete case, where the compact Lie group $G \subset U_N$ to be liberated is finite, the situation is a bit more rigid, but in many interesting cases the liberation procedure works, and we have a free quantum group $G^+ \subset U_N^+$. Quite remarkably, these latter quantum groups G^+, while being trivially non-finite, are somehow of "continuous nature".

In relation with these latter quantum groups, the central object here is the quantum permutation group S_N^+, constructed by Wang in [93]. There has been a lot of work on S_N^+ and its subgroups, and our purpose here will be that of explaining this material.

In order to construct the quantum group S_N^+, let us look first at S_N. We can regard S_N as being the group of permutation matrices, $S_N \subset O_N$. This is quite useful, because we obtain in this way N^2 coordinate functions on S_N, which are given by:

$$u_{ij}(\sigma) = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

Now observe that these N^2 functions separate the points of S_N. Thus, we obtain by the Stone-Weierstrass theorem that we have:

$$C(S_N) = \left\langle (u_{ij})_{i,j=1,\ldots,N} \right\rangle$$

Another obvious remark about the coordinates u_{ij} is that these are certain characteristic functions, and so are projections ($p^2 = p = p^*$) in the operator algebra sense. Moreover, the matrix $u = (u_{ij})$ that they form is "magic", in the sense that these projections sum up to 1, on each row and each column. By combining this with the previous remark, and with a bit more work, we are led to a presentation result, as follows:

$$C(S_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N} \Big| u = \text{magic}\right)$$

Here by C^*_{comm} we mean universal commutative C^*-algebra, and all this follows from the above remarks, and from the Gelfand theorem.

Quite remarkably, the group operations m, u, i, or rather their functional analytic transposes Δ, ε, S, can be recaptured in a very simple way, as follows:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

In short, what we have here is a fully satisfactory functional analytic description of S_N.

We can now go ahead, and construct S_N^+. The definition is very simple, just by lifting the commutativity property from the above picture of S_N, as follows:

$$C(S_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u = \text{magic}\right)$$

As for the quantum group structure, this comes by definition from maps Δ, ε, S, which can be constructed exactly as in the classical case, as follows:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

Moreover, once this done, we can talk about subgroups $G \subset S_N^+$, in the obvious way, and a whole theory of "quantum permutation groups" can be developed.

More generally, we can talk about quantum symmetry groups S_X^+ of finite noncommutative spaces X. Let us recall indeed that, according to the general operator algebra philosophy, such a finite space X appears as dual of a finite dimensional algebra A, which in turn can be shown to be a direct sum of matrix algebras:

$$C(X) = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

The above construction of S_N^+ can be generalized in this setting, and we obtain a quantum symmetry group of X. If we denote by $N = \sum_i N_i^2$ the cardinality of our finite quantum space X, the corresponding quantum symmetry group S_X^+ appears as follows:

$$S_X^+ \subset U_N^+ \quad , \quad N = |X|$$

This generalization is quite interesting, and often provides good explanations for results regarding the quantum groups S_N^+ themselves. As an example here, it is known that S_N^+ with $N \geq 4$ has the same fusion rules as SO_3. But this is best understood in the quantum symmetry group setting, via a pair of results, stating that: (1) the fusion rules for S_X^+ with $|X| \geq 4$ are independent of X, and (2) for $X = M_2$ we have $S_X^+ = SO_3$.

Summarizing, we are interested here in S_N^+ and its closed subgroups $G \subset S_N^+$, and more generally in S_X^+ with $|X| < \infty$, and its closed subgroups $G \subset S_X^+$. Moreover, we are interested in both basic and advanced theory, in algebraic and analytic aspects, and in generalities and applications. We will discuss here all this material.

The present text is organized as follows: in 1-3 we review the basic theory of quantum permutations, in 4-6 we discuss more advanced aspects, in 7-9 we discuss the quantum reflections, and in 10-12 we discuss arbitrary quantum permutation groups.

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1. Quantum groups

Generally speaking, a quantum group is a "noncommutative space" with group type structure. The quantum permutation groups, and other quantum groups that we will be interested in here, and not finite, but rather compact. So, in order to introduce them, we need a good formalism of "noncommutative compact spaces".

There are several such formalisms, and a particularly simple and beautiful and powerful one, which is exactly what we need for our quantum permutation group purposes, is provided by the C^*-algebra theory. The result that we will need is as follows:

Theorem 1.1. *Consider the category of C^*-algebras, meaning the category of complex algebras with unit A, having a norm $||.||$ making them Banach algebras, and an involution $*$, related to the norm by the formula $||aa^*|| = ||a||^2$, for any $a \in A$.*

1. *Given a compact space X, the algebra $C(X) = \{f : X \to \mathbb{C} \text{ continuous}\}$ is a C^*-algebra, with norm $||f|| = \sup_{x\in X} |f(x)|$ and involution $f^*(x) = \overline{f(x)}$.*
2. *Any commutative C^*-algebra is of the form $C(X)$, with the space $X = Spec(A)$ appearing as the space of characters $\chi : A \to \mathbb{C}$.*
3. *Given a Hilbert space H, the algebra $B(H) = \{T : H \to H \text{ linear, bounded}\}$ is a C^*-algebra, with $||T|| = \sup_{||x||=1} ||Tx||$, and $< Tx, y >=< x, T^*y >$.*
4. *More generally, and closed $*$-subalgebra $A \subset B(H)$ is a C^*-algebra. Conversely, any C^*-algebra appears in this way, for a certain Hilbert space H.*

In view of this, given an arbitrary C^*-algebra A, we agree to write $A = C(X)$, and call X a noncommutative compact space. Also, in the case $A \subset B(H)$, the weak closure of A, with respect to $T_n \to T \iff T_n x \to Tx, \forall x \in H$, will be denoted $L^\infty(X)$.

Proof. All this is not very complicated, the idea being as follows:

(1) This is clear from definitions. Indeed, it is well-known, and elementary to prove, that $C(X)$ is complete with respect to the sup norm. As for the formula $||ff^*|| = ||f||^2$, this is something trivial, because on both sides we obtain $\sup_{x\in X} |f(x)|^2$.

(2) Given a commutative C^*-algebra A, the character space $X = \{\chi : A \to \mathbb{C}\}$ is compact, and we have an evaluation morphism $ev : A \to C(X)$. The tricky point, which follows from basic spectral theory, is to prove that ev is indeed isometric.

(3) This is something well-known, and elementary. Indeed, given a Cauchy sequence $\{T_n\}$, we can set $Tx = \lim_{n\to\infty} T_n x$ for any $x \in H$, and we have then $T_n \to T$. As for the formula $||TT^*|| = ||T||^2$, this can be proved by double inequality.

(4) In the commutative case, $A = C(X)$, we can set $H = L^2(X)$, and we have then an embedding $A \subset B(H)$, given by $a \to (b \to ab)$. In general the idea is the same, the only technical point being the construction of an integration functional $\int : A \to \mathbb{C}$.

Regarding the last part, this is something rather informal, and at the beginner level, because there are some functoriality issues with the constructions there . We will be back to this, with full details and explanations, in the compact quantum group case. □

All the above was of course quite brief, but the complete proof of all this, which takes around 10-15 pages, can be found in any operator algebra book.

We are ready now to introduce the compact quantum groups. The axioms here, due to Woronowicz [99], and slightly modified for our purposes, are as follows:

Definition 1.2. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u \in M_N(A)$ whose coefficients generate A, such that the formulae*

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}^*$$

define morphisms of C^*-algebras $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{C}$, $S : A \to A^{opp}$.

The morphisms Δ, ε, S are called comultiplication, counit and antipode. Observe that, since we have $A =< u_{ij} >$, if these maps exist, they are unique. We will see in a moment that these maps satisfy the usual Hopf algebra axioms, along with $S^2 = id$.

As another comment, the tensor product used in the definition of Δ can be any C^*-algebra tensor product. In order to get rid of redundancies, coming from this and from amenability issues, we will divide anyway anything by an equivalence relation.

Finally, as a last technical comment, the use of the opposite algebra in the definition of S is really needed, and this for covering the group duals. This is something standard in Hopf algebras, which follows by examining the proof of Proposition 1.3 below.

Our claim now is that any Woronowicz algebra A can be written as follows, with G being a compact matrix quantum group, Γ being a finitely generated discrete quantum group, and with these quantum groups being dual to each other:

$$A = C(G) = C^*(\Gamma)$$

We say that A is cocommutative when $\Sigma\Delta = \Delta$, where $\Sigma(a \otimes b) = b \otimes a$ is the flip. We have then the following result, which basically justifies our claim:

Proposition 1.3. *The following are Woronowicz algebras:*

(1) *$C(G)$, with $G \subset U_N$ compact Lie group. Here the structural maps are:*

$$\begin{aligned}\Delta(\varphi) &= (g,h) \to \varphi(gh)\\ \varepsilon(\varphi) &= \varphi(1)\\ S(\varphi) &= g \to \varphi(g^{-1})\end{aligned}$$

(2) *$C^*(\Gamma)$, with $F_N \to \Gamma$ finitely generated group. Here the structural maps are:*

$$\begin{aligned}\Delta(g) &= g \otimes g\\ \varepsilon(g) &= 1\\ S(g) &= g^{-1}\end{aligned}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

Proof. In both cases, we have to exhibit a certain matrix u. For the first assertion, we can use the matrix $u=(u_{ij})$ formed by matrix coordinates of G, given by:

$$g=\begin{pmatrix}u_{11}(g)&\ldots&u_{1N}(g)\\ \vdots&&\vdots\\ u_{N1}(g)&\ldots&u_{NN}(g)\end{pmatrix}$$

For the second assertion, we can use the diagonal matrix formed by generators:

$$u=\begin{pmatrix}g_1&&0\\ &\ddots&\\ 0&&g_N\end{pmatrix}$$

Finally, regarding the last assertion, in the commutative case this follows from the Gelfand result, Theorem 1.1 (2) above. In the cocommutative case this is something more complicated, requiring as well a norm discussion. We will be back to this. □

In order to fully justify our quantum group claims, we will need as well:

Proposition 1.4. *Assuming that $G\subset U_N$ is abelian, we have an identification of Woronowicz algebras $C(G)=C^*(\Gamma)$, with Γ being the Pontrjagin dual of G:*

$$\Gamma=\{\chi:G\to\mathbb T\}$$

Conversely, assuming that $F_N\to\Gamma$ is abelian, we have an identification of Woronowicz algebras $C^*(\Gamma)=C(G)$, with G being the Pontrjagin dual of Γ:

$$G=\{\chi:\Gamma\to\mathbb T\}$$

Thus, the Woronowicz algebras which are both commutative and cocommutative are exactly those of type $A=C(G)=C^*(\Gamma)$, with G,Γ being abelian, in Pontrjagin duality.

Proof. All this follows from Gelfand duality, Theorem 1.1 (2) above, because the characters of a group algebra are in correspondence with the characters of the group. □

We can now formulate the following definition, complementing Definition 1.2:

Definition 1.5. *We make the following conventions:*

(1) *We write $A=C(G)=C^*(\Gamma)$, and call G compact quantum group, and Γ discrete quantum group. Also, we call G,Γ dual to each other, and write $G=\widehat{\Gamma},\Gamma=\widehat{G}$.*
(2) *We agree to identify $(A,u)=(B,v)$ when we have an isomorphism of $*$-algebras $<u_{ij}>\simeq<v_{ij}>$ mapping coordinates to coordinates, $u_{ij}\to v_{ij}$.*

In this definition (1) comes from the above discussion, Proposition 1.3 and Proposition 1.4 above, and fully justifies our axioms, terminology and formalism in general. As for (2), this is something which is needed, in order to avoid troubles with topological tensor products, amenability, and other analytic issues. We will be back to this.

Let us discuss now some tools for studying the Woronowicz algebras, and the underlying compact and discrete quantum groups. First, we have the following result:

Proposition 1.6. *Let (A,u) be a Woronowicz algebra.*

(1) *Δ,ε satisfy the usual axioms for a comultiplication and a counit, namely:*

$$\begin{aligned}(\Delta\otimes id)\Delta &= (id\otimes\Delta)\Delta\\(\varepsilon\otimes id)\Delta &= (id\otimes\varepsilon)\Delta=id\end{aligned}$$

(2) *S satisfies the antipode axiom, on the $*$-algebra generated by entries of u:*

$$m(S\otimes id)\Delta=m(id\otimes S)\Delta=\varepsilon(.)1$$

(3) *In addition, the square of the antipode is the identity, $S^2=id$.*

Proof. As a first observation, the result holds in the commutative case, $A=C(G)$ with $G\subset U_N$. Indeed, here we know from Proposition 1.3 that Δ,ε,S appear as functional analytic transposes of the multiplication, unit and inverse maps m,u,i:

$$\Delta=m^T\quad,\quad\varepsilon=u^T\quad,\quad S=i^T$$

With these remark in hand, the various conditions in the statement on Δ,ε,S come by transposition from the group axioms satisfied by m,u,i, namely:

$$\begin{aligned}m(m\times id) &= m(id\times m)\\m(u\times id) &= m(id\times u)=id\\m(i\times id)\delta &= m(id\times i)\delta=1\end{aligned}$$

Observe that the condition $S^2=id$ is satisfied too, coming by transposition from the formula $i^2=id$, which corresponds to the following formula, for group elements:

$$(g^{-1})^{-1}=g$$

The result holds as well in the cocommutative case, $A=C^*(\Gamma)$ with $F_N\to\Gamma$, trivially. In general now, the two comultiplication axioms follow from:

$$\begin{aligned}(\Delta\otimes id)\Delta(u_{ij}) &= (id\otimes\Delta)\Delta(u_{ij})=\sum_{kl}u_{ik}\otimes u_{kl}\otimes u_{lj}\\(\varepsilon\otimes id)\Delta(u_{ij}) &= (id\otimes\varepsilon)\Delta(u_{ij})=u_{ij}\end{aligned}$$

As for the antipode axiom, the verification here is similar, as follows:

$$\begin{aligned}m(S\otimes id)\Delta(u_{ij}) &= \sum_k u_{ki}^*u_{kj}=(u^*u)_{ij}=\delta_{ij}\\m(id\otimes S)\Delta(u_{ij}) &= \sum_k u_{ik}u_{jk}^*=(uu^*)_{ij}=\delta_{ij}\end{aligned}$$

Finally, we have $S^2(u_{ij})=u_{ij}$, and so $S^2=id$ everywhere, as claimed. □

As a conclusion to this, we can do as many things with Δ, ε, S in the quantum group setting as we can do with m, u, i in the group setting. The only thing is that we have to talk about algebras of functions instead of spaces, and transpose everywhere.

In the compact Lie group case, in order to reach to more advanced results, we have to do either representation theory, or Lie algebras. The situation is quite similar for quantum groups: we won't get very far with Δ, ε, S, and we need more advanced tools.

The quantum groups G that we are interested in are of matrix type, and the somewhat philosophical question is whether they are of Lie type as well. The anwser here is rather "no", or to be more precise, it is "no" at the beginner level. There is absolutely no way of linearizing G, by some simple geometric method. We will be back to this.

In short, we are left with representation theory, as a main potential tool. Fortunately, things here are very satisfactory, and explaining all this will be our next purpose.

Let us start our study with the following definition:

Definition 1.7. *Given a Woronowicz algebra A, we call corepresentation of it any unitary matrix $v \in M_n(A)$ satisfying the same conditions are those satisfied by u, namely:*

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

We also say that v is a representation of the underlying compact quantum group G, and a corepresentation of the underlying discrete quantum group Γ.

In the commutative case, $A = C(G)$ with $G \subset U_N$, we obtain in this way the finite dimensional unitary smooth representations $v : G \to U_n$, as follows:

$$v(g) = \begin{pmatrix} v_{11}(g) & \ldots & v_{1n}(g) \\ \vdots & & \vdots \\ v_{n1}(g) & \ldots & v_{nn}(g) \end{pmatrix}$$

In the cocommutative case, $A = C^*(\Gamma)$ with $F_N \to \Gamma$, we will see in a moment that we obtain in this way the formal sums of elements of Γ, possibly rotated by a unitary.

As a first result regarding the representations, we have:

Proposition 1.8. *The corepresentations are subject to the following operations:*

1. *Making sums, $v + w = diag(v, w)$.*
2. *Making tensor products, $(v \otimes w)_{ia,jb} = v_{ij} w_{ab}$.*
3. *Taking conjugates, $(\bar{v})_{ij} = v_{ij}^*$.*
4. *Rotating by a unitary, $v \to UvU^*$.*

Proof. The fact that $v+w$ is unitary is clear. Regarding now $v\otimes w$, this can be written in leg-numbering notation as $v\otimes w=v_{13}w_{23}$, so its unitarity is clear as well.

In order to check that $\bar{v}$ is unitary, we can use the antipode. Indeed, by regarding the antipode as an antimultiplicative map $S:A\to A$, we have, as desired:

$$(\bar{v}v^t)_{ij}=\sum_k v_{ik}^*v_{jk}=\sum_k S(v_{kj}^*v_{ki})=S((v^*v)_{ji})=\delta_{ij}$$

$$(v^t\bar{v})_{ij}=\sum_k v_{ki}v_{kj}^*=\sum_k S(v_{jk}v_{ik}^*)=S((vv^*)_{ji})=\delta_{ij}$$

Finally, the fact that UvU^* is unitary is clear. As for the verification of the comultiplicativity axioms, involving Δ,ε,S, this is elementary and routine, in all cases. □

As a consequence of the above result, we can formulate:

Definition 1.9. *We denote by $u^{\otimes k}$, with $k=\circ\bullet\bullet\circ\ldots$ being a colored integer, the various tensor products between $u,\bar{u}$, indexed according to the rules*

$$u^{\otimes\emptyset}=1\quad,\quad u^{\otimes\circ}=u\quad,\quad u^{\otimes\bullet}=\bar{u}$$

and multiplicativity, $u^{\otimes kl}=u^{\otimes k}\otimes u^{\otimes l}$, and call them Peter-Weyl corepresentations.

Here are a few examples of such corepresentations, namely those coming from the colored integers of length 2, to be often used in what follows:

$$u^{\otimes\circ\circ}=u\otimes u\quad,\quad u^{\otimes\bullet\bullet}=\bar{u}\otimes\bar{u}$$

$$u^{\otimes\circ\bullet}=u\otimes\bar{u}\quad,\quad u^{\otimes\bullet\circ}=\bar{u}\otimes u$$

We will be back to these corepresentations later on, the idea being that any irreducible corepresentation appears inside such a Peter-Weyl corepresentation $u^{\otimes k}$.

In order to do representation theory, we first need to know how to integrate over G. And we have here the following key result, due to Woronowicz [99]:

Theorem 1.10. *Any Woronowicz algebra $A=C(G)$ has a unique Haar integration,*

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

which can be constructed by starting with any faithful positive form $\varphi\in A^*$, and setting

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation $v\in M_n(\mathbb{C})\otimes A$ we have

$$\left(id\otimes\int_G\right)v=P$$

where P is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb{C}^n|v\xi=\xi\}$.

Proof. Following [99], this can be done in 3 steps, as follows:

(1) Given $\varphi \in A^*$, our claim is that the following limit converges, for any $a \in A$:

$$\int_\varphi a = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \varphi^{*k}(a)$$

Indeed, by linearity we can assume that a is the coefficient of corepresentation, $a = (\tau \otimes id)v$. But in this case, an elementary computation shows that we have the following formula, where P_φ is the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi)v$:

$$\left(id \otimes \int_\varphi\right) v = P_\varphi$$

(2) Since $v\xi = \xi$ implies $[(id \otimes \varphi)v]\xi = \xi$, we have $P_\varphi \geq P$, where P is the orthogonal projection onto the space $Fix(v) = \{\xi \in \mathbb{C}^n | v\xi = \xi\}$. The point now is that when $\varphi \in A^*$ is faithful, by using a positivity trick, one can prove that we have $P_\varphi = P$. Thus our linear form $\int_\varphi$ is independent of φ, and is given on coefficients $a = (\tau \otimes id)v$ by:

$$\left(id \otimes \int_\varphi\right) v = P$$

(3) With the above formula in hand, the left and right invariance of $\int_G = \int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [99]. □

In the classical case, where $A = C(G)$ with $G \subset U_N$, we obtain in this way the Haar integration over G. As for the group dual case, where $A = C^*(\Gamma)$ with $F_N \to \Gamma$, here the integration is given by the following formula, on the group elements $g \in \Gamma$:

$$\int_{\widehat{\Gamma}} g = \delta_{g1}$$

We can now develop a Peter-Weyl type theory for the corepresentations. We will need a number of straightforward definitions and results. Let us begin with:

Definition 1.11. *Given two corepresentations $v \in M_n(A), w \in M_m(A)$, we set*

$$Hom(v,w) = \left\{T \in M_{m\times n}(\mathbb{C})\Big| Tv = wT\right\}$$

and we use the following conventions:

1. *We use the notations $Fix(v) = Hom(1,v)$, and $End(v) = Hom(v,v)$.*
2. *We write $v \sim w$ when $Hom(v,w)$ contains an invertible element.*
3. *We say that v is irreducible, and write $v \in Irr(G)$, when $End(v) = \mathbb{C}1$.*

In the classical case $A = C(G)$ we obtain the usual notions concerning the representations. Observe also that in the group dual case we have $g \sim h$ when $g = h$. Finally, observe that $v \sim w$ means that v, w are conjugated by an invertible matrix.

Here are a few basic results, regarding the above Hom spaces:

Proposition 1.12. *We have the following results:*

(1) $T \in Hom(u, v), S \in Hom(v, w) \implies ST \in Hom(u, w)$.
(2) $S \in Hom(p, q), T \in Hom(v, w) \implies S \otimes T \in Hom(p \otimes v, q \otimes w)$.
(3) $T \in Hom(v, w) \implies T^* \in Hom(w, v)$.

In other words, the Hom spaces form a tensor $*$-category.

Proof. The proofs are all elementary, as follows:

(1) By using our assumptions $Tu = vT$ and $Sv = Ws$ we obtain, as desired:

$$STu = SvT = wST$$

(2) Assume indeed that we have $Sp = qS$ and $Tv = wT$. With tensor product notations, as in the proof of Proposition 1.8 above, we have:

$$(S \otimes T)(p \otimes v) = S_1T_2p_{13}v_{23} = (Sp)_{13}(Tv)_{23}$$

$$(q \otimes w)(S \otimes T) = q_{13}w_{23}S_1T_2 = (qS)_{13}(wT)_{23}$$

The quantities on the right being equal, this gives the result.

(3) By conjugating, and then using the unitarity of v, w, we obtain, as desired:

$$\begin{aligned} Tv = wT &\implies v^*T^* = T^*w^* \\ &\implies vv^*T^*w = vT^*w^*w \\ &\implies T^*w = vT^* \end{aligned}$$

Finally, the last assertion follows from definitions, and from the obvious fact that, in addition to (1,2,3) above, the Hom spaces are linear spaces, and contain the units. □

Finally, in order to formulate the Peter-Weyl results, we will need as well:

Proposition 1.13. *The characters of the corepresentations, given by*

$$\chi_v = \sum_i v_{ii}$$

behave as follows, in respect to the various operations:

$$\chi_{v+w} = \chi_v + \chi_w \quad , \quad \chi_{v \otimes w} = \chi_v \chi_w \quad , \quad \chi_{\bar{v}} = \chi_v^*$$

In addition, given two equivalent corepresentations, $v \sim w$, we have $\chi_v = \chi_w$.

Proof. The three formulae in the statement are all clear from definitions. Regarding now the last assertion, assuming that we have $v = T^{-1}wT$, we obtain:

$$\chi_v = Tr(v) = Tr(T^{-1}wT) = Tr(w) = \chi_w$$

We conclude that $v \sim w$ implies $\chi_v = \chi_w$, as claimed. □

Consider the dense $*$-subalgebra $\mathcal{A}\subset A$ generated by the coefficients of the fundamental corepresentation u, and endow it with the following scalar product:

$$<a,b>=\int_G ab^*$$

With this convention, we have the following fundamental result, from [99]:

Theorem 1.14. *We have the following Peter-Weyl type results:*

(1) *Any corepresentation decomposes as a sum of irreducible corepresentations.*
(2) *Each irreducible corepresentation appears inside a certain $u^{\otimes k}$.*
(3) $\mathcal{A}=\bigoplus_{v\in Irr(A)}M_{\dim(v)}(\mathbb{C})$*, the summands being pairwise orthogonal.*
(4) *The characters of irreducible corepresentations form an orthonormal system.*

Proof. All these results are from [99], the idea being as follows:

(1) Given a corepresentation $v\in M_n(A)$, consider its interwiner algebra:

$$End(v)=\{T\in M_n(\mathbb{C})|Tv=vT\}$$

We know from Proposition 1.12 that this is a finite dimensional C^*-algebra, and due to this fact, it is standard to work out a decomposition as follows:

$$End(v)=M_{n_1}(\mathbb{C})\oplus\ldots\oplus M_{n_r}(\mathbb{C})$$

To be more precise, such a decomposition appears by writing the unit of our algebra as a sum of minimal projections, as follows, and then working out the details:

$$1=p_1+\ldots+p_r$$

But this decomposition allows us to define subcorepresentations $v_i\subset v$, which are irreducible, so we obtain a decomposition of type $v=v_1+\ldots+v_r$, as desired.

(2) Consider indeed the Peter-Weyl corepresentations, $u^{\otimes k}$ with k colored integer, defined by $u^{\otimes\emptyset}=1$, $u^{\otimes\circ}=u$, $u^{\otimes\bullet}=\bar{u}$ and multiplicativity. The coefficients of these corepresentations span the dense algebra $\mathcal{A}$, and by using (1), this gives the result.

(3) Here the direct sum decomposition, which is technically a $*$-coalgebra isomorphism, follows from (2). As for the second assertion, this follows from the fact that $(id\otimes\int_G)v$ is the orthogonal projection P_v onto the space $Fix(v)$, for any corepresentation v.

(4) Let us define indeed the character of $v\in M_n(A)$ to be the matrix trace, $\chi_v=Tr(v)$. Since this character is a coefficient of v, the orthogonality assertion follows from (3). As for the norm 1 claim, this follows once again from $(id\otimes\int_G)v=P_v$. □

We refer to [99] for full details on all the above, and for some applications as well. Let us just record here the fact that in the cocommutative case, we obtain from (4) that the irreducible corepresentations must be all 1-dimensional, and so that we must have $A=C^*(\Gamma)$ for some discrete group Γ, as mentioned in Proposition 1.3 above.

As a first consequence of the above results, once again by basically following [99], we have the following result, dealing with functional analysis aspects:

Theorem 1.15. *Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of A_{full} is faithful.*
2. *The projection map $A_{full} \to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red}.*
4. *We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

Proof. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial, and conversely, a counit map $\varepsilon : A_{red} \to \mathbb{C}$ produces an isomorphism $A_{red} \to A_{full}$, via a formula of type $(\varepsilon \otimes id)\Phi$. See [99].

(3) $\iff$ (4) Here $\implies$ is clear, coming from $\varepsilon(N - Re(\chi(u))) = 0$, and the converse can be proved by doing some functional analysis. Once again, we refer here to [99]. $\square$

With these results in hand, we can formulate, as a refinement of Definition 1.5:

Definition 1.16. *Given a Woronowicz algebra A, we formally write as before*

$$A = C(G) = C^*(\Gamma)$$

and by GNS construction with respect to the Haar functional, we write as well

$$A'' = L^\infty(G) = L(\Gamma)$$

with G being a compact quantum group, and Γ being a discrete quantum group.

In other words, by using the Haar functional we construct the Hilbert space $H = L^2(A)$, then by using the GNS construction from Theorem 1.1 (4) we obtain an embedding $A \subset B(H)$, so we can talk afterwards about the von Neumann algebra $A'' \subset B(H)$. With all this taken up to the standard equivalence relation for Woronowicz algebras, i.e. isomorphism of $*$-algebras of coordinates, which amounts in identifying $A_{full} = A_{red}$.

As in the discrete group case, the most interesting criterion for amenability is the Kesten one, from Theorem 1.15 (4). This leads us into computing character laws:

Proposition 1.17. *Given a Woronowicz algebra (A, u), consider its main character:*

$$\chi = \sum_i u_{ii}$$

1. *The moments of χ are the numbers $M_k = \dim(Fix(u^{\otimes k}))$.*
2. *When $u \sim \bar{u}$ the law of χ is a real measure, supported by $\sigma(\chi)$.*
3. *The notion of coamenability of A depends only on $law(\chi)$.*

Proof. All this is elementary, the idea being as follows:

(1) This follows indeed from Peter-Weyl theory.

(2) When $u \sim \bar{u}$ we have $\chi = \chi^*$, which gives the result.

(3) This follows from from Theorem 1.15 (4), and from (2) applied to $u + \bar{u}$. $\square$

All this is quite interesting, because it tells us that, regardless on whether we want to understand the representation theory of our compact quantum group G, or the analytic aspects of its discrete dual Γ, we must compute the fixed point spaces $Fix(u^{\otimes k})$.

The computation of these spaces is a delicate algebra problem, related to results of Schur-Weyl, Brauer and Tannaka. In order to get started, the first idea is to replace the series of fixed point spaces $F_k = Fix(u^{\otimes k})$ by the double series of Hom spaces:

$$C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$$

Indeed, by Frobenius, computing $\{F_k\}$ is the same as computing $\{C_{kl}\}$. But computing $\{C_{kl}\}$ is simpler than computing $\{F_k\}$, because these spaces form a category. We can use here the following version of Tannakian duality, due to Woronowicz [100]:

Theorem 1.18. *The following operations are inverse to each other:*

1. *The construction $A \to C$, which associates to any Woronowicz algebra A the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
2. *The construction $C \to A$, which associates to any tensor category C the Woronowicz algebra A presented by the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

Proof. This is something quite deep, going back to [100] in a slightly different form, and to [68] in the simplified form presented above. The idea is as follows:

(1) We have indeed a construction $A \to C$ as above, whose output is a tensor C^*-subcategory with duals of the tensor C^*-category of Hilbert spaces.

(2) We have as well a construction $C \to A$ as above, simply by dividing the free $*$-algebra on N^2 variables by the relations in the statement.

Regarding now the bijection claim, some elementary algebra shows that $C = C_{A_C}$ implies $A = A_{C_A}$, and also that $C \subset C_{A_C}$ is automatic. Thus we are left with proving $C_{A_C} \subset C$. But this latter inclusion can be proved indeed, by doing a lot of algebra, and using von Neumann's bicommutant theorem, in finite dimensions. See [68]. $\square$

All the above was of course quite short. We generally recommend here Woronowicz's papers [99], [100], under the assumption $S^2 = id$, which simplifies many things. We will be back to the above results, and in particular to Tannakian duality, with concrete illustrations and applications, later on, once we will have some examples to study.

As a last piece of general theory, let us discuss fusion rules, and Cayley graphs. In the group dual case, $A = C^*(\Gamma)$ with $F_N \to \Gamma$, the elements of Γ can be recovered as being the irreducible corepresentations of A, which are all 1-dimensional, and their product and inversion corresponds to the tensor product and conjugation of corepresentations.

This suggests that in the general case, where $A = C^*(\Gamma)$ with Γ discrete quantum group, the structure of Γ, or rather its combinatorics, comes from the fusion rules on $Irr(A)$. This is indeed the case, and as a basic result here, we have:

Proposition 1.19. *Let (A, u) be a Woronowicz algebra, and assume, by enlarging if necessary u, that we have $1 \in u = \bar{u}$. The formula*

$$d(v, w) = \min\left\{k \in \mathbb{N}\Big|1 \subset \bar{v} \otimes w \otimes u^{\otimes k}\right\}$$

defines then a distance on $Irr(A)$, which coincides with the geodesic distance on the associated Cayley graph. Moreover, the moments of the main character,

$$\int_G \chi^k = \dim\left(Fix(u^{\otimes k})\right)$$

count the loops based at 1, having lenght k, on the corresponding Cayley graph.

Proof. Observation first the result holds indeed in the group dual case, where $A = C^*(\Gamma)$ with $\Gamma =< S >$ being a finitely generated discrete group. Indeed, our normalization condition $1 \in u = \bar{u}$ means that the generating set must satisfy $1 \in S = S^{-1}$. But this is precisely the usual normalization condition for the discrete groups.

In general now, the fact that the lengths are finite follows from Peter-Weyl theory. The symmetry axiom is clear as well, and the triangle inequality is elementary to establish as well. Finally, the last assertion, regarding the moments, is elementary as well. □

It is possible to build on the above result, notably with a theory of growth for the discrete quantum groups. We will discuss this later, once we will have examples.

Let us discuss now the basic examples of compact and discrete quantum groups. We know so far that the compact quantum groups include the usual compact Lie groups, $G \subset U_N$, and the abstract duals $G = \widehat{\Gamma}$ of the finitely generated groups $F_N \to \Gamma$.

Equivalently, we know that the discrete quantum groups include the finitely generated groups $F_N \to \Gamma$, and the abstract duals $\Gamma = \widehat{G}$ of the compact Lie groups, $G \subset U_N$.

We can combine these examples by performing basic operations, as follows:

Proposition 1.20. *The class of Woronowicz algebras is stable under taking:*

1. *Tensor products, $A = A' \otimes A''$, with $u = u' + u''$. At the quantum group level we obtain usual products, $G = G' \times G''$ and $\Gamma = \Gamma' \times \Gamma''$.*
2. *Free products, $A = A' * A''$, with $u = u' + u''$. At the quantum group level we obtain dual free products $G = G' \,\hat{*}\, G''$ and free products $\Gamma = \Gamma' * \Gamma''$.*

Proof. Everything here is clear from definitions. In addition to this, let us mention as well that we have the formulae $\int_{A' \otimes A''} = \int_{A'} \otimes \int_{A''}$ and $\int_{A' * A''} = \int_{A'} * \int_{A''}$, and that the corepresentations of the products can be explicitely computed. See [92]. □

Here are some further basic operations, once again from [92]:

Proposition 1.21. *The class of Woronowicz algebras is stable under taking:*

1. *Subalgebras $A' =< u'_{ij} >\subset A$, with u' being a corepresentation of A. At the quantum group level we obtain quotients $G \to G'$ and subgroups $\Gamma' \subset \Gamma$.*
2. *Quotients $A \to A' = A/I$, with I being a Hopf ideal, $\Delta(I) \subset A \otimes I + I \otimes A$. At the quantum group level we obtain subgroups $G' \subset G$ and quotients $\Gamma \to \Gamma'$.*

Proof. Once again, everything is clear, and we have as well some straightforward supplementary results, regarding integration and corepresentations. See [92]. □

Finally, here are two more operations, which are of key importance:

Proposition 1.22. *The class of Woronowicz algebras is stable under taking:*

1. *Projective versions, $PA =< w_{ia,jb} >\subset A$, where $w = u \otimes \bar{u}$. At the quantum group level we obtain projective versions, $G \to PG$ and $P\Gamma \subset \Gamma$.*
2. *Free complexifications, $\tilde{A} =< zu_{ij} >\subset C(\mathbb{T}) * A$. At the quantum group level we obtain free complexifications, denoted $\widetilde{G}$ and $\widetilde{\Gamma}$.*

Proof. This is clear from the previous results, because the projective version is a particular csse of the subalgebra construction, and the free complexification appears by combining the free product operation and the subalgebra construction. See [92]. □

As already mentioned, we can use the commutative or cocommutative Woronowicz algebras as input for these constructions, in order to obtain Woronowicz algebras which are not commutative, nor cocommutative. We will be back to these constructions.

Once again following [92] and subsequent papers, let us discuss now a number of truly "new" quantum groups, obtained by liberating and twisting. In what regards the liberation, and the half-liberation as well, the very first result is as follows:

Theorem 1.23. *The following universal algebras are Woronowicz algebras,*

$$\begin{aligned} C(O_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \bar{u}, u^t = u^{-1}\right) \\ C(U_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^* = u^{-1}, u^t = \bar{u}^{-1}\right) \end{aligned}$$

and the same goes for the following quotient algebras,

$$\begin{aligned} C(O_N^*) &= C(O_N^+)\Big/\left\langle abc = cba\Big|\forall a,b,c \in \{u_{ij}\}\right\rangle \\ C(U_N^*) &= C(U_N^+)\Big/\left\langle abc = cba\Big|\forall a,b,c \in \{u_{ij}, u_{ij}^*\}\right\rangle \end{aligned}$$

so the underlying spaces O_N^+, U_N^+ and O_N^*, U_N^* are compact quantum groups.

Proof. The first assertion follows from the elementary fact that if a matrix $u = (u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$u^{\Delta}_{ij} = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u^{\varepsilon}_{ij} = \delta_{ij} \quad , \quad u^S_{ij} = u^*_{ji}$$

Thus, we can define morphisms Δ, ε, S as in Definition 1.2, by using the universality property of $C(O_N^+)$, $C(U_N^+)$. As for the second assertion, the proof here is similar, based on the fact that if the entries of u satisfy $abc = cba$, then so do the entries of $u^{\Delta}, u^{\varepsilon}, u^S$. □

We will see later on that the liberation procedure $G \to G^+$ applies to many other compact Lie groups. As for the half-liberation procedure $G \to G^*$, this can be actually performed without the need of liberating first, and we will discuss this later.

Our first task is to verify that Theorem 1.23 provides us indeed with new quantum groups. For this purpose, we can use the notion of diagonal torus:

Proposition 1.24. *Given a closed subgroup $G \subset U_N^+$, consider its diagonal torus, which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda =< g_1, \ldots, g_N >$ is the discrete group generated by the elements $g_i = u_{ii}$, which are unitaries inside $C(T)$.

Proof. Since u is unitary, its diagonal entries $g_i = u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^*-completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. □

We can now distinguish between our various quantum groups, as follows:

Theorem 1.25. *The diagonal tori of the basic unitary quantum groups are*

$$\begin{matrix} U_N & \to & U_N^* & \to & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ O_N & \to & O_N^* & \to & O_N^+ \end{matrix} \qquad \to \qquad \begin{matrix} \widehat{\mathbb{Z}^N} & \to & \widehat{\mathbb{Z}^{\circ N}} & \to & \widehat{\mathbb{Z}^{*N}} \\ \uparrow & & \uparrow & & \uparrow \\ \widehat{\mathbb{Z}_2^N} & \to & \widehat{\mathbb{Z}_2^{\circ N}} & \to & \widehat{\mathbb{Z}_2^{*N}} \end{matrix}$$

with $\circ$ standing for the half-classical product operation for groups.

Proof. This is clear for U_N^+, where on the diagonal we obtain the biggest group dual, namely $\widehat{F_N}$. For the other quantum groups this follows by taking quotients, which corresponds to taking quotients as well, at the level of the diagonal torus dual $\Lambda = \widehat{T}$. □

Let us discuss now the representation theory of these quantum groups. We are especially interested in computing the associated Tannakian categories:

$$C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$$

For O_N, U_N the results are well-known since Brauer, and the idea will be that O_N^*, U_N^* and O_N^+, U_N^+ will be subject as well to Brauer type theorems. In order to formulate our results, we use the modern notion of "easiness", from [32]:

Definition 1.26. *A closed subgroup $G \subset U_N^+$ is called easy when we have*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

for any colored integers k, l, for certain sets of partitions $D(k,l) \subset P(k,l)$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with the Kronecker type symbols $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

To be more precise here, let $P(k,l)$ be the set of partitions between an upper row of k points, and a lower row of l points. Our claim is that given $N \in \mathbb{N}$, any partition $\pi \in P(k,l)$ produces a linear map between tensor powers of $\mathbb{C}^N$, as follows:

$$T_\pi : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$$

Indeed, if we denote by $e_1, \ldots, e_N$ the standard basis of $\mathbb{C}^N$, we can define T_π by the formula in Definition 1.26, with the Kronecker symbols appearing there being computed by putting the multi-indices i, j on the legs of π, in the obvious way. If all the blocks of π contain equal indices we set $\delta_\pi = 1$, and if not, we set $\delta_\pi = 0$.

With this definition for T_π, we can talk about easy quantum groups, as above.

All this might seem a bit complicated, at a first glance, but it is not. We cannot expect C to come from something simpler than partitions. Thus, philosophically, easiness means that "the Tannakian category appears in the simplest possible way: from partitions".

With this notion in hand, we can recover and extend Brauer's result, as follows:

Theorem 1.27. *The basic unitary quantum groups are all easy, with*

$$\begin{array}{ccccc} U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \end{array} \quad : \quad \begin{array}{ccccc} \mathcal{P}_2 & \longleftarrow & \mathcal{P}_2^* & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow & & \downarrow \\ P_2 & \longleftarrow & P_2^* & \longleftarrow & NC_2 \end{array}$$

being the associated categories of partitions $D \subset P$.

Proof. This is something that requires some work, the idea being as follows:

(1) O_N^+. Consider the set NC_2 of all noncrossing pairings. It is routine to check that $span(T_\pi|\pi\in NC_2)$ is a Tannakian category, and also that this category is the smallest possible one allowed by the Tannakian axioms, in the $u=\bar{u}$ setting. Thus, the associated quantum group must be the biggest subgroup $G\subset O_N^+$, which is O_N^+ itself.

(2) O_N. Since $O_N\subset O_N^+$ appears by adding the commutation relations $ab=ba$ between coordinates, which are implemented by the linear map $T_{\slash\!\!\!\backslash}$ coming from the basic crossing $\slash\!\!\!\backslash$, we obtain here the category $<NC_2,\slash\!\!\!\backslash>=P_2$ of all pairings.

(3) O_N^*. Here we obtain the category $<NC_2,\slash\!\!\!|\!\!\!\backslash>=P_2^*$ of pairings having the property that, when legs are labelled clockwise $\circ\bullet\circ\bullet\ldots$, each string connects $\circ-\bullet$.

(4) U_N^+,U_N,U_N^*. The situation is similar here, but due to $u\neq\bar{u}$ everything is now colored, and we obtain in all cases pairings which are "matching", in the sense that the vertical strings connect $\circ-\circ$ or $\bullet-\bullet$, and the horizontal ones connect $\circ-\bullet$. $\square$

Here are some concrete consequences of the above result:

Theorem 1.28. *The quantum groups O_N^+,U_N^+ have the following properties:*

(1) *We have isomorphisms as follows, up to the standard equivalence relation:*

$$PO_N^+=PU_N^+\quad,\quad\widetilde{O}_N^+=U_N^+$$

(2) *The fusion rules for O_N^+ are the same as the Clebsch-Gordan rules for SU_2:*

$$r_k\otimes r_l=r_{|k-l|}+r_{|k-l|+2}+\ldots+r_{k+l}$$

(3) *Those for U_N^+ are as follows, with the representations being indexed by $\mathbb{N}*\mathbb{N}$:*

$$r_k\otimes r_l=\sum_{k=xy,l=\bar{y}z}r_{xz}$$

(4) *The main characters follow the Wigner semicircle and Voiculescu circular law:*

$$\chi\sim\begin{cases}\gamma_1&\text{for }O_N^+\ ,\ N\geq2\\ \Gamma_1&\text{for }U_N^+\ ,\ N\geq2\end{cases}$$

(5) *With $N\to\infty$, the truncated characters follow the t-versions of these laws:*

$$\chi_t\sim\begin{cases}\gamma_t&\text{for }O_N^+\ ,\ N\to\infty\\ \Gamma_t&\text{for }U_N^+\ ,\ N\to\infty\end{cases}$$

Proof. All this follows from our Brauer type results, via standard techniques. We will be back to this. Let us mention that some similar results regarding O_N^*,U_N^* are available as well, and also that we can twist everything at $q=-1$ too. We will be back to this. $\square$

2. Quantum permutations

Welcome to quantum permutations. The rest of this book is dedicated to them. And, good news, the presentation will be far less intense than that in the previous section, which was meant to be a quick introduction to the quantum groups, survey style.

Our aim now will be that of explaining things in detail. However, this book being at the same time an introduction to the subject, and a survey, we will have to make some compromises, and sometimes omit details, and refer to the literature.

In order to get started, let us look at the usual symmetric group S_N. This is the permutation group of $\{1,\ldots,N\}$, but since our general philosophy here is that of looking at algebraic groups $G\subset U_N$, we will rather regard S_N as being the permutation group of the N coordinates axes of $\mathbb R^N$. We are led in this way to the following result:

Proposition 2.1. *Consider the symmetric group S_N, viewed as the permutation group of the N coordinate axes of $\mathbb R^N$. The coordinate functions on $S_N\subset O_N$ are then given by*

$$u_{ij}=\chi\left(\sigma\in G\Big|\sigma(j)=i\right)$$

and the matrix $u=(u_{ij})$ that these functions form is magic, in the sense that its entries are projections ($p^2=p^*=p$), summing up to 1 on each row and each column.

Proof. Everything here follows from definitions. The formula of the coordinates u_{ij} is obviously the good one, and the fact that $u=(u_{ij})$ is magic is clear too. □

With a bit more effort, we obtain the following nice characterization of S_N:

Theorem 2.2. *The algebra of functions on S_N has the following presentation,*

$$C(S_N)=C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\text{magic}\right)$$

and the multiplication, unit and inverse map of S_N appear from the maps

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}\quad,\quad\varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}$$

defined at the algebraic level, by transposing.

Proof. This is something elementary as well. Indeed, the universal algebra in the statement is a commutative C^*-algebra, so by the Gelfand theorem it must be of the form $C(X)$, with X being a certain compact space. Now since we have coordinates $u_{ij}:X\to\mathbb R$, we conclude that we have $X\subset M_N(\mathbb R)$. Moreover, since we know that these coordinates from a magic matrix, we deduce from this that we have $X=S_N$. □

Summarizing, our idea of looking at S_N as a real algebraic group, $S_N\subset O_N$, has led us to a very simple description of the associated algebra $C(S_N)$.

Following now Wang [93], we can liberate S_N, simply by lifting the commutativity condition in Theorem 2.2. To be more precise, we have the following result:

Theorem 2.3. *The following universal algebra*

$$C(S_N^+)=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\text{magic}\right)$$

has a comultiplication, counit and antipode map, defined as follows,

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}\quad,\quad\varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}$$

so its spectrum S_N^+ is a compact quantum group, called quantum permutation group.

Proof. As a first observation, the universal algebra in the statement is indeed well-defined, because the projection conditions $p^2=p^*=p$ satisfied by the coordinates give $||u_{ij}||\leq 1$ for any i,j, so the universal C^*-norm on the underlying $*$-algebra is bounded.

In order to construct now Δ,ε,S, consider the following matrices:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}$$

Since u is magic, so are these matrices, and so we can define indeed Δ,ε,S by the formulae in the statement, by using the universality of the algebra $C(S_N^+)$.

As a conclusion, the algebra $C(S_N^+)$ satisfies Woronowicz's axioms from Definition 1.2 above, and so its abstract spectrum S_N^+ is a compact quantum group, as claimed. □

Our first task is to make sure that Theorem 2.3 produces indeed a new quantum group, which does not collapse to S_N. Quite surprisingly, this is indeed the case:

Theorem 2.4. *We have an embedding $S_N\subset S_N^+$, given at the algebra level by:*

$$u_{ij}\to\chi\left(\sigma\Big|\sigma(j)=i\right)$$

This is an isomorphism at $N\leq 3$, but not at $N\geq 4$, where S_N^+ is not classical, nor finite.

Proof. The fact that we have indeed an embedding as above follows from Theorem 2.2. Observe that in fact more is true, because Theorem 2.2 and Theorem 2.3 give:

$$C(S_N)=C(S_N^+)\Big/\Big\langle ab=ba\Big\rangle$$

Thus, the inclusion $S_N\subset S_N^+$ is a "liberation", in the sense that S_N is the classical version of S_N^+. We will often use this basic fact, in what follows.

Regarding now the second assertion, we can prove this in four steps, as follows:

<u>Case $N=2$</u>. The fact that S_2^+ is indeed classical, and hence collapses to S_2, is trivial, because the 2×2 magic matrices are as follows, with p being a projection:

$$U=\begin{pmatrix}p&1-p\\1-p&p\end{pmatrix}$$

Indeed, this shows that the entries of U commute. Thus $C(S_2^+)$ is commutative, and so equals its biggest commutative quotient, which is $C(S_2)$. Thus, $S_2^+ = S_2$.

Case $N = 3$. By using the same argument as in the $N = 2$ case, and the symmetries of the problem, it is enough to check that u_{11}, u_{22} commute. But this follows from:

$$\begin{aligned} u_{11}u_{22} &= u_{11}u_{22}(u_{11} + u_{12} + u_{13}) \\ &= u_{11}u_{22}u_{11} + u_{11}u_{22}u_{13} \\ &= u_{11}u_{22}u_{11} + u_{11}(1 - u_{21} - u_{23})u_{13} \\ &= u_{11}u_{22}u_{11} \end{aligned}$$

Indeed, by applying the involution to this formula, we obtain from this that we have $u_{22}u_{11} = u_{11}u_{22}u_{11}$ as well, and so we get $u_{11}u_{22} = u_{22}u_{11}$, as desired.

Case $N = 4$. Consider the following matrix, with p, q being projections:

$$U = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

This matrix is magic, and we can choose p, q as for the algebra $< p, q >$ to be noncommutative and infinite dimensional. We conclude that $C(S_4^+)$ is noncommutative and infinite dimensional as well, and so S_4^+ is non-classical and infinite, as claimed.

Case $N \geq 5$. Here we can use the standard embedding $S_4^+ \subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u \to \begin{pmatrix} u & 0 \\ 0 & 1_{N-4} \end{pmatrix}$$

Indeed, with this in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N \geq 5$ has these two properties as well. See [93]. $\square$

The above result is quite surprising, and understanding all this will be our next goal. As a first observation, we are not wrong with our formalism, because we have as well:

Theorem 2.5. *The quantum permutation group S_N^+ acts on the set $X = \{1, \ldots, N\}$, the corresponding coaction map $\Phi : C(X) \to C(X) \otimes C(S_N^+)$ being given by:*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

In fact, S_N^+ is the biggest compact quantum group acting on X, by leaving the counting measure invariant, in the sense that $(tr \otimes id)\Phi = tr(.)1$, where $tr(e_i) = \frac{1}{N}, \forall i$.

Proof. Our claim is that given a compact matrix quantum group G, the formula $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ defines a morphism of algebras, which is a coaction map, leaving the trace invariant, precisely when the matrix $u = (u_{ij})$ is a magic corepresentation of $C(G)$.

Indeed, let us first determine when Φ is multiplicative. We have:

$$\Phi(e_i)\Phi(e_k) = \sum_{jl} e_j e_l \otimes u_{ji}u_{lk} = \sum_j e_j \otimes u_{ji}u_{jk}$$

$$\Phi(e_i e_k) = \delta_{ik}\Phi(e_i) = \delta_{ik}\sum_j e_j \otimes u_{ji}$$

We conclude that the multiplicativity of Φ is equivalent to the following conditions:

$$u_{ji}u_{jk} = \delta_{ik}u_{ji} \quad , \quad \forall i,j,k$$

Regarding now the unitality of Φ, we have the following formula:

$$\Phi(1) = \sum_i \Phi(e_i) = \sum_{ij} e_j \otimes u_{ji} = \sum_j e_j \otimes \left(\sum_i u_{ji}\right)$$

Thus Φ is unital when the following conditions are satisfied:

$$\sum_i u_{ji} = 1 \quad , \quad \forall j$$

Finally, the fact that Φ is a $*$-morphism translates into:

$$u_{ij} = u_{ij}^* \quad , \quad \forall i,j$$

Summing up, in order for $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ to be a morphism of C^*-algebras, the elements u_{ij} must be projections, summing up to 1 on each row of u. Regarding now the preservation of the trace condition, observe that we have:

$$(tr \otimes id)\Phi(e_i) = \frac{1}{N}\sum_j u_{ji}$$

Thus the trace is preserved precisely when the elements u_{ij} sum up to 1 on each of the columns of u. We conclude from this that $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ is a morphism of C^*-algebras preserving the trace precisely when u is magic, and since the coaction conditions on Φ are equivalent to the fact that u must be a corepresentation, this finishes the proof of our claim. But this claim proves all the assertions in the statement. □

Summarizing, the quantum permutation group S_N^+ appears as a natural free analogue of the usual permutation group S_N, from several possible viewpoints. This quantum group is by definition compact, and at $N \geq 4$ we know that it is not finite.

In order to study now S_N^+, we can use various methods from section 1 above. Let us begin with some basic algebraic results, in connection with the product operations, with the diagonal tori, and with the notion of half-liberation as well:

Theorem 2.6. *The quantum groups S_N^+ have the following properties:*

1. *We have $S_N^+ \hat{*} S_M^+ \subset S_{N+M}^+$, for any N, M.*
2. *In particular, we have an embedding $\widehat{D_\infty} \subset S_4^+$.*
3. *$S_4 \subset S_4^+$ are distinguished by their spinned diagonal tori.*
4. *The half-classical version $S_N^* = S_N^+ \cap O_N^*$ collapses to S_N.*

Proof. These results are all elementary, the proofs being as follows:

(1) If we denote by u, v the fundamental corepresentations of $C(S_N^+), C(S_M^+)$, the fundamental corepresentation of $C(S_N^+ \hat{*} S_M^+)$ is by definition:

$$w = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

But this matrix is magic, because both u, v are magic. Thus by universality of $C(S_{N+M}^+)$ we obtain a quotient map $C(S_{N+M}^+) \to C(S_N^+ \hat{*} S_M^+)$, as desired.

(2) This result, which refines our $N = 4$ trick from the proof of Theorem 2.4, follows from (1) with $N = M = 2$. Indeed, we have the following computation:

$$\begin{aligned} S_2^+ \hat{*} S_2^+ &= S_2 \hat{*} S_2 \\ &= \mathbb{Z}_2 \hat{*} \mathbb{Z}_2 \\ &= \widehat{\mathbb{Z}_2} \hat{*} \widehat{\mathbb{Z}_2} \\ &= \widehat{\mathbb{Z}_2 * \mathbb{Z}_2} \\ &= \widehat{D_\infty} \end{aligned}$$

Here we have used the formula $\widehat{\Gamma} \hat{*} \widehat{\Lambda} = \widehat{\Gamma * \Lambda}$ for discrete groups, which is clear from definitions, plus at the end the standard identification $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$.

(3) As a first observation here, the quantum groups $S_4 \subset S_4^+$ are not distinguished by their diagonal torus, which is $\{1\}$ for both of them. However, according to the general results of Woronowicz in [99], the group dual $\widehat{D_\infty} \subset S_4^+$ that we found in (2) must be a subgroup of the diagonal torus of (S_4^+, FuF^*), for a certain unitary $F \in U_4$.

Now since this group dual $\widehat{D_\infty}$ is not classical, it cannot be a subgroup of the diagonal torus of (S_4, FuF^*). Thus, the diagonal torus spinned by F distinguishes $S_4 \subset S_4^+$.

(4) Consider the quantum group $S_N^* = S_N^+ \cap O_N^*$, whose coordinates satisfy $abc = cba$. In order to prove that we have $S_N^* = S_N$, we can use the fact that for a magic matrix, the entries in each row sum up to 1. Indeed, by using $abc = cba$, and making c vary over a full row of u, we obtain by summing $ab = ba$, and so $S_N^* = S_N$, as claimed. □

Summarizing, we have some advances, including a more conceptual explanation for our main observation so far, namely $S_4^+ \neq S_4$. We will be back to all this material, which by the way might seem quite wizarding, at a first glance. This is how algebra goes.

Let us discuss now the representation theory of S_N^+, which will eventually lead to a clarification of all this. Our main result here, which is quite conceptual, will be the fact that $S_N\subset S_N^+$ is a liberation of easy quantum groups. We will derive as well some explicit consequences of this, in the $N\to\infty$ limit. More technical aspects, regarding the case where $N\in\mathbb N$ is fixed, will be discussed later on, in section 3 below.

We have already seen the definition and basic properties of easiness, in section 1 above. However, the presentation there was lightning quick, and focusing on the general complex case. Here we will just need the real case, which is simpler to explain, so let us start by discussing in detail all this material. Following [32], let us formulate:

Definition 2.7. *Let $P(k,l)$ be the set of partitions between an upper row of k points, and a lower row of l points. A set $D=\bigsqcup_{k,l}D(k,l)$ with $D(k,l)\subset P(k,l)$ is called a category of partitions when it has the following properties:*

1. *Stability under the horizontal concatenation, $(\pi,\sigma)\to[\pi\sigma]$.*
2. *Stability under the vertical concatenation, $(\pi,\sigma)\to[^\sigma_\pi]$.*
3. *Stability under the upside-down turning, $\pi\to\pi^*$.*
4. *Each set $P(k,k)$ contains the identity partition $||\ldots||$.*
5. *The set $P(0,2)$ contains the semicircle partition $\cap$.*

As a basic example, we have P itself. Other basic examples include the category of pairings P_2, or the categories NC,NC_2 of noncrossing partitions, and pairings.

The relation with the Tannakian categories and duality comes from:

Proposition 2.8. *Each $\pi\in P(k,l)$ produces a linear map $T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$,*

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not. The assignement $\pi\to T_\pi$ is categorical, in the sense that we have

$$T_\pi\otimes T_\sigma=T_{[\pi\sigma]}\quad,\quad T_\pi T_\sigma=N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}\quad,\quad T_\pi^*=T_{\pi^*}$$

where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.

Proof. The concatenation axiom follows from the following computation:

$$\begin{aligned}
&(T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\,T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})
\end{aligned}$$

The composition axiom follows from the following computation:

$$\begin{aligned}
& T_\pi T_\sigma(e_{i_1}\otimes\ldots\otimes e_{i_p})\\
= & \sum_{j_1\ldots j_q}\delta_\sigma\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\sum_{k_1\ldots k_r}\delta_\pi\begin{pmatrix}j_1&\ldots&j_q\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
= & \sum_{k_1\ldots k_r}N^{c(\pi,\sigma)}\delta_{[^\sigma_\pi]}\begin{pmatrix}i_1&\ldots&i_p\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
= & N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}(e_{i_1}\otimes\ldots\otimes e_{i_p})
\end{aligned}$$

Finally, the involution axiom follows from the following computation:

$$\begin{aligned}
& T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q})\\
= & \sum_{i_1\ldots i_p}< T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q}),e_{i_1}\otimes\ldots\otimes e_{i_p}> e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & \sum_{i_1\ldots i_p}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & T_{\pi^*}(e_{j_1}\otimes\ldots\otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. □

In relation with the quantum groups, we have the following result, from [32]:

Theorem 2.9. *Each category of partitions $D=(D(k,l))$ produces a family of compact quantum groups $G=(G_N)$, one for each $N\in\mathbb N$, via the formula*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

which produces a Tannakian category, and the Tannakian duality correspondence.

Proof. This follows indeed from Woronowicz's Tannakian duality, in its "soft" form from [68], as explained in Theorem 1.18 above. Indeed, let us set:

$$C(k,l)=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

By using the axioms in Definition 2.7, and the categorical properties of the operation $\pi\to T_\pi$, from Proposition 2.8 above, we deduce that $C=(C(k,l))$ is a Tannakian category. Thus the Tannakian duality applies, and gives the result. □

We can now formulate the following key definition:

Definition 2.10. *A compact quantum group G_N is called easy when we have*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

for any colored integers k,l, for a certain category of partitions $D\subset P$.

In other words, a compact quantum group is called easy when its Tannakian category appears in the simplest possible way: from a category of partitions. The terminology is quite natural, because Tannakian duality is basically our only serious tool.

Observe that the category D is not unique, for instance because at $N = 1$ all the categories of partitions produce the same easy quantum group, namely $G_1 = \{1\}$. We will be back to this issue on several occasions, with various results about it.

In relation now with our quantum groups, here is our main result:

Theorem 2.11. *The quantum permutation and rotation groups are all easy,*

$$\begin{tikzcd}
S_N^+ \arrow[r] & O_N^+ \\
S_N \arrow[u] \arrow[r] & O_N \arrow[u]
\end{tikzcd}
\quad : \quad
\begin{tikzcd}
NC \arrow[d] & NC_2 \arrow[l] \arrow[d] \\
P & P_2 \arrow[l]
\end{tikzcd}$$

with the corresponding categories of partitions being those on the right.

Proof. This is something quite fundamental, the proof being as follows:

(1) O_N^+. Consider the Tannakian category of O_N^+, formed by the following spaces:

$$C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$$

By using Proposition 2.8, consider as well the following Tannakian category:

$$D = span\left(T_\pi \Big| \pi \in NC_2\right)$$

We want to prove that we have $C = D$. In one sense, this follows from:

$$\begin{aligned}
u^t = u^{-1} \quad & \implies \quad T_\cap \in C \\
& \implies \quad < T_\cap > \subset C \\
& \implies \quad span\left(T_\pi \Big| \pi \in < \cap >\right) \subset C \\
& \implies \quad D \subset C
\end{aligned}$$

In the other sense, Tannakian duality tells us that associated to D is a certain closed subgroup $G \subset O_N^+$. But since Tannakian duality is contravariant, at the level of categories $G \subset O_N^+$ translates into $C \subset D$. Thus we have $C = D$, and we are done.

(2) O_N. Since $O_N \subset O_N^+$ appears by adding the commutation relations $ab = ba$ between coordinates, which are implemented by the linear map $T_{\slash\!\!\!\backslash}$ coming from the basic crossing $\slash\!\!\!\backslash$, this group is indeed easy, coming from the following category:

$$< NC_2, \slash\!\!\!\backslash >= P_2$$

Alternatively, if this argument was too fast, the above proof for O_N^+ can be simply rewritten, by adding at each step the basic crossing $\slashed{\times}$, next to the semicircle $\cap$.

(3) S_N^+. We know that the algebra $C(S_N^+)$ appears as follows:

$$C(S_N^+)=C(O_N^+)\Big/\Big\langle u=\text{magic}\Big\rangle$$

In order to interpret the magic condition, consider the fork partition:

$$Y\in P(2,1)$$

The linear map associated to this fork partition Y is then given by:

$$T_Y(e_i\otimes e_j)=\delta_{ij}e_i$$

Thus, in usual matrix notation, this linear map is given by:

$$T_Y=(\delta_{ijk})_{i,jk}$$

Now given a corepresentation u, we have the following formulae:

$$(T_Yu^{\otimes 2})_{i,jk}=\sum_{lm}(T_Y)_{i,lm}(u^{\otimes 2})_{lm,jk}=u_{ij}u_{ik}$$

$$(uT_Y)_{i,jk}=\sum_l u_{il}(T_Y)_{l,jk}=\delta_{jk}u_{ij}$$

We conclude that we have the following equivalence:

$$T_Y\in Hom(u^{\otimes 2},u)\iff u_{ij}u_{ik}=\delta_{jk}u_{ij},\forall i,j,k$$

The condition on the right being equivalent to the magic condition, we obtain:

$$C(S_N^+)=C(O_N^+)\Big/\Big\langle T_Y\in Hom(u^{\otimes 2},u)\Big\rangle$$

Thus S_N^+ is indeed easy, the corresponding category of partitions being:

$$D=<Y>=NC$$

(4) S_N. Here there is no need for new computations, because we have $S_N=S_N^+\cap O_N$, which at the categorial level means that S_N is easy, coming from:

$$<NC,P_2>=P$$

Alternatively, if this was too fast, we can rewrite the proof for S_N^+ or O_N, by adding at each step the basic crossing/fork next respectively to the fork/basic crossing. □

As already mentioned in section 1, in the context of the unitary quantum groups, this kind of easiness result has a massive number of applications. We will explore these applications in what follows, gradually. Let us start with something philosophical:

Theorem 2.12. *The inclusions $O_N\subset O_N^+$ and $S_N\subset S_N^+$ are liberation operations in the easy quantum group sense, given by*

$$D_{G^+}=D_G\cap NC$$

at the level of the associated categories of partitions.

Proof. This is clear indeed from Theorem 2.11 above, and from the trivial equalities $NC_2=P_2\cap NC$ and $NC=P\cap NC$, connecting the categories found there. $\square$

Let us get now into the real thing, namely classification of the irreducible representations, computation of their fusion rules, and of the associated Cayley graph, plus computation of the laws of characters, and other probabilistic questions.

As explained in section 1 above, all these problems are related, and their solution basically requires the knowledge of the associated Tannakian category, given by:

$$C_{kl}=Hom(u^{\otimes k},u^{\otimes l})$$

But in the easy case, where our quantum group G comes from a category of partitions D, and which covers our 4 main examples, this problem is half-solved, because:

$$C_{kl}=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

The remaining half-problem to be solved is that of investigating the linear independence properties of the maps T_π, and then deriving explicit consequences from this.

Let us begin with some standard combinatorics, as follows:

Definition 2.13. *Let $P(k)$ be the set of partitions of $\{1,\ldots,k\}$, and let $\pi,\sigma\in P(k)$.*

1. *We write $\pi\leq\sigma$ if each block of π is contained in a block of σ.*
2. *We let $\pi\vee\sigma\in P(k)$ be the partition obtained by superposing π,σ.*

Also, we denote by $|.|$ the number of blocks of the partitions $\pi\in P(k)$.

As an illustration here, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and we have $||\leq\sqcap$. Also, at $k=3$ we have $P(3)=\{|||,\sqcap|,\sqcap\hspace{-3.5mm}|\ ,|\sqcap,\sqcap\hspace{-0.7mm}\sqcap\}$, and the order relation is as follows:

$$|||\ \leq\ \sqcap|\ ,\ \sqcap\hspace{-3.5mm}|\ ,\ |\sqcap\ \leq\ \sqcap\hspace{-0.7mm}\sqcap$$

Observe also that we have $\pi,\sigma\leq\pi\vee\sigma$, and that $\pi\vee\sigma$ is the smallest partition with this property. Due to this fact, $\pi\vee\sigma$ is called supremum of π,σ.

Now back to quantum groups, and to the questions that we want to solve, by Frobenius duality it is enough to study the partitions having no upper legs. We have:

Proposition 2.14. *The vectors $\xi_\pi = T_\pi$ with $\pi \in P(k)$ are given by*

$$\xi_\pi = \sum_{i_1 \ldots i_k} \delta_\pi(i_1, \ldots, i_k)\, e_{i_1} \otimes \ldots \otimes e_{i_k}$$

and their scalar products are given by the formula

$$< \xi_\pi, \xi_\sigma >= N^{|\pi \vee \sigma|}$$

where $\vee$ is the superposition operation, and $|.|$ is the number of blocks.

Proof. According to the formula of the vectors ξ_π, we have:

$$\begin{aligned} < \xi_\pi, \xi_\sigma > &= \sum_{i_1 \ldots i_k} \delta_\pi(i_1, \ldots, i_k)\delta_\sigma(i_1, \ldots, i_k) \\ &= \sum_{i_1 \ldots i_k} \delta_{\pi \vee \sigma}(i_1, \ldots, i_k) \\ &= N^{|\pi \vee \sigma|} \end{aligned}$$

Thus, we have obtained the formula in the statement. □

In order to study the Gram matrix $G_{\pi\sigma} = N^{|\pi \vee \sigma|}$, and more specifically to compute its determinant, we will use several standard facts about the partitions. We have:

Definition 2.15. *The Möbius function of any lattice, and so of P, is given by*

$$\mu(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ -\sum_{\pi \leq \tau < \sigma} \mu(\pi, \tau) & \text{if } \pi < \sigma \\ 0 & \text{if } \pi \not\leq \sigma \end{cases}$$

with the construction being performed by recurrence.

As an illustration here, let us go back to the set of 2-point partitions, $P(2) = \{||, \sqcap\}$. We have by definition $\mu(||, ||) = \mu(\sqcap, \sqcap) = 1$. Also, we know that we have $|| < \sqcap$, with no intermediate partition in between, and so the above recurrence procedure gives:

$$\mu(||, \sqcap) = -\mu(||, ||) = -1$$

Finally, we have $\sqcap \not\leq ||$, and so $\mu(\sqcap, ||) = 0$. Thus, as a conclusion, the Möbius matrix $M_{\pi\sigma} = \mu(\pi, \sigma)$ of the lattice $P(2) = \{||, \sqcap\}$ is as follows:

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The interest in the Möbius function comes from the Möbius inversion formula:

$$f(\sigma) = \sum_{\pi \leq \sigma} g(\pi) \implies g(\sigma) = \sum_{\pi \leq \sigma} \mu(\pi, \sigma) f(\pi)$$

In linear algebra terms, the statement and proof of this formula are as follows:

Theorem 2.16. *The inverse of the adjacency matrix of P, given by*

$$A_{\pi\sigma}=\begin{cases}1&\text{if }\pi\leq\sigma\\ 0&\text{if }\pi\not\leq\sigma\end{cases}$$

is the Möbius matrix of P, given by $M_{\pi\sigma}=\mu(\pi,\sigma)$.

Proof. This is well-known, coming for instance from the fact that A is upper triangular. Indeed, when inverting, we are led into the recurrence from Definition 2.15. $\square$

As a first illustration, for $P(2)$ the formula $M=A^{-1}$ appears as follows:

$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}$$

Also, for $P(3)=\{|||,\sqcap|,\sqcap\!\!\!|\,,|\sqcap,\sqcap\!\sqcap\}$ the formula $M=A^{-1}$ reads:

$$\begin{pmatrix}1&-1&-1&-1&2\\0&1&0&0&-1\\0&0&1&0&-1\\0&0&0&1&-1\\0&0&0&0&1\end{pmatrix}=\begin{pmatrix}1&1&1&1&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}^{-1}$$

Now back to our Gram matrix considerations, we have the following result:

Proposition 2.17. *The Gram matrix of the vectors ξ_π with $\pi\in P(k)$,*

$$G_{\pi\sigma}=N^{|\pi\vee\sigma|}$$

decomposes as a product of upper/lower triangular matrices, $G=AL$, where

$$L(\pi,\sigma)=\begin{cases}N(N-1)\ldots(N-|\pi|+1)&\text{if }\sigma\leq\pi\\ 0&\text{otherwise}\end{cases}$$

and where $A=M^{-1}$ is the adjacency matrix of $P(k)$.

Proof. Given a multi-index $i=(i_1,\ldots,i_k)$, let us denote by $\ker i\in P(k)$ the partition collecting the equal indices of i. With this convention, we have:

$$\begin{aligned}N^{|\pi\vee\sigma|}&=\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i\geq\pi\vee\sigma\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i=\tau\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}N(N-1)\ldots(N-|\tau|+1)\end{aligned}$$

According to Theorem 2.16 and to the definition of A, L, this formula reads:

$$G_{\pi\sigma} = \sum_{\tau \geq \pi} L_{\tau\sigma} = \sum_{\tau} A_{\pi\tau} L_{\tau\sigma} = (AL)_{\pi\sigma}$$

Thus, we obtain in this way the formula in the statement. □

As an illustration for the above result, at $k = 2$ we have $P(2) = \{||, \sqcap\}$, and the above formula $G = AL$ appears as follows:

$$\begin{pmatrix} N^2 & N \\ N & N \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^2 - N & 0 \\ N & N \end{pmatrix}$$

At $k = 3$ we have $P(3) = \{|||, \sqcap|, \sqcap\!\!\!\!\,|\,, |\sqcap, \sqcap\!\sqcap\}$, which leads to a similar formula.

With the above result in hand, we can now investigate the linear independence properties of the vectors ξ_π. To be more precise, we have the following result:

Theorem 2.18. *The determinant of the Gram matrix $G_{\pi\sigma} = N^{|\pi \vee \sigma|}$ is given by:*

$$\det(G) = \prod_{\pi \in P(k)} \frac{N!}{(N - |\pi|)!}$$

In particular, for $N \geq k$, the vectors $\{\xi_\pi | \pi \in P(k)\}$ are linearly independent.

Proof. According to the formula in Proposition 2.17 above, we have:

$$\det(G) = \det(A)\det(L)$$

Now if we order $P(k)$ as above, with respect to the number of blocks, and then lexicographically, we see that A is upper triangular, and that L is lower triangular.

Thus $\det(A)$ can be computed simply by making the product on the diagonal, and we obtain 1. As for $\det(L)$, this can computed as well by making the product on the diagonal, and we obtain the number in the statement, with the technical remark that in the case $N < k$ the convention is that we obtain a vanishing determinant. See [24]. □

Now back to the laws of characters, we can formulate:

Theorem 2.19. *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D = (D(k, l))$, the asymptotic moments of the main character are given by*

$$\lim_{N \to \infty} \int_{G_N} \chi^k = \# D(k)$$

where $D(k) = D(\emptyset, k)$, with the limiting sequence on the left consisting of certain integers, and being stationary at least starting from the k-th term.

Proof. According to the Peter-Weyl theory, and to the definition of easiness, the moments of the main character are given by the following formula:

$$\begin{aligned}\int_{G_N}\chi^k &= \int_{G_N}\chi_{u^{\otimes k}}\\ &= \dim\left(Fix(u^{\otimes k})\right)\\ &= \dim\left(span\left(\xi_\pi\Big|\pi\in D(k)\right)\right)\end{aligned}$$

By using now the linear independence result from Theorem 2.18 above, with $N\to\infty$ we obtain the formula in the statement. $\square$

Let us see now what happens for our 4 main quantum groups, from Theorem 2.11 above. The result here, obtained by counting partitions, is as follows:

Theorem 2.20. *The asymptotic k-moments for the main examples of quantum permutation and quantum rotation groups are given by*

$$\begin{matrix} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{matrix} \qquad : \qquad \begin{matrix} C_k & \text{---} & C_{k/2} \\ | & & | \\ B_k & \text{---} & k!! \end{matrix}$$

with the numbers on the right being as follows,

(1) $k!!=1.3.5\ldots(k-3)(k-1)$,
(2) $B_k=|P(k)|$ *are the Bell numbers,*
(3) $C_k=\frac{1}{k+1}\binom{2k}{k}$ *are the Catalan numbers,*

and with the conventions $k!!=0$ and $C_{k/2}=0$ for $k\notin 2\mathbb{N}$.

Proof. According to Theorem 2.11 and Theorem 2.19, the asymptotic moments in question appear by counting the following sets of partitions:

$$\begin{matrix} NC(k) & \longleftarrow & NC_2(k) \\ \downarrow & & \downarrow \\ P(k) & \longleftarrow & P_2(k) \end{matrix}$$

By these counting questions are all standard, as follows:

(1) Regarding $k!!=|P_2(k)|$, this formula is clear, because we have $k-1$ choices for the pair of 1, then $k-3$ choices for the pair of the next number, and so on.

(2) Regarding $B_k = |P(k)|$, there is nothing much to be done here, because these numbers, called Bell numbers, cannot be explicitely computed.

(3) Regarding $C_{k/2} = |NC_2(k)|$, which are the Catalan numbers, these can be explicitely computed by recurrence, and we obtain $C_k = \frac{1}{k+1}\binom{2k}{k}$, as stated.

(4) Regarding $C_k = |NC(k)|$, this formula can be established either by recurrence, or deduced from (3), via fattening/shrinking. $\square$

As a comment here, we are definitely on the good way, because the numbers appearing in the above statement are the main numbers in combinatorics.

By doing now some calculus, we can compute the asymptotic laws of characters:

Theorem 2.21. *The asymptotic laws of characters for the main examples of quantum permutation and quantum rotation groups are given by*

$$\begin{tikzcd} S_N^+ \arrow[r] & O_N^+ \\ S_N \arrow[u] \arrow[r] & O_N \arrow[u] \end{tikzcd} \quad : \quad \begin{tikzcd} \pi_1 \arrow[r, dash] & \gamma_1 \\ p_1 \arrow[u, dash] \arrow[r, dash] & g_1 \arrow[u, dash] \end{tikzcd}$$

with the measures on the right being as follows:

(1) $g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$ *is the Gaussian law of parameter* 1.
(2) $p_1 = \frac{1}{e}\sum_p \frac{\delta_p}{p!}$ *is the Poisson law of parameter* 1.
(3) $\gamma_1 = \frac{1}{2\pi}\sqrt{4-x^2}\,dx$ *is the Wigner semicircle law of parameter* 1.
(4) $\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$ *is the Marchenko-Pastur law of parameter* 1.

Proof. This follows indeed from Theorem 2.20, by doing some calculus:

(1) By partial integration, we have the following formula:

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^k e^{-x^2/2}\,dx = (k-1)\times\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^{k-2} e^{-x^2/2}\,dx$$

Thus the moments of g_1 satisfy the same recurrence as the numbers $k!!$.

(2) The moments of the Poisson law p_1 are the following numbers:

$$M_k = \frac{1}{e}\sum_{p\in\mathbb{N}} \frac{p^k}{k!}$$

Computations show that the recurrence is the same as for the Bell numbers B_k.

(3) The moment generating function for the semicircle law γ_1 is given by:

$$f(z)=\frac{1}{2\pi}\int_{-2}^2\frac{\sqrt{4-x^2}}{1-zx}\,dx$$

By doing some computations, the coefficients of f are the Catalan numbers.

(4) The moment generating function for the Marchenko-Pastur law π_1 is:

$$f(z)=\frac{1}{2\pi}\int_0^4\frac{\sqrt{4x^{-1}-1}}{1-zx}\,dx$$

By computation, we obtain the generating series of the Catalan numbers. $\square$

Summarizing, the representation theory of our basic quantum groups is something extremely simple and fundamental, in the $N\to\infty$ limit.

We will see in the next section that the results in the free case can be improved, with the convergences there being actually stationary, starting from $N=2$.

Also, we will see later on that the above results can be extended to the case of truncated characters, with the limiting $N\to\infty$ measures being p_t,g_t,π_t,γ_t, with $t\in(0,1]$.

3. Representation theory

We have seen so far that the inclusion $S_N \subset S_N^+$, as well as its companion inclusion $O_N \subset O_N^+$, are liberations in the sense of easy quantum groups, and that some interesting representation theory consequences, in the $N \to \infty$ limit, can be derived from this.

We discuss here the case where $N \in \mathbb{N}$ is fixed. Among the problems to be solved, we must classify the irreducible representations, compute their fusion rules, and the Cayley graphs as well, and also study the laws of characters at fixed values of N.

We will get as well into the structure of the subgroups $G \subset S_N^+$. These are not easy, in general, but we will present a result refining the Tannakian duality for them, stating that the spaces $P_k = Fix(u^{\otimes k})$ form a planar algebra in the sense of Jones [57].

In order to get started, we need a lot of preliminaries, the lineup being von Neumann algebras, II_1 factors, subfactors of II_1 factors, and finally planar algebras.

Let us go back to the operator algebra $B(H)$, from Theorem 1.1. Inspired by its properties there, we can formulate as well, in parallel to the C^*-algebra theory:

Definition 3.1. *A von Neumann algebra is a $*$-algebra of operators $A \subset B(H)$ satisfying one of the following equivalent conditions:*

(1) *A is closed under the weak topology, making each $T \to Tx$ continuous.*
(2) *A is equal to its bicommutant, $A = A''$.*

Here the equivalence between (1) and (2) is von Neumann's bicommutant theorem, whose proof uses basic spectral theory and functional analysis.

The basic theory of the von Neumann algebras is as follows:

Theorem 3.2. *The von Neumann algebras have the following properties:*

(1) *They are special types of C^*-algebras. In fact, they are exactly the C^*-algebras of operators $A \subset B(H)$ having a predual A_*.*
(2) *In the commutative case, they are the algebras of type $A = L^\infty(X)$, with X measured space, represented on $H = L^2(X)$, up to a multiplicity.*
(3) *If we write the center as $Z(A) = L^\infty(X)$, then we have a decomposition of type $A = \int_X A_x\, dx$, with the fibers A_x being factors, $Z(A_x) = \mathbb{C}$.*

Proof. This is something standard, the idea being as follows:

(1) We know that von Neumann implies C^*, and the fact that the converse fails follows from (2) below. As for the A_* result, this is something technical, due to Sakai.

(2) It is clear, via basic measure theory, that $L^\infty(X)$ is indeed a von Neumann algebra on $H = L^2(X)$. The converse can be proved as well, by using spectral theory.

(3) This holds in finite dimensions, where $A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C})$. In general, this is von Neumann's reduction theory result, based on advanced functional analysis. □

We can see now the exact difference between C^*-algebras and von Neumann algebras. While the C^*-algebras are the algebras of the form $C(X)$, with X being a noncommutative compact space, the von Neumann algebras are the algebras of the form $L^\infty(X)$, with X being a noncommutative measured space. Both these algebras are very useful.

At a more advanced level now, we know from Theorem 3.2 (3) that things basically reduce to "factors". And, regarding these factors, we have:

Theorem 3.3. *The von Neumann factors, $Z(A)=\mathbb{C}$, have the following properties:*

(1) *They can be fully classified in terms of II_1 factors, which are by definition those satisfying $\dim A=\infty$, and having a faithful trace $tr:A\to\mathbb{C}$.*
(2) *The II_1 factors enjoy the "continuous dimension geometry" property, in the sense that the traces of their projections can take any values in $[0,1]$.*
(3) *Among the II_1 factors, the smallest one is the Murray-von Neumann hyperfinite factor R, obtained as an inductive limit of matrix algebras.*

Proof. This is something quite heavy, the idea being as follows:

(1) This comes from results of Murray-von Neumann and Connes, the idea being that the other factors can be basically obtained via crossed product constructions.

(2) This is subtle functional analysis, with the rational traces being relatively easy to obtain, and with the irrational ones coming from limiting arguments.

(3) Once again, heavy results, by Murray-von Neumann and Connes, the idea being that any finite dimensional construction always leads to the same factor, called R. $\square$

All the above was of course quite brief, but there are many books on the subject, a standard reference here being the book by Blackadar. We recommend as well the original papers of Murray-von Neumann and Connes, which are as must-read.

Let us discuss now subfactor theory, following Jones' paper [55]. Jones looked at the inclusions of II_1 factors $A\subset B$, called subfactors, which are quite natural objects in quantum physics. Given such an inclusion, we can talk about its index:

Definition 3.4. *The index of an inclusion of II_1 factors $A\subset B$ is the quantity*

$$[B:A]=\dim_A B\in[1,\infty]$$

constructed by using the Murray-von Neumann continuous dimension theory.

The discovery of Jones is that each such subfactor produces a representation of the Temperley-Lieb algebra $TL_N\subset B(H)$. As a consequence, the index is quantized:

$$N\in\left\{4\cos^2\left(\frac{\pi}{n}\right)\Big|n\in\mathbb{N}\right\}\cup[4,\infty]$$

In order to discuss all this, which is useful for us, we first need to talk about the Temperley-Lieb algebra. This algebra, discovered by Temperley and Lieb in the context of general statistical mechanics [86], has a very simple definition, as follows:

Definition 3.5. *The Temperley-Lieb algebra of index $N \in [1, \infty)$ is defined as*

$$TL_N(k) = span(NC_2(k,k))$$

with product given by vertical concatenation, with the rule $\bigcirc = N$.

In other words, the algebra $TL_N(k)$, depending on parameters $k \in \mathbb{N}$ and $N \in [1, \infty)$, is the formal linear span of the pairings $\pi \in NC_2(k,k)$. The product operation is obtained by linearity, for the pairings which span $TL_N(k)$ this being the usual vertical concatenation, with the conventions that things go "from top to bottom", and that each floating circle that might appear when concatenating is replaced by a scalar factor, equal to N.

In order to explain now Jones' result, it is better to relabel our subfactor as $A_0 \subset A_1$. We can construct the orthogonal projection $e_1 : A_1 \to A_0$, and set $A_2 = < A_1, e_1 >$.

This remarkable procedure, called "basic construction", can be iterated, and we obtain in this way a whole tower of II_1 factors, as follows:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots\ldots$$

Quite surprisingly, this construction leads to a link with the Temperley-Lieb algebra TL_N, and with many other things, which can be summarized as follows:

Theorem 3.6. *Let $A_0 \subset A_1$ be an inclusion of II_1 factors.*

1. *The sequence of projections $e_1, e_2, e_3, \ldots \in B(H)$ produces a representation of the Temperley-Lieb algebra $TL_N \subset B(H)$, where $N = [A_1, A_0]$.*
2. *The index $N = [A_1, A_0]$, which is a Murray-von Neumann continuous quantity $N \in [1, \infty]$, must satisfy $N \in \{4\cos^2(\frac{\pi}{n}) | n \in \mathbb{N}\} \cup [4, \infty]$.*

Proof. This is something quite tricky, the idea being as follows:

(1) The idea here is that the functional analytic study of the basic construction leads to the conclusion that the sequence of projections $e_1, e_2, e_3, \ldots \in B(H)$ behaves algebrically exactly as the sequence of diagrams $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \in TL_N$ given by:

$$\varepsilon_1 = {}^{\cup}_{\cap} \quad , \quad \varepsilon_2 = |{}^{\cup}_{\cap} \quad , \quad \varepsilon_3 = ||{}^{\cup}_{\cap} \quad , \quad \ldots$$

But these diagrams generate TL_N, and so we have an embedding $TL_N \subset B(H)$, where H is the Hilbert space where our subfactor $A_0 \subset A_1$ lives, as claimed.

(2) This is something quite surprising, which follows from (1), via some clever positivity considerations, involving the Perron-Frobenius theorem. In fact, the subfactors having index $N \in [1, 4]$ can be classified by ADE diagrams, and the obstruction $N = 4\cos^2(\frac{\pi}{n})$ itself comes from the fact that N must be the squared norm of such a graph. □

Quite remarkably, the above result is just the "tip of the iceberg". One can prove indeed that the planar algebra structure of TL_N, taken in an intuitive sense, extends to a planar algebra structure on the sequence of commutants $P_k = A_0' \cap A_k$.

In order to discuss this key result, that we will need as well, let us start with:

Definition 3.7. *The planar algebras are defined as follows:*

(1) *A k-tangle is a rectangle in the plane, also called box, with $2k$ marked points on its boundary, containing r small boxes, each having $2k_i$ marked points, and with the $2k+\sum 2k_i$ marked points being connected by noncrossing strings.*
(2) *A planar algebra is a sequence of finite dimensional vector spaces $P=(P_k)$, together with linear maps $P_{k_1}\otimes\ldots\otimes P_{k_r}\to P_k$, one for each k-tangle, such that the gluing of tangles corresponds to the composition of linear maps.*

As basic example of a planar algebra, we have the Temperley-Lieb algebra TL_N. Indeed, putting $TL_N(k_i)$ diagrams into the small r boxes of a k-tangle clearly produces a $TL_N(k)$ diagram, and so we have indeed a planar algebra, of somewhat "trivial" type.

In general, the planar algebras are more complicated than this, and we will be back later with some explicit examples. However, the idea is very simple, namely "the elements of a planar algebra are not necessarily diagrams, but they behave like diagrams".

In relation now with subfactors, the result, which extends Theorem 3.6 (1) above, and which was found by Jones in [57], almost 20 years after [55], is as follows:

Theorem 3.8. *Given a subfactor $A_0\subset A_1$, the collection $P=(P_k)$ of linear spaces*

$$P_k=A_0'\cap A_k$$

has a planar algebra structure, extending the planar algebra structure of TL_N.

Proof. As a first observation, since $e_1:A_1\to A_0$ commutes with A_0 we have $e_1\in P_2'$, and by translation we obtain $e_1,\ldots,e_{k-1}\in P_k$ for any k, and so $TL_N\subset P$.

The point now is that the planar algebra structure of TL_N, obtained by composing diagrams, can be shown to extend into an abstract planar algebra structure of P.

This is something quite heavy, and we will not get into details here. See [57]. $\square$

As it was the case with other things, our explanations here were very brief. For all this, and more, we recommend Jones' papers [55], [56], [57], along with [86].

Getting back now to quantum groups, all this is very interesting for us. Let us begin with a key technical result, making the connection with our questions:

Proposition 3.9. *Consider the representation $i:TL_N(k)\to B((\mathbb C^N)^{\otimes k})$, $\pi\to T_\pi$.*

(1) *We have $Tr(T_\pi)=N^{loops(<\pi>)}$, where $\pi\to<\pi>$ is the closing operation.*
(2) *The linear form $\tau=Tr\circ i:TL_N(k)\to\mathbb C$ is a faithful positive trace.*
(3) *The representation $i:TL_N(k)\to B((\mathbb C^N)^{\otimes k})$ is faithful.*

Proof. By using the categorical properties of the construction $\pi\to T_\pi$, from Proposition 2.8 above, we conclude that we have indeed a representation as follows:

$$i:TL_N(k)\to B((\mathbb C^N)^{\otimes k})\quad,\quad \pi\to T_\pi$$

Regarding now (1-3), these basically come from Theorem 3.6, as follows:

(1) This follows indeed from the following computation:

$$\begin{aligned} Tr(T_\pi) &= \sum_{i_1\ldots i_k}\delta_\pi\begin{pmatrix}i_1\ldots i_k\\ i_1\ldots i_k\end{pmatrix}\\ &= \#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker\begin{pmatrix}i_1\ldots i_k\\ i_1\ldots i_k\end{pmatrix}\geq\pi\right\}\\ &= N^{loops(<\pi>)} \end{aligned}$$

(2) The traciality of τ is clear, because Tr is tracial. Regarding now the faithfulness, this is best viewed via the formula $\tau(\pi)=N^{loops(<\pi>)}$. Indeed, in the subfactor context, the Temperley-Lieb trace appears as a II_1 factor trace, and so it is faithful.

(3) This follows from (2) above, via a standard positivity argument. □

We can use the above result in the quantum group context, as follows:

Proposition 3.10. *We have an isomorphism as follows, given by $\pi\to T_\pi$,*

$$TL_N(k)\simeq End(u^{\otimes k})\subset B((\mathbb{C}^N)^{\otimes k})$$

with u being the fundamental representation of the quantum group O_N^+.

Proof. We know from easiness, Theorem 2.11 above, that the algebra of intertwiners of $u^{\otimes k}$ appears from the partitions in $NC_2(k,k)$, as follows:

$$End(u^{\otimes k})=span\left(T_\pi\Big|\pi\in NC_2(k,k)\right)\subset B((\mathbb{C}^N)^{\otimes k})$$

On the other hand, the faithfulness result from Proposition 3.9 tells us that we have a concrete realization of the Temperley-Lieb algebra, as follows:

$$TL_N(k)=span(NC_2(k,k))\subset B((\mathbb{C}^N)^{\otimes k})\quad,\quad \pi\to T_\pi$$

Thus, we are led to the conclusion in the statement. □

We can work out now the representation theory of O_N^+, as follows:

Theorem 3.11. *The quantum groups O_N^+ with $N\geq 2$ have the following properties:*

1. *The even moments of the main character are the Catalan numbers, $\int\chi^{2k}=C_k$.*
2. *The main character follows the Wigner semicircle law, $\chi\sim\gamma_1$.*
3. *The fusion rules are $r_k\otimes r_l=r_{|k-l|}+r_{|k-l|+2}+\ldots+r_{k+l}$, as for SU_2.*
4. *The dimensions are $\dim(r_k)=\frac{q^{k+1}-q^{-k-1}}{q-q^{-1}}$, where $q^2-Nq+1=0$.*

Proof. The proof of this result, going back to [1], is as follows:

(1) We have indeed the following computation, based on Proposition 3.10:

$$\int_{O_N^+}\chi^{2k}=\dim(End(u^{\otimes k}))=\dim TL_N(k)=\#NC_2(k,k)=C_k$$

(2) This follows from (1), as explained in the proof of Theorem 2.21 (3) above.

(3) Let $\{\chi_k\}_{k\in\mathbb{N}}$ be the characters of the irreducible representations of SU_2. These characters span a complex subalgebra $A\subset C(SU_2)$, which is isomorphic to $\mathbb{C}[X]$, via $X\to\chi_1$. We can find integers $c_{kl}\in\mathbb{N}$ such that $c_{kk}=1$ and:

$$\chi_1^k=\sum_{l=0}^k c_{kl}\chi_l$$

Also, we can define a morphism $\Psi:A\to C(O_N^+)$ by $\chi_1\to f_1$, where f_1 is the character of the fundamental representation of O_N^+. The elements $f_k=\Psi(\chi_k)$ verify then:

$$f_kf_l=f_{|k-l|}+f_{|k-l|+2}+\ldots+f_{k+l}$$

We prove now by recurrence on k that each f_k is the character of an irreducible corepresentation r_k of $C(O_N^+)$, non-equivalent to $r_0,\ldots,r_{k-1}$. At $k=0,1$ this is clear.

Assume now that the result holds at $k-1$. We have $f_{k-2}f_1=f_{k-3}+f_{k-1}$, and so we get $r_{k-2}\otimes r_1=r_{k-3}+r_{k-1}$, which gives $r_{k-1}\subset r_{k-2}\otimes r_1$. Now since r_{k-2} is irreducible, by Frobenius reciprocity we have $r_{k-2}\subset r_{k-1}\otimes r_1$, so there exists a representation r_k such that $r_{k-1}\otimes r_1=r_{k-2}+r_k$. Since $f_{k-1}f_1=f_{k-2}+f_k$, the character of r_k is f_k.

It remains to prove that r_k is irreducible, and non-equivalent to $r_1,\ldots,r_{k-1}$. For this purpose, observe that we have an inequality as follows:

$$\sum_{l=0}^k c_{kl}^2\leq\dim(End(u^{\otimes k}))=C_k$$

Indeed, the inequality on the left comes from the fact that we have $f_1^k=\sum_{l=0}^k c_{kl}f_l$, with the remark that the equality case holds precisely when r_k is irreducible, and non-equivalent to $r_1,\ldots,r_{k-1}$. As for the equality on the right, this comes from (1).

On the other hand, a standard computation involving SU_2 and the Clebsch-Gordan rules shows that we must have overall equality. Thus, we obtain the result.

(4) The dimension formula there is clear by recurrence. □

In order to pass now to quantum permutations, we can use the following well-known trick, relating noncrossing pairings to arbitrary noncrossing partitions:

Proposition 3.12. *We have a bijection $NC(k)\simeq NC_2(2k)$, constructed as follows:*

1. *The application $NC(k)\to NC_2(2k)$ is the "fattening" one, obtained by doubling all the legs, and doubling all the strings as well.*
2. *Its inverse $NC_2(2k)\to NC(k)$ is the "shrinking" application, obtained by collapsing pairs of consecutive neighbors.*

Proof. The fact that the two operations in the statement are indeed inverse to each other is clear, by computing the corresponding two compositions, with the remark that the construction of the fattening operation requires the partitions to be noncrossing. □

At the level of the associated Gram matrices, the result is as follows:

Proposition 3.13. *The Gram matrices of $NC_2(2k), NC(k)$ are related as follows, where $\pi \to \pi'$ is the shrinking operation, and Δ_{kn} is the diagonal of G_{kn}:*

$$G_{2k,n}(\pi,\sigma) = n^k(\Delta_{kn}^{-1}G_{k,n^2}\Delta_{kn}^{-1})(\pi',\sigma')$$

In particular, we have $\det(G_{kN}) \neq 0$ for any $N = n^2 \geq 4$, and so the family of vectors $\{\xi_\pi|\pi \in NC(k)\} \subset (\mathbb{C}^N)^{\otimes k}$ is linearly independent.

Proof. In the context of the general fattening and shrinking bijection from Proposition 3.12 above, it is elementary to see that we have:

$$|\pi \vee \sigma| = k + 2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|$$

We therefore have the following formula, valid for any $n \in \mathbb{N}$:

$$n^{|\pi\vee\sigma|} = n^{k+2|\pi'\vee\sigma'|-|\pi'|-|\sigma'|}$$

Thus, we obtain the formula in the statement. Now by applying the determinant, we obtain from this of formula of the following type, with $C > 0$ being a constant:

$$\det(G_{2k,n}) = C \cdot \det(G_{k,n^2})$$

Since we know from Proposition 3.9 above that we have $\det(G_{2k,n}) \neq 0$, we conclude that we have as well $\det(G_{k,n^2}) \neq 0$, as claimed. $\square$

We can work out now the representation theory of S_N^+, as follows:

Theorem 3.14. *The quantum groups S_N^+ with $N \geq 4$ have the following properties:*

1. *The moments of the main character are the Catalan numbers, $\int \chi^k = C_k$.*
2. *The main character follows the Marchenko-Pastur law, $\chi \sim \pi_1$.*
3. *The fusion rules are $r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+1} + \ldots + r_{k+l}$, as for SO_3.*
4. *The dimensions are $\dim(r_k) = \frac{q^{k+1}-q^{-k}}{q-1}$, where $q^2 - (N-2)q + 1 = 0$.*

Proof. We know from Proposition 3.13 that the vectors $\{\xi_\pi|\pi \in NC(k)\} \subset (\mathbb{C}^N)^{\otimes k}$ are linearly independent, and by using this, the proof, from [1], goes as follows:

(1) We have indeed the following computation, based on the above:

$$\int_{S_N^+} \chi^k = \dim(Fix(u^{\otimes k})) = \#NC(k) = \#NC_2(2k) = C_k$$

(2) This follows from (1), as explained in the proof of Theorem 2.21 (4) above.

(3) This is standard, by using the moment formula in (1), and the known theory of SO_3. Let indeed $A = span(\chi_k|k \in \mathbb{N})$ be the algebra of characters of SO_3. We can define a morphism $\Psi : A \to C(S_N^+)$ by $\chi_1 \to f_1 - 1$, where f_1 is the character of the fundamental representation of S_N^+. The elements $f_k = \Psi(\chi_k)$ verify then:

$$f_kf_l = f_{|k-l|} + f_{|k-l|+1} + \ldots + f_{k+l}$$

We prove now by recurrence on k that each f_k is the character of an irreducible corepresentation r_k of $C(S_N^+)$, non-equivalent to $r_0, \ldots, r_{k-1}$. At $k = 0, 1$ this is clear.

Assume now that the result holds at $k-1$. By integrating characters we have then $r_{k-2}, r_{k-1} \subset r_{k-1} \otimes r_1$, exactly as for SO_3, so there exists a corepresentation r_k such that $r_{k-1} \otimes r_1 = r_{k-2} + r_{k-1} + r_k$. Once again by integrating characters, we conclude that r_k is irreducible, and non-equivalent to $r_1, \ldots, r_{k-1}$, as for SO_3. This proves our claim.

Finally, since any irreducible representation of S_N^+ must appear in some tensor power of u, and we have a formula for decomposing each $u^{\otimes k}$ into sums of representations r_l, we conclude that these representations r_l are all the irreducible representations of S_N^+.

(4) The dimension formula there is clear by recurrence. □

The above results are quite surprising, and there are many things that can be said about O_N^+, S_N^+, in analogy with SU_2, SO_3. However, all this is quite technical, needing some solid algebraic knowledge, and we defer the discussion here to section 4 below.

Let us record however the following simple consequence of the above results:

Theorem 3.15. *The quantum groups O_N^+, S_N^+ have the following properties:*

(1) *O_2^+, S_4^+ are coamenable, and of polynomial growth.*
(2) *O_N^+, S_N^+ with $N \geq 3, 5$ are not coamenable, and have exponential growth.*

Proof. The various coamenability assertions follow from the Kesten criterion from Theorem 1.15 (4), the support of the spectral measure of χ being respectively:

$$supp(\gamma_1) = [-2, 2] \quad , \quad supp(\pi_1) = [0, 4]$$

As for the growth assertions, which can be of course improved with explicit exponents and so on, these follow from the fact that the corresponding Cayley graphs are $\mathbb{N}$. □

In the remainder of this section we keep developing useful general theory, in relation with subfactors and planar algebras. We first have the following result, from [3]:

Theorem 3.16. *Assume that a closed subgroup $G \subset U_N^+$ acts minimally on a II_1 factor A, in the sense that we have $(A^G)' \cap A = \mathbb{C}$.*

(1) *In the case $G \subset S_N^+$, the inclusion $A^G \subset (\mathbb{C}^N \otimes A)^G$ is a subfactor of index N, whose planar algebra is $P_k = Fix(u^{\otimes k})$.*
(2) *In the case $G \subset O_N^+$, the inclusion $A^G \subset (M_N(\mathbb{C}) \otimes A)^G$ is a subfactor of index N^2, whose planar algebra is $P_k = End(u^{\otimes k})$.*
(3) *In the case $G \subset U_N^+$, the inclusion $A^G \subset (M_N(\mathbb{C}) \otimes A)^G$ is a subfactor of index N^2, whose planar algebra is $P_k = End(u \otimes \bar{u} \otimes u \otimes \bar{u} \otimes \ldots)$, k terms.*

Proof. These results look quite similar, and they are indeed particular cases of some more general results, to be gradually discussed, later on. The idea is as follows:

(1) The above statements are particular cases of a general statement, involving group actions on finite dimensional algebras, and the associated fixed point subfactors:

$$G \curvearrowright B \implies A^G \subset (B \otimes A)^G$$

We will discuss all this in section 4 below, the idea being that with $B = C(X)$ we need an action $G \curvearrowright X$, and so a generalized quantum permutation group, $G \subset S_X^+$.

(2) Even more generally, we can consider quantum groups acting on inclusions of finite dimensional algebras, and we have a construction of the following type:

$$G \curvearrowright (B_0 \subset B_1) \implies (B_0 \otimes A)^G \subset (B_1 \otimes A)^G$$

We will discuss this in section 4 below as well, the idea being that with $B_i = C(X_i)$ we have a fibration $X_1 \to X_0$, and so we need an action $G \curvearrowright X_1$, leaving X_0 invariant.

(3) Let us briefly explain, however, what happens in the general case, where we have a subfactor as above. Consider the Jones tower for the inclusion $B_0 \subset B_1$:

$$B_0 \subset B_1 \subset B_2 \subset B_3 \subset \ldots\ldots$$

It is elementary to check that the Jones tower for our subfactor is as follows:

$$(B_0 \otimes A)^G \subset (B_1 \otimes A)^G \subset (B_2 \otimes A)^G \subset \ldots\ldots$$

The point now is that when looking at the commutants, the factor A dissapears in the computation, and so the associated planar algebra is given by:

$$P_k = \left[(B_0 \otimes A)^G\right]' \cap (B_k \otimes A)^G = (B_0' \cap B_k)^G$$

This result is something quite general, and with $B_0 = \mathbb{C}$ we obtain the various statements formulated above. As already mentioned, we will be back to this. $\square$

Generally speaking, the about result is something quite heavy, which is mainly useful in connection with subfactors, and with von Neumann algebras in general.

For concrete applications to quantum groups, the interesting statements are those regarding the planar algebras. Our purpose in what follows will be that of explaining in detail these results, and especially (1), and establishing some converse results too.

As a first observation, the construction $G \to P$ being contravariant, we have implications as follows, with $\mathcal{S}_N, \mathcal{T}_N$ being respectively the planar algebras associated to the trivial group $G = \{1\}$, in each of the cases under investigation:

$$G \subset S_N^+ \implies P \subset \mathcal{S}_N$$

$$G \subset O_N^+/G \subset U_N^+ \implies P \subset \mathcal{T}_N$$

Thus, our first goal will be that of explaining the precise definition of $\mathcal{S}_N, \mathcal{T}_N$. These two algebras, called spin planar algebra, and tensor planar algebra, are among the "simplest" planar algebras, and their construction goes back to Jones' paper [57].

Let us begin with the construction of $\mathcal{T}_N$. This is as follows:

Definition 3.17. *The tensor planar algebra $\mathcal{T}_N$ is the sequence of vector spaces*

$$P_k = M_N(\mathbb{C})^{\otimes k}$$

with the multilinear maps associated to the various k-tangles

$$T_\pi : P_{k_1} \otimes \ldots \otimes P_{k_r} \to P_k$$

being given by the following formula, in multi-index notation,

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_r}) = \sum_j \delta_\pi(i_1, \ldots, i_r : j) e_j$$

with the Kronecker symbols δ_π being 1 if the indices fit, and being 0 otherwise.

In other words, we are using here a construction which is very similar to the construction $\pi \to T_\pi$ from easy quantum group theory. We put the indices of the basic tensors on the marked points of the small boxes, in the obvious way, and the coefficients of the output tensor are then given by Kronecker symbols, exactly as in the easy case.

In what regards now the spin planar algebra $\mathcal{S}_N$, the construction here is quite similar, but using this time the algebra $\mathbb{C}^N$ instead of the algebra $M_N(\mathbb{C})$.

There is one subtlety, however, coming from the fact that the general planar algebra formalism, from Definition 3.7 above, requires the tensors to have even length. Note that this was automatic for $\mathcal{T}_N$, where the tensors of $M_N(\mathbb{C})$ have length 2.

In the case of the spin planar algebra $\mathcal{S}_N$, where we want the vector spaces to be $P_k = (\mathbb{C}^N)^{\otimes k}$, we must double the indices of the tensors, in the following way:

Definition 3.18. *We write the standard basis of $(\mathbb{C}^N)^{\otimes k}$ in $2 \times k$ matrix form,*

$$e_{i_1 \ldots i_k} = \begin{pmatrix} i_1 & i_1 & i_2 & i_2 & i_3 & \ldots & \ldots \\ i_k & i_k & i_{k-1} & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

by duplicating the indices, and then writing them clockwise, starting from top left.

We will be back in a second with examples of this. Now with this convention in hand, we can formulate the construction of $\mathcal{S}_N$, also from [57], as follows:

Definition 3.19. *The spin planar algebra $\mathcal{S}_N$ is the sequence of vector spaces*

$$P_k = (\mathbb{C}^N)^{\otimes k}$$

written as above, with the multilinear maps associated to the various k-tangles

$$T_\pi : P_{k_1} \otimes \ldots \otimes P_{k_r} \to P_k$$

being given by the following formula, in multi-index notation,

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_r}) = \sum_j \delta_\pi(i_1, \ldots, i_r : j) e_j$$

with the Kronecker symbols δ_π being 1 if the indices fit, and being 0 otherwise.

In other words, we are using exactly the same construction as for the tensor planar algebra $\mathcal{T}_N$, which was itself very related to the easy quantum group formalism, but with $M_N(\mathbb{C})$ replaced by $\mathbb{C}^N$, with the indices doubled, as in Definition 3.18.

As already mentioned, the reasons for doubling basically come from the general planar algebra formalism, from Definition 3.7 above, which requires the tensors to have even length. At a more conceptual level, all this is related to the fattening and shrinking operations from Proposition 3.12 above. We will be back to this.

The planar calculus for tensors is quite simple, and doesn't really require diagrams. It suffices to imagine that the way various indices appear, travel around and dissapear is by following some obvious strings connecting them. Here are some illustrating examples:

Example 3.20. *Identity, multiplication, inclusion.*

The identity 1_k is the (k,k)-tangle having vertical strings only. The solutions of $\delta_{1_k}(x,y)=1$ being the pairs of the form (x,x), this tangle 1_k acts by the identity:

$$1_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

The multiplication M_k is the (k,k,k)-tangle having 2 input boxes, one on top of the other, and vertical strings only. It acts in the following way:

$$M_k\left(\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}\otimes\begin{pmatrix}l_1&\ldots&l_k\\ m_1&\ldots&m_k\end{pmatrix}\right)=\delta_{j_1m_1}\ldots\delta_{j_km_k}\begin{pmatrix}l_1&\ldots&l_k\\ i_1&\ldots&i_k\end{pmatrix}$$

The inclusion I_k is the $(k,k+1)$-tangle which looks like 1_k, but has one more vertical string, at right of the input box. Given x, the solutions of $\delta_{I_k}(x,y)=1$ are the elements y obtained from x by adding to the right a vector of the form $\binom{l}{l}$, and so:

$$I_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\sum_l\begin{pmatrix}j_1&\ldots&j_k&l\\ i_1&\ldots&i_k&l\end{pmatrix}$$

Observe that I_k is an inclusion of algebras, and that the various I_k are compatible with each other. The inductive limit of the algebras $\mathcal{S}_N(k)$ is a graded algebra, denoted $\mathcal{S}_N$.

Example 3.21. *Expectation, Jones projection.*

The expectation U_k is the $(k+1,k)$-tangle which looks like 1_k, but has one more string, connecting the extra 2 input points, both at right of the input box:

$$U_k\begin{pmatrix}j_1&\ldots&j_k&j_{k+1}\\ i_1&\ldots&i_k&i_{k+1}\end{pmatrix}=\delta_{i_{k+1}j_{k+1}}\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

Observe that U_k is a bimodule morphism with respect to I_k.

The Jones projection E_k is a $(0,k+2)$-tangle, having no input box. There are k vertical strings joining the first k upper points to the first k lower points, counting from left to right. The remaining upper 2 points are connected by a semicircle, and the remaining lower 2 points are also connected by a semicircle. We have the following formula:

$$E_k(1) = \sum_{ijl} \begin{pmatrix} i_1 & \dots & i_k & j & j \\ i_1 & \dots & i_k & l & l \end{pmatrix}$$

The elements $e_k = N^{-1} E_k(1)$ are projections, and define a representation of the infinite Temperley-Lieb algebra of index N inside the inductive limit algebra $\mathcal{S}_N$.

Example 3.22. *Rotation.*

The rotation R_k is the (k,k)-tangle which looks like 1_k, but the first 2 input points are connected to the last 2 output points, and the same happens at right:

$$R_k = \begin{matrix} \Cap\,|\,|\,|\,\| \\ \|\qquad\;\| \\ \|\,|\,|\,|\,\Cup \end{matrix}$$

The action of R_k on the standard basis is by rotation of the indices, as follows:

$$R_k(e_{i_1 \dots i_k}) = e_{i_2 i_3 \dots i_k i_1}$$

Thus R_k acts by an order k linear automorphism of $\mathcal{S}_N(k)$, also called rotation.

There are many other interesting examples of k-tangles, but in view of our present purposes, we can actually stop here, due to the following useful fact:

Theorem 3.23. *The multiplications, inclusions, expectations, Jones projections, and rotations generate the set of all tangles, via the gluing operation.*

Proof. This is something well-known and elementary, obtained by "chopping" the various planar tangles into small pieces, as in the above list. See [57]. □

Finally, in order for things to be complete, we must talk as well about the $*$-structure. Once again this is constructed as in the easy quantum group calculus, as follows:

$$\begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix}^* = \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix}$$

Summarizing, the sequence of vector spaces $\mathcal{S}_N(k) = C(X^k)$ has a planar $*$-algebra structure, called spin planar algebra of index $N = |X|$. See [57].

Let us get now to the second part of our program, namely proving that any quantum permutation group $G \subset S_N^+$ produces a planar subalgebra of $\mathcal{S}_N$.

As already mentioned, this is something which follows from Theorem 3.16, but in view of concrete applications, we would like to have a purely algebraic proof of this fact.

Following [4], the precise statement, and its proof, are as follows:

Theorem 3.24. *Given a quantum permutation group $G\subset S_N^+$, consider the associated coaction map on $C(X)$, where $X=\{1,\ldots,N\}$,*

$$\Phi:C(X)\to C(X)\otimes C(G)\quad,\quad e_j\to\sum_j e_j\otimes u_{ji}$$

and then consider the tensor powers of this coaction, which are the following linear maps:

$$\Phi^k:C(X^k)\to C(X^k)\otimes C(G)\quad,\quad e_{i_1\ldots i_k}\to\sum_{j_1\ldots j_k}e_{j_1\ldots j_k}\otimes u_{j_1i_1}\ldots u_{j_ki_k}$$

The fixed point spaces of these latter coactions, which are by definition the spaces

$$P_k=\left\{x\in C(X^k)\Big|\Phi^k(x)=1\otimes x\right\}$$

are given by $P_k=Fix(u^{\otimes k})$, and form a planar subalgebra of $\mathcal{S}_N$.

Proof. Since the map Φ is a coaction, coming from the corepresentation u, its tensor powers Φ^k are coactions too, coming fron the corepresentations $u^{\otimes k}$, and at the level of the fixed point algebras we have the following formula, which is standard:

$$P_k=Fix(u^{\otimes k})$$

In order to prove now the planar algebra assertion, we will use Theorem 3.23.

Consider the rotation R_k. Rotating, then applying Φ^k, and rotating backwards by R_k^{-1} is the same as applying Φ^k, then rotating each k-fold product of coefficients of Φ.

Thus the elements obtained by rotating, then applying Φ^k, or by applying Φ^k, then rotating, differ by a sum of Dirac masses tensored with commutators in $A=C(G)$:

$$\Phi^kR_k(x)-(R_k\otimes id)\Phi^k(x)\in C(X^k)\otimes[A,A]$$

Now let $\int_A$ be the Haar functional of A, and consider the conditional expectation onto the fixed point algebra P_k, which is given by the following formula:

$$\phi_k=\left(id\otimes\int_A\right)\Phi^k$$

The square of the antipode being the identity, the Haar integration $\int_A$ is a trace, so it vanishes on commutators. Thus R_k commutes with ϕ_k:

$$\phi_kR_k=R_k\phi_k$$

The commutation relation $\phi_kT=T\phi_l$ holds in fact for any (l,k)-tangle T. These tangles are called annular, and the proof is by verification on generators of the annular category. In particular we obtain, for any annular tangle T:

$$\phi_kT\phi_l=T\phi_l$$

We conclude from this that the annular category is contained in the suboperad $\mathcal{P}' \subset \mathcal{P}$ of the planar operad consisting of tangles T satisfying the following condition, where $\phi = (\phi_k)$, and where $i(.)$ is the number of input boxes:

$$\phi T \phi^{\otimes i(T)} = T \phi^{\otimes i(T)}$$

On the other hand the multiplicativity of Φ^k gives $M_k \in \mathcal{P}'$. Since $\mathcal{P}$ is generated by multiplications and annular tangles, it follows that we have $\mathcal{P}' = P$.

Thus for any tangle T the corresponding multilinear map between spaces $P_k(X)$ restricts to a multilinear map between spaces P_k. In other words, the action of the planar operad $\mathcal{P}$ restricts to P, and makes it a subalgebra of $\mathcal{S}_N$, as claimed. □

As a third and last result now, also from [4], completing our study, we have:

Theorem 3.25. *Given a planar subalgebra $Q \subset \mathcal{S}_N$, there is a unique quantum permutation group $G \subset S_N^+$ whose associated planar algebra is Q.*

Proof. This will follow by applying Tannakian duality to the annular category over Q. This is constructed as follows. Let n, m be positive integers. To any element $T_{n+m} \in Q_{n+m}$ we associate a linear map $L_{nm}(T_{n+m}) : P_n(X) \to P_m(X)$ in the following way:

$$L_{nm}\begin{pmatrix} |\,|\,| \\ T_{n+m} \\ |\,|\,| \end{pmatrix} : \begin{pmatrix} | \\ a_n \\ | \end{pmatrix} \to \begin{pmatrix} |\ \ |\ \ \cap \\ \ \ T_{n+m}| \\ |\,|\ \ |\ \ | \\ a_n|\ \ |\ \ | \\ \cup\ \ \ |\ \ | \end{pmatrix}$$

That is, we consider the planar $(n, n+m, m)$-tangle having an small input n-box, a big input $n+m$-box and an output m-box, with strings as on the picture of the right. This defines a certain multilinear map, as follows:

$$P_n(X) \otimes P_{n+m}(X) \to P_m(X)$$

Now let us put the element T_{n+m} in the big input box. We obtain in this way a certain linear map $P_n(X) \to P_m(X)$, that we call L_{nm}.

The above picture corresponds to $n = 1$ and $m = 2$. This is illustrating whenever $n \leq m$, suffices to imagine that in the general case all strings are multiple.

If $n > m$ there are $n+m$ strings of a_n which connect to the $n+m$ lower strings of T_{n+m}, and the remaining $n-m$ ones go to the upper right side and connect to the $n-m$ strings on top right of T_{n+m}. Here is the picture for $n = 2$ and $m = 1$:

$$L_{nm}\begin{pmatrix} |\,|\,| \\ T_{n+m} \\ |\,|\,| \end{pmatrix} : \begin{pmatrix} |\,| \\ a_n \\ |\,| \end{pmatrix} \to \begin{pmatrix} |\ \ \Cap \\ T_{n+m}\| \\ \ \ \ \|\ \ \||| \\ \ \ \ \ \ a_n||| \\ \ \ \ \ \ \ \ \Cup| \\ \ \ \ \ \ \ \ \ \ | \end{pmatrix}$$

This problem with two cases $n \leq m$ and $n > m$ can be avoided by using an uniform approach, with discs with marked points instead of boxes. See [57].

Consider the linear spaces formed by such maps:

$$Q_{nm} = \{L_{nm}(T_{n+m}) : P_n(X) \to P_m(X) \mid T_{n+m} \in Q_{n+m}\}$$

Pictures show that these spaces form a tensor C^*-subcategory of the tensor C^*-category of linear maps between tensor powers of the Hilbert space $H = C(X)$. If j is the antilinear map from $C(X)$ to itself given by $j\binom{i}{i} = \binom{i}{i}$ for any i, then the elements $t_j(1)$ and $t_{j^{-1}}(1)$ constructed by Woronowicz in [100] are both equal to the unit of $Q_2 = Q_{02}$. In other words, the tensor C^*-category has conjugation, and Tannakian duality in [100] applies.

We get a pair (H, v) consisting of a unital Hopf C^*-algebra H and a unitary corepresentation v of H on $C(X)$, such that the following equalities hold, for any m, n:

$$Hom(v^{\otimes m}, v^{\otimes n}) = Q_{mn}$$

We prove that v is a magic biunitary. We have $Hom(1, v^{\otimes 2}) = Q_{02} = Q_2$, so the unit of Q_2 must be a fixed vector of $v^{\otimes 2}$. But $v^{\otimes 2}$ acts on the unit of Q_2 as follows:

$$v^{\otimes 2}(1) = v^{\otimes 2}\left(\sum_i \begin{pmatrix} i & i \\ i & i \end{pmatrix}\right) = \sum_{ikl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes v_{ki}v_{li} = \sum_{kl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes (vv^t)_{kl}$$

From $v^{\otimes 2}(1) = 1 \otimes 1$ ve get that vv^t is the identity matrix. Together with the unitarity of v, this gives the following formulae:

$$v^t = v^* = v^{-1}$$

Consider the Jones projection $E_1 \in Q_3$. After isotoping $L_{21}(E_1)$ looks as follows:

$$L_{21}\left(\begin{matrix} |\cup \\ |\cap \end{matrix}\right) : \begin{pmatrix} | & | \\ i & i \\ j & j \\ | & | \end{pmatrix} \to \begin{pmatrix} | & & \\ i & i & \supset \\ j & j & \\ | & & \end{pmatrix} = \delta_{ij} \begin{pmatrix} | \\ i \\ i \\ | \end{pmatrix}$$

In other words, the linear map $M = L_{21}(E_1)$ is the multiplication $\delta_i \otimes \delta_j \to \delta_{ij}\delta_i$:

$$M \begin{pmatrix} i & i \\ j & j \end{pmatrix} = \delta_{ij} \begin{pmatrix} i \\ i \end{pmatrix}$$

Consider now the following element of $C(X) \otimes H$:

$$\begin{aligned} (M \otimes id)v^{\otimes 2}\left(\begin{pmatrix} i & i \\ j & j \end{pmatrix} \otimes 1\right) &= (M \otimes id)\left(\sum_{kl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes v_{ki}v_{lj}\right) \\ &= \sum_k \begin{pmatrix} k \\ k \end{pmatrix} \delta_k \otimes v_{ki}v_{kj} \end{aligned}$$

Since $M\in Q_{21}=Hom(v^{\otimes 2},v)$, this equals the following element of $C(X)\otimes H$:

$$\begin{aligned}v(M\otimes id)\left(\begin{pmatrix}i&i\\j&j\end{pmatrix}\otimes 1\right)&=&v\left(\delta_{ij}\begin{pmatrix}i\\i\end{pmatrix}\delta_i\otimes 1\right)\\&=&\sum_k\begin{pmatrix}k\\k\end{pmatrix}\delta_k\otimes\delta_{ij}v_{ki}\end{aligned}$$

Thus $v_{ki}v_{kj}=\delta_{ij}v_{ki}$ for any i,j,k. With $i=j$ we get $v_{ki}^2=v_{ki}$, and together with the formula $v^t=v^*$ this shows that all entries of v are self-adjoint projections. With $i\neq j$ we get $v_{ki}v_{kj}=0$, so projections on each row of v are orthogonal to each other. Together with $v^t=v^{-1}$ this shows that each row of v is a partition of unity with self-adjoint projections.

The antipode is given by the formula $(id\otimes S)v=v^*$. But v^* is the transpose of v, so we can apply S to the formulae saying that rows of v are partitions of unity, and we get that columns of v are also partitions of unity. Thus v is a magic biunitary.

Consider the planar algebra P associated to v. We have the following equalities:

$$Hom(1,v^{\otimes n})=P_n$$

Thus $P_n=Q_n$ for any n and this proves the existence assertion.

As for uniqueness, let (K,w) be another pair corresponding to Q. The functorial properties of Tannakian duality give a morphism $f:(H,v)\to(K,w)$. Since morphisms increase spaces of fixed points we have the following inclusions:

$$Q_k=Hom(1,v^{\otimes k})\subset Hom(1,w^{\otimes k})=Q_k$$

We must have equality for any k, and by using Frobenius reciprocity and a basis of coefficients of irreducible corepresentations, and Peter-Weyl theory, we see that f must be an isomorphism on this basis, and we are done. □

Summarizing, we are done with the case $G\subset S_N$, with a full explanation of the planar algebra assertion in Theorem 3.16 (1), coming with a direct proof, and with a useful converse statement as well, obtained by suitably modifying Tannakian duality.

Still in relation with Theorem 3.16, but in connection with (2,3) there, in the case $G\subset U_N^+$, the spaces $P_k=End(u\otimes\bar{u}\otimes u\otimes\bar{u}\otimes\ldots)$ can be shown to form a subalgebra of the tensor planar algebra $\mathcal{T}_N$. Any subalgebra $P\subset\mathcal{T}_N$ appears in this way, the correspondence with the subgroups of $PO_N^+=PU_N^+$ being bijective. See [5].

4. Symmetry groups

We have seen in the previous section, as a main result, that S_N^+ with $N \geq 4$ has the same fusion rules as SO_3. This is something quite surprising. Although our explanations were quite conceptual, with $N \geq 4$ being regarded as a Jones index, and with everything coming from the magics of the Temperley-Lieb algebra, the occurrence of SO_3 in this quantum permutation business, supposed to be "discrete", remains quite mysterious.

In this section we investigate the quantum permutation groups S_X^+ of the finite noncommutative spaces X. Besides providing a useful generalization of our results regarding S_N^+, this will eventually explain the connection with SO_3, in an elegant way. As a bonus, we will obtain as well a conceptual result on the connection between S_N^+ and O_N^+.

In order to get started, let us first talk about finite noncommutative spaces. According to our general Gelfand duality philosophy, these spaces should be the abstract duals of the finite dimensional C^*-algebras. And these latter algebras are subject to:

Theorem 4.1. *Let B be a finite dimensional C^*-algebra.*

1. *We can write $1 = p_1 + \ldots + p_k$, with $p_i \in B$ central minimal projections.*
2. *Each of the linear spaces $B_i = p_i B p_i$ is a non-unital $*$-subalgebra of B.*
3. *We have a non-unital $*$-algebra sum decomposition $B = B_1 \oplus \ldots \oplus B_k$.*
4. *We have unital $*$-algebra isomorphisms $B_i \simeq M_{n_i}(\mathbb{C})$, where $n_i = rank(p_i)$.*
5. *Thus, we have a C^*-algebra isomorphism $B \simeq M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$.*

Proof. This is something well-known, the idea being as follows:
(1) This is rather a definition.
(2) This follows indeed from $p_i^2 = p_i^* = p_i$.
(3) The various verifications here are routine.
(4) This follows from the minimality assumption on the p_i.
(5) This is the final conclusion, which follows from (3,4). □

We can now formulate our definition, as follows:

Definition 4.2. *A finite noncommutative space X is the abstract dual of a finite dimensional C^*-algebra B, according to the following formula:*

$$C(X) = B$$

The number of elements of such a space is by definition $|X| = \dim B$. By decomposing the algebra B, as above, we have a formula of the following type:

$$C(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

With $n_1 = \ldots = n_k = 1$ we obtain in this way the space $X = \{1, \ldots, k\}$. Also, when $k = 1$ the equation is $C(X) = M_n(\mathbb{C})$, and the solution will be denoted $X = M_n$.

In order to talk now about the quantum symmetry group S_X^+, the situation is a bit more complicated than before, because the magic condition has no simple extension to this setting. Thus, we must use coactions, and a result similar to Theorem 2.5.

Now recall from Theorem 2.5 that, in order for things to work, we must endow our space X with its counting measure. In general, this can be done as follows:

Definition 4.3. *We endow each finite noncommutative space X with its counting measure, corresponding as the algebraic level to the integration functional*

$$tr : C(X) \to B(l^2(X)) \to \mathbb{C}$$

obtained by applying the regular representation, and then the normalized matrix trace.

To be more precise, consider the algebra $B = C(X)$, which is by definition finite dimensional. We can make act B on itself, by left multiplication:

$$\pi : B \to \mathcal{L}(B) \quad , \quad a \to (b \to ab)$$

The target of π being a matrix algebra, $\mathcal{L}(B) \simeq M_N(\mathbb{C})$ with $N = \dim B$, we can further compose with the normalized matrix trace, and we obtain tr:

$$tr = \frac{1}{N} Tr \circ \pi$$

As basic examples, for both $X = \{1, \ldots, N\}$ and $X = M_N$ we obtain the usual trace. In general, with $C(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$, the weights of tr are $n_i^2 / \sum_i n_i^2$.

Let us study now the quantum group actions $G \curvearrowright X$. We denote by μ, η the multiplication and unit map of the algebra $C(G)$. Following [1], we first have:

Proposition 4.4. *Consider a linear map $\Phi : C(X) \to C(X) \otimes C(G)$, written as*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

with $\{e_i\}$ being a linear space basis of $C(X)$, orthonormal with respect to tr.

1. *Φ is a linear space coaction $\iff$ u is a corepresentation.*
2. *Φ is multiplicative $\iff$ $\mu \in Hom(u^{\otimes 2}, u)$.*
3. *Φ is unital $\iff$ $\eta \in Hom(1, u)$.*
4. *Φ leaves invariant tr $\iff$ $\eta \in Hom(1, u^*)$.*
5. *If these conditions hold, Φ is involutive $\iff$ u is unitary.*

Proof. This is a bit similar to the proof of Theorem 2.5 above, as follows:

(1) There are two axioms to be processed here. First, we have:

$$\begin{aligned}(id\otimes\Delta)\Phi=(\Phi\otimes id)\Phi\quad&\iff\quad\sum_je_j\otimes\Delta(u_{ji})=\sum_k\Phi(e_k)\otimes u_{ki}\\&\iff\quad\sum_je_j\otimes\Delta(u_{ji})=\sum_{jk}e_j\otimes u_{jk}\otimes u_{ki}\\&\iff\quad\Delta(u_{ji})=\sum_ku_{jk}\otimes u_{ki}\end{aligned}$$

As for the axiom involving the counit, here we have as well, as desired:

$$\begin{aligned}(id\otimes\varepsilon)\Phi=id\quad&\iff\quad\sum_j\varepsilon(u_{ji})e_j=e_i\\&\iff\quad\varepsilon(u_{ji})=\delta_{ji}\end{aligned}$$

(2) We have the following formula:

$$\begin{aligned}\Phi(e_i)&=\sum_je_j\otimes u_{ji}\\&=\left(\sum_{ij}e_{ji}\otimes u_{ji}\right)(e_i\otimes1)\\&=u(e_i\otimes1)\end{aligned}$$

By using this formula, we obtain the following identity:

$$\begin{aligned}\Phi(e_ie_k)&=u(e_ie_k\otimes1)\\&=u(\mu\otimes id)(e_i\otimes e_k\otimes1)\end{aligned}$$

On the other hand, we have as well the following identity, as desired:

$$\begin{aligned}\Phi(e_i)\Phi(e_k)&=\sum_{jl}e_je_l\otimes u_{ji}u_{lk}\\&=(\mu\otimes id)\sum_{jl}e_j\otimes e_l\otimes u_{ji}u_{lk}\\&=(\mu\otimes id)\left(\sum_{ijkl}e_{ji}\otimes e_{lk}\otimes u_{ji}u_{lk}\right)(e_i\otimes e_k\otimes1)\\&=(\mu\otimes id)u^{\otimes2}(e_i\otimes e_k\otimes1)\end{aligned}$$

(3) The formula $\Phi(e_i)=u(e_i\otimes1)$ found above gives by linearity $\Phi(1)=u(1\otimes1)$, which shows that Φ is unital precisely when $u(1\otimes1)=1\otimes1$, as desired.

(4) This follows from the following computation, by applying the involution:

$$\begin{aligned}(tr\otimes id)\Phi(e_i)=tr(e_i)1 &\iff \sum_j tr(e_j)u_{ji}=tr(e_i)1\\ &\iff \sum_j u_{ji}^*1_j=1_i\\ &\iff (u^*1)_i=1_i\\ &\iff u^*1=1\end{aligned}$$

(5) Assuming that (1-4) are satisfied, and that Φ is involutive, we have:

$$\begin{aligned}(u^*u)_{ik} &= \sum_l u_{li}^*u_{lk}\\ &= \sum_{jl} tr(e_j^*e_l)u_{ji}^*u_{lk}\\ &= (tr\otimes id)\sum_{jl} e_j^*e_l\otimes u_{ji}^*u_{lk}\\ &= (tr\otimes id)(\Phi(e_i)^*\Phi(e_k))\\ &= (tr\otimes id)\Phi(e_i^*e_k)\\ &= tr(e_i^*e_k)1\\ &= \delta_{ik}\end{aligned}$$

Thus $u^*u=1$, and since we know from (1) that u is a corepresentation, it follows that u is unitary. The proof of the converse is standard too, by using similar tricks. □

Following now [1], we have the following result, extending the basic theory of S_N^+ from the previous section to the present finite noncommutative space setting:

Theorem 4.5. *Given a finite noncommutative space X, there is a universal compact quantum group S_X^+ acting on X, leaving the counting measure invariant. We have*

$$C(S_X^+)=C(U_N^+)\Big/\Big\langle\mu\in Hom(u^{\otimes2},u),\eta\in Fix(u)\Big\rangle$$

where $N=|X|$ and where μ,η are the multiplication and unit maps of $C(X)$. For $X=\{1,\ldots,N\}$ we have $S_X^+=S_N^+$. Also, for the space $X=M_2$ we have $S_X^+=SO_3$.

Proof. This result is from [1], the idea being as follows:

(1) This follows from Proposition 4.4 above, by using the standard fact that the complex conjugate of a corepresentation is a corepresentation too.

(2) Regarding now the main example, for $X=\{1,\ldots,N\}$ we obtain indeed the quantum permutation group S_N^+, due to the abstract result in Theorem 2.5 above.

(3) In order to do now the computation for $X = M_2$, we use some standard facts about SU_2, SO_3. We have an action by conjugation $SU_2 \curvearrowright M_2(\mathbb{C})$, and this action produces, via the canonical quotient map $SU_2 \to SO_3$, an action $SO_3 \curvearrowright M_2(\mathbb{C})$.

On the other hand, it is routine to check, by using arguments like those in the proof of Theorem 2.4 at $N = 2, 3$, that any action $G \curvearrowright M_2(\mathbb{C})$ must come from a classical group. We conclude that the action $SO_3 \curvearrowright M_2(\mathbb{C})$ is universal, as claimed. $\square$

Regarding now the representation theory of these generalized quantum permutation groups S_X^+, the result here, also from [1], is very similar to the one for S_N^+, as follows:

Theorem 4.6. *The quantum groups S_X^+ have the following properties:*

1. *The associated Tannakian categories are $TL(N)$, with $N = |X|$.*
2. *The main character follows the Marchenko-Pastur law π_1, when $N \geq 4$.*
3. *The fusion rules for S_X^+ with $|X| \geq 4$ are the same as for SO_3.*

Proof. Once again this result is from [1], the idea being as follows:

(1) Our first claim is that the fundamental representation is equivalent to its adjoint, $u \sim \bar{u}$. Indeed, let us go back to the coaction formula from Proposition 4.4:

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

We can pick our orthogonal basis $\{e_i\}$ to be the stadard multimatrix basis of $C(X)$, so that we have $e_i^* = e_{i^*}$, for a certain involution $i \to i^*$ on the index set. With this convention made, by conjugating the above formula of $\Phi(e_i)$, we obtain:

$$\Phi(e_{i^*}) = \sum_j e_{j^*} \otimes u_{ji}^*$$

Now by interchanging $i \leftrightarrow i^*$ and $j \leftrightarrow j^*$, this latter formula reads:

$$\Phi(e_i) = \sum_j e_j \otimes u_{j^*i^*}^*$$

We therefore conclude, by comparing with the original formula, that we have:

$$u_{ji}^* = u_{j^*i^*}$$

But this shows that we have an equivalence $u \sim \bar{u}$, as claimed. Now with this result in hand, the proof goes as for the proof for S_N^+, from the previous section. To be more precise, the result follows from the fact that the multiplication and unit of any complex algebra, and in particular of $C(X)$, can be modelled by the following two diagrams:

$$m = |\cup| \quad , \quad u = \cap$$

Indeed, this is certainly true algebrically, and this is something well-known. As in what regards the $*$-structure, things here are fine too, because our choice for the trace from

Definition 4.3 leads to the following formula, which must be satisfied as well:

$$\mu\mu^* = N \cdot id$$

But the above diagrams m, u generate the Temperley-Lieb algebra $TL(N)$, as stated.

(2) The proof here is exactly as for S_N^+, by using moments. To be more precise, according to (1) these moments are the Catalan numbers, which are the moments of π_1.

(3) Once again same proof as for S_N^+, by using the fact that the moments of χ are the Catalan numbers, which naturally leads to the Clebsch-Gordan rules. □

It is quite clear now that our present formalism, and the above results, provide altogether a good and conceptual explanation for our SO_3 result regarding S_N^+. To be more precise, we can merge and reformulate the above results in the following way:

Theorem 4.7. *The quantun groups S_X^+ have the following properties:*

1. *For $X = \{1, \ldots, N\}$ we have $S_X^+ = S_N^+$.*
2. *For the space $X = M_N$ we have $S_X^+ = PO_N^+ = PU_N^+$.*
3. *In particular, for the space $X = M_2$ we have $S_X^+ = SO_3$.*
4. *The fusion rules for S_X^+ with $|X| \geq 4$ are independent of X.*
5. *Thus, the fusion rules for S_X^+ with $|X| \geq 4$ are the same as for SO_3.*

Proof. This is basically a compact form of what has been said above, with a new result added, and with some technicalities left aside:

(1) This is something that we know from Theorem 4.5.

(2) This is new, the idea being as follows. First of all, we know from Theorem 1.28 above that the inclusion $PO_N^+ \subset PU_N^+$ is an isomorphism, with this coming from the free complexification formula $\widetilde{O}_N^+ = U_N^+$, but we will actually reprove this result.

Consider the standard vector space action $U_N^+ \curvearrowright \mathbb{C}^N$, and then its adjoint action $PU_N^+ \curvearrowright M_N(\mathbb{C})$. By universality of $S_{M_N}^+$, we have inclusions as follows:

$$PO_N^+ \subset PU_N^+ \subset S_{M_N}^+$$

On the other hand, the main character of O_N^+ with $N \geq 2$ being semicircular, the main character of PO_N^+ must be Marchenko-Pastur. Thus the inclusion $PO_N^+ \subset S_{M_N}^+$ has the property that it keeps fixed the law of main character, and by Peter-Weyl theory we conclude that this inclusion must be an isomorphism, as desired.

(3) This is something that we know from Theorem 4.5, and that can be deduced as well from (2), by using the formula $PO_2^+ = SO_3$, which is something elementary.

(4) This is something that we know from Theorem 4.6.

(5) This follows from (3,4), as already pointed out in Theorem 4.6. □

Summarizing, we have now a good explanation for the occurrence of SO_3, in connection with quantum permutation questions. Philosophically, the idea is that S_X^+ does not depend that much on X, and so in order to obtain results, it is enough to take $X = M_2$, where the corresponding symmetry group is simply $S_X^+ = SO_3$, and then to conclude.

Let us focus now on the case $N = 4$, where the similarity between S_N^+ and SO_3 is even more striking, because the irreducible representations have the same dimensions. According to the above philosophical considerations, the link comes as follows:

$$\{1,2,3,4\} \sim M_2 \implies S_4^+ \sim SO_3$$

This is of course quite philosophical, but it is possible to get beyond this, with a very precise result, stating that S_4^+ is a twist of SO_3. Let us start with:

Definition 4.8. *$C(SO_3^{-1})$ is the universal C^*-algebra generated by the entries of a 3×3 orthogonal matrix $a = (a_{ij})$, with the following relations:*

(1) *Skew-commutation: $a_{ij}a_{kl} = \pm a_{kl}a_{ij}$, with sign $+$ if $i \neq k, j \neq l$, and $-$ otherwise.*
(2) *Twisted determinant condition: $\Sigma_{\sigma \in S_3} a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} = 1$.*

We should mention here that this looks a bit like a Drinfeld-Jimbo twist at $q = -1$, but it is not. There are in fact many other instances of this phenomenon, with the correct $q = -1$ twists being known from the compact quantum group literature.

Normally, our first task would be to prove that $C(SO_3^{-1})$ is a Woronowicz algebra. This is of course possible, by doing some computations, but we will not need to do these computations, because the result follows from the following result, from [9]:

Theorem 4.9. *We have an isomorphism of compact quantum groups*

$$S_4^+ = SO_3^{-1}$$

given by the Fourier transform over the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Consider indeed the matrix $a^+ = diag(1, a)$, corresponding to the action of SO_3^{-1} on $\mathbb{C}^4$, and apply to it the Fourier transform over the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$:

$$u = \frac{1}{4}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&a_{11}&a_{12}&a_{13}\\0&a_{21}&a_{22}&a_{23}\\0&a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}$$

It is routine to check that this matrix is magic, and vice versa, i.e. that the Fourier transform over K converts the relations in Definition 4.8 into the magic relations in Proposition 2.1. Thus, we obtain the identification from the statement. $\square$

The above result is very useful for investigating S_4^+ and its closed subgroups $G \subset S_4^+$. We have the following classification result, also from [9]:

Theorem 4.10. *The closed subgroups of $S_4^+ = SO_3^{-1}$ are as follows:*

(1) *Infinite quantum groups: S_4^+, O_2^{-1}, $\widehat{D}_\infty$.*
(2) *Finite groups: S_4, and its subgroups.*
(3) *Finite group twists: S_4^{-1}, A_5^{-1}.*
(4) *Series of twists: D_{2n}^{-1} $(n \geq 3)$, DC_n^{-1} $(n \geq 2)$.*
(5) *A group dual series: $\widehat{D}_n$, with $n \geq 3$.*

Moreover, these quantum groups are subject to an ADE classification result.

Proof. The idea here is that the classification result can be obtained by taking some inspiration from the McKay classification of the subgroups of SO_3. See [9]. □

An interesting extension of the $S_4^+ = SO_3^{-1}$ result comes by looking at the general case $N = n^2$, with $n \in \mathbb{N}$. We will prove that we have a twisting result, as follows:

$$PO_n^+ = (S_N^+)^\sigma$$

This will actually appear as a particular case, with $F = \mathbb{Z}_n \times \mathbb{Z}_n$, with standard Fourier cocycle, of a more general twisting result, involving a finite group F:

$$S_{\widehat{F}_\sigma}^+ = (S_{\widehat{F}}^+)^\sigma$$

In order to explain this material, from [14], which is quite technical, requiring good algebraic knowledge, let us begin with some generalities. We first have:

Proposition 4.11. *Given a finite group F, the algebra $C(S_{\widehat{F}}^+)$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g, h \in F$, with the following relations:*

$$x_{1g} = x_{g1} = \delta_{1g} \quad , \quad x_{s,gh} = \sum_{t \in F} x_{st^{-1},g} x_{th} \quad , \quad x_{gh,s} = \sum_{t \in F} x_{gt^{-1}} x_{h,ts}$$

The comultiplication, counit and antipode are given by the formulae

$$\Delta(x_{gh}) = \sum_{s \in F} x_{gs} \otimes x_{sh} \quad , \quad \varepsilon(x_{gh}) = \delta_{gh} \quad , \quad S(x_{gh}) = x_{h^{-1}g^{-1}}$$

on the standard generators x_{gh}.

Proof. This follows indeed from a direct verification, based either on Theorem 4.5 above, or on its equivalent formulation from Wang's paper [92]. □

Let us discuss now the twisted version of the above result. Consider a 2-cocycle on F, which is by definition a map $\sigma : F \times F \to \mathbb{C}^*$ satisfying:

$$\sigma_{gh,s}\sigma_{gh} = \sigma_{g,hs}\sigma_{hs} \quad , \quad \sigma_{g1} = \sigma_{1g} = 1$$

Given such a cocycle, we can construct the associated twisted group algebra $C(\widehat{F}_\sigma)$, as being the vector space $C(\widehat{F}) = C^*(F)$, with product as follows:

$$e_g e_h = \sigma_{gh} e_{gh}$$

We have then the following generalization of Proposition 4.11:

Proposition 4.12. *The algebra $C(S_{\widehat{F}_\sigma}^+)$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g,h\in G$, with the relations $x_{1g}=x_{g1}=\delta_{1g}$ and:*

$$\sigma_{gh}x_{s,gh}=\sum_{t\in F}\sigma_{st^{-1},t}x_{st^{-1},g}x_{th}\quad,\quad\sigma_{gh}^{-1}x_{gh,s}=\sum_{t\in F}\sigma_{t^{-1},ts}^{-1}x_{gt^{-1}}x_{h,ts}$$

The comultiplication, counit and antipode are given by the formulae

$$\Delta(x_{gh})=\sum_{s\in F}x_{gs}\otimes x_{sh}\quad,\quad\varepsilon(x_{gh})=\delta_{gh}\quad,\quad S(x_{gh})=\sigma_{h^{-1}h}\sigma_{g^{-1}g}^{-1}x_{h^{-1}g^{-1}}$$

on the standard generators x_{gh}.

Proof. Once again, this follows from a direct verification. Note that by using the cocycle identities we obtain $\sigma_{gg^{-1}}=\sigma_{g^{-1}g}$, needed in the proof. □

In what follows, we will prove that the quantum groups $S_{\widehat{F}}^+$ and $S_{\widehat{F}_\sigma}^+$ are related by a cocycle twisting operation. Let us begin with some preliminaries.

Let H be a Hopf algebra. We use the Sweedler notation $\Delta(x)=\sum x_1\otimes x_2$. Recall that a left 2-cocycle is a convolution invertible linear map $\sigma:H\otimes H\to\mathbb{C}$ satisfying:

$$\sigma_{x_1y_1}\sigma_{x_2y_2,z}=\sigma_{y_1z_1}\sigma_{x,y_2z_2}\quad,\quad\sigma_{x1}=\sigma_{1x}=\varepsilon(x)$$

Note that σ is a left 2-cocycle if and only if σ^{-1}, the convolution inverse of σ, is a right 2-cocycle, in the sense that we have:

$$\sigma_{x_1y_1,z}^{-1}\sigma_{x_1y_2}^{-1}=\sigma_{x,y_1z_1}^{-1}\sigma_{y_2z_2}^{-1}\quad,\quad\sigma_{x1}^{-1}=\sigma_{1x}^{-1}=\varepsilon(x)$$

Given a left 2-cocycle σ on H, one can form the 2-cocycle twist H^σ as follows. As a coalgebra, $H^\sigma=H$, and an element $x\in H$, when considered in H^σ, is denoted $[x]$. The product in H^σ is defined, in Sweedler notation, by:

$$[x][y]=\sum\sigma_{x_1y_1}\sigma_{x_3y_3}^{-1}[x_2y_2]$$

Note that the cocycle condition ensures the fact that we have indeed a Hopf algebra. Note also that the coalgebra isomorphism $H\to H^\sigma$ given by $x\to[x]$ commutes with the respective Haar integrals, as soon as H has a Haar integral.

We are now in position to state and prove our main theorem:

Theorem 4.13. *If F is a finite group and σ is a 2-cocycle on F, the Hopf algebras*

$$C(S_{\widehat{F}}^+)\quad,\quad C(S_{\widehat{F}_\sigma}^+)$$

are 2-cocycle twists of each other, in the above sense.

Proof. In order to prove this result, we use the following Hopf algebra map:

$$\pi : C(S_{\widehat{F}}^+) \to C(\widehat{F}) \quad , \quad x_{gh} \to \delta_{gh} e_g$$

Our 2-cocycle $\sigma : F \times F \to \mathbb{C}^*$ can be extended by linearity into a linear map as follows, which is a left and right 2-cocycle in the above sense:

$$\sigma : C(\widehat{F}) \otimes C(\widehat{F}) \to \mathbb{C}$$

Consider now the following composition:

$$\alpha = \sigma(\pi \otimes \pi) : C(S_{\widehat{F}}^+) \otimes C(S_{\widehat{F}}^+) \to C(\widehat{F}) \otimes C(\widehat{F}) \to \mathbb{C}$$

Then α is a left and right 2-cocycle, because it is induced by a cocycle on a group algebra, and so is its convolution inverse α^{-1}. Thus we can construct the twisted algebra $C(S_{\widehat{F}}^+)^{\alpha^{-1}}$, and inside this algebra we have the following computation:

$$\begin{aligned}
[x_{gh}][x_{rs}] &= \alpha^{-1}(x_g, x_r)\alpha(x_h, x_s)[x_{gh}x_{rs}] \\
&= \sigma_{gr}^{-1}\sigma_{hs}[x_{gh}x_{rs}]
\end{aligned}$$

By using this, we obtain the following formula:

$$\begin{aligned}
\sum_{t\in F} \sigma_{st^{-1},t}[x_{st^{-1},g}][x_{th}] &= \sum_{t\in F} \sigma_{st^{-1},t}\sigma_{st^{-1},t}^{-1}\sigma_{gh}[x_{st^{-1},g}x_{th}] \\
&= \sigma_{gh}[x_{s,gh}]
\end{aligned}$$

Similarly, we have the following formula:

$$\sum_{t\in F} \sigma_{t^{-1},ts}^{-1}[x_{g,t^{-1}}][x_{h,ts}] = \sigma_{gh}^{-1}[x_{gh,s}]$$

We deduce from this that there exists a Hopf algebra map, as follows:

$$\Phi : C(S_{\widehat{F}_\sigma}^+) \to C(S_{\widehat{F}}^+)^{\alpha^{-1}} \quad , \quad x_{gh} \to [x_{g,h}]$$

This map is clearly surjective, and is injective as well, by a standard fusion semiring argument, because both Hopf algebras have the same fusion semiring. □

Summarizing, we have proved our main twisting result. Our purpose in what follows will be that of working out versions and particular cases of it. We first have:

Proposition 4.14. *If F is a finite group and σ is a 2-cocycle on F, then*

$$\Phi(x_{g_1h_1}\ldots x_{g_mh_m}) = \Omega(g_1,\ldots,g_m)^{-1}\Omega(h_1,\ldots,h_m)x_{g_1h_1}\ldots x_{g_mh_m}$$

with the coefficients on the right being given by the formula

$$\Omega(g_1,\ldots,g_m) = \prod_{k=1}^{m-1}\sigma_{g_1\ldots g_k,g_{k+1}}$$

is a coalgebra isomorphism $C(S_{\widehat{F}_\sigma}^+) \to C(S_{\widehat{F}}^+)$, commuting with the Haar integrals.

Proof. This is indeed just a technical reformulation of Theorem 4.13. ☐

Here is another useful result, that we will need in what follows:

Theorem 4.15. *Let $X \subset F$ be such that $\sigma_{gh} = 1$ for any $g, h \in X$, and consider the subalgebra $B_X \subset C(S^+_{\widehat{F}_\sigma})$ generated by the elements x_{gh}, with $g, h \in X$. Then we have an injective algebra map $\Phi_0 : B_X \to C(S^+_{\widehat{F}})$, given by $x_{g,h} \to x_{g,h}$.*

Proof. With the notations in the proof of Theorem 4.13, we have the following equality in $C(S^+_{\widehat{F}})^{\alpha^{-1}}$, for any $g_i, h_i, r_i, s_i \in X$:

$$[x_{g_1h_1} \ldots x_{g_ph_p}] \cdot [x_{r_1s_1} \ldots x_{r_qs_q}] = [x_{g_1h_1} \ldots x_{g_ph_p} x_{r_1s_1} \ldots x_{r_qs_q}]$$

Now Φ_0 can be defined to be the composition of $\Phi_{|B_X}$ with the linear isomorphism $C(S^+_{\widehat{F}})^{\alpha^{-1}} \to C(S^+_{\widehat{F}})$ given by $[x] \to x$, and is clearly an injective algebra map. ☐

Let us discuss now some concrete applications of the general results established above. Consider the group $F = \mathbb{Z}_n^2$, let $w = e^{2\pi i/n}$, and consider the following map:

$$\sigma : F \times F \to \mathbb{C}^* \quad , \quad \sigma_{(ij)(kl)} = w^{jk}$$

It is easy to see that σ is a bicharacter, and hence a 2-cocycle on F. Thus, we can apply our general twisting result, to this situation.

In order to understand what is the formula that we obtain, we must do some computations. Let E_{ij} with $i, j \in \mathbb{Z}_n$ be the standard basis of $M_n(\mathbb{C})$. We have:

Proposition 4.16. *The linear map given by*

$$\psi(e_{(i,j)}) = \sum_{k=0}^{n-1} w^{ki} E_{k,k+j}$$

defines an isomorphism of algebras $\psi : C(\widehat{F}_\sigma) \simeq M_n(\mathbb{C})$.

Proof. Consider indeed the following linear map:

$$\psi'(E_{ij}) = \frac{1}{n} \sum_{k=0}^{n-1} w^{-ik} e_{(k,j-i)}$$

It is routine to check that both ψ, ψ' are morphisms of algebras, and that these maps are inverse to each other. In particular, ψ is an isomorphism of algebras, as stated. ☐

Next in line, we have the following result:

Proposition 4.17. *The algebra map given by*

$$\varphi(u_{ij}u_{kl})=\frac{1}{n}\sum_{a,b=0}^{n-1}w^{ai-bj}x_{(a,k-i),(b,l-j)}$$

defines a Hopf algebra isomorphism $\varphi:C(S_{M_n}^+)\simeq C(S_{\widehat{F}_\sigma}^+)$.

Proof. Consider the universal coactions on the two algebras in the statement:

$$\begin{aligned}\alpha:M_n(\mathbb{C})&\to M_n(\mathbb{C})\otimes C(S_{M_n}^+)\\ \beta:C(\widehat{F}_\sigma)&\to C(\widehat{F}_\sigma)\otimes C(S_{\widehat{F}_\sigma}^+)\end{aligned}$$

In terms of the standard bases, these coactions are given by:

$$\begin{aligned}\alpha(E_{ij})&=\sum_{kl}E_{kl}\otimes u_{ki}u_{lj}\\ \beta(e_{(i,j)})&=\sum_{kl}e_{(k,l)}\otimes x_{(k,l),(i,j)}\end{aligned}$$

We use now the identification $C(\widehat{F}_\sigma)\simeq M_n(\mathbb{C})$ from Proposition 4.16. This identification produces a coaction map, as follows:

$$\gamma:M_n(\mathbb{C})\to M_n(\mathbb{C})\otimes C(S_{\widehat{F}_\sigma}^+)$$

Now observe that this map is given by the following formula:

$$\gamma(E_{ij})=\frac{1}{n}\sum_{ab}E_{ab}\otimes\sum_{kr}w^{ar-ik}x_{(r,b-a),(k,j-i)}$$

By comparing with the formula of α, we obtain the isomorphism in the statement. □

We will need one more result of this type, as follows:

Proposition 4.18. *The algebra map given by*

$$\rho(x_{(a,b),(i,j)})=\frac{1}{n^2}\sum_{klrs}w^{ki+lj-ra-sb}p_{(r,s),(k,l)}$$

defines a Hopf algebra isomorphism $\rho:C(S_{\widehat{F}}^+)\simeq C(S_F^+)$.

Proof. We have a Fourier transform isomorphism, as follows:

$$C(\widehat{F})\simeq C(F)$$

Thus the algebras in the statement are indeed isomorphic. □

As a conclusion to all this, we have the following result:

Theorem 4.19. *Let $n \geq 2$ and $w = e^{2\pi i/n}$. Then*

$$\Theta(u_{ij}u_{kl}) = \frac{1}{n}\sum_{ab=0}^{n-1} w^{-a(k-i)+b(l-j)}p_{ia,jb}$$

defines a coalgebra isomorphism

$$C(PO_n^+) \to C(S_{n^2}^+)$$

commuting with the Haar integrals.

Proof. We recall from Theorem 4.7 (2) that we have identifications as follows, where the projective version of (A,u) is the pair (PA, v), with $PA =< v_{ij} >$ and $v = u \otimes \bar{u}$:

$$PO_n^+ = PU_n^+ = S_{M_n}^+$$

With this in hand, the result follows from Theorem 4.13 and Proposition 4.14, by combining them with the various isomorphisms established above. □

Here is a useful version of the above result, that we will need later on:

Theorem 4.20. *The following two algebras are isomorphic, via $u_{ij}^2 \to X_{ij}$:*

1. *The algebra generated by the variables $u_{ij}^2 \in C(O_n^+)$.*
2. *The algebra generated by $X_{ij} = \frac{1}{n}\sum_{a,b=1}^n p_{ia,jb} \in C(S_{n^2}^+)$*

Proof. We have $\Theta(u_{ij}^2) = X_{ij}$, so it remains to prove that if B is the subalgebra of $C(S_{M_n}^+)$ generated by the variables u_{ij}^2, then $\Theta_{|B}$ is an algebra morphism. Let us set:

$$X = \{(i,0)|i \in \mathbb{Z}_n\} \subset \mathbb{Z}_n^2$$

Then X satisfies the assumption in Theorem 4.14, and $\varphi(B) \subset B_X$. Thus by Theorem 4.15, the map $\Theta_{|B} = \rho F_0 \varphi_{|B}$ is indeed an algebra morphism. □

We will be back to this in section 6 below, with some probabilistic consequences.

As an overall conclusion, the twisting formula $S_4^+ = SO_3^{-1}$ ultimately comes from something of type $X_4 \simeq M_2$, where $X_4 = \{1,2,3,4\}$ and $M_2 = Spec(M_2(\mathbb{C}))$, and at $N \geq 5$ there are some extensions of this, and notably when $N = n^2$ with $n \geq 3$.

Let us go back now to the quantum groups S_X^+, and develop some general theory. In connection with planar algebra and subfactors, we have the following result:

Theorem 4.21. *Assume that a closed subgroup $G \subset S_X^+$ acts minimally on a II_1 factor A, in the sense that we have $(A^G)' \cap A = \mathbb{C}$, the inclusion*

$$A^G \subset (B \otimes A)^G$$

with $B = C(X)$, is a subfactor of index $N = |X|$. The associated planar algebra is

$$P_k = Fix(u^{\otimes k}) \subset \mathcal{S}_X(k)$$

and any subalgebra of the generalized spin planar algebra $\mathcal{S}_X$ is of this form.

Proof. This is something quite heavy, the idea being as follows:

(1) As a first observation, the various subfactor assertions generalize those in Theorem 3.16 above, which can be recovered by setting $B=\mathbb{C}^N$ and $B=M_N(\mathbb{C})$.

Regarding now the proof, the factoriality is clear. In order to compute the commutants, consider the Jones tower for the inclusion $(B_0\subset B_1)=(\mathbb{C}\subset B)$:

$$B_0\subset B_1\subset B_2\subset B_3\subset\ldots\ldots$$

It is elementary to check that the Jones tower for our subfactor is as follows:

$$(B_0\otimes A)^G\subset(B_1\otimes A)^G\subset(B_2\otimes A)^G\subset\ldots\ldots$$

The point now is that when looking at the commutants, the factor A dissapears in the computation, and so the associated planar algebra is given by:

$$P_k=\left[(B_0\otimes A)^G\right]'\cap(B_k\otimes A)^G=(B_0'\cap B_k)^G$$

Via some standard identifications, we obtain the formula in the statement, namely:

$$P_k=Fix(u^{\otimes k})$$

(2) In what regards the last assertion, this is something more technical. We already know from section 3 that the result holds for $B=\mathbb{C}^N$, in the sense that any subalgebra $P\subset\mathcal{S}_N$ of the spin planar algebra appears from a closed subgroup $G\subset S_N^+$.

As mentioned at the end of section 3, the same method, based on Tannakian duality, gives a similar result in the tensor planar algebra case, stating that any subalgebra $P\subset\mathcal{T}_N$ of the tensor planar algebra appears from a quantum group, as follows:

$$G\subset PO_N^+=PU_N^+$$

The point now is that these results can be unified, by using Jones' construction from [58] of the planar algebra of a bipartite graph. To be more precise, let us define the generalized spin planar algebra $\mathcal{S}_X$ to be the algebra associated as in [58] to the bipartite graph of the inclusion $\mathbb{C}\subset B$. Given a subfactor as in the statement, we have:

$$P\subset\mathcal{S}_X$$

Now by using Tannakian duality, as in the cases $B=\mathbb{C}^N$ and $B=M_N(\mathbb{C})$, which lead to results regarding the subalgebras $P\subset\mathcal{S}_N$ and $P\subset\mathcal{T}_N$, we obtain a general result, stating that any subalgebra $P\subset\mathcal{S}_X$ must come from a subgroup $G\subset S_X^+$.

(3) Finally, let us mention that all this can be still further generalized, by considering arbitrary fixed point subfactors, of the following type:

$$G\curvearrowright(B_0\subset B_1)\implies(B_0\otimes A)^G\subset(B_1\otimes A)^G$$

The computation of the relative commutants can be done as in (1) above, and the corresponding planar algebra is a subalgebra of the planar algebra associated as in [58] to the bipartite graph of the inclusion $B_0\subset B_1$. Moreover, we have as well a Tannakian result, involving subalgebras of this latter planar algebra. We will be back to this. $\square$

Let us discuss now some applications of all this to the subfactors associated to the commuting squares. The subject here is quite technical. Let us start with:

Definition 4.22. *A commuting square in the sense of subfactor theory is a commuting diagram of finite dimensional algebras with traces, as follows,*

$$\begin{tikzcd} C_{01} \arrow[r] & C_{11} \\ C_{00} \arrow[r] \arrow[u] & C_{10} \arrow[u] \end{tikzcd}$$

having the property that the conditional expectations $C_{11} \to C_{01}$ and $C_{11} \to C_{10}$ commute, and their product is the conditional expectations $C_{11} \to C_{00}$.

The idea now is that, under some suitable extra mild assumptions, any such square C produces a subfactor of the hyperfinite II_1 factor R. Indeed, by performing the basic construction, in finite dimensions, and we obtain a whole array of squares, as follows:

$$\begin{tikzcd} A_0 & A_1 & A_2 & \\ C_{02} \arrow[r] \arrow[u, dotted] & C_{12} \arrow[r] \arrow[u, dotted] & C_{22} \arrow[r, dotted] \arrow[u, dotted] & B_2 \\ C_{01} \arrow[r] \arrow[u] & C_{11} \arrow[r] \arrow[u] & C_{21} \arrow[r, dotted] \arrow[u] & B_1 \\ C_{00} \arrow[r] \arrow[u] & C_{10} \arrow[r] \arrow[u] & C_{20} \arrow[r, dotted] \arrow[u] & B_0 \end{tikzcd}$$

Here the various A, B letters stand for the von Neumann algebras obtained in the limit, which are all isomorphic to the hyperfinite II_1 factor R, and we have:

Theorem 4.23. *In the context of the above diagram, the following happen:*

1. *$A_0 \subset A_1$ is a subfactor, and $\{A_i\}$ is the Jones tower for it.*
2. *The corresponding planar algebra is given by $A_0' \cap A_k = C_{01}' \cap C_{k0}$.*
3. *A similar result holds for the "horizontal" subfactor $B_0 \subset B_1$.*

Proof. Here (1) is something quite routine, (2) is a subtle result, called Ocneanu compactness theorem, and (3) follows from (1,2), by flipping the diagram. □

The commuting squares having $\mathbb{C}$ in the lower left corner can be investigated with quantum groups techniques, and in particular we have the following result:

Theorem 4.24. *Any commuting square having $\mathbb{C}$ in the lower left corner, and in particular the commuting squares associated to the vertex and spin models, come from compact quantum groups, and the associated subfactors are fixed point subfactors.*

Proof. Once again, this is something heavy. The idea is that the initial commuting square, and the whole array of squares that can be obtained from it, can be written as:

$$\begin{tikzcd}
D_0\otimes_G E_\infty & D_1\otimes_G E_\infty & D_2\otimes_G E_\infty & \\
D_0\otimes_G E_2 \arrow[r] \arrow[u, dotted] & D_1\otimes_G E_2 \arrow[r] \arrow[u, dotted] & D_2\otimes_G E_2 \arrow[r, dotted] \arrow[u, dotted] & D_\infty\otimes_G E_2 \\
D_0\otimes_G E_1 \arrow[r] \arrow[u] & D_1\otimes_G E_1 \arrow[r] \arrow[u] & D_2\otimes_G E_1 \arrow[r, dotted] \arrow[u] & D_\infty\otimes_G E_1 \\
D_0\otimes_G E_0 \arrow[r] \arrow[u] & D_1\otimes_G E_0 \arrow[r] \arrow[u] & D_2\otimes_G E_0 \arrow[r, dotted] \arrow[u] & D_\infty\otimes_G E_0
\end{tikzcd}$$

With a bit more work, consisting in converting the $\otimes_G$ products into fixed point algebras, this shows that we obtain indeed fixed point subfactors. See [3]. □

All this is of course a bit heavy, but, as already mentioned on several occasions, and notably in the proof of Theorem 4.21, can be still further generalized. The statement here, which is advanced, generalizing all that has being said so far, is as follows:

Theorem 4.25. *Consider a fibration $X\to Y$ of finite noncommutative spaces, which corresponds by definition to a Markov inclusion of finite dimensional algebras:*

$$C(Y)\subset C(X)$$

1. *The category of compact quantum groups acting on X in a measure-preserving way, and by leaving Y invariant, has a universal object, denoted $S_{X\to Y}^+$.*
2. *This quantum group $S_{X\to Y}^+$ appears via the relations coming from m,u,e, the multiplication and unit of $C(X)$, and the expectation onto $C(Y)$.*
3. *The planar algebra of $S_{X\to Y}^+$ is the Fuss-Catalan algebra. In general, the planar algebra of $G\subset S_{X\to Y}^+$ appears as subalgebra of the spin-type algebra $\mathcal{S}_{X\to Y}$.*
4. *Associated to any closed subgroup $G\subset S_{X\to Y}^+$, acting minimally on a II_1 factor A, is a fixed point subfactor $(C(Y)\otimes A)^G\subset(C(X)\otimes A)^G$.*
5. *The general theory of the above planar algebras and subfactors is similar to the theory from the $Y=\{.\}$ case, discussed above.*

Proof. This is quite technical, but rather routine, by generalizing the $Y=\{.\}$ case, discussed above. We refer here to [4], [58], and to subsequent papers. □

5. Laws of characters

Let us go back to the diagram in Theorem 2.21, describing the asymptotic laws of the main characters $\chi = \sum_i u_{ii}$, for the basic quantum permutation and rotation groups:

$$\begin{matrix} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{matrix} \quad : \quad \begin{matrix} \pi_1 & - & \gamma_1 \\ | & & | \\ p_1 & - & g_1 \end{matrix}$$

The measures $p_1, g_1, \pi_1, \gamma_1$ on the right are the Poisson, Gaussian, Marchenko-Pastur and Wigner laws of parameter 1, given by the following formulae:

$$\begin{matrix} \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx & - & \frac{1}{2\pi}\sqrt{4-x^2}\,dx \\ | & & | \\ \frac{1}{e}\sum_p \frac{\delta_p}{p!} & - & \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx \end{matrix}$$

All this is very nice, and quite conceptual, but at a more advanced level, there are a number of troubles with this, as follows:

(1) According to the more specialized results from section 3, dealing with the case where $N \in \mathbb{N}$ is fixed, the convergence is in fact stationary for S_N^+ starting from $N = 4$, and for O_N^+ starting from $N = 2$. This is in stark contrast with the classical case, where the convergence for S_N, O_N is definitely not stationary.

(2) Standard free probability shows that π_1, γ_1 are indeed the free analogues of p_1, g_1. However, at a more advanced level, that of the Bercovici-Pata bijection, the correct statement is that the free convolution semigroups $\{\pi_t\}, \{\gamma_t\}$ are the free analogues of the convolution semigroups $\{p_t\}, \{g_t\}$. Thus, we need a parameter $t > 0$.

Our purpose in this section will be to discuss a “fix” for this. The idea will be that of looking at truncated characters, with respect to a parameter $t \in (0, 1]$:

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

We will see that the asymptotic laws of these variables are respectively $p_t, g_t, \pi_t, \gamma_t$, and that the convergence at generic $t \in (0, 1]$ is not stationary, thus fixing both (1,2).

As a bonus, all this will get us into advanced representation theory and free probability. We will explore further aspects in this section, and in the next section as well.

In order to get started, let us recall the main motivations for the computation of the law of the main character. These can be summarized as follows:

Theorem 5.1. *Given a Woronowicz algebra (A,u), the law of the main character*

$$\chi=\sum_{i=1}^N u_{ii}$$

with respect to the Haar integration has the following properties:

(1) *The moments of χ are the numbers $M_k=\dim(Fix(u^{\otimes k}))$.*
(2) *M_k counts as well the lenght p loops at 1, on the Cayley graph of A.*
(3) *$law(\chi)$ is the Kesten measure of the associated discrete quantum group.*
(4) *When $u\sim\bar{u}$ the law of χ is a usual measure, supported on $[-N,N]$.*
(5) *The algebra A is amenable precisely when $N\in supp(law(Re(\chi)))$.*
(6) *Any morphism $f:(A,u)\to(B,v)$ must increase the numbers M_k.*
(7) *Such a morphism f is an isomorphism when $law(\chi_u)=law(\chi_v)$.*

Proof. All this is very standard, the idea being as follows:

(1) This comes from the Peter-Weyl type theory in [99], which tells us the number of fixed points of $v=u^{\otimes k}$ can be recovered by integrating the character $\chi_v=\chi_u^k$.

(2) This is something true, and well-known, for $A=C^*(\Gamma)$, with $\Gamma=<g_1,\ldots,g_N>$ being a discrete group. In general, the proof is quite similar.

(3) This is actually the definition of the Kesten measure, in the case $A=C^*(\Gamma)$, with $\Gamma=<g_1,\ldots,g_N>$ being a discrete group. In general, this follows from (2).

(4) The equivalence $u\sim\bar{u}$ translates into $\chi_u=\chi_u^*$, and this gives the first assertion. As for the support claim, this follows from $uu^*=1\implies||u_{ii}||\leq1$, for any i.

(5) This is the Kesten amenability criterion, which can be established as in the classical case, $A=C^*(\Gamma)$, with $\Gamma=<g_1,\ldots,g_N>$ being a discrete group.

(6) This is something elementary, which follows from (1) above, and from the fact that the morphisms of Woronowicz algebras increase the spaces of fixed points.

(7) This follows by using (6), and the Peter-Weyl type theory from [99], the idea being that if f is not injective, then it must strictly increase one of the spaces $Fix(u^{\otimes k})$. $\square$

Summarizing, regardless of our precise motivations and philosophy, computing the law of $\chi=\sum_i u_{ii}$ is a central and luminous question, and the "main problem" to be solved. Here is as well some extra motivation from this, coming from subfactor theory:

Proposition 5.2. *In the quantum permutation group case, $G\subset S_N^+$, the Poincaré series of the associated planar algebra $P_k=(Fix(u^{\otimes k}))$, which is by definition the series*

$$f(z)=\sum_{k=0}^\infty\dim(P_k)z^k$$

appears as Stieltjes transform of the law of the main character $\chi=\sum_i u_{ii}$.

Proof. This is more or less an empty statement, coming from the fact that the moments of χ are the numbers $M_k = \dim(P_k)$, that we know from Theorem 5.1 (1). $\square$

As a side comment here, following Jones' influential paper [57] and its various follow-ups, that we will not get into here, computing the Poincaré series of the planar algebras, and further manipulating them, is the "main problem" in subfactor theory.

Let us recall as well how the law of χ can be computed in the easy case:

Theorem 5.3. *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D = (D(k,l))$, the asymptotic moments of the main character are given by*

$$\lim_{N\to\infty} \int_{G_N} \chi^k = \#D(k)$$

where $D(k) = D(\emptyset, k)$, with the limiting sequence on the left consisting of certain integers, and being stationary at least starting from the k-th term.

Proof. This is something that we already know, from Theorem 2.19, the idea being that by Peter-Weyl theory, and by easiness, the moments of χ are given by:

$$\int_{G_N} \chi^k = \dim\left(Fix(u^{\otimes k})\right) = \dim\left(span\left(\xi_\pi \Big| \pi \in D(k)\right)\right)$$

Now since the vectors ξ_π become linearly independent with $N \to \infty$, as established in Theorem 2.18 via Gram determinants, we obtain the formula in the statement. $\square$

Finally, let us recall how the above result applies to our main quantum groups:

Theorem 5.4. *The asymptotic character laws for the basic quantum groups are*

$$\begin{tikzcd} S_N^+ \arrow[r] & O_N^+ \\ S_N \arrow[u] \arrow[r] & O_N \arrow[u] \end{tikzcd} \quad : \quad \begin{tikzcd} \pi_1 \arrow[r, dash] & \gamma_1 \\ p_1 \arrow[u, dash] \arrow[r, dash] & g_1 \arrow[u, dash] \end{tikzcd}$$

with $p_1, g_1, \pi_1, \gamma_1$ being the Poisson, Gaussian, Marchenko-Pastur and Wigner laws.

Proof. This is something that we already know, from Theorem 2.21, coming from the fact that the above quantum groups are all easy, the categories and moments being:

$$\begin{tikzcd} NC \arrow[d] & NC_2 \arrow[l] \arrow[d] \\ P & P_2 \arrow[l] \end{tikzcd} \quad : \quad \begin{tikzcd} C_k \arrow[r, dash] \arrow[d, dash] & C_{k/2} \arrow[d, dash] \\ B_k \arrow[r, dash] & k!! \end{tikzcd}$$

The moments on the right being those of $p_1, g_1, \pi_1, \gamma_1$, this gives the result. $\square$

Summarizing, the computation of $law(\chi)$ is a very interesting question, and in the simplest case, where our quantum group is easy, the $N \to \infty$ computation can be done by counting partitions, and then by recapturing the measure from its moments.

In order to have a better understanding of this, and prior to any further computation, we must do some probability theory. The Gaussian laws g_t and Poisson laws p_t appear via the Central Limit Theorem (CLT) and the Poisson Limit Theorem (PLT), and our first task will be that of explaining these results. The first of them is as follows:

Theorem 5.5 (CLT)**.** *Given random variables $f_1, f_2, f_3, \ldots \in L^\infty(X)$ which are i.i.d., centered, and with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n f_i \sim g_t$$

where g_t is the Gaussian law of parameter t, given by $g_t = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dx$.

Proof. We use the well-known fact that the log of the Fourier transform $F_f(x) = \mathbb{E}(e^{ixf})$ linearizes the convolution. The Fourier transform of the variable in the statement is:

$$\begin{aligned}
F(x) &= \left[F_f\left(\frac{x}{\sqrt{n}}\right)\right]^n\\
&= \left[1 - \frac{tx^2}{2n} + O(n^{-2})\right]^n\\
&\simeq e^{-tx^2/2}
\end{aligned}$$

On the other hand, the Fourier transform of g_t is given by:

$$\begin{aligned}
F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-y^2/2t+ixy}dy\\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-(y/\sqrt{2t}-\sqrt{t/2}ix)^2-tx^2/2}dy\\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-z^2-tx^2/2}dz\\
&= e^{-tx^2/2}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Regarding now the Poisson Limit Theorem (PLT), this is as follows:

Theorem 5.6 (PLT)**.** *We have the following convergence, in moments,*

$$\left(\left(1-\frac{t}{n}\right)\delta_0 + \frac{t}{n}\,\delta_1\right)^{*n} \to p_t$$

where p_t is the Poisson law of parameter $t > 0$, given by $p_t = e^{-t}\sum_k \frac{t^k\delta_k}{k!}$.

Proof. Once again, we use the fact the log of the Fourier transform $F_f(z) = \mathbb{E}(e^{izf})$ linearizes the convolution. The Fourier transform of the variable in the statement is:

$$\begin{aligned} F(x) &= \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\right)^n \\ &= \lim_{n\to\infty}\left(1+\frac{(e^{ix}-1)t}{n}\right)^n \\ &= \exp\left((e^{ix}-1)t\right) \end{aligned}$$

On the other hand, the Fourier transform of p_t is given by:

$$\begin{aligned} F_{p_t}(x) &= e^{-t}\sum_k \frac{t^k}{k!}e^{ikx} \\ &= e^{-t}\sum_k \frac{(e^{ix}t)^k}{k!} \\ &= \exp(-t)\exp(e^{ix}t) \\ &= \exp\left((e^{ix}-1)t\right) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

In order to discuss now the free version of these results, we first need to talk about moments, laws and freeness. Let us start with the following definition:

Definition 5.7. *Let A be a $*$-algebra, given with a trace tr.*

(1) *The elements $a \in A$ are called random variables.*
(2) *The moments of such a variable are the numbers $M_k(a) = tr(a^k)$.*
(3) *The law of such a variable is the functional $\mu_a : P \to tr(P(a))$.*

Here $k = \circ\bullet\bullet\circ\ldots$ is by definition a colored integer, and the powers a^k are defined by the formulae $a^\emptyset = 1, a^\circ = a, a^\bullet = a^*$ and multiplicativity. As for the polynomial P, this is a noncommuting $*$-polynomial in one variable, $P \in \mathbb{C} < X, X^* >$.

Observe that the law is uniquely determined by the moments, because:

$$P(X) = \sum_k \lambda_k X^k \implies \mu_a(P) = \sum_k \lambda_k M_k(a)$$

Let us discuss now the independence, and its noncommutative versions. As a starting point here, we have the following notion:

Definition 5.8. *We call two $*$-subalgebras $B, C \subset A$ independent when the following condition is satisfied, for any $x \in B$ and $y \in C$:*

$$tr(x) = tr(y) = 0 \implies tr(xy) = 0$$

Also, two variables $b, c \in A$ are called independent when the $*$-algebras that they generate $B =< b >$ and $C =< c >$ are independent, in the above sense.

It is possible to develop some theory here, but all this is ultimately not very interesting, being just an abstract generalization of usual probability theory.

As a much more interesting notion, we have the following definition:

Definition 5.9. *Given a pair (A,tr), we call two subalgebras $B,C\subset A$ free when the following condition is satisfied, for any $x_i\in B$ and $y_i\in C$:*

$$tr(x_i)=tr(y_i)=0\implies tr(x_1y_1x_2y_2\ldots)=0$$

Also, two noncommutative random variables $b,c\in A$ are called free when the C^*-algebras $B=<b>,C=<c>$ that they generate inside A are free, in the above sense.

Thus, freeness appears by definition as a kind of "free analogue" of independence.

As a basic result now regarding these notions, and providing us with examples, we have:

Proposition 5.10. *We have the following results, valid for group algebras:*

1. *$C^*(\Gamma),C^*(\Lambda)$ are independent inside $C^*(\Gamma\times\Lambda)$.*
2. *$C^*(\Gamma),C^*(\Lambda)$ are free inside $C^*(\Gamma*\Lambda)$.*

Proof. In order to prove these results, we can use the fact that each group algebra is spanned by the corresponding group elements. Thus, it is enough to check the independence and freeness formulae on group elements, and this is in turn trivial. □

In short, we have now a notion of freeness, dealing with noncommutativity itself, in its most pure form, where there are no algebraic relations at all. This is very nice, and philosophically speaking, this should normally be enough, as a motivation.

Next, we need an analogue of the Fourier transform, or rather of the log of the Fourier transform. The result here, due to Voiculescu [88], is as follows:

Theorem 5.11. *Given a probability measure μ, define its R-transform as follows:*

$$G_\mu(\xi)=\int_\mathbb{R}\frac{d\mu(t)}{\xi-t}\implies G_\mu\left(R_\mu(\xi)+\frac{1}{\xi}\right)=\xi$$

The free convolution operation is then linearized by the R-transform.

Proof. The proof here, which is quite tricky, is in three steps, as follows:

Step 1. In order to model the free convolution operation, the best is to use the monoid algebra $C^*(\mathbb{N}*\mathbb{N})$. Indeed, we have some freeness here, a bit in the same way as for the above group algebras $C^*(\Gamma*\Lambda)$, and the point is that the variables of type $S^*+f(S)$, with $S\in C^*(\mathbb{N})$ being the shift, and with $f\in\mathbb{C}[X]$ being a polynomial, are easily seen to model in moments all the distributions $\mu:\mathbb{C}[X]\to\mathbb{C}$.

Step 2. Now let $f, g \in \mathbb{C}[X]$ and consider the variables $S^* + f(S)$ and $T^* + g(T)$, where $S, T \in C^*(\mathbb{N} * \mathbb{N})$ are the shifts corresponding to the generators of $\mathbb{N} * \mathbb{N}$. These variables are free, and by using a 45° argument, their sum has the same law as $S^* + (f + g)(S)$.

Step 3. Thus the operation $\mu \to f$ linearizes the free convolution. We are therefore left with a computation inside $C^*(\mathbb{N})$, which is elementary, and whose conclusion is that $R_\mu = f$ can be recaptured from μ via the Cauchy transform G_μ, as in the statement. □

We are now ready to state and prove the free CLT:

Theorem 5.12 (FCLT)**.** *Given self-adjoint variables $x_1, x_2, x_3, \ldots$, which are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \gamma_t$$

where γ_t is the Wigner semicircle law of parameter t, having density $\frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$.

Proof. We follow the same idea as in the proof of the CLT, Theorem 5.5 above.

At $t = 1$, the R-transform of the variable in the statement can be computed by using the linearization property from Theorem 5.11, and is given by:

$$R(\xi) = nR_x\left(\frac{\xi}{\sqrt{n}}\right) \simeq \xi$$

On the other hand, some standard computations show that the Cauchy transform of the Wigner semicircle law γ_1 satisfies the following equation:

$$G_{\gamma_1}\left(\xi + \frac{1}{\xi}\right) = \xi$$

Thus we have $R_{\gamma_1}(\xi) = \xi$, which by the way follows as well from $\frac{S^*+S}{2} \sim \gamma_1$, and this gives the result. The passage to the general case, $t > 0$, is routine, by dilation. □

We can state and prove as well the free PLT, as follows:

Theorem 5.13 (FPLT)**.** *We have the following convergence, in moments,*

$$\left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\,\delta_1\right)^{\boxplus n} \to \pi_t$$

the limiting measure being the Marchenko-Pastur law of parameter $t > 0$,

$$\pi_t = \max(1 - t, 0)\delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter $t > 0$.

Proof. The Cauchy transform of the measure $\mu = \left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\delta_1$ is given by:

$$G_\mu(\xi) = \left(1 - \frac{t}{n}\right)\frac{1}{\xi} + \frac{t}{n} \cdot \frac{1}{\xi - 1}$$

Thus the equation satisfied by $R = R_{\mu^{\boxplus n}}(y) = nR_\mu(y)$ is as follows:

$$\left(1 - \frac{t}{n}\right)\frac{1}{y^{-1} + R/n} + \frac{t}{n} \cdot \frac{1}{y^{-1} + R/n - 1} = y$$

By multiplying by n/y, and rearranging terms, this equation can be written as:

$$\frac{t + yR}{1 + yR/n} = \frac{t}{1 + yR/n - y}$$

With $n \to \infty$ we obtain $t + yR = \frac{t}{1-y}$, and so $R = \frac{t}{1-y} = R_{\pi_t}$, as desired. $\square$

As in the classical case, there are several interesting generalizations of the above results. We will be back to all this, gradually, in what follows.

As a conclusion to this, let us formulate the following philosophical statement:

Theorem 5.14. *The main limiting results in classical and free probability are*

$$\begin{matrix} \mathrm{FPLT} & - & \mathrm{FCLT} \\ | & & | \\ \mathrm{PLT} & - & \mathrm{CLT} \end{matrix} \quad \to \quad \begin{matrix} \pi_t & - & \gamma_t \\ | & & | \\ p_t & - & g_t \end{matrix}$$

the limiting measures being Gaussian, Poisson, Wigner and Marchenko-Pastur.

Proof. This follows indeed by putting together all the above results, classical and free, and with $g_t, p_t, \gamma_t, \pi_t$ being respectively the measures in the statement. $\square$

Summarizing, we have now a much better understanding of our quantum group scheme, from Theorem 5.4 above:

$$\begin{matrix} S_N^+ & \to & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \to & O_N \end{matrix} \quad : \quad \begin{matrix} \pi_1 & - & \gamma_1 \\ | & & | \\ p_1 & - & g_1 \end{matrix}$$

Indeed, the laws on the right are not some "random" probability measures, but rather the main laws in classical and free probability, coming from Theorem 5.14 at $t = 1$.

In order to get beyond this, and reach to an even better understanding of all this, we must still discuss the introduction of a parameter $t > 0$.

As already mentioned in the beginning of this section, this can be done by truncating the main character. However, before getting into this, let us do some more free probability, and understand the meaning of this parameter $t > 0$, in the abstract setting.

Given a noncommutative random variable a, we can define its classical cumulants $k_n(a)$ and its free cumulants $\kappa_n(a)$ by the following formulae:

$$\log F_a(\xi) = \sum_n k_n(a)\xi^n \quad , \quad R_a(\xi) = \sum_n \kappa_n(a)\xi^n$$

With this notion in hand, we can define then more general quantities $k_\pi(a), \kappa_\pi(a)$, depending on partitions $\pi \in P(k)$, by multiplicativity over the blocks.

We have then the following result:

Theorem 5.15. *We have the classical and free moment-cumulant formulae*

$$M_k(a) = \sum_{\pi \in P(k)} k_\pi(a) \quad , \quad M_k(a) = \sum_{\pi \in NC(k)} \kappa_\pi(a)$$

where $k_\pi(a), \kappa_\pi(a)$ are the generalized cumulants and free cumulants of a.

Proof. This is standard, by using the formulae of F_a, R_a, or by doing some direct combinatorics, based on the Möbius inversion formula from Theorem 2.16. See [91]. $\square$

In connection now with the main laws in Theorem 5.14, we have:

Theorem 5.16. *The moments of the main limiting laws are given by $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$, where $D \subset P$ are the following categories of partitions,*

$$\begin{array}{ccc} \pi_t & - & \gamma_t \\ \updownarrow & & \updownarrow \\ p_t & - & g_t \end{array} \quad : \quad \begin{array}{ccc} NC & - & NC_2 \\ \updownarrow & & \updownarrow \\ P & - & P_2 \end{array}$$

and on the vertical we have the Bercovici-Pata bijection, which states that "the classical cumulants of the classical measures equal the free cumulants of the free measures".

Proof. Regarding the first assertion, this is something that we already know at $t = 1$, from Theorem 2.21 above. The proof at general values of $t > 0$, as well as the proof of the last assertion, are standard combinatorics, based on Theorem 5.15 above. $\square$

We can now upgrade Theorem 5.14, into a final statement, as follows:

Theorem 5.17. *The main limiting results are as follows, with $g_t, p_t, \gamma_t, \pi_t$ being the Gaussian and Poisson semigroups, and Wigner and Marchenko-Pastur free semigroups,*

$$\begin{matrix} \mathrm{FPLT} & - & \mathrm{FCLT} \\ | & & | \\ \mathrm{PLT} & - & \mathrm{CLT} \end{matrix} \quad \to \quad \begin{matrix} \pi_t & - & \gamma_t \\ | & & | \\ p_t & - & g_t \end{matrix}$$

which are related by the Bercovici-Pata bijection, which states that "the classical cumulants of the classical measures are equal to the free cumulants of the free measures".

Proof. This follows indeed by putting together all the above results, classical and free, and with the moment and cumulant computations being standard. □

Summarizing, we have now a perfect understanding of the liberation operation, at the purely probabilistic level, and as a main remark, this requires a parameter $t > 0$.

Let us discuss now the introduction of a parameter $t > 0$ in all this, in order to solve the problems mentioned in the beginning of this section. We will be using:

Definition 5.18. *Associated to any Woronowicz algebra (A, u) are the variables*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

with $t \in (0, 1]$, called truncations of the main character.

In order to understand what these variables are about, let us first investigate the symmetric group S_N. The result here, which is extremely beautiful, is as follows:

Theorem 5.19. *Consider the symmetric group S_N, regarded as a compact group of matrices, $S_N \subset O_N$, via the standard permutation matrices.*

1. *The main character $\chi \in C(S_N)$, defined as usual as $\chi = \sum_i u_{ii}$, counts the number of fixed points, $\chi(\sigma) = \#\{i|\sigma(i) = i\}$.*
2. *The probability for a permutation $\sigma \in S_N$ to be a derangement, meaning to have no fixed points at all, becomes, with $N \to \infty$, equal to $1/e$.*
3. *The law of the main character $\chi \in C(S_N)$ becomes with $N \to \infty$ a Poisson law of parameter 1, with respect to the counting measure.*
4. *In fact, the law of any truncated character $\chi_t = \sum_{i=1}^{[tN]} u_{ii}$ becomes with $N \to \infty$ a Poisson law of parameter t, with respect to the counting measure.*

Proof. This is something very classical, the proof being as follows:

(1) We have indeed the following computation:

$$\chi(\sigma)=\sum_i u_{ii}(\sigma)=\sum_i\delta_{\sigma(i)i}=\#\left\{i\Big|\sigma(i)=i\right\}$$

(2) This is best viewed by using the inclusion-exclusion principle. Let us set:

$$S_N^{i_1\ldots i_k}=\left\{\sigma\in S_N\Big|\sigma(i_1)=i_1,\ldots,\sigma(i_k)=i_k\right\}$$

By using the inclusion-exclusion principle, we have:

$$\mathbb{P}(\chi=0)=\frac{1}{N!}\left(|S_N|-\sum_i|S_N^i|+\sum_{i<j}|S_N^{ij}|-\ldots+(-1)^N\sum_{i_1<\ldots<i_N}|S_N^{i_1\ldots i_N}|\right)$$

Now since $|S_N^{i_1\ldots i_k}|=(N-k)!$ for any $i_1<\ldots<i_k$, we obtain from this:

$$\mathbb{P}(\chi=0)=1-\frac{1}{1!}+\frac{1}{2!}-\ldots+(-1)^{N-1}\frac{1}{(N-1)!}+(-1)^N\frac{1}{N!}$$

Since on the right we have the expansion of $\frac{1}{e}$, we conclude that we have:

$$\lim_{N\to\infty}\mathbb{P}(\chi=0)=\frac{1}{e}$$

(3) This follows by generalizing the computation in (2). To be more precise, a similar application of the inclusion-exclusion principle gives the following formula:

$$\lim_{N\to\infty}\mathbb{P}(\chi=k)=\frac{1}{k!e}$$

Thus, we obtain in the limit a Poisson law of parameter 1, as stated.

(4) As a first observation, and in analogy with the formula in (1) above, the truncated characters count as well certain fixed points, as follows:

$$\chi(\sigma)=\sum_{i=1}^{[tN]}u_{ii}(\sigma)=\sum_{i=1}^{[tN]}\delta_{\sigma(i)i}=\#\left\{i\in\{1,\ldots,[tN]\}\Big|\sigma(i)=i\right\}$$

Regarding now the computation of the law of χ_t, this follows by generalizing the computation in (3). Indeed, an application of the inclusion-exclusion principle gives:

$$\lim_{N\to\infty}\mathbb{P}(\chi_t=k)=\frac{t^k}{k!e^t}$$

Thus, we obtain in the limit a Poisson law of parameter t, as stated. □

Summarizing, the truncated characters χ_t are objects which are quite interesting, and whose laws are worth computing, independently of our abstract motivations.

In general now, and in particular in what regards O_N,S_N^+,O_N^+, here there is no simple trick as for S_N, and we must use general integration methods, from [44], [96].

First, we have the following very general result:

Theorem 5.20. *Assuming that $A = C(G)$ has Tannakian category $C = (C(k,l))$, the Haar integration over G is given by the Weingarten type formula*

$$\int_G u_{i_1j_1}^{e_1}\ldots u_{i_kj_k}^{e_k} = \sum_{\pi,\sigma\in D_k} \delta_\pi(i)\delta_\sigma(j)W_k(\pi,\sigma)$$

for any colored integer $k = e_1 \ldots e_k$ and any multi-indices i,j, where D_k is a linear basis of $C(\emptyset,k)$, $\delta_\pi(i) =< \pi, e_{i_1}\otimes\ldots\otimes e_{i_k} >$, and $W_k = G_k^{-1}$, with $G_k(\pi,\sigma) =< \pi,\sigma >$.

Proof. We know from section 1 above that the integrals in the statement form altogether the orthogonal projection P^k onto the space $Fix(u^{\otimes k}) = span(D_k)$. Consider now the following linear map, with $D_k = \{\xi_k\}$ being as in the statement:

$$E(x) = \sum_{\pi\in D_k} < x,\xi_\pi > \xi_\pi$$

By a standard linear algebra computation, it follows that we have $P = WE$, where W is the inverse on $span(T_\pi|\pi \in D_k)$ of the restriction of E. But this restriction is the linear map given by G_k, and so W is the linear map given by W_k, and this gives the result. □

In the easy quantum group case, the above formula simplifies, as follows:

Theorem 5.21. *For an easy quantum group $G \subset O_N^+$, coming from a category of partitions $D = (D(k,l))$, we have the Weingarten integration formula*

$$\int_G u_{i_1j_1}\ldots u_{i_kj_k} = \sum_{\pi,\sigma\in D(k)} \delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

for any $k \in \mathbb{N}$ and any i,j, where $D(k) = D(\emptyset,k)$, δ are usual Kronecker symbols, and $W_{kN} = G_{kN}^{-1}$, with $G_{kN}(\pi,\sigma) = N^{|\pi\vee\sigma|}$, where $|.|$ is the number of blocks.

Proof. With notations from Theorem 5.20, the Kronecker symbols are given by:

$$\begin{aligned}\delta_{\xi_\pi}(i) &= < \xi_\pi, e_{i_1}\otimes\ldots\otimes e_{i_k} >\\ &= \delta_\pi(i_1,\ldots,i_k)\end{aligned}$$

The Gram matrix being as well the correct one, we obtain the result. See [17]. □

With the above formula in hand, we can go back now to the question of computing the laws of truncated characters. First, we have the following formula, from [17]:

Theorem 5.22. *The moments of truncated characters are given by the formula*

$$\int_G (u_{11}+\ldots+u_{ss})^k = Tr(W_{kN}G_{ks})$$

where G_{kN} and $W_{kN} = G_{kN}^{-1}$ are the associated Gram and Weingarten matrices.

Proof. We have indeed the following computation:

$$\begin{aligned}
\int_G (u_{11}+\ldots+u_{ss})^k &= \sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\int u_{i_1i_1}\ldots u_{i_ki_k}\\
&= \sum_{\pi,\sigma\in D(k)}W_{kN}(\pi,\sigma)\sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\delta_\pi(i)\delta_\sigma(i)\\
&= \sum_{\pi,\sigma\in D(k)}W_{kN}(\pi,\sigma)G_{ks}(\sigma,\pi)\\
&= Tr(W_{kN}G_{ks})
\end{aligned}$$

Thus, we have obtained the formula in the statement. □

In order to process now the above formula, things are quite technical, and won't work well in general. We must impose here an uniformity condition, as follows:

Theorem 5.23. *For an easy quantum group $G=(G_N)$, coming from a category of partitions $D\subset P$, the following conditions are equivalent:*

1. *$G_{N-1}=G_N\cap U_{N-1}^+$, via the embedding $U_{N-1}^+\subset U_N^+$ given by $u\to diag(u,1)$.*
2. *$G_{N-1}=G_N\cap U_{N-1}^+$, via the N possible diagonal embeddings $U_{N-1}^+\subset U_N^+$.*
3. *D is stable under the operation which consists in removing blocks.*

If these conditions are satisfied, we say that $G=(G_N)$ is "uniform".

Proof. We use the general easiness theory from section 1 above.

(1) $\iff$ (2) This is something standard, coming from the inclusion $S_N\subset G_N$, which makes everything S_N-invariant. The result follows as well from the proof of (1) $\iff$ (3) below, which can be converted into a proof of (2) $\iff$ (3), in the obvious way.

(1) $\iff$ (3) Given a subgroup $K\subset U_{N-1}^+$, with fundamental corepresentation u, consider the $N\times N$ matrix $v=diag(u,1)$. Our claim is that for any $\pi\in P(k)$ we have:

$$\xi_\pi\in Fix(v^{\otimes k})\iff \xi_{\pi'}\in Fix(v^{\otimes k'}),\forall\pi'\in P(k'),\pi'\subset\pi$$

In order to prove this, we must study the condition on the left. We have:

$$\begin{aligned}
\xi_\pi\in Fix(v^{\otimes k}) &\iff (v^{\otimes k}\xi_\pi)_{i_1\ldots i_k}=(\xi_\pi)_{i_1\ldots i_k},\forall i\\
&\iff \sum_j(v^{\otimes k})_{i_1\ldots i_k,j_1\ldots j_k}(\xi_\pi)_{j_1\ldots j_k}=(\xi_\pi)_{i_1\ldots i_k},\forall i\\
&\iff \sum_j\delta_\pi(j_1,\ldots,j_k)v_{i_1j_1}\ldots v_{i_kj_k}=\delta_\pi(i_1,\ldots,i_k),\forall i
\end{aligned}$$

Now let us recall that our corepresentation has the special form $v=diag(u,1)$. We conclude from this that for any index $a\in\{1,\ldots,k\}$, we must have:

$$i_a=N\implies j_a=N$$

With this observation in hand, if we denote by i',j' the multi-indices obtained from i,j obtained by erasing all the above $i_a=j_a=N$ values, and by $k'\leq k$ the common length of these new multi-indices, our condition becomes:

$$\sum_{j'}\delta_\pi(j_1,\ldots,j_k)(v^{\otimes k'})_{i'j'}=\delta_\pi(i_1,\ldots,i_k),\forall i$$

Here the index j is by definition obtained from j' by filling with N values. In order to finish now, we have two cases, depending on i, as follows:

Case 1. Assume that the index set $\{a|i_a=N\}$ corresponds to a certain subpartition $\pi'\subset\pi$. In this case, the N values will not matter, and our formula becomes:

$$\sum_{j'}\delta_\pi(j'_1,\ldots,j'_{k'})(v^{\otimes k'})_{i'j'}=\delta_\pi(i'_1,\ldots,i'_{k'})$$

Case 2. Assume now the opposite, namely that the set $\{a|i_a=N\}$ does not correspond to a subpartition $\pi'\subset\pi$. In this case the indices mix, and our formula reads:

$$0=0$$

Thus, we are led to $\xi_{\pi'}\in Fix(v^{\otimes k'})$, for any subpartition $\pi'\subset\pi$, as claimed.

Now with this claim in hand, the result follows from Tannakian duality. $\square$

At the level of examples, the uniformity axiom is something very natural and useful, substantially cutting from complexity, and we have the following result, from [32]:

Theorem 5.24. *The classical and free uniform orthogonal easy quantum groups, with inclusions between them, are as follows,*

$$\begin{tikzcd}
 & H_N^+ \arrow[rr] & & O_N^+ \\
S_N^+ \arrow[rr] \arrow[ur] & & B_N^+ \arrow[ur] & \\
 & H_N \arrow[rr] \arrow[uu, dotted] & & O_N \arrow[uu, dotted] \\
S_N \arrow[rr] \arrow[ur] \arrow[uu, dotted] & & B_N \arrow[ur] \arrow[uu, dotted] &
\end{tikzcd}$$

with $H_N=\mathbb{Z}_2\wr S_N$ and $B_N=\{U\in O_N|\sum_iu_{ij}=\sum_ju_{ij}=1\}$ being the hyperoctahedral and bistochastic groups, and with H_N^+,B_N^+ being their free analogues.

Proof. The above quantum groups are indeed easy and uniform, the corresponding categories of partitions being as follows, with 12 standing for "singletons and pairings":

$$\begin{tikzcd}
 & NC_{even} \arrow[dl] \arrow[dd, dotted] & & NC_2 \arrow[ll] \arrow[dl] \arrow[dd, dotted] \\
NC \arrow[dd, dotted] & & NC_{12} \arrow[ll] \arrow[dd, dotted] & \\
 & P_{even} \arrow[dl] & & P_2 \arrow[ll] \arrow[dl] \\
P & & P_{12} \arrow[ll] &
\end{tikzcd}$$

Regarding now the classification, consider an easy quantum group $S_N \subset G_N \subset O_N$. This most come from a category $P_2 \subset D \subset P$, and by doing some combinatorics, we can see that D is uniquely determined by the subset $L \subset \mathbb{N}$ consisting of the sizes of the blocks of the partitions in D, with the admissible sets being as follows:

(1) $L = \{2\}$, producing O_N.
(2) $L = \{1, 2\}$, producing B_N.
(3) $L = \{2, 4, 6, \ldots\}$, producing H_N.
(4) $L = \{1, 2, 3, \ldots\}$, producing S_N.

In the free case, $S_N^+ \subset G_N \subset O_N^+$, the situation is quite similar, the admissible sets being once again the above ones, producing this time $O_N^+, B_N^+, H_N^+, S_N^+$. See [32]. □

By getting back now to the truncated characters, we have the following result:

Theorem 5.25. *For a uniform easy quantum group $G = (G_N)$, we have the formula*

$$\lim_{N \to \infty} \int_{G_N} \chi_t^k = \sum_{\pi \in D(k)} t^{|\pi|}$$

with $D \subset P$ being the associated category of partitions.

Proof. We use the general moment formula from Theorem 5.22 above. With $s = [tN]$, this formula becomes:

$$\int_{G_N} \chi_t^k = Tr(W_{kN} G_{k[tN]})$$

The point now is that in the uniform case the Gram and Weingarten matrices are asymptotically diagonal, and this leads to the formula in the statement. See [17], [18]. □

We can now improve our quantum group results, as follows:

Theorem 5.26. *The asymptotic laws of the truncated characters* $\chi_t = \sum_{i=1}^{[tN]}$ *for the quantum rotation and permutation groups are*

$$\begin{tikzcd} S_N^+ \arrow[r, dash] \arrow[d, leftrightarrow] & O_N^+ \arrow[d, leftrightarrow] \\ S_N \arrow[r, dash] & O_N \end{tikzcd} \quad : \quad \begin{tikzcd} \pi_t \arrow[r, dash] \arrow[d, leftrightarrow] & \gamma_t \arrow[d, leftrightarrow] \\ p_t \arrow[r, dash] & g_t \end{tikzcd}$$

with $g_t, p_t, \gamma_t, \pi_t$ *being the Gaussian, Poisson, Wigner, Marchenko-Pastur laws, and with the vertical arrows coming from liberation, and from the Bercovici-Pata bijection.*

Proof. The first assertion follows from Theorem 5.16 and Theorem 5.25, and the second assertion is something that we already know, from Theorem 5.17. □

Summarizing, we have completed the program outlined in the beginning of this section, and everything is now very well understood, and flawless.

6. Analytic aspects

We have seen so far that, in what concerns the probability theory on classical or quantum groups, the very first problem which appears, and which is of key importance, is that of computing the laws of characters, and more generally of truncated characters:

$$\chi=\sum_i u_{ii}\quad,\quad \chi_t=\sum_{i=1}^{[tN]}u_{ii}$$

For the quantum rotation and permutation groups, this problem can be investigated by using easiness and combinatorics, and satisfactory results in this sense, which are in tune with free probability theory, can be obtained in the $N\to\infty$ limit.

That was for the basic theory. In this section we discuss more advanced aspects, regarding the case where $N\in\mathbb N$ is fixed, or variables which are more general than the truncated characters χ_t, or regarding both, more advanced variables at fixed $N\in\mathbb N$.

Technically speaking, this requires either a good understanding of the Weingarten formula, with estimates for the entries of the Weingarten matrix, or some alternative integration formulae. Let us begin with the latter, with some tricks concerning S_N,O_N.

In what regards the symmetric group S_N, we have already seen that the laws of truncated characters can be computed by using the inclusion-exclusion principle.

In fact, the Weingarten formula is not really needed for the symmetric group, because the computations can always be done with elementary combinatorics.

The arbitrary integrals over S_N can be in fact computed as follows:

Theorem 6.1. *Consider the symmetric group S_N, together with its standard matrix coordinates $u_{ij}=\chi(\sigma\in S_N|\sigma(j)=i)$. We have the formula*

$$\int_{S_N}u_{i_1j_1}\ldots u_{i_kj_k}=\begin{cases}\frac{(N-|\ker i|)!}{N!}&\text{if }\ker i=\ker j\\ 0&\text{otherwise}\end{cases}$$

where $\ker i$ denotes as usual the partition of $\{1,\ldots,k\}$ whose blocks collect the equal indices of i, and where $|.|$ denotes the number of blocks.

Proof. According to the definition of u_{ij}, the integrals in the statement are given by:

$$\int_{S_N}u_{i_1j_1}\ldots u_{i_kj_k}=\frac{1}{N!}\#\left\{\sigma\in S_N\Big|\sigma(j_1)=i_1,\ldots,\sigma(j_k)=i_k\right\}$$

Since the existence of $\sigma\in S_N$ as above requires $i_m=i_n\iff j_m=j_n$, this integral vanishes when $\ker i\neq\ker j$. As for the case $\ker i=\ker j$, if we denote by $b\in\{1,\ldots,k\}$ the number of blocks of this partition, we have $N-b$ points to be sent bijectively to $N-b$ points, and so $(N-b)!$ solutions, and the integral is $\frac{(N-b)!}{N!}$, as claimed. $\square$

As an illustration for the above formula, we can recover the computation of the asymptotic laws of the truncated characters χ_t. This was already done, first by using the inclusion-exclusion principle, and then by using the Weingarten formula. Here is now our third proof, based on the above formula, and which is probably the best:

Theorem 6.2. *For the symmetric group $S_N \subset O_N$, regarded as a compact group of matrices, $S_N \subset O_N$, via the standard permutation matrices, the truncated character*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

counts the number of fixed points among $\{1, \ldots, [tN]\}$, and its law with respect to the counting measure becomes, with $N \to \infty$, a Poisson law of parameter t.

Proof. The first assertion comes from $u_{ij} = \chi(\sigma|\sigma(j) = i)$. Regarding now the second assertion, we use Theorem 6.1. With S_{kb} being the Stirling numbers, we have:

$$\begin{aligned}
\int_{S_N} \chi_t^k &= \sum_{i_1 \ldots i_k = 1}^{[tN]} \int_{S_N} u_{i_1 i_1} \ldots u_{i_k i_k} \\
&= \sum_{\pi \in P(k)} \frac{[tN]!}{([tN] - |\pi|!)} \cdot \frac{(N - |\pi|!)}{N!} \\
&= \sum_{b=1}^{[tN]} \frac{[tN]!}{([tN] - b)!} \cdot \frac{(N - b)!}{N!} \cdot S_{kb}
\end{aligned}$$

In particular with $N \to \infty$ we obtain the following formula:

$$\lim_{N \to \infty} \int_{S_N} \chi_t^k = \sum_{b=1}^{k} S_{kb} t^b$$

But this is a Poisson(t) moment, and so we are done. $\square$

Summarizing, the integration formula in Theorem 6.1 is extremely simple and efficient. We will regularly use it in what follows, for other questions, to be formulated later.

Regarding now the orthogonal group O_N, here the situation is more complicated, and the Weingarten formula is generally needed. In fact, explicitely integrating over O_N, U_N was historically the main motivation for developing the Weingarten calculus.

However, in certain special situations, we can use the following trick:

Theorem 6.3. *Each row of coordinates on O_N has the same joint distribution as the sequence of coordinates on the real sphere S^{N-1},*

$$(u_{i1}, \ldots, u_{iN}) \sim (x_1, \ldots, x_N)$$

and the same happens for the columns.

Proof. Given an index $i \in \{1, \ldots, N\}$, our claim is that we have an embedding as follows, which commutes with the corresponding uniform integration functionals:

$$C(S^{N-1}) \subset C(O_N) \quad , \quad x_j \to u_{ij}$$

In order to prove this claim, consider the subalgebra $C(S) \subset C(O_N)$ generated by the variables u_{ij}, with i being fixed, and with $j = 1, \ldots, N$. Since these N variables are real, and their squares sum up to 1, we have a quotient map, as follows:

$$C(S^{N-1}) \to C(S) \subset C(O_N) \quad , \quad x_j \to u_{ij}$$

Now observe that $S \subset S^{N-1}$ must be an isomorphism, because by Gram-Schmidt we can complete any vector of S^{N-1} into an orthogonal matrix. Thus, the above composition of morphisms is an embedding. As for the commutation with the uniform integration functionals, this follows from the fact that we have an action $O_N \curvearrowright S$. □

Summarizing, in what regards the law of the individual coordinates $u_{ij} \in C(O_N)$, and more generally the joint law of coordinates belonging to the same row or column of $u = (u_{ij})$, we can use here spherical integrals, instead of the Weingarten formula.

In view of this, let us discuss now the integration over S^{N-1}. We first have:

Proposition 6.4. *We have the formula*

$$\int_{S^{N-1}} x_{i_1} \ldots x_{i_k} \, dx = 0$$

unless each x_i appears an even number of times.

Proof. This follows from the fact that for any i we have an automorphism given by $x_i \to -x_i$. Indeed, this automorphism must preserve the trace, so if x_i appears an odd number of times, the integral in the statement satisfies $I = -I$, so $I = 0$. □

In order to compute now the nonzero integrals, we will use polar coordinates and the Fubini theorem, which reduce the computation to the case $N = 2$.

So, let us start with the case $N = 2$. Here we have the following result:

Proposition 6.5. *The integrals over $S^2 = \mathbb{T}$ are given by the formula*

$$\frac{2}{\pi} \int_0^{\pi/2} \cos^p t \sin^q t \, dt = \left(\frac{2}{\pi}\right)^{\delta(p,q)} \frac{p!!q!!}{(p+q+1)!!}$$

where $\delta(a, b) = 0$ if both a, b are even, and $\delta(a, b) = 1$ otherwise.

Proof. This follows indeed by double recurrence, using a partial integration. □

In general, the nonzero integrals over S^{N-1} can be computed as follows:

Theorem 6.6. *For any $k_1,\ldots,k_p\in\mathbb N$ we have*

$$\int_{S^{N-1}}\left|x_1^{k_1}\ldots x_p^{k_p}\right|dx=\left(\frac{2}{\pi}\right)^{\Sigma(k_1,\ldots,k_p)}\frac{(N-1)!!k_1!!\ldots k_p!!}{(N+\Sigma k_i-1)!!}$$

with $\Sigma=[odds/2]$ if N is odd and $\Sigma=[(odds+1)/2]$ if N is even, where "odds" denotes the number of odd numbers in the sequence $k_1,\ldots,k_p$.

Proof. The formula holds indeed at $N=2$, due to Proposition 6.5 above, with the remark that the δ symbols used there are given by the following formula:

$$\delta(a,b)=\left[\frac{odds(a,b)+1}{2}\right]$$

In general now, the integral in the statement can be written in spherical coordinates, in the following way:

$$I=\frac{2^N}{V}\int_0^{\pi/2}\ldots\int_0^{\pi/2}x_1^{k_1}\ldots x_N^{k_N}J\,dt_1\ldots dt_{N-1}$$

Here V is the volume of the sphere, J is the Jacobian, and the 2^N factor comes from the restriction to the $1/2^N$ part of the sphere where all the coordinates are positive.

The normalization constant in front of the integral is:

$$\frac{2^N}{V}=\frac{2^N}{N\pi^{N/2}}\cdot\Gamma\left(\frac{N}{2}+1\right)=\left(\frac{2}{\pi}\right)^{[N/2]}(N-1)!!$$

As for the unnormalized integral, this is given by:

$$\begin{aligned}I'=\int_0^{\pi/2}\ldots\int_0^{\pi/2}\quad&(\cos t_1)^{k_1}\\&(\sin t_1\cos t_2)^{k_2}\\&\ldots\\&(\sin t_1\sin t_2\ldots\sin t_{N-2}\cos t_{N-1})^{k_{N-1}}\\&(\sin t_1\sin t_2\ldots\sin t_{N-2}\sin t_{N-1})^{k_N}\\&\sin^{N-2}t_1\sin^{N-3}t_2\ldots\sin^2t_{N-3}\sin t_{N-2}\\&dt_1\ldots dt_{N-1}\end{aligned}$$

By rearranging the terms, we get:

$$\begin{aligned} I' &= \int_0^{\pi/2}\cos^{k_1}t_1\sin^{k_2+\ldots+k_N+N-2}t_1\,dt_1\\ &\quad\int_0^{\pi/2}\cos^{k_2}t_2\sin^{k_3+\ldots+k_N+N-3}t_2\,dt_2\\ &\quad\ldots\\ &\quad\int_0^{\pi/2}\cos^{k_{N-2}}t_{N-2}\sin^{k_{N-1}+k_N+1}t_{N-2}\,dt_{N-2}\\ &\quad\int_0^{\pi/2}\cos^{k_{N-1}}t_{N-1}\sin^{k_N}t_{N-1}\,dt_{N-1}\end{aligned}$$

Now by using the formula at $N=2$, we get:

$$\begin{aligned} I' &= \frac{\pi}{2}\cdot\frac{k_1!!(k_2+\ldots+k_N+N-2)!!}{(k_1+\ldots+k_N+N-1)!!}\left(\frac{2}{\pi}\right)^{\delta(k_1,k_2+\ldots+k_N+N-2)}\\ &\quad\frac{\pi}{2}\cdot\frac{k_2!!(k_3+\ldots+k_N+N-3)!!}{(k_2+\ldots+k_N+N-2)!!}\left(\frac{2}{\pi}\right)^{\delta(k_2,k_3+\ldots+k_N+N-3)}\\ &\quad\ldots\\ &\quad\frac{\pi}{2}\cdot\frac{k_{N-2}!!(k_{N-1}+k_N+1)!!}{(k_{N-2}+k_{N-1}+k_N+2)!!}\left(\frac{2}{\pi}\right)^{\delta(k_{N-2},k_{N-1}+k_N+1)}\\ &\quad\frac{\pi}{2}\cdot\frac{k_{N-1}!!k_N!!}{(k_{N-1}+k_N+1)!!}\left(\frac{2}{\pi}\right)^{\delta(k_{N-1},k_N)}\end{aligned}$$

In this expression most of the factorials cancel, and the δ exponents on the right sum up to the following number:

$$\Delta(k_1,\ldots,k_N)=\sum_{i=1}^{N-1}\delta(k_i,k_{i+1}+\ldots+k_N+N-i-1)$$

In other words, with this notation, the above formula reads:

$$\begin{aligned} I' &= \left(\frac{\pi}{2}\right)^{N-1}\frac{k_1!!k_2!!\ldots k_N!!}{(k_1+\ldots+k_N+N-1)!!}\left(\frac{2}{\pi}\right)^{\Delta(k_1,\ldots,k_N)}\\ &= \left(\frac{2}{\pi}\right)^{\Delta(k_1,\ldots,k_N)-N+1}\frac{k_1!!k_2!!\ldots k_N!!}{(k_1+\ldots+k_N+N-1)!!}\\ &= \left(\frac{2}{\pi}\right)^{\Sigma(k_1,\ldots,k_N)-[N/2]}\frac{k_1!!k_2!!\ldots k_N!!}{(k_1+\ldots+k_N+N-1)!!}\end{aligned}$$

Here the formula relating Δ to Σ follows from a number of simple observations, the first of which is the following one: due to obvious parity reasons, the sequence of δ numbers appearing in the definition of Δ cannot contain two consecutive zeros.

Together with $I=(2^N/V)I'$, this gives the formula in the statement. □

As a remark, the exponent Σ appearing in the statement of Theorem 6.6 can be written as well in the following compact form:

$$\Sigma(k_1,\ldots,k_p)=\left[\frac{N+odds+1}{2}\right]-\left[\frac{N+1}{2}\right]$$

However, for concrete applications, the writing in Theorem 6.6 is more convenient.

As an application, we can compute the laws of the individual coordinates $u_{ij}\in C(O_N)$, and work out their asymptotics. The result here is as follows:

Theorem 6.7. *The individual coordinates $u_{ij}\in C(O_N)$ follow the same law as the coordinates $x_i\in C(S^{N-1})$. This common law, called hyperspherical, has as moments*

$$M_k=\frac{(N-1)!!k!!}{(N+k-1)!!}$$

and with $N\to\infty$, the hyperspherical variables rescaled by $1/\sqrt{N}$ become normal.

Proof. The first assertion follows from Theorem 6.3 above. In order to compute now the moments of the hyperspherical law, we can use Proposition 6.4 and Theorem 6.6.

Indeed, Proposition 6.4 with $i_1=\ldots=i_k$ tells us that the odd moments vanish. As for the even moments, Theorem 6.6 with $k_1=k\in 2\mathbb{N}$ and $k_2=\ldots=k_p=0$ gives:

$$\int_{S^{N-1}}x_1^k\,dx=\frac{(N-1)!!k!!}{(N+k-1)!!}$$

With our usual convention that the double factorials $k!!$ make vanish the expression they appear in, when k is odd, this gives the formula in the statement.

Regarding now the last assertion, observe that with $N\to\infty$ we have:

$$\int_{S^{N-1}}x_1^k\,dx\simeq N^{k/2}k!!$$

Thus, by rescaling by $1/\sqrt{N}$, we obtain the following formula:

$$\int_{S^{N-1}}\left(\frac{x_1}{\sqrt{N}}\right)^k dx\simeq k!!$$

It follows that we have $x_1/\sqrt{N}\sim g_1$ with $N\to\infty$, as desired. □

All this is quite interesting, and there are some further applications of Theorem 6.3 and Theorem 6.6. However, as already mentioned, and unlike in the case of the symmetric group S_N, for the group O_N we have to use in general the Weingarten formula.

Let us discuss now the case of the quantum groups S_N^+ and O_N^+. Here there is no simple combinatorial or geometric trick, and the Weingarten formula is the only integration tool that we have. However, this is not an issue, because this formula is after all something quite elementary, and working out the combinatorics always leads to results.

As a first question regarding S_N^+, O_N^+, let us discuss the computation and asymptotics of the free hyperspherical and free hypergeometric laws. Regarding the free hyperspherical laws, these are by definition the laws of the variables $u_{ij} \in C(O_N^+)$.

We should mention that it is possible to talk as well about a free real sphere $S^{N-1}_{\mathbb{R},+}$, and about a free analogue of Theorem 6.3, but in practice this is not very interesting, because integrating over $S^{N-1}_{\mathbb{R},+}$ requires using the Weingarten formula for O_N^+, or at least there is no known alternative trick, and so we cannot get to new results in this way.

Summarizing, the problem is that of computing the law of $u_{ij} \in C(O_N^+)$, using the Weingarten formula. This is something non-trivial, and the result here is as follows:

Theorem 6.8. *The moments of the free hyperspherical law are given by*

$$\int_{O_N^+} u_{ij}^{2l} = \frac{1}{(N+1)^l} \cdot \frac{q+1}{q-1} \cdot \frac{1}{l+1} \sum_{r=-l-1}^{l+1} (-1)^r \binom{2l+2}{l+r+1} \frac{r}{1+q^r}$$

where $q \in [-1, 0)$ is such that $q + q^{-1} = -N$.

Proof. The idea is that $u_{ij} \in C(O_N^+)$ has the same law as a certain variable $w \in C(SU_2^q)$, with $q \in [-1, 0)$ being as above, and this latter variable can be modelled by an explicit operator on $l^2(\mathbb{N})$, whose law can be computed by using advanced calculus.

Let us first explain the relation between O_N^+ and SU_2^q. To any matrix $F \in GL_N(\mathbb{R})$ satisfying $F^2 = 1$ we associate the following universal algebra:

$$C(O_F^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N} \Big| u = F\bar{u}F = \text{unitary}\right)$$

Observe that $O_{I_N}^+ = O_N^+$. In general, the above algebra satisfies Woronowicz's generalized axioms in [99], which do not include the antipode condition $S^2 = id$.

At $N = 2$, up to a trivial equivalence relation on the matrices F, and on the quantum groups O_F^+, we can assume that F is as follows, with $q \in [-1, 0)$:

$$F = \begin{pmatrix} 0 & \sqrt{-q} \\ 1/\sqrt{-q} & 0 \end{pmatrix}$$

Our claim is that for this matrix we have $O_F^+ = SU_2^q$. Indeed, the relations $u = F\bar{u}F$ tell us that u must be of the following special form:

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

Thus $C(O_F^+)$ is the universal algebra generated by two elements α, γ, with the relations making the above matrix u unitary. But these unitarity conditions are:

$$\alpha\gamma = q\gamma\alpha$$

$$\alpha\gamma^* = q\gamma^*\alpha$$
$$\gamma\gamma^* = \gamma^*\gamma$$
$$\alpha^*\alpha + \gamma^*\gamma = 1$$
$$\alpha\alpha^* + q^2\gamma\gamma^* = 1$$

We recognize here the relations in [99] defining the algebra $C(SU_2^q)$, and it follows that we have an isomorphism of Hopf C^*-algebras:

$$C(O_F^+) \simeq C(SU_2^q)$$

Now back to the general case, let us try to understand the integration over O_F^+. Given $\pi \in NC_2(2k)$ and $i = (i_1, \ldots, i_{2k})$, we set:

$$\delta_\pi^F(i) = \prod_{s\in\pi} F_{i_{s_l} i_{s_r}}$$

Here the product is over all the strings $s = \{s_l \curvearrowright s_r\}$ of π. Our claim is that the following family of vectors, with $\pi \in NC_2(2k)$, spans the space of fixed vectors of $u^{\otimes 2k}$, for the quantum group O_F^+:

$$\xi_\pi = \sum_i \delta_\pi^F(i) e_{i_1} \otimes \ldots \otimes e_{i_{2k}}$$

Indeed, having $\xi_\cap$ fixed by $u^{\otimes 2}$ is equivalent to assuming that $u = F\bar{u}F$ is unitary.

By using now the above vectors, we obtain the following Weingarten formula:

$$\int_{O_F^+} u_{i_1j_1} \ldots u_{i_{2k}j_{2k}} = \sum_{\pi\sigma} \delta_\pi^F(i)\delta_\sigma^F(j) W_{kN}(\pi,\sigma)$$

With these preliminaries in hand, let us start the computation. Let $N \in \mathbb{N}$, and consider the number $q \in [-1, 0)$ satisfying $q + q^{-1} = -N$. Our claim is that we have:

$$\int_{O_N^+} \varphi(\sqrt{N+2}\, u_{ij}) = \int_{SU_2^q} \varphi(\alpha + \alpha^* + \gamma - q\gamma^*)$$

Indeed, the moments of the variable on the left are given by:

$$\int_{O_N^+} u_{ij}^{2k} = \sum_{\pi\sigma} W_{kN}(\pi,\sigma)$$

On the other hand, the moments of the variable on the right, which in terms of the fundamental corepresentation $v = (v_{ij})$ is given by $w = \sum_{ij} v_{ij}$, are given by:

$$\int_{SU_2^q} w^{2k} = \sum_{ij} \sum_{\pi\sigma} \delta_\pi^F(i)\delta_\sigma^F(j) W_{kN}(\pi,\sigma)$$

We deduce that $w/\sqrt{N+2}$ has the same moments as u_{ij}, which proves our claim.

In order to do now the computation over SU_2^q, we can use a matrix model due to Woronowicz [99], where the standard generators α,γ are mapped as follows:

$$\begin{aligned}\pi_u(\alpha)e_k &= \sqrt{1-q^{2k}}e_{k-1}\\ \pi_u(\gamma)e_k &= uq^ke_k\end{aligned}$$

Here $u\in\mathbb T$ is a parameter, and (e_k) is the standard basis of $l^2(\mathbb N)$. The point with this representation is that it allows the computation of the Haar functional. Indeed, if D is the diagonal operator given by $D(e_k)=q^{2k}e_k$, then the formula is as follows:

$$\int_{SU_2^q}x=(1-q^2)\int_{\mathbb T}tr(D\pi_u(x))\frac{du}{2\pi iu}$$

With the above model in hand, the law of the variable that we are interested in is as follows, where $M(e_k)=e_{k+1}+q^k(u-qu^{-1})e_k+(1-q^{2k})e_{k-1}$:

$$\int_{SU_2^q}\varphi(\alpha+\alpha^*+\gamma-q\gamma^*)=(1-q^2)\int_{\mathbb T}tr(D\varphi(M))\frac{du}{2\pi iu}$$

The point now is that the integral on the right can be computed, by using advanced calculus methods, and this gives the result. □

The computation of the joint free hyperspherical laws remains an open problem. Open as well is the question of finding a more conceptual proof for the above formula.

Following now [15], let us discuss now the free hypergeometric laws. We will use here the twisting result established in section 4 above, which is also from [15].

We know from that twisting result that we have, at the probabilistic level:

Theorem 6.9. *The following two algebras are isomorphic, via $u_{ij}^2\to X_{ij}$,*

1. *The algebra generated by the variables $u_{ij}^2\in C(O_n^+)$,*
2. *The algebra generated by $X_{ij}=\frac{1}{n}\sum_{a,b=1}^np_{ia,jb}\in C(S_{n^2}^+)$,*

and this isomorphism commutes with the respective Haar integration functionals.

Proof. This follows indeed from the general twisting result from section 4. □

As pointed out in [15], it is possible to derive as well this result directly, by using the Weingarten formula, and manipulations on the partitions. Let us start with:

Proposition 6.10. *We have a bijection $NC(k)\simeq NC_2(2k)$, constructed as follows:*

1. *The application $NC(k)\to NC_2(2k)$ is the "fattening" one, obtained by doubling all the legs, and doubling all the strings as well.*
2. *Its inverse $NC_2(2k)\to NC(k)$ is the "shrinking" application, obtained by collapsing pairs of consecutive neighbors.*

Proof. The fact that the two operations in the statement are indeed inverse to each other is clear, by computing the corresponding two compositions, with the remark that the construction of the fattening operation requires the partitions to be noncrossing. □

We have the following key observation:

Theorem 6.11. *The Gram matrices of $NC_2(2k), NC(k)$ are related as follows,*

$$G_{2k,n}(\pi,\sigma)=n^k(\Delta_{kn}^{-1}G_{k,n^2}\Delta_{kn}^{-1})(\pi',\sigma')$$

where $\pi\to\pi'$ is the shrinking operation, and Δ_{kn} is the diagonal of G_{kn}.

Proof. In the context of Proposition 6.10, it is elementary to see that we have:

$$|\pi\vee\sigma|=k+2|\pi'\vee\sigma'|-|\pi'|-|\sigma'|$$

We therefore have the following formula, valid for any $n\in\mathbb N$:

$$n^{|\pi\vee\sigma|}=n^{k+2|\pi'\vee\sigma'|-|\pi'|-|\sigma'|}$$

Thus, we obtain the formula in the statement. □

We can now basically reprove Theorem 6.9, as follows:

Theorem 6.12. *The following families of variables have the same joint law,*

(1) $\{u_{ij}^2\}\in C(O_n^+)$,
(2) $\{X_{ij}=\frac{1}{n}\sum_{ab}p_{ia,jb}\}\in C(S_{n^2}^+)$,

where $u=(u_{ij})$ and $p=(p_{ia,jb})$ are the corresponding fundamental corepresentations.

Proof. As already mentioned, this result can be obtained via twisting methods. An alternative approach is by using the Weingarten formula for our two quantum groups, and the shrinking operation $\pi\to\pi'$. Indeed, we obtain the following moment formulae:

$$\begin{aligned}\int_{O_n^+}u_{ij}^{2k}&=\sum_{\pi,\sigma\in NC_2(2k)}W_{2k,n}(\pi,\sigma)\\ \int_{S_{n^2}^+}X_{ij}^k&=\sum_{\pi,\sigma\in NC_2(2k)}n^{|\pi'|+|\sigma'|-k}W_{k,n^2}(\pi',\sigma')\end{aligned}$$

According to Theorem 6.11 the summands coincide, and so the moments are equal, as desired. The proof in general, dealing with joint moments, is similar. □

In what follows we will be interested in single variables. We have here:

Theorem 6.13. *The free hypergeometric variable*

$$X_{ij}=\frac{1}{n}\sum_{a,b=1}^n u_{ia,jb}\in C(S_{n^2}^+)$$

has the same law as the squared free hyperspherical variable $u_{ij}^2\in C(O_n^+)$.

Proof. This follows indeed from Theorem 6.12 above. □

The variables X_{ij} appearing above have the following generalization:

Definition 6.14. *The noncommutative random variable*

$$X(n,m,N)=\sum_{i=1}^n\sum_{j=1}^m u_{ij}\in C(S_N^+)$$

is called free hypergeometric, of parameters (n,m,N).

The terminology comes from the fact that the variable $X'(n,m,N)$, defined as above, but over the algebra $C(S_N)$, follows a hypergeometric law of parameters (n,m,N).

Here is an exploration of the basic asymptotic properties of these laws:

Theorem 6.15. *The free hypergeometric laws have the following properties:*

(1) *Let $n,m,N\to\infty$, with $\frac{nm}{N}\to t\in(0,\infty)$. Then the law of $X(n,m,N)$ converges to Marchenko-Pastur law π_t.*
(2) *Let $n,m,N\to\infty$, with $\frac{n}{N}\to\nu\in(0,1)$ and $\frac{m}{N}\to 0$. Then the law of $S(n,m,N)=(X(n,m,N)-m\nu)/\sqrt{m\nu(1-\nu)}$ converges to the semicircle law γ_1.*

Proof. This is standard, by using the Weingarten formula, as follows:

(1) From the Weingarten formula, we have:

$$\int X(n,m,N)^k=\sum_{\pi,\sigma\in NC(k)}W_{kN}(\pi,\sigma)n^{|\pi|}m^{|\sigma|}$$

The point now is that we have the following estimate:

$$W_{kN}(\pi,\sigma)=\begin{cases}N^{-|\pi|}+O(N^{-|\pi|-1})&\text{if }\pi=\sigma\\ O(N^{|\pi\vee\sigma|-|\pi|-|\sigma|})&\text{if }\pi\neq\sigma\end{cases}$$

It follows that we have:

$$W_{kN}(\pi,\sigma)n^{|\pi|}m^{|\sigma|}\to\begin{cases}t^{|\pi|}&\text{if }\pi=\sigma\\ 0&\text{if }\pi\neq\sigma\end{cases}$$

Thus the k-th moment of $X(n,m,N)$ converges to $\sum_{\pi\in NC(k)}\lambda^{|\pi|}$, which is the k-th moment of the Marchenko-Pastur law π_t, and we are done.

(2) We need to show that the free cumulants satisfy:

$$\kappa^{(p)}[S(n,m,N),\ldots,S(n,m,N)]\to\begin{cases}1&\text{if }p=2\\ 0&\text{if }p\neq 2\end{cases}$$

The case $p=1$ is trivial, so suppose $p\geq 2$. We have:

$$\begin{aligned}
&\kappa^{(p)}[S(n,m,N),\ldots,S(n,m,N)]\\
=\ &(m\nu(1-\nu))^{-p/2}\kappa^{(p)}[X(n,m,N),\ldots,X(n,m,N)]
\end{aligned}$$

On the other hand, from the Weingarten formula, we have:

$$\begin{aligned}
&\kappa^{(p)}[X(n,m,N),\ldots,X(n,m,N)]\\
=&\sum_{w\in NC(p)}\mu_p(w,1_p)\prod_{V\in w}\sum_{\pi_V,\sigma_V\in NC(V)}W_{NC(V),N}(\pi_V,\sigma_V)n^{|\pi_V|}m^{|\sigma_V|}\\
=&\sum_{w\in NC(p)}\mu_p(w,1_p)\prod_{V\in w}\sum_{\pi_V,\sigma_V\in NC(V)}(N^{-|\pi_V|}\mu_{|V|}(\pi_V,\sigma_V)+O(N^{-|\pi_V|-1}))n^{|\pi_V|}m^{|\sigma_V|}\\
=&\sum_{\substack{\pi,\sigma\in NC(p)\\ \pi\leq\sigma}}(N^{-|\pi|}\mu_p(\pi,\sigma)+O(N^{-|\pi|-1}))n^{|\pi|}m^{|\sigma|}\sum_{\substack{w\in NC(p)\\ \sigma\leq w}}\mu_p(w,1_p)
\end{aligned}$$

We use now the following standard identity:

$$\sum_{\substack{w\in NC(p)\\ \sigma\leq w}}\mu_p(w,1_p)=\begin{cases}1&\text{if }\sigma=1_p\\ 0&\text{if }\sigma\neq 1_p\end{cases}$$

This gives the following formula for the cumulants:

$$\kappa^{(p)}[X(n,m,N),\ldots,X(n,m,N)]=m\sum_{\pi\in NC(p)}(N^{-|\pi|}\mu_p(\pi,1_p)+O(N^{-|\pi|-1}))n^{|\pi|}$$

It follows that for $p\geq 3$ we have, as desired:

$$\kappa^{(p)}[S(n,m,N),\ldots,S(n,m,N)]\to 0$$

As for the remaining case $p=2$, here we have:

$$\begin{aligned}
\kappa^{(2)}[S(n,m,N),S(n,m,N)]&\to\frac{1}{\nu(1-\nu)}\sum_{\pi\in NC(2)}\nu^{|\pi|}\mu_2(\pi,1_2)\\
&=\frac{1}{\nu(1-\nu)}\left(\nu-\nu^2\right)\\
&=1
\end{aligned}$$

This gives the result. □

Summarizing, the twisting result relating $O_n^+,S_{n^2}^+$ is very interesting, analytically speaking, and the results that can be obtained in this way have no classical counterpart.

Let us discuss now another family of interesting results, this time regarding Gram matrix determinants. We first recall from section 2 above that for S_N we have:

Theorem 6.16. *The determinant of the Gram matrix for S_N is given by*

$$\det(G_{kN})=\prod_{\pi\in P(k)}\frac{N!}{(N-|\pi|)!}$$

with our usual convention that negative factorials make the expression vanish.

Proof. As explained in the proof of Theorem 2.18 above, this comes from a formula of type $G_{kN}=AL$, with A being upper triangular, and L being lower triangular. □

The interesting feature of the above formula is the fact that we have a decomposition over all the partitions associated to S_N, with each partition contributing to $\det(G_{kN})$.

As explained in [24], this is part of a more general phenomenon, involving the easy quantum groups in general. We will discuss here the case of O_N, and then of S_N^+,O_N^+.

In what concerns O_N, here the combinatorics is that of the Young diagrams. We denote by $|.|$ the number of boxes, and we use quantity f^λ, which gives the number of standard Young tableaux of shape λ. With these conventions, the result is as follows:

Theorem 6.17. *The determinant of the Gram matrix for O_N is given by*

$$\det(G_{kN})=\prod_{|\lambda|=k/2}f_N(\lambda)^{f^{2\lambda}}$$

where $f_N(\lambda)=\prod_{(i,j)\in\lambda}(N+2j-i-1)$.

Proof. This follows from the various results of Collins and Matsumoto and Zinn-Justin. Indeed, it is known from them that the Gram matrix is diagonalizable, as follows:

$$G_{kN}=\sum_{|\lambda|=k/2}f_N(\lambda)P_{2\lambda}$$

Here $1=\Sigma P_{2\lambda}$ is the standard partition of unity associated to the Young diagrams having $k/2$ boxes, and the coefficients $f_N(\lambda)$ are those in the statement.

Now since we have $Tr(P_{2\lambda})=f^{2\lambda}$, this gives the result. □

Let P_r be the Chebycheff polynomials, given by $P_0=1,P_1=X$ and $P_{r+1}=XP_r-P_{r-1}$. Consider also the following numbers, depending on $k,r\in\mathbb{Z}$:

$$f_{kr}=\binom{2k}{k-r}-\binom{2k}{k-r-1}$$

We set $f_{kr}=0$ for $k\notin\mathbb{Z}$. The following key result was proved in [47]:

Theorem 6.18. *The determinant of the Gram matrix for O_N^+ is given by*

$$\det(G_{kN})=\prod_{r=1}^{[k/2]}P_r(N)^{d_{k/2,r}}$$

where $d_{kr}=f_{kr}-f_{k+1,r}$.

Proof. As already mentioned, the result is from [47]. We present below a short proof. The result holds when k is odd, all the exponents being 0, so we assume that k is even.

Step 1. We establish a useful formula of type $G_{kN}(\pi,\sigma)=< f_\pi, f_\sigma >$.

For this purpose, let Γ be a locally finite bipartite graph, with distinguished vertex 0 and adjacency matrix A, and let μ be an eigenvector of A, with eigenvalue N.

Let L_k be the set of length k loops $l=l_1\ldots l_k$ based at 0, and $H_k=span(L_k)$. For $\pi\in\mathcal P_{o^+}(k)$ define $f_\pi\in H_k$ by:

$$f_\pi=\sum_{l\in L_k}\left(\prod_{i\sim_\pi j}\delta(l_i,l_j^o)\gamma(l_i)\right)l$$

Here $e\to e^o$ is the edge reversing, and the "spin factor" is $\gamma=\sqrt{\mu(t)/\mu(s)}$, where s,t are the source and target of the edges. The point is that we have:

$$G_{kN}(\pi,\sigma)=< f_\pi, f_\sigma >$$

We refer to [58] for details regarding all this.

Step 2. With a suitable choice of (Γ,μ), we obtain a fomula of type $G_{kN}=T_{kN}T_{kN}^t$.

Indeed, let us choose $\Gamma=\mathbb N$ to be the Cayley graph of O_N^+, and the eigenvector entries $\mu(r)$ to be the Chebycheff polynomials $P_r(N)$, i.e. the orthogonal polynomials for O_N^+.

In this case, we have a bijection $\mathcal P_{o^+}(k)\to L_k$, constructed as follows. For $\pi\in\mathcal P_{o^+}(k)$ and $0\le i\le k$ we define $h_\pi(i)$ to be the number of $1\le j\le i$ which are joined by π to a number strictly larger than i. We then define a loop $l(\pi)=l(\pi)_1\ldots l(\pi)_k$, where $l(\pi)_i$ is the edge from $h_\pi(i-1)$ to $h_\pi(i)$. Consider now the following matrix:

$$T_{kN}(\pi,\sigma)=\prod_{i\sim_\pi j}\delta(l(\sigma)_i,l(\sigma)_j^o)\gamma(l(\sigma)_i)$$

We have $f_\pi=\sum_\sigma T_{kn}(\pi,\sigma)\cdot l(\sigma)$, so we obtain as desired $G_{kN}=T_{kN}T_{kN}^t$.

Step 3. We show that, with suitable conventions, T_{kN} is lower triangular.

Indeed, consider the partial order on $\mathcal P_{o^+}(k)$ given by $\pi\le\sigma$ if $h_\pi(i)\le h_\sigma(i)$ for $i=1,\ldots,k$. Our claim is that $\sigma\not\le\pi$ implies $T_{kN}(\pi,\sigma)=0$.

Indeed, suppose that $\sigma\not\le\pi$, and let j be the least number with $h_\sigma(j)>h_\pi(j)$. Note that we must have $h_\sigma(j-1)=h_\pi(j-1)$ and $h_\sigma(j)=h_\pi(j)+2$. It follows that we have $i\sim_\pi j$ for some $i<j$. From the definitions of T_{kn} and $l(\sigma)$, if $T_{kn}(\pi,\sigma)\neq0$ then we must have $h_\sigma(i-1)=h_\sigma(j)=h_\pi(j)+2$. But we also have $h_\pi(i-1)=h_\pi(j)$, so that $h_\sigma(i-1)=h_\pi(i-1)+2$, which contradicts the minimality of j.

Step 4. End of the proof, by computing the determinant of T_{kN}.

Since T_{kN} is lower triangular we have:

$$\det(T_{kN})=\prod_\pi T_{kN}(\pi,\pi)=\prod_\pi\prod_{i\sim_\pi j}\sqrt{\frac{P_{h_{\pi(i)}}}{P_{h_{\pi(i)-1}}}}=\prod_{r=1}^{k/2}P_r^{e_{kr}/2}$$

Here the exponents appearing on the right are by definition as follows:

$$e_{kr}=\sum_\pi\sum_{i\sim_\pi j}\delta_{h_\pi(i),r}-\delta_{h_\pi(i),r+1}$$

Our claim now, which finishes the proof, is that for $1\leq r\leq k/2$ we have:

$$\sum_\pi\sum_{i\sim_\pi j}\delta_{h_\pi(i)r}=f_{k/2,r}$$

Indeed, note that the left term counts the number of times that the edge $(r,r+1)$ appears in all loops in L_k. Define a shift operator S on the edges of Γ by $S(s,t)=(s+1,t+1)$. Given a loop $l=l_1\ldots l_k$ and $1\leq s\leq k$ with $l_s=(r,r+1)$, define a path $S^r(l_s)\ldots S^r(l_k)l_{s-1}^o\ldots l_1^o$. Observe that this is a path in Γ from $2r$ to 0 whose first edge is $(2r,2r+1)$ and first reaches $r-1$ after $k-s+1$ steps.

Conversely, given a path $f_1\ldots f_k$ in Γ from $2r$ to 0 whose first edge is $(2r,2r+1)$ and first reaches $r-1$ after s steps, define a loop $f_k^o\ldots f_s^oS^{-r}(f_1)\ldots S^{-r}(f_{s-1})$. Observe that this is a loop in Γ based at 0 whose $k-s+1$ edge is $(r,r+1)$.

These two operations are inverse to each other, so we have established a bijection between k-loops in Γ based at 0 whose s-th edge is $(r,r+1)$ and k-paths in Γ from $2r$ to 0 whose first edge is $(2r,2r+1)$ and which first reach $r-1$ after $k-s+1$ steps.

It follows that the left hand side is equal to the number of paths in $\Gamma=\mathbb{N}$ from $2r$ to 0 whose first edge is $(2r,2r+1)$. By the usual reflection trick, this is the difference of binomials defining $f_{k/2,r}$, and we are done. $\square$

Regarding now the quantum group S_N^+, we have here:

Theorem 6.19. *The determinant of the Gram matrix for S_N^+ is given by*

$$\det(G_{kN})=(\sqrt{N})^{a_k}\prod_{r=1}^kP_r(\sqrt{N})^{d_{kr}}$$

where $a_k=\sum_{\pi\in\mathcal{P}(k)}(2|\pi|-k)$.

Proof. We use the shrinking operation $\pi\to\widetilde{\pi}$, obtained by collapsing neighbors. We have the following formula:

$$|\pi\vee\sigma|=k/2+2|\widetilde{\pi}\vee\widetilde{\sigma}|-|\widetilde{\pi}|-|\widetilde{\sigma}|$$

In terms of Gram matrices, if we denote by G' the Gram matrix for O_N^+, we have the following formula, with $D_{kN}=diag(N^{|\widetilde{\pi}|/2-k/4})$:

$$G_{kN}=D_{kN}G'_{2k,\sqrt{N}}D_{kN}$$

With this formula in hand, the result follows from Theorem 6.18. □

We refer to [24] for further computations of this type, and for comments.

7. Finite graphs

We have seen that the quantum permutation groups S_N^+ are understood quite well. In what follows we explore, with similar methods, some of the subgroups $G\subset S_N^+$.

Many interesting examples of quantum permutation groups appear as particular cases of the following general construction from [4], involving finite graphs:

Proposition 7.1. *Given a finite graph Z, with adjacency matrix $d\in M_N(0,1)$, the following construction produces a quantum permutation group,*

$$C(G^+(Z))=C(S_N^+)\Big/\Big\langle du=ud\Big\rangle$$

whose classical version $G(Z)$ is the usual automorphism group of Z.

Proof. The fact that we have a quantum group comes from the fact that $du=ud$ reformulates as $d\in End(u)$, which makes it clear that we are dividing by a Hopf ideal.

Regarding the second assertion, we must establish here the following equality:

$$C(G(Z))=C(S_N)\Big/\Big\langle du=ud\Big\rangle$$

For this purpose, observe that with $u_{ij}=\chi(\sigma|\sigma(j)=i)$, as in Proposition 2.1 above, which is the same as saying that $u_{ij}(\sigma)=\delta_{\sigma(j)i}$, we have:

$$(du)_{ij}(\sigma)=\sum_k d_{ik}u_{kj}(\sigma)=\sum_k d_{ik}\delta_{\sigma(j)k}=d_{i\sigma(j)}$$

$$(ud)_{ij}(\sigma)=\sum_k u_{ik}(\sigma)d_{kj}=\sum_k \delta_{\sigma(k)i}d_{kj}=d_{\sigma^{-1}(i)j}$$

Thus $du=ud$ reformulates as $d_{ij}=d_{\sigma(i)\sigma(j)}$, and we are led to the usual notion of an action of a permutation group on Z, which leaves invariant the edges, as claimed. □

Let us work out some basic examples. We have the following result:

Theorem 7.2. *The construction $Z\to G^+(Z)$ has the following properties:*

1. *For the N-point graph, having no edges at all, we obtain S_N^+.*
2. *For the N-simplex, having edges everywhere, we obtain as well S_N^+.*
3. *We have $G^+(Z)=G^+(Z^c)$, where Z^c is the complementary graph.*
4. *For a disconnected union, we have $G^+(Z)\,\hat{*}\,G^+(T)\subset G^+(Z\sqcup T)$.*
5. *For the square, we obtain a non-classical, proper subgroup of S_4^+.*

Proof. All these results are elementary, the proofs being as follows:

(1) This follows from definitions, because here we have $d=0$.

(2) Here $d=\mathbb{I}$ is the all-one matrix, and since the magic condition gives $u\mathbb{I}=\mathbb{I}u=N\mathbb{I}$, we conclude that $du=ud$ is automatic in this case, and so $G^+(Z)=S_N^+$.

(3) The adjacency matrices of Z, Z^c being related by the formula $d_Z + d_{Z^c} = \mathbb{I}$, we can use here the above formula $u\mathbb{I} = \mathbb{I}u = N\mathbb{I}$, and we conclude that $d_Z u = u d_Z$ is equivalent to $d_{Z^c}u = ud_{Z^c}$. Thus, we obtain $G^+(Z) = G^+(Z^c)$, as claimed.

(4) The adjacency matrix of a disconnected union is given by $d_{Z\sqcup T} = diag(d_Z, d_T)$. Now let $w = diag(u, v)$ be the fundamental corepresentation of $G^+(Z)\,\hat{*}\,G^+(T)$. Since $d_Z u = ud_Z$ and $d_T v = vd_T$ imply $d_{Z\sqcup T}w = wd_{Z\sqcup T}$, this gives the result.

(5) We know from (3) that we have $G^+(\square) = G^+(|\ |)$, and we know as well from (4) that we have $\mathbb{Z}_2\,\hat{*}\,\mathbb{Z}_2 \subset G^+(|\ |)$. It follows that $G^+(\square)$ is non-classical. Finally, the inclusion $G^+(\square) \subset S_4^+$ is indeed proper, because $S_4 \subset S_4^+$ does not act on the square. $\square$

Summarizing, our notion of quantum automorphism group is quite interesting, and as a basic example, coming from the empty graph, we have S_N^+ itself. Also, for the simplest non-trivial graph, namely the square $\square$, we are led into some interesting questions.

In order to further advance, and to explicitely compute various quantum automorphism groups, we can use the spectral decomposition of d, as follows:

Proposition 7.3. *A closed subgroup $G \subset S_N^+$ acts on a graph Z precisely when*

$$P_\lambda u = uP_\lambda \quad , \quad \forall\lambda \in \mathbb{R}$$

where $d = \sum_\lambda \lambda \cdot P_\lambda$ is the spectral decomposition of the adjacency matrix of Z.

Proof. Since $d \in M_N(0,1)$ is a symmetric matrix, we can consider indeed its spectral decomposition, $d = \sum_\lambda \lambda \cdot P_\lambda$. We have then the following formula:

$$< d >= span\left\{P_\lambda\Big|\lambda \in \mathbb{R}\right\}$$

But this shows that we have the following equivalence:

$$d \in End(u) \iff P_\lambda \in End(u), \forall\lambda \in \mathbb{R}$$

Thus, we are led to the conclusion in the statement. $\square$

In order to exploit this, we will often combine it with the following standard fact:

Proposition 7.4. *Consider a closed subgroup $G \subset S_N^+$, with associated coaction map $\Phi : \mathbb{C}^N \to \mathbb{C}^N \otimes C(G)$. For a linear subspace $V \subset \mathbb{C}^N$, the following are equivalent:*

(1) *The magic matrix $u = (u_{ij})$ commutes with P_V.*
(2) *V is invariant, in the sense that $\Phi(V) \subset V \otimes C(G)$.*

Proof. Let $P = P_V$. For any $i \in \{1, \ldots, N\}$ we have the following formula:

$$\Phi(P(e_i)) = \Phi\left(\sum_k P_{ki}e_k\right) = \sum_{jk} P_{ki}e_j \otimes u_{jk} = \sum_j e_j \otimes (uP)_{ji}$$

On the other hand the linear map $(P \otimes id)\Phi$ is given by a similar formula:

$$(P \otimes id)(\Phi(e_i)) = \sum_k P(e_k) \otimes u_{ki} = \sum_{jk} P_{jk} e_j \otimes u_{ki} = \sum_j e_j \otimes (Pu)_{ji}$$

Thus $uP = Pu$ is equivalent to $\Phi P = (P \otimes id)\Phi$, and the conclusion follows. □

We have as well the following useful complementary result, from [4]:

Proposition 7.5. *Let $p \in M_N(\mathbb{C})$ be a matrix, and consider its "color" decomposition, obtained by setting $(p_c)_{ij} = 1$ if $p_{ij} = c$ and $(p_c)_{ij} = 0$ otherwise:*

$$p = \sum_{c \in \mathbb{C}} c \cdot p_c$$

Then $u = (u_{ij})$ commutes with p if and only if it commutes with all matrices p_c.

Proof. Since the multiplication map $M : e_i \otimes e_j \to e_i e_j$ and the counit map $C : e_i \to e_i \otimes e_i$ intertwine $u, u^{\otimes 2}$, their iterations $M^{(k)}, C^{(k)}$ intertwine $u, u^{\otimes k}$, and so we have:

$$p^{(k)} = M^{(k)} p^{\otimes k} C^{(k)} = \sum_{c \in \mathbb{C}} c^k p_c \in End(u)$$

Let $S = \{c \in \mathbb{C} | p_c \neq 0\}$, and $f(c) = c$. By Stone-Weierstrass we have $S =< f >$, and so for any $e \in S$ the Dirac mass at e is a linear combination of powers of f:

$$\delta_e = \sum_k \lambda_k f^k = \sum_k \lambda_k \left(\sum_{c \in S} c^k \delta_c \right) = \sum_{c \in S} \left(\sum_k \lambda_k c^k \right) \delta_c$$

The corresponding linear combination of matrices $p^{(k)}$ is given by:

$$\sum_k \lambda_k p^{(k)} = \sum_k \lambda_k \left(\sum_{c \in S} c^k p_c \right) = \sum_{c \in S} \left(\sum_k \lambda_k c^k \right) p_c$$

The Dirac masses being linearly independent, in the first formula all coefficients in the right term are 0, except for the coefficient of δ_e, which is 1. Thus the right term in the second formula is p_e, and it follows that we have $p_e \in End(u)$, as claimed. □

The above results can be combined, and we are led to the following statement:

Theorem 7.6. *A closed subgroup $G \subset S_N^+$ acts on a graph Z precisely when $u = (u_{ij})$ commutes with all the matrices coming from the color-spectral decomposition of d.*

Proof. This follows by combining Proposition 7.3 and Proposition 7.5, with the "color-spectral" decomposition in the statement referring to what comes out by succesively doing the color and spectral decomposition, until the process stabilizes. □

The above statement might seem in need of some further discussion, and axiomatization, in what regards the two operations used there. In answer to all this, the point is that we are in fact doing planar algebras. We have the following result, from [4]:

Theorem 7.7. *The planar algebra associated to $G^+(Z)$ is equal to the planar algebra generated by d, viewed as a 2-box in the spin planar algebra $\mathcal{S}_N$, with $N = |Z|$.*

Proof. We recall from section 3 above that any quantum permutation group $G \subset S_N^+$ produces a subalgebra $P \subset \mathcal{S}_N$ of the spin planar algebra, given by:

$$P_k = Fix(u^{\otimes k})$$

In our case, the idea is that $G = G^+(Z)$ comes via the relation $d \in End(u)$, but we can view this relation, via Frobenius duality, as a relation of type $\xi_d \in Fix(u^{\otimes 2})$.

Indeed, let us view the adjacency matrix $d \in M_N(0,1)$ as a 2-box in $\mathcal{S}_N$, by using the canonical identification between $M_N(\mathbb{C})$ and the algebra of 2-boxes $\mathcal{S}_N(2)$:

$$(d_{ij}) \leftrightarrow \sum_{ij} d_{ij} \begin{pmatrix} i & i \\ j & j \end{pmatrix}$$

Let P be the planar algebra associated to $G^+(Z)$ and let Q be the planar algebra generated by d. The action of $v^{\otimes 2}$ on d viewed as a 2-box is given by:

$$v^{\otimes 2}\left(\sum_{ij} d_{ij} \begin{pmatrix} i & i \\ j & j \end{pmatrix}\right) = \sum_{ijkl} d_{ij} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes v_{ki}v_{lj} = \sum_{kl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes (vdv^t)_{kl}$$

Since v is a magic unitary commuting with d we have $vdv^t = dvv^t = d$. This means that d, viewed as a 2-box, is in the algebra P_2 of fixed points of $v^{\otimes 2}$. Thus $Q \subset P$.

For $P \subset Q$ we use the duality found in section 3. Let indeed (B, w) be the pair whose associated planar algebra is Q. The same computation with w at the place of v shows that w commutes with d. Thus we have a morphism $A \to B$ sending $v_{ij} \to w_{ij}$. Since morphisms increase spaces of fixed points we have the following inclusions:

$$P_k = Hom(1, v^{\otimes k}) \subset Hom(1, w^{\otimes k}) = Q_{0k} = Q_k$$

It follows that we have $P \subset Q$, and we are done. $\square$

With the above results in hand, it is quite clear that our assumption that $d \in M_N(0,1)$ is the adjacency matrix of a usual graph Z is somehow unnatural, and that we can look at more general objects. We can consider for instance general permutation quantum groups of the following type, depending on an arbitrary matrix $d \in M_N(\mathbb{C})$:

$$C(G^+(Z)) = C(S_N^+)\Big/\Big\langle du = ud\Big\rangle$$

Here Z stands for the combinatorial object associated to d, namely the complete graph having as vertices $\{1, \ldots, N\}$, with each oriented edge $i \to j$ colored by $d_{ij} \in \mathbb{C}$.

Generally speaking, the theory extends well to this setting, and we have analogues of the above results, some valid for any $d \in M_N(\mathbb{C})$, and some other valid under the asumption $d = d^*$. We refer to [4] and subsequent papers for a full discussion here.

We can of course further enlarge our formalism, by looking at subgroups $G^+(Z) \subset S_X^+$ of the quantum symmetry groups S_X^+ from section 4, of the following type:

$$C(G^+(Z)) = C(S_X^+) \Big/ \Big\langle du = ud \Big\rangle$$

To be more precise, X is here a finite noncommutative space, coming from a finite dimensional algebra $B = C(X)$. With $N = |X| = \dim B$, the other piece of data is a matrix $d \in M_N(\mathbb{C})$. As for Z, this stands for the graph-type structure (X, d).

With these abstract issues discussed, so let us get back now to concrete things. As a basic application of the above results, we can further study $G^+(\square)$, as follows:

Theorem 7.8. *The quantum automorphism group of the N-cycle is as follows:*
(1) *At $N \neq 4$ we have $G^+(Z) = D_N$.*
(2) *At $N = 4$ we have $D_4 \subset G^+(Z) \subset S_4^+$, with proper inclusions.*

Proof. We already know that the results hold at $N \leq 4$, so let us assume $N \geq 5$.

Given a N-th root of unity, $w^N = 1$, the vector $\xi = (w^i)$ is an eigenvector of d, with eigenvalue $w + w^{N-1}$. Now by taking $w = e^{2\pi i/N}$, it follows that $1, f, f^2, \ldots, f^{N-1}$ are eigenvectors of d. More precisely, the invariant subspaces of d are as follows, with the last subspace having dimension 1 or 2 depending on the parity of N:

$$\mathbb{C}1,\ \mathbb{C}f \oplus \mathbb{C}f^{N-1},\ \mathbb{C}f^2 \oplus \mathbb{C}f^{N-2}, \ldots$$

Consider now the associated coaction $\Phi : \mathbb{C}^N \to \mathbb{C}^N \otimes C(G)$, and write:

$$\Phi(f) = f \otimes a + f^{N-1} \otimes b$$

By taking the square of this equality we obtain:

$$\Phi(f^2) = f^2 \otimes a^2 + f^{N-2} \otimes b^2 + 1 \otimes (ab + ba)$$

It follows that $ab = -ba$, and that $\Phi(f^2)$ is given by the following formula:

$$\Phi(f^2) = f^2 \otimes a^2 + f^{N-2} \otimes b^2$$

By multiplying this with $\Phi(f)$ we obtain:

$$\Phi(f^3) = f^3 \otimes a^3 + f^{N-3} \otimes b^3 + f^{N-1} \otimes ab^2 + f \otimes ba^2$$

Now since $N \geq 5$ implies that $1, N-1$ are different from $3, N-3$, we must have $ab^2 = ba^2 = 0$. By using this and $ab = -ba$, we obtain by recurrence on k that:

$$\Phi(f^k) = f^k \otimes a^k + f^{N-k} \otimes b^k$$

In particular at $k=N-1$ we obtain:

$$\Phi(f^{N-1})=f^{N-1}\otimes a^{N-1}+f\otimes b^{N-1}$$

On the other hand we have $f^*=f^{N-1}$, so by applying $*$ to $\Phi(f)$ we get:

$$\Phi(f^{N-1})=f^{N-1}\otimes a^*+f\otimes b^*$$

Thus $a^*=a^{N-1}$ and $b^*=b^{N-1}$. Together with $ab^2=0$ this gives:

$$(ab)(ab)^*=abb^*a^*=ab^Na^{N-1}=(ab^2)b^{N-2}a^{N-1}=0$$

From positivity we get from this $ab=0$, and together with $ab=-ba$, this shows that a,b commute. On the other hand $C(G)$ is generated by the coefficients of Φ, which are powers of a,b, and so $C(G)$ must be commutative, and we obtain the result. $\square$

Summarizing, this was a bad attempt in understanding $G^+(\square)$, which appears to be "exceptional" among the quantum symmetry groups of the N-cycles.

An alternative approach to $G^+(\square)$ comes by regarding the square as the $N=2$ particular case of the N-hypercube $\square_N$. Indeed, the usual symmetry group of $\square_N$ is the hyperoctahedral group H_N, so we should have a formula of type $G(\square)=H_2^+$.

Quite surprisingly, we will see that $G^+(\square_N)$ is in fact a twist of O_N. So, in order to discuss this material, we first need some twisting preliminaries:

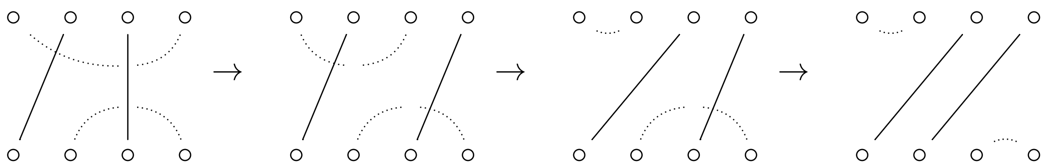
Theorem 7.9. *There is a signature map $\varepsilon:P_{even}\to\{-1,1\}$, given by $\varepsilon(\tau)=(-1)^c$, where c is the number of switches needed to make τ noncrossing. In addition:*

1. *For $\tau\in S_k$, this is the usual signature.*
2. *For $\tau\in P_2$ we have $(-1)^c$, where c is the number of crossings.*
3. *For $\tau\leq\pi\in NC_{even}$, the signature is 1.*

Proof. The fact that ε is indeed well-defined comes from the fact that the number c in the statement is well-defined modulo 2, which is standard combinatorics.

In order to prove the remaining assertion, observe that any partition $\tau\in P(k,l)$ can be put in "standard form", by ordering its blocks according to the appearence of the first leg in each block, counting clockwise from top left, and then by performing the switches as for block 1 to be at left, then for block 2 to be at left, and so on.

Here is an example of such an algorithmic switching operation, with block 1 being first put at left, by using two switches, then with block 2 left unchanged, and then with block 3 being put at left as well, but at right of blocks 1 and 2, with one switch:



With this convention, the proof of the remaining assertions is as follows:

(1) For $\tau\in S_k$ the standard form is $\tau'=id$, and the passage $\tau\to id$ comes by composing with a number of transpositions, which gives the signature.

(2) For a general $\tau\in P_2$, the standard form is of type $\tau'=|\ldots|^{\cup\ldots\cup}_{\cap\ldots\cap}$, and the passage $\tau\to\tau'$ requires c mod 2 switches, where c is the number of crossings.

(3) Assuming that $\tau\in P_{even}$ comes from $\pi\in NC_{even}$ by merging a certain number of blocks, we can prove that the signature is 1 by proceeding by recurrence. □

With the above result in hand, we can now formulate:

Definition 7.10. *Associated to a partition $\pi\in P_{even}(k,l)$ is the linear map*

$$\bar{T}_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\bar{\delta}_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

where $\bar{\delta}_\pi\in\{-1,0,1\}$ is $\bar{\delta}_\pi=\varepsilon(\tau)$ if $\tau\geq\pi$, and $\bar{\delta}_\pi=0$ otherwise, with $\tau=\ker(^i_j)$.

In other words, what we are doing here is to add signatures to the usual formula of T_π. Indeed, observe that the usual formula for T_π can be written as folllows:

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j:\ker(^i_j)\geq\pi}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

Now by inserting signs, coming from the signature map $\varepsilon:P_{even}\to\{\pm1\}$, we are led to the following formula, which coincides with the one from Definition 7.10:

$$\bar{T}_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{\tau\geq\pi}\varepsilon(\tau)\sum_{j:\ker(^i_j)=\tau}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

We will be back later to this analogy, with more details on what can be done with it. For the moment, we must first prove a key categorical result, as follows:

Proposition 7.11. *The assignement $\pi\to\bar{T}_\pi$ is categorical, in the sense that*

$$\bar{T}_\pi\otimes\bar{T}_\sigma=\bar{T}_{[\pi\sigma]},\qquad \bar{T}_\pi\bar{T}_\sigma=N^{c(\pi,\sigma)}\bar{T}_{[^\sigma_\pi]},\qquad \bar{T}_\pi^*=\bar{T}_{\pi^*}$$

where $c(\pi,\sigma)$ are certain positive integers.

Proof. In order to prove this result we can go back to the proof from the easy case, and insert signs, where needed. We have to check three conditions, as follows:

<u>1. Concatenation.</u> It is enough to check the following formula:

$$\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\right)\varepsilon\left(\ker\begin{pmatrix}k_1\ldots k_r\\ l_1\ldots l_s\end{pmatrix}\right)=\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p&k_1\ldots k_r\\ j_1\ldots j_q&l_1\ldots l_s\end{pmatrix}\right)$$

Let us denote by τ,ν the partitions on the left, so that the partition on the right is of the form $\rho\leq[\tau\nu]$. Now by switching to the noncrossing form, $\tau\to\tau'$ and $\nu\to\nu'$, the partition on the right transforms into $\rho\to\rho'\leq[\tau'\nu']$. Now since $[\tau'\nu']$ is noncrossing, we can use Theorem 7.9 (3), and we obtain the result.

2. Composition. Here we must establish the following formula:

$$\varepsilon\left(\ker\begin{pmatrix}i_1\dots i_p\\ j_1\dots j_q\end{pmatrix}\right)\varepsilon\left(\ker\begin{pmatrix}j_1\dots j_q\\ k_1\dots k_r\end{pmatrix}\right)=\varepsilon\left(\ker\begin{pmatrix}i_1\dots i_p\\ k_1\dots k_r\end{pmatrix}\right)$$

Let τ,ν be the partitions on the left, so that the partition on the right is of the form $\rho\leq[^\tau_\nu]$. Our claim is that we can jointly switch τ,ν to the noncrossing form. Indeed, we can first switch as for $\ker(j_1\dots j_q)$ to become noncrossing, and then switch the upper legs of τ, and the lower legs of ν, as for both these partitions to become noncrossing.

Now observe that when switching in this way to the noncrossing form, $\tau\to\tau'$ and $\nu\to\nu'$, the partition on the right transforms into $\rho\to\rho'\leq[^{\tau'}_{\nu'}]$. Now since $[^{\tau'}_{\nu'}]$ is noncrossing, we can apply Theorem 7.9 (3), and we obtain the result.

3. Involution. Here we must prove the following formula:

$$\bar{\delta}_\pi\begin{pmatrix}i_1\dots i_p\\ j_1\dots j_q\end{pmatrix}=\bar{\delta}_{\pi^*}\begin{pmatrix}j_1\dots j_q\\ i_1\dots i_p\end{pmatrix}$$

But this is clear from the definition of $\bar{\delta}_\pi$, and we are done. □

As a conclusion, our construction $\pi\to\bar{T}_\pi$ has all the needed properties for producing quantum groups, via Tannakian duality. So, we can now formulate:

Theorem 7.12. *Given a category of partitions $D\subset P_{even}$, the construction*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\bar{T}_\pi\Big|\pi\in D(k,l)\right)$$

produces via Tannakian duality a quantum group $\bar{G}_N\subset O_N^+$, for any $N\in\mathbb{N}$.

Proof. This follows indeed from the Tannakian results from section 1 above, exactly as in the easy case, by using this time Proposition 7.11 as technical ingredient. □

We can unify the easy quantum groups, or at least the examples coming from categories $D\subset P_{even}$, with the quantum groups constructed above, as follows:

Definition 7.13. *A closed subgroup $G\subset O_N^+$ is called q-easy, or quizzy, with deformation parameter $q=\pm1$, when its tensor category appears as follows,*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\dot{T}_\pi\Big|\pi\in D(k,l)\right)$$

for a certain category of partitions $D\subset P_{even}$, where $\dot{T}=\bar{T},T$ for $q=-1,1$. The Schur-Weyl twist of G is the quizzy quantum group $\bar{G}\subset O_N^+$ obtained via $q\to-q$.

We can now twist the orthogonal group. The result here is as follows:

Theorem 7.14. *The twist of O_N is obtained by replacing the relations $ab=ba$ with $ab=\pm ba$, with anticommutation on rows and columns, and commutation otherwise.*

Proof. The basic crossing, $\ker\binom{ij}{ji}$ with $i \neq j$, comes from the transposition $\tau \in S_2$, so its signature is -1. As for its degenerated version $\ker\binom{ii}{ii}$, this is noncrossing, so here the signature is 1. We conclude that the linear map associated to the basic crossing is:

$$\bar{T}_{\slash\!\!\!\backslash}(e_i \otimes e_j) = \begin{cases} -e_j \otimes e_i & \text{for } i \neq j \\ e_j \otimes e_i & \text{otherwise} \end{cases}$$

We can proceed now as in the untwisted case, and since the intertwining relations coming from $\bar{T}_{\slash\!\!\!\backslash}$ correspond to the relations defining $\bar{O}_N$, we obtain the result. $\square$

Getting back now to graphs, recall that our question was that of computing the quantum symmetry group of $\square_N$. And we have here the following result, from [13]:

Theorem 7.15. *The quantum symmetry group of the N-hypercube is*

$$G^+(\square_N) = \bar{O}_N$$

with the corresponding coaction map on the vertex set being the map

$$\Phi : C^*(\mathbb{Z}_2^N) \to C^*(\mathbb{Z}_2^N) \otimes C(\bar{O}_N) \quad , \quad g_i \to \sum_j g_j \otimes u_{ji}$$

via the standard identification $\square_N = \widehat{\mathbb{Z}_2^N}$.

Proof. We use here the fact that the cube $\square_N$, when regarded as a graph, is the Cayley graph of the group $\mathbb{Z}_2^N$. The eigenvectors and eigenvalues of $\square_N$ are as follows:

$$\begin{aligned} v_{i_1\ldots i_N} &= \sum_{j_1\ldots j_N} (-1)^{i_1j_1+\ldots+i_Nj_N} g_1^{j_1} \ldots g_N^{j_N} \\ \lambda_{i_1\ldots i_N} &= (-1)^{i_1} + \ldots + (-1)^{i_N} \end{aligned}$$

Modulo some standard computations, explained in [13], it is enough to construct a map Φ as in the statement. For this purpose, consider the following variables:

$$G_i = \sum_j g_j \otimes u_{ji}$$

We must show that these variables satisfy the same relations as the generators $g_j \in \mathbb{Z}_2^N$. The self-adjointness being automatic, the relations to be checked are therefore:

$$G_i^2 = 1 \quad , \quad G_iG_j = G_jG_i$$

We have the following formulae:

$$\begin{aligned} G_i^2 &= \sum_{kl} g_k g_l \otimes u_{ik}u_{il} = 1 + \sum_{k<l} g_k g_l \otimes (u_{ik}u_{il} + u_{il}u_{ik}) \\ [G_i, G_j] &= \sum_{k<l} g_k g_l \otimes (u_{ik}u_{jl} - u_{jk}u_{il} + u_{il}u_{jk} - u_{jl}u_{ik}) \end{aligned}$$

From the first relation we obtain $ab = 0$ for $a \neq b$ on the same row of u, and by using the antipode, the same happens for the columns. From the second relation we obtain:

$$[u_{ik}, u_{jl}] = [u_{jk}, u_{il}] \quad , \forall k \neq l$$

Now by applying the antipode we obtain $[u_{lj}, u_{ki}] = [u_{li}, u_{kj}]$, and by relabelling, this gives $[u_{ik}, u_{jl}] = [u_{il}, u_{jk}]$ for $j \neq i$. Thus for $i \neq j, k \neq l$ we must have $[u_{ik}, u_{jl}] = [u_{jk}, u_{il}] = 0$, and we are therefore led to $G \subset \bar{O}_N$, as claimed. $\square$

In connection with the various extensions of our formalism, regarding colored graphs, or finite metric spaces, let us record as well the following result, also from [13]:

Theorem 7.16. *The quantum isometry group of the N-hypercube, regarded as a finite metric subspace of $\mathbb{R}^N$, is the twisted orthogonal group $\bar{O}_N$.*

Proof. We recall, from the discussion after Theorem 7.7, that the quantum symmetry group of a finite colored graph is produced by $du = ud$, with $d \in M_N(\mathbb{C})$ being the adjacency matrix. This construction applies in particular to the finite metric spaces, which can be regarded as complete graphs, with the edges colored by their lengths.

The distance matrix of the cube has a color decomposition as follows:

$$d = d_1 + \sqrt{2}d_2 + \sqrt{3}d_3 + \ldots + \sqrt{N}d_N$$

Since the powers of d_1 can be computed by counting loops on the cube, we have formulae as follows, with $x_{ij} \in \mathbb{N}$ being certain positive integers:

$$\begin{aligned}
d_1^2 &= x_{21}1_N + x_{22}d_2 \\
d_1^3 &= x_{31}1_N + x_{32}d_2 + x_{33}d_3 \\
&\ldots \\
d_1^N &= x_{N1}1_N + x_{N2}d_2 + x_{N3}d_3 + \ldots + x_{NN}d_N
\end{aligned}$$

But this shows that we have the following equality of algebras:

$$< d > = < d_1 >$$

Now since d_1 is the adjacency matrix of $\square_N$, viewed as graph, this proves our claim, and we obtain the result from Theorem 7.15. $\square$

As a comment here, the similarity between Theorem 7.15 and Theorem 7.16 is part of a wider phenomenon, the idea being that the various "familiar" finite metric spaces have the same classical and quantum automorphism groups as the graphs that are usually used for representing them. For more on this topic, we refer to [4] and related papers.

Our purpose now is to understand which representation of O_N produces by twisting the magic representation of $\bar{O}_N$. In order to solve this question, we will need:

Proposition 7.17. *The Fourier transform over $\mathbb{Z}_2^N$ is the map*

$$\alpha : C(\mathbb{Z}_2^N) \to C^*(\mathbb{Z}_2^N) \quad , \quad \delta_{g_1^{i_1}\ldots g_N^{i_N}} \to \frac{1}{2^N}\sum_{j_1\ldots j_N}(-1)^{<i,j>}g_1^{j_1}\ldots g_N^{j_N}$$

with the usual convention $<i,j>=\sum_k i_kj_k$, and its inverse is the map

$$\beta : C^*(\mathbb{Z}_2^N) \to C(\mathbb{Z}_2^N) \quad , \quad g_1^{i_1}\ldots g_N^{i_N} \to \sum_{j_1\ldots j_N}(-1)^{<i,j>}\delta_{g_1^{j_1}\ldots g_N^{j_N}}$$

with all the exponents being binary, $i_1,\ldots,i_N,j_1,\ldots,j_N\in\{0,1\}$.

Proof. Observe first that the group $\mathbb{Z}_2^N$ can be written as follows:

$$\mathbb{Z}_2^N = \left\{g_1^{i_1}\ldots g_N^{i_N}\Big|i_1,\ldots,i_N\in\{0,1\}\right\}$$

Thus both α,β are well-defined, and it is elementary to check that both are morphisms of algebras. We have as well $\alpha\beta=\beta\alpha=id$, coming from the standard formula:

$$\frac{1}{2^N}\sum_{j_1\ldots j_N}(-1)^{<i,j>} = \prod_{k=1}^N\left(\frac{1}{2}\sum_{j_r}(-1)^{i_rj_r}\right) = \delta_{i0}$$

Thus we have indeed a pair of inverse Fourier morphisms, as claimed. □

As an illustration here, at $N=1$, with the notation $\mathbb{Z}_2=\{1,g\}$, the map α is given by $\delta_1\to\frac{1}{2}(1+g)$, $\delta_g\to\frac{1}{2}(1-g)$ and its inverse β is given by $1\to\delta_1+\delta_g$, $g\to\delta_1-\delta_g$.

By using now these Fourier transforms, we obtain following formula:

Theorem 7.18. *The magic unitary for the embedding $\bar{O}_N\subset S_{2^N}^+$ is given by*

$$w_{i_1\ldots i_N,k_1\ldots k_N} = \frac{1}{2^N}\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}(-1)^{<i+k_b,j>}\left(\frac{1}{N}\right)^{\#(0\in j)}u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}$$

where $k_b=(k_{b_1},\ldots,k_{b_N})$, with respect to multi-indices $i,k\in\{0,1\}^N$ as above.

Proof. By composing the coaction map Φ from Theorem 7.15 with the above Fourier transform isomorphisms α,β, we have a diagram as follows:

$$\begin{tikzcd} C^*(\mathbb{Z}_2^N) \arrow[r, "\Phi"] & C^*(\mathbb{Z}_2^N)\otimes C(\bar{O}_N) \arrow[d, "\beta\otimes id"] \\ C(\mathbb{Z}_2^N) \arrow[u, "\alpha"] \arrow[r, dotted, "\Psi"] & C(\mathbb{Z}_2^N)\otimes C(\bar{O}_N) \end{tikzcd}$$

In order to compute the composition on the bottom Ψ, we first recall from Theorem 7.15 above that the coaction map Φ is defined by the formula $\Phi(g_b)=\sum_a g_a\otimes u_{ab}$, for any $a\in\{1,\ldots,N\}$. Now by making products of such quantities, we obtain the following global formula for Φ, valid for any exponents $i_1,\ldots,i_N\in\{1,\ldots,N\}$:

$$\Phi(g_1^{i_1}\ldots g_N^{i_N})=\left(\frac{1}{N}\right)^{\#(0\in i)}\sum_{b_1\ldots b_N}g_{b_1}^{i_1}\ldots g_{b_N}^{i_N}\otimes u_{1b_1}^{i_1}\ldots u_{Nb_N}^{i_N}$$

The term on the right can be put in "standard form" as follows:

$$g_{b_1}^{i_1}\ldots g_{b_N}^{i_N}=g_1^{\sum_{b_x=1}i_x}\ldots g_N^{\sum_{b_x}i_x}$$

We therefore obtain the following formula for the coaction map Φ:

$$\Phi(g_1^{i_1}\ldots g_N^{i_N})=\left(\frac{1}{N}\right)^{\#(0\in i)}\sum_{b_1\ldots b_N}g_1^{\sum_{b_x=1}i_x}\ldots g_N^{\sum_{b_x=N}i_x}\otimes u_{1b_1}^{i_1}\ldots u_{Nb_N}^{i_N}$$

Now by applying the Fourier transforms, we obtain the following formula:

$$\begin{aligned}
&\Psi(\delta_{g_1^{i_1}\ldots g_N^{i_N}})\\
=\ &(\beta\otimes id)\Phi\left(\frac{1}{2^N}\sum_{j_1\ldots j_N}(-1)^{<i,j>}g_1^{j_1}\ldots g_N^{j_N}\right)\\
=\ &\frac{1}{2^N}\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}(-1)^{<i,j>}\left(\frac{1}{N}\right)^{\#(0\in j)}\beta\left(g_1^{\sum_{b_x=1}j_x}\ldots g_N^{\sum_{b_x=N}j_x}\right)\otimes u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}
\end{aligned}$$

By using now the formula of β from Proposition 7.17, we obtain:

$$\begin{aligned}
\Psi(\delta_{g_1^{i_1}\ldots g_N^{i_N}})\ =\ &\frac{1}{2^N}\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}\sum_{k_1\ldots k_N}\left(\frac{1}{N}\right)^{\#(0\in j)}\\
&(-1)^{<i,j>}(-1)^{<(\sum_{b_x=1}j_x,\ldots,\sum_{b_x=N}j_x),(k_1,\ldots,k_N)>}\\
&\delta_{g_1^{k_1}\ldots g_N^{k_N}}\otimes u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}
\end{aligned}$$

Now observe that, with the notation $k_b=(k_{b_1},\ldots,k_{b_N})$, we have:

$$<(\sum_{b_x=1}j_x,\ldots,\sum_{b_x=N}j_x),(k_1,\ldots,k_N)>=<j,k_b>$$

Thus, we obtain the following formula for our map Ψ:

$$\Psi(\delta_{g_1^{i_1}\ldots g_N^{i_N}})=\frac{1}{2^N}\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}\sum_{k_1\ldots k_N}(-1)^{<i+k_b,j>}\left(\frac{1}{N}\right)^{\#(0\in j)}\delta_{g_1^{k_1}\ldots g_N^{k_N}}\otimes u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}$$

But this gives the formula in the statement for the corresponding magic unitary, with respect to the basis $\{\delta_{g_1^{i_1}\ldots g_N^{i_N}}\}$ of the algebra $C(\mathbb{Z}_2^N)$, and we are done. $\square$

We can now solve our original question, namely understanding where the magic representation of $\bar{O}_N$ really comes from, with the following final answer to it:

Theorem 7.19. *The magic representation of $\bar{O}_N$, coming from its action on the N-cube, corresponds to the antisymmetric representation of O_N, via twisting.*

Proof. This follows from the formula of w in Theorem 7.18, by computing the character, and then interpreting the result via twisting, as follows:

(1) By applying the trace to the formula of w, we obtain:

$$\chi=\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}\left(\frac{1}{2^N}\sum_{i_1\ldots i_N}(-1)^{<i+i_b,j>}\right)\left(\frac{1}{N}\right)^{\#(0\in j)}u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}$$

(2) By computing the Fourier sum in the middle, we are led to the following formula, with binary indices $j_1,\ldots,j_N\in\{0,1\}$, and plain indices $b_1,\ldots,b_N\in\{1,\ldots,N\}$:

$$\chi=\sum_{j_1\ldots j_N}\sum_{b_1\ldots b_N}\left(\frac{1}{N}\right)^{\#(0\in j)}\delta_{j_1,\sum_{b_x=1}j_x}\ldots\delta_{j_N,\sum_{b_x=N}j_x}u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}$$

(3) With the notation $r=\#(1\in j)$ we obtain a decomposition of type $\chi=\sum_{r=0}^N\chi_r$, with the variables χ_r being as follows:

$$\chi_r=\frac{1}{N^{N-r}}\sum_{\#(1\in j)=r}\sum_{b_1\ldots b_N}\delta_{j_1,\sum_{b_x=1}j_x}\ldots\delta_{j_N,\sum_{b_x=N}j_x}u_{1b_1}^{j_1}\ldots u_{Nb_N}^{j_N}$$

(4) Consider now the set $A\subset\{1,\ldots,N\}$ given by $A=\{a|j_a=1\}$. The binary multi-indices $j\in\{0,1\}^N$ satisfying $\#(1\in j)=r$ being in bijection with such subsets A, satisfying $|A|=r$, we can replace the sum over j with a sum over such subsets A.

We therefore obtain a formula as follows, where j is the index corresponding to A:

$$\chi_r=\frac{1}{N^{N-r}}\sum_{|A|=r}\sum_{b_1\ldots b_N}\delta_{j_1,\sum_{b_x=1}j_x}\ldots\delta_{j_N,\sum_{b_x=N}j_x}\prod_{a\in A}u_{ab_a}$$

(5) Let us identify b with the corresponding function $b:\{1,\ldots,N\}\to\{1,\ldots,N\}$, via $b(a)=b_a$. Then for any $p\in\{1,\ldots,N\}$ we have:

$$\delta_{j_p,\sum_{b_x=p}j_x}=1\iff|b^{-1}(p)\cap A|=\chi_A(p)\ (\mathrm{mod}\ 2)$$

We conclude that the multi-indices $b\in\{1,\ldots,N\}^N$ which effectively contribute to the sum are those coming from the functions satisfying $b<A$. Thus, we have:

$$\chi_r=\frac{1}{N^{N-r}}\sum_{|A|=r}\sum_{b<A}\prod_{a\in A}u_{ab_a}$$

(6) We can further split each χ_r over the sets $A\subset\{1,\ldots,N\}$ satisfying $|A|=r$, and the point is that for each of these sets we have:

$$\frac{1}{N^{N-r}}\sum_{b<A}\prod_{a\in A}u_{ab_a}=\sum_{\sigma\in S_N^A}\prod_{a\in A}u_{a\sigma(a)}$$

Thus, the magic character of $\bar{O}_N$ is given by $\chi=\sum_{r=0}^N\chi_r$, where:

$$\chi_r=\sum_{|A|=r}\sum_{\sigma\in S_N^A}\prod_{a\in A}u_{a\sigma(a)}$$

(7) The twisting operation $O_N\to\bar{O}_N$ makes correspond the following products:

$$\varepsilon(\sigma)\prod_{a\in A}u_{a\sigma(a)}\to\prod_{a\in A}u_{a\sigma(a)}$$

Now by summing over sets A and permutations σ, we conclude that the twisting operation $O_N\to\bar{O}_N$ makes correspond the following quantities:

$$\sum_{|A|=r}\sum_{\sigma\in S_N^A}\varepsilon(\sigma)\prod_{a\in A}u_{a\sigma(a)}\to\sum_{|A|=r}\sum_{\sigma\in S_N^A}\prod_{a\in A}u_{a\sigma(a)}$$

Thus, we are led to the conclusion in the statement. □

Let us go back now to the square problem. In order to present the correct, final solution to it, the idea will be that to look at the quantum group $G^+(|\ |)$ instead, which is equal to it, according to Theorem 7.2 (3). We will need the following result, from [36]:

Theorem 7.20. *Given closed subgroups $G\subset U_N^+$, $H\subset S_k^+$, with fundamental corepresentations u,v, the following construction produces a closed subgroup of U_{Nk}^+:*

$$C(G\wr_*H)=(C(G)^{*k}*C(H))/<[u_{ij}^{(a)},v_{ab}]=0>$$

In the case where G,H are classical, the classical version of $G\wr_*H$ is the usual wreath product $G\wr H$. Also, when G is a quantum permutation group, so is $G\wr_*H$.

Proof. Consider indeed the matrix $w_{ia,jb}=u_{ij}^{(a)}v_{ab}$, over the quotient algebra in the statement. Then w is unitary, and in the case $G\subset S_N^+$, this matrix is magic.

With these observations in hand, it is routine to check that $G\wr_*H$ is indeed a quantum group, with fundamental corepresentation w, by constructing maps Δ,ε,S as in Definition 1.2, and in the case $G\subset S_N^+$, we obtain in this way a closed subgroup of S_{Nk}^+. See [36]. □

We refer to [7], [36], [84] for more details regarding the above construction.

With this notion in hand, we can now formulate a non-trivial result, as follows:

Theorem 7.21. *Given a connected graph Z, and $k\in\mathbb N$, we have the formulae*

$$G(kZ)=G(Z)\wr S_k\quad,\quad G^+(kZ)=G^+(Z)\wr_*S_k^+$$

where $kZ=Z\sqcup\ldots\sqcup Z$ is the k-fold disjoint union of Z with itself.

Proof. The first formula is something well-known, which follows as well from the second formula, by taking the classical version. Regarding now the second formula, it is elementary to check that we have an inclusion as follows, for any finite graph Z:

$$G^+(Z)\wr_*S_k^+\subset G^+(kZ)$$

Regarding now the reverse inclusion, which requires Z to be connected, this follows by doing some matrix analysis, by using the commutation with u. To be more precise, let us denote by w the fundamental corepresentation of $G^+(kZ)$, and set:

$$u_{ij}^{(a)}=\sum_b w_{ia,jb}\quad,\quad v_{ab}=\sum_i v_{ab}$$

It is then routine to check, by using the fact that Z is indeed connected, that we have here magic unitaries, as in the definition of the free wreath products. Thus we obtain the reverse inclusion $G^+(kZ)\subset G^+(Z)\wr_*S_k^+$, and this gives the result. See [7]. □

We are led in this way to the following result:

Theorem 7.22. *Consider the graph consisting of N segments.*

(1) *Its symmetry group is the hyperoctahedral group $H_N=\mathbb Z_2\wr S_N$.*
(2) *Its quantum symmetry group is the quantum group $H_N^+=\mathbb Z_2\wr_*S_N^+$.*

Proof. Here the first assertion is clear from definitions, with the remark that the relation with the formula $H_N=G(\square_N)$ comes by viewing the N segments as being the $[-1,1]$ segments on each of the N coordinate axes of $\mathbb R^N$. Indeed, a symmetry of the N-cube is the same as a symmetry of the N segments, and so $G(\square_N)=\mathbb Z_2\wr S_N$, as desired.

As for the second assertion, this follows from Theorem 7.21 above, applied to the segment graph. Observe also that (2) implies (1), by taking the classical version. □

Now back to the square, we have $G^+(\square)=H_2^+$, and our claim is that this is the "good" and final formula. In order to prove this, we must work out the easiness theory for H_N,H_N^+, and find a compatibility there. We first have the following result:

Proposition 7.23. *The algebra $C(H_N^+)$ can be presented in two ways, as follows:*

(1) *As the universal algebra generated by the entries of a $2N\times2N$ magic unitary having the "sudoku" pattern $w=\begin{pmatrix}a&b\\ b&a\end{pmatrix}$, with a,b being square matrices.*
(2) *As the universal algebra generated by the entries of a $N\times N$ orthogonal matrix which is "cubic", in the sense that $u_{ij}u_{ik}=u_{ji}u_{ki}=0$, for any $j\neq k$.*

As for $C(H_N)$, this has similar presentations, among the commutative algebras.

Proof. We must prove that the algebras A_s, A_c coming from (1,2) coincide.

We can define a morphism $A_c \to A_s$ by the following formula:

$$\varphi(u_{ij}) = a_{ij} - b_{ij}$$

We construct now the inverse morphism. Consider the following elements:

$$\alpha_{ij} = \frac{u_{ij}^2 + u_{ij}}{2} \quad , \quad \beta_{ij} = \frac{u_{ij}^2 - u_{ij}}{2}$$

These are projections, and the following matrix is a sudoku unitary:

$$M = \begin{pmatrix} (\alpha_{ij}) & (\beta_{ij}) \\ (\beta_{ij}) & (\alpha_{ij}) \end{pmatrix}$$

Thus we can define a morphism $A_s \to A_c$ by the following formula:

$$\psi(a_{ij}) = \frac{u_{ij}^2 + u_{ij}}{2} \quad , \quad \psi(b_{ij}) = \frac{u_{ij}^2 - u_{ij}}{2}$$

We check now the fact that ψ, φ are indeed inverse morphisms:

$$\psi\varphi(u_{ij}) = \psi(a_{ij} - b_{ij}) = \frac{u_{ij}^2 + u_{ij}}{2} - \frac{u_{ij}^2 - u_{ij}}{2} = u_{ij}$$

As for the other composition, we have the following computation:

$$\varphi\psi(a_{ij}) = \varphi\left(\frac{u_{ij}^2 + u_{ij}}{2}\right) = \frac{(a_{ij} - b_{ij})^2 + (a_{ij} - b_{ij})}{2} = a_{ij}$$

A similar computation gives $\varphi\psi(b_{ij}) = b_{ij}$, which completes the proof. □

We can now work out the easiness property of H_N, H_N^+, with respect to the cubic representations, and we are led to the following result, which is fully satisfactory:

Theorem 7.24. *The quantum groups H_N, H_N^+ are both easy, as follows:*

(1) *H_N corresponds to the category P_{even}.*
(2) *H_N^+ corresponds to the category NC_{even}.*

Proof. These assertions follow indeed from the fact that the cubic relations are implemented by the one-block partition in $P(2,2)$, which generates NC_{even}. □

As a final conclusion now, to the long story told here, the correct analogue of the hyperoctahedral group H_N is the quantum group H_N^+ constructed above, with $H_N \to H_N^+$ being a liberation, in the sense of easy quantum group theory.

8. Reflection groups

We have seen in the previous section that the correct analogue of the hyperoctahedral group $H_N = \mathbb{Z}_2 \wr S_N$ is the quantum group $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$. These key quantum groups belong in fact to series, depending on a parameter $s \in \mathbb{N} \cup \{\infty\}$, as follows:

$$H_N^s = \mathbb{Z}_s \wr S_N \quad , \quad H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$$

We discuss here, following [6], [33], the algebraic and analytic structure of these latter quantum groups. The main motivation comes from the cases $s = 1, 2, \infty$, where we recover respectively S_N, S_N^+ and H_N, H_N^+, and the full reflection groups K_N, K_N^+.

Let us start with a brief discussion concerning the classical case. The result that we will need, which is well-known and elementary, is as follows:

Proposition 8.1. *The group $H_N^s = \mathbb{Z}_s \wr S_N$ of $N \times N$ permutation-like matrices having as nonzero entries the s-th roots of unity is as follows:*

1. *$H_N^1 = S_N$ is the symmetric group.*
2. *$H_N^2 = H_N$ is the hyperoctahedral group.*
3. *$H_N^\infty = K_N$ is the group of unitary permutation-like matrices.*

Proof. Everything here is clear from definitions. □

Let us mention as well that the groups H_N^s are part of a more general series H_N^{sd}, depending on an extra parameter $d|s$, via the condition $\det(M)^d = 1$, which is the series of complex reflection groups. We will be back to this in the next section.

The free analogues of the reflection groups H_N^s can be constructed as follows:

Definition 8.2. *$C(H_N^{s+})$ is the universal C^*-algebra generated by N^2 normal elements u_{ij}, subject to the following relations,*

1. *$u = (u_{ij})$ is unitary,*
2. *$u^t = (u_{ji})$ is unitary,*
3. *$p_{ij} = u_{ij}u_{ij}^*$ is a projection,*
4. *$u_{ij}^s = p_{ij}$,*

with Woronowicz algebra maps Δ, ε, S constructed by universality.

Here we allow the value $s = \infty$, with the convention that the last axiom simply disappears in this case. Observe that at $s < \infty$ the normality condition is actually redundant. This is because a partial isometry a subject to the relation $aa^* = a^s$ is normal.

As a first result, making the connection with H_N^s, we have:

Theorem 8.3. *We have an inclusion $H_N^s \subset H_N^{s+}$, which is a liberation, in the sense that the classical version of H_N^{s+}, obtained by dividing by the commutator ideal, is H_N^s.*

Proof. This follows as for $O_N \subset O_N^+$ or $S_N \subset S_N^+$, by using the Gelfand theorem. $\square$

In analogy with Proposition 8.1 above, we have the following result:

Proposition 8.4. *The algebras $C(H_N^{s+})$ with $s = 1, 2, \infty$, and their presentation relations in terms of the entries of the matrix $u = (u_{ij})$, are as follows.*

1. *For $C(H_N^{1+}) = C(S_N^+)$, the matrix u is magic: all its entries are projections, summing up to 1 on each row and column.*
2. *For $C(H_N^{2+}) = C(H_N^+)$ the matrix u is cubic: it is orthogonal, and the products of pairs of distinct entries on the same row or the same column vanish.*
3. *For $C(H_N^{\infty+}) = C(K_N^+)$ the matrix u is unitary, its transpose is unitary, and all its entries are normal partial isometries.*

Proof. Here (1) and (2) follow from definitions and from standard operator algebra tricks, and (3) is just a translation of the definition of $C(H_N^{s+})$, at $s = \infty$. $\square$

Let us prove now that H_N^{s+} with $s < \infty$ is a quantum permutation group. For this purpose, we must change the fundamental representation. Let us start with:

Definition 8.5. *A (s, N)-sudoku matrix is a magic unitary of size sN, of the form*

$$m = \begin{pmatrix} a^0 & a^1 & \ldots & a^{s-1} \\ a^{s-1} & a^0 & \ldots & a^{s-2} \\ \vdots & \vdots & & \vdots \\ a^1 & a^2 & \ldots & a^0 \end{pmatrix}$$

where $a^0, \ldots, a^{s-1}$ are $N \times N$ matrices.

The basic examples of such sudoku matrices come from the group H_n^s. Indeed, with $w = e^{2\pi i/s}$, each of the N^2 matrix coordinates $u_{ij} : H_N^s \to \mathbb{C}$ takes values in the set $\{0\} \cup \{1, w, \ldots, w^{s-1}\}$, hence decomposes as follows:

$$u_{ij} = \sum_{r=0}^{s-1} w^r a_{ij}^r$$

Here each a_{ij}^r is a function taking values in $\{0, 1\}$, and so a projection in the C^*-algebra sense, and it follows from definitions that these projections form a sudoku matrix.

With this notion in hand, we have the following result:

Theorem 8.6. *The following happen:*

1. *The algebra $C(H_N^s)$ is isomorphic to the universal commutative C^*-algebra generated by the entries of a (s, N)-sudoku matrix.*
2. *The algebra $C(H_N^{s+})$ is isomorphic to the universal C^*-algebra generated by the entries of a (s, N)-sudoku matrix.*

Proof. The first assertion follows from the second one, via Theorem 8.3. In order to prove now the second assertion, consider the universal algebra in the statement, namely:

$$A = C^*\left(a_{ij}^p \,\Big|\, (a_{ij}^{q-p})_{pi,qj} = (s,N) - \text{sudoku}\right)$$

Consider also the algebra $C(H_N^{s+})$. According to Definition 8.2, this is presented by certain relations R, that we will call here level s cubic conditions:

$$C(H_N^{s+}) = C^*\left(u_{ij} \,\Big|\, u = N \times N \text{ level } s \text{ cubic}\right)$$

We will construct a pair of inverse morphisms between these algebras.

(1) Our first claim is that $U_{ij} = \sum_p w^{-p} a_{ij}^p$ is a level s cubic unitary. Indeed, by using the sudoku condition, the verification of (1-4) in Definition 8.2 is routine.

(2) Our second claim is that the elements $A_{ij}^p = \frac{1}{s}\sum_r w^{rp} u_{ij}^r$, with the convention $u_{ij}^0 = p_{ij}$, form a level s sudoku unitary. Once again, the proof here is routine.

(3) According to the above, we can define a morphism $\Phi : C(H_N^{s+}) \to A$ by the formula $\Phi(u_{ij}) = U_{ij}$, and a morphism $\Psi : A \to C(H_N^{s+})$ by the formula $\Psi(a_{ij}^p) = A_{ij}^p$.

(4) We check now the fact that Φ, Ψ are indeed inverse morphisms:

$$\begin{aligned}
\Psi\Phi(u_{ij}) &= \sum_p w^{-p} A_{ij}^p \\
&= \frac{1}{s}\sum_p w^{-p} \sum_r w^{rp} u_{ij}^r \\
&= \frac{1}{s}\sum_{pr} w^{(r-1)p} u_{ij}^r \\
&= u_{ij}
\end{aligned}$$

As for the other composition, we have the following computation:

$$\begin{aligned}
\Phi\Psi(a_{ij}^p) &= \frac{1}{s}\sum_r w^{rp} U_{ij}^r \\
&= \frac{1}{s}\sum_r w^{rp} \sum_q w^{-rq} a_{ij}^q \\
&= \frac{1}{s}\sum_q a_{ij}^q \sum_r w^{r(p-q)} \\
&= a_{ij}^p
\end{aligned}$$

Thus we have an isomorphism $C(H_N^{s+}) = A$, as claimed. $\square$

Let us discuss now the interpretation of H_N^s, H_N^{s+} as classical and quantum symmetry groups of graphs. We will need the following simple fact:

Proposition 8.7. *A $sN\times sN$ magic unitary commutes with the matrix*

$$\Sigma=\begin{pmatrix}0&I_N&0&\ldots&0\\ 0&0&I_N&\ldots&0\\ \vdots&\vdots&&\ddots&\\ 0&0&0&\ldots&I_N\\ I_N&0&0&\ldots&0\end{pmatrix}$$

if and only if it is a sudoku matrix in the sense of Definition 8.5.

Proof. This follows from the fact that commutation with Σ means that the matrix is circulant. Thus, we obtain the sudoku relations from Definition 8.5 above. □

Now let $\circlearrowleft_s$ be the oriented cycle with s vertices, and consider the graph $N\circlearrowleft_s$ consisting of N disjoint copies of it. Observe that, with a suitable labeling of the vertices, the adjacency matrix of this graph is the above matrix Σ. We obtain from this:

Theorem 8.8. *We have the following results:*

(1) *H_N^s is the symmetry group of $N\circlearrowleft_s$.*
(2) *H_N^{s+} is the quantum symmetry group of $N\circlearrowleft_s$.*

Proof. Here (1) follows from definitions, and (2) follows from Theorem 8.6 and Proposition 8.7, because $C(H_N^{s+})$ is the quotient of $C(S_{sN}^+)$ by the relations making the fundamental corepresentation commute with the adjacency matrix of $N\circlearrowleft_s$. □

Next in line, we must talk about wreath products. We have here:

Theorem 8.9. *We have the following results:*

(1) *$H_N^s=\mathbb{Z}_s\wr S_N$.*
(2) *$H_N^{s+}=\mathbb{Z}_s\wr_* S_N^+$.*

Proof. This follows from Theorem 8.8 and from the following formulae, valid for any connected graph X, and explained in the previous section, applied to $N\circlearrowleft_s$:

$$\begin{aligned}G(NX)&=G(X)\wr S_N\\ G^+(NX)&=G^+(X)\wr_* S_N^+\end{aligned}$$

Alternatively, (1) follows from definitions, and (2) can be proved directly, by constructing a pair of inverse morphisms. For details here, we refer to [33]. □

Regarding now the easiness property of the quantum groups H_N^s, H_N^{s+}, we already know that this happens at $s=1,2$. In general, we have the following result, from [6]:

Theorem 8.10. *The quantum groups H_N^s, H_N^{s+} are easy, the corresponding categories*

$$P^s \subset P \quad , \quad NC^s \subset NC$$

consisting of partitions having the property

$$\# \circ - \# \bullet = 0(s)$$

as a weighted sum, in each block.

Proof. Observe that the result holds at $s = 1$, trivially, and at $s = 2$ as well, where our condition is equivalent to $\# \circ + \# \bullet = 0(2)$, in each block. In general, this follows as in the proof of Theorem 7.24, by using the one-block partition in $P(s, s)$. See [6]. □

The above proof was of course quite brief, but we will not be really interested here in the case $s \geq 3$, which is quite technical. In fact, the above result, dealing with the general case $s \in \mathbb{N}$, is here for providing an introduction to the case $s = \infty$, where we have:

Theorem 8.11. *The quantum groups K_N, K_N^+ are easy, the corresponding categories*

$$\mathcal{P}_{even} \subset P \quad , \quad \mathcal{NC}_{even} \subset NC$$

consisting of partitions having the property

$$\#\circ = \#\bullet$$

as a weighted equality, in each block.

Proof. This follows from Theorem 8.10 above, or rather by proving the result directly, a bit as in the $s = 1, 2$ cases, because the $s = \infty$ case is needed first, in order to discuss the general case, $s \in \mathbb{N} \cup \{\infty\}$. For details here, we refer once again to [6]. □

Let us discuss now, following [33], the classification of the irreducible representations of H_N^{s+}, and the computation of their fusion rules. For this purpose, let us go back to the elements u_{ij}, p_{ij} in Definition 8.2 above. We recall that, as a consequence of Proposition 8.4, the matrix $p = (p_{ij})$ is a magic unitary. We first have the following result:

Proposition 8.12. *The elements u_{ij} and p_{ij} satisfy:*

(1) $p_{ij} u_{ij} = u_{ij}$.
(2) $u_{ij}^* = u_{ij}^{s-1}$.
(3) $u_{ij} u_{ik} = 0$ *for* $j \neq k$.

Proof. We use the fact that in a C^*-algebra, $aa^* = 0$ implies $a = 0$.

(1) This follows from the following computation, with $a = (p_{ij} - 1)u_{ij}$:

$$aa^* = (p_{ij} - 1)p_{ij}(p_{ij} - 1) = 0$$

(2) With $a = u_{ij}^* - u_{ij}^{s-1}$ we have $aa^* = 0$, which gives the result.
(3) With $a = u_{ij} u_{ik}$ we have $aa^* = 0$, which gives the result. □

In what follows, we make the convention $u_{ij}^0 = p_{ij}$. We have then:

Theorem 8.13. *The algebra $C(H_N^{s+})$ has a family of N-dimensional corepresentations $\{u_k|k\in\mathbb{Z}\}$, satisfying the following conditions:*

(1) $u_k=(u_{ij}^k)$ *for any* $k\geq 0$.
(2) $u_k=u_{k+s}$ *for any* $k\in\mathbb{Z}$.
(3) $\bar{u}_k=u_{-k}$ *for any* $k\in\mathbb{Z}$.

Proof. Let us set $u_k=(u_{ij}^k)$. By using Proposition 8.12 (3), we have:

$$\Delta(u_{ij}^k)=\sum_{l_1\ldots l_k}u_{il_1}\ldots u_{il_k}\otimes u_{l_1j}\ldots u_{l_kj}=\sum_l u_{il}^k\otimes u_{lj}^k$$

We have as well $\varepsilon(u_{ij}^k)=\delta_{ij}$ and $S(u_{ij}^k)=u_{ji}^{*k}$, trivially, so we are done with (1). Regarding now (2), this follows once again from Proposition 8.10 (3), as follows:

$$u_{ij}^{k+s}=u_{ij}^k u_{ij}^s=u_{ij}^k p_{ij}=u_{ij}^k$$

Finally (3) follows from Proposition 8.12 (2), and we are done. □

Let us compute now the intertwiners between the various tensor products between the above corepresentations u_i. For this purpose, we make the assumption $N\geq 4$, which brings linear independence. In order to simplify the notations, we will use:

Definition 8.14. *For $i_1,\ldots,i_k\in\mathbb{Z}$ we use the notation*

$$u_{i_1\ldots i_k}=u_{i_1}\otimes\ldots\otimes u_{i_k}$$

where $\{u_i|i\in\mathbb{Z}\}$ are the corepresentations in Theorem 8.13.

Observe that in the particular case $i_1,\ldots,i_k\in\{\pm1\}$, we obtain in this way all the possible tensor products between $u=u_1$ and $\bar{u}=u_{-1}$, known by [99] to contain any irreducible corepresentation of $C(H_N^{s+})$. Here is now our main result:

Theorem 8.15. *We have the following equality of linear spaces*

$$Hom(u_{i_1\ldots i_k},u_{j_1\ldots j_l})=span\left\{T_p\Big|p\in NC_s(i_1\ldots i_k,j_1\ldots j_l)\right\}$$

where the set on the right consists of elements of $NC(k,l)$ having the property that in each block, the sum of i indices equals the sum of j indices, modulo s.

Proof. This result is from [33], the idea of the proof being as follows:

(1) Our first claim is that, in order to prove $\supset$, we may restrict attention to the case $k=0$. This follow indeed from the Frobenius duality isomorphism.

(2) Our second claim is that, in order to prove $\supset$ in the case $k=0$, we may restrict attention to the one-block partitions. Indeed, this follows once again from a standard trick. Consider the following disjoint union:

$$NC_s=\bigcup_{k=0}^\infty\bigcup_{i_1\ldots i_k}NC_s(0,i_1\ldots i_k)$$

This is a set of labeled partitions, having property that each $p \in NC_s$ is noncrossing, and that for $p \in NC_s$, any block of p is in NC_s. But it is well-known that under these assumptions, the global algebraic properties of NC_s can be checked on blocks.

(3) Proof of $\supset$. According to the above considerations, we just have to prove that the vector associated to the one-block partition in $NC(l)$ is fixed by $u_{j_1\ldots j_l}$, when:

$$s|j_1 + \ldots + j_l$$

Consider the standard generators $e_{ab} \in M_N(\mathbb{C})$, acting on the basis vectors by $e_{ab}(e_c) = \delta_{bc}e_a$. The corepresentation $u_{j_1\ldots j_l}$ is given by the following formula:

$$u_{j_1\ldots j_l} = \sum_{a_1\ldots a_l}\sum_{b_1\ldots b_l} u^{j_1}_{a_1b_1}\ldots u^{j_l}_{a_lb_l} \otimes e_{a_1b_1} \otimes \ldots \otimes e_{a_lb_l}$$

As for the vector associated to the one-block partition, this is $\xi_l = \sum_b e_b^{\otimes l}$. By using now several times the relations in Proposition 8.12, we obtain, as claimed:

$$\begin{aligned}
u_{j_1\ldots j_l}(1 \otimes \xi_l) &= \sum_{a_1\ldots a_l}\sum_b u^{j_1}_{a_1b}\ldots u^{j_l}_{a_lb} \otimes e_{a_1} \otimes \ldots \otimes e_{a_l}\\
&= \sum_{ab} u^{j_1+\ldots+j_l}_{ab} \otimes e_a^{\otimes l}\\
&= 1 \otimes \xi_l
\end{aligned}$$

(4) Proof of $\subset$. The spaces on the right in the statement form a Tannakian category in the sense of [100], so they correspond to a certain Woronowicz algebra A.

This algebra is by definition the maximal model for the Tannakian category. In other words, it comes with a family of corepresentations $\{v_i\}$, such that:

$$Hom(v_{i_1\ldots i_k}, v_{j_1\ldots j_l}) = \mathrm{span}\left\{T_p\Big|p \in NC_s(i_1\ldots i_k, j_1\ldots j_l)\right\}$$

On the other hand, the inclusion $\supset$ that we just proved shows that $C(H_N^{s+})$ is a model for the category. Thus we have a quotient map $A \to C(H_N^{s+})$, mapping $v_i \to u_i$.

But this latter map can be shown to be an isomorphism, by suitably adapting the proof from the $s = 1$ case, for the quantum permutation group S_N^+. See [6], [33]. □

As an illustration for the above result, we have the following statement:

Theorem 8.16. *The basic corepresentations $u_0, \ldots, u_{s-1}$ are as follows:*

(1) *$u_1, \ldots, u_{s-1}$ are irreducible.*
(2) *$u_0 = 1 + r_0$, with r_0 irreducible.*
(3) *$r_0, u_1, \ldots, u_{s-1}$ are distinct.*

Proof. We apply Theorem 8.15 with $k = l = 1$ and $i_1 = i, j_1 = j$. This gives:

$$\dim(Hom(u_i, u_j)) = \#NC_s(i, j)$$

We have two candidates for the elements of $NC_s(i,j)$, namely the two partitions in $NC(1,1)$. So, consider these two partitions, with the points labeled by i,j:

$$p=\left\{\begin{matrix}i\\ |\\ |\\ j\end{matrix}\right\}\qquad\qquad q=\left\{\begin{matrix}i\\ |\\ \\ |\\ j\end{matrix}\right\}$$

We have to check for each of these partitions if the sum of i indices equals or not the sum of j indices, modulo s, in each block. The answer is as follows:

$$\begin{aligned}p\in NC_s(i,j)&\iff i=j\\ q\in NC_s(i,j)&\iff i=j=0\end{aligned}$$

By collecting together these two answers, we obtain:

$$\#NC_s(i,j)=\begin{cases}0&\text{if } i\neq j\\ 1&\text{if } i=j\neq 0\\ 2&\text{if } i=j=0\end{cases}$$

Now (1) follows from the second equality, (2) follows from the third equality and from the fact that we have $1\in u_s$, and (3) follows from the first equality. $\square$

Let us record as well, as a second consequence, the following result:

Theorem 8.17. *We have the formula*

$$\#(1\in u_{i_1}\otimes\ldots\otimes u_{i_k})=\#NC_s(i_1\ldots i_k)$$

where the set on the right consists of noncrossing partitions of $\{1,\ldots,k\}$ having the property that the sum of indices in each block is a multiple of s.

Proof. This is clear indeed from Theorem 8.15 above. $\square$

With these ingredients in hand, we can now compute the fusion semiring for H_N^{s+}. The result here, once again from [33], is as follows:

Theorem 8.18. *Let $F=<\mathbb{Z}_s>$ be the monoid formed by the words over $\mathbb{Z}_s$, with involution $(i_1\ldots i_k)^-=(-i_k)\ldots(-i_1)$, and with fusion product given by:*

$$(i_1\ldots i_k)\cdot(j_1\ldots j_l)=i_1\ldots i_{k-1}(i_k+j_1)j_2\ldots j_l$$

The irreducible representations of H_N^{s+} can then be labeled r_x with $x\in F$, such that the involution and fusion rules are $\bar{r}_x=r_{\bar{x}}$ and

$$r_x\otimes r_y=\sum_{x=vz,y=\bar{z}w}r_{vw}+r_{v\cdot w}$$

and such that we have $r_i=u_i-\delta_{i0}1$ for any $i\in\mathbb{Z}_s$.

Proof. This basically follows from Theorem 8.15, the idea being as follows:

(1) Consider the monoid $A = \{a_x | x \in F\}$, with multiplication $a_x a_y = a_{xy}$. We denote by $\mathbb{N}A$ the set of linear combinations of elements in A, with coefficients in $\mathbb{N}$, and we endow it with fusion rules as in the statement:

$$a_x \otimes a_y = \sum_{x=vz, y=\bar{z}w} a_{vw} + a_{v\cdot w}$$

With these notations, $(\mathbb{N}A, +, \otimes)$ is a semiring. We will use as well the set $\mathbb{Z}A$, formed by the linear combinations of elements of A, with coefficients in $\mathbb{Z}$. The above tensor product operation extends to $\mathbb{Z}A$, and $(\mathbb{Z}A, +, \otimes)$ is a ring.

(2) Our claim is that the fusion rules on $\mathbb{Z}A$ can be uniquely described by conversion formulae as follows, with C being positive integers, and D being integers:

$$a_{i_1} \otimes \ldots \otimes a_{i_k} = \sum_l \sum_{j_1 \ldots j_l} C_{i_1 \ldots i_k}^{j_1 \ldots j_l} a_{j_1 \ldots j_l}$$

$$a_{i_1 \ldots i_k} = \sum_l \sum_{j_1 \ldots j_l} D_{i_1 \ldots i_k}^{j_1 \ldots j_l} a_{j_1} \otimes \ldots \otimes a_{j_l}$$

Indeed, the existence and uniqueness of such decompositions follow from the definition of the tensor product operation, and by recurrence over k for the D coefficients.

(3) Our claim is that there is a unique morphism of rings $\Phi : \mathbb{Z}A \to R$, such that $\Phi(a_i) = r_i$ for any i. Indeed, consider the following elements of R:

$$r_{i_1 \ldots i_k} = \sum_l \sum_{j_1 \ldots j_l} D_{i_1 \ldots i_k}^{j_1 \ldots j_l} r_{j_1} \otimes \ldots \otimes r_{j_l}$$

In case we have a morphism as claimed, we must have $\Phi(a_x) = r_x$ for any $x \in F$. Thus our morphism is uniquely determined on A, so it is uniquely determined on $\mathbb{Z}A$.

In order to prove the existence, we can set $\Phi(a_x) = r_x$ for any $x \in F$, then extend Φ by linearity to the whole $\mathbb{Z}A$. Since Φ commutes with the above conversion formulae, which describe the fusion rules, it is indeed a morphism.

(4) Our claim is that Φ commutes with the linear forms $x \to \#(1 \in x)$. Indeed, by linearity we just have to check the following equality:

$$\#(1 \in a_{i_1} \otimes \ldots \otimes a_{i_k}) = \#(1 \in r_{i_1} \otimes \ldots \otimes r_{i_k})$$

Now remember that the elements r_i are defined as $r_i = u_i - \delta_{i0}1$. So, consider the elements $c_i = a_i + \delta_{i0}1$. Since the operations $r_i \to u_i$ and $a_i \to c_i$ are of the same nature, by linearity the above formula is equivalent to:

$$\#(1 \in c_{i_1} \otimes \ldots \otimes c_{i_k}) = \#(1 \in u_{i_1} \otimes \ldots \otimes u_{i_k})$$

Now by using Theorem 8.15, what we have to prove is:

$$\#(1\in c_{i_1}\otimes\ldots\otimes c_{i_k})=\#NC_s(i_1\ldots i_k)$$

In order to prove this formula, consider the product on the left:

$$P=(a_{i_1}+\delta_{i_10}1)\otimes(a_{i_2}+\delta_{i_20}1)\otimes\ldots\otimes(a_{i_k}+\delta_{i_k0}1)$$

This quantity can be computed by using the fusion rules on A. A recurrence on k shows that the final components of type a_x will come from the different ways of grouping and summing the consecutive terms of the sequence $(i_1,\ldots,i_k)$, and removing some of the sums which vanish modulo s, as to obtain the sequence x. But this can be encoded by families of noncrossing partitions, and in particular the 1 components will come from the partitions in $NC_s(i_1\ldots i_k)$. Thus $\#(1\in P)=\#NC_s(i_1\ldots i_k)$, as claimed.

(5) Our claim now is that Φ is injective. Indeed, this follows from the result in the previous step, by using a standard positivity argument:

$$\begin{aligned}\Phi(\alpha)=0 &\implies \Phi(\alpha\alpha^*)=0\\ &\implies \#(1\in\Phi(\alpha\alpha^*))=0\\ &\implies \#(1\in\alpha\alpha^*)=0\\ &\implies \alpha=0\end{aligned}$$

Here α is arbitrary in the domain of Φ, we use the notation $a_x^*=a_{\bar{x}}$, where $a\to\#(1,a)$ is the unique linear extension of the operation consisting of counting the number of 1's. Observe that this latter linear form is indeed positive definite, according to the identity $\#(1,a_xa_y^*)=\delta_{xy}$, which is clear from the definition of the product of $\mathbb{Z}A$.

(6) Our claim is that we have $\Phi(A)\subset R_{irr}$. This is the same as saying that $r_x\in R_{irr}$ for any $x\in F$, and we will prove it by recurrence on the length of x.

Assume that the assertion is true for all the words of length $<k$, and consider an arbitrary length k word, $x=i_1\ldots i_k$. We have:

$$a_{i_1}\otimes a_{i_2\ldots i_k}=a_x+a_{i_1+i_2,i_3\ldots i_k}+\delta_{i_1+i_2,0}a_{i_3\ldots i_k}$$

By applying Φ to this decomposition, we obtain:

$$r_{i_1}\otimes r_{i_2\ldots i_k}=r_x+r_{i_1+i_2,i_3\ldots i_k}+\delta_{i_1+i_2,0}r_{i_3\ldots i_k}$$

We have the following computation, which is valid for $y=i_1+i_2,i_3\ldots i_k$, as well as for $y=i_3\ldots i_k$ in the case $i_1+i_2=0$:

$$\begin{aligned}\#(r_y\in r_{i_1}\otimes r_{i_2\ldots i_k}) &= \#(1,r_{\bar{y}}\otimes r_{i_1}\otimes r_{i_2\ldots i_k})\\ &= \#(1,a_{\bar{y}}\otimes a_{i_1}\otimes a_{i_2\ldots i_k})\\ &= \#(a_y\in a_{i_1}\otimes a_{i_2\ldots i_k})\\ &= 1\end{aligned}$$

Moreover, we know from the previous step that we have $r_{i_1+i_2,i_3\ldots i_k} \neq r_{i_3\ldots i_k}$, so we conclude that the following formula defines an element of R^+:

$$\alpha = r_{i_1} \otimes r_{i_2\ldots i_k} - r_{i_1+i_2,i_3\ldots i_k} - \delta_{i_1+i_2,0} r_{i_3\ldots i_k}$$

On the other hand, we have $\alpha = r_x$, so we conclude that we have $r_x \in R^+$. Finally, the irreducibility of r_x follows from the following computation:

$$\begin{aligned}\#(1 \in r_x \otimes \bar{r}_x) &= \#(1 \in r_x \otimes r_{\bar{x}}) \\ &= \#(1 \in a_x \otimes a_{\bar{x}}) \\ &= \#(1 \in a_x \otimes \bar{a}_x) \\ &= 1\end{aligned}$$

(7) Summarizing, we have constructed an injective ring morphism $\Phi : \mathbb{Z}A \to R$, having the property $\Phi(A) \subset R_{irr}$. The remaining fact to be proved, namely that we have $\Phi(A) = R_{irr}$, is clear from the general results in [99]. Indeed, since each element of $\mathbb{N}A$ is a sum of elements in A, by applying Φ we get that each element in $\Phi(\mathbb{N}A)$ is a sum of irreducible corepresentations in $\Phi(A)$. But since $\Phi(\mathbb{N}A)$ contains all the tensor powers between the fundamental corepresentation and its conjugate, we get $\Phi(A) = R_{irr}$, and we are done. $\square$

For further results regarding representation theory, we refer to [33].

Let us discuss now the computation of the asymptotic laws of characters. We begin with a discussion for H_N, from [14], which has its own interest:

Theorem 8.19. *The asymptotic law of χ_t for the group H_N is given by*

$$b_t = e^{-t} \sum_{k=-\infty}^{\infty} \delta_k \sum_{p=0}^{\infty} \frac{(t/2)^{|k|+2p}}{(|k|+p)!p!}$$

where δ_k is the Dirac mass at $k \in \mathbb{Z}$.

Proof. We regard the hyperoctahedral group H_N as being the symmetry group of the graph $I_N = \{I^1, \ldots, I^N\}$ formed by N segments. The diagonal coefficients are then:

$$u_{ii}(g) = \begin{cases} 0 & \text{if } g \text{ moves } I^i \\ +1 & \text{if } g \text{ fixes } I^i \\ -1 & \text{if } g \text{ returns } I^i \end{cases}$$

Let $s = [tN]$, and denote by $\uparrow g, \downarrow g$ the number of segments among $\{I^1, \ldots, I^s\}$ which are fixed, respectively returned by an element $g \in H_N$. With this notation, we have:

$$u_{11} + \ldots + u_{ss} = \uparrow g - \downarrow g$$

We denote by P_N probabilities computed over the group H_N. The density of the law of $u_{11}+\ldots+u_{ss}$ at a point $k\geq 0$ is given by the following formula:

$$\begin{aligned}
D(k) &= P_N(\uparrow g-\downarrow g=k)\\
&= \sum_{p=0}^\infty P_N(\uparrow g=k+p,\downarrow g=p)
\end{aligned}$$

Assume first that we have $t=1$. We use the fact that the probability of $\sigma\in S_N$ to have no fixed points is asymptotically $1/e$. Thus the probability of $\sigma\in S_N$ to have m fixed points is asymptotically $1/(em!)$. In terms of probabilities over H_N, we obtain:

$$\begin{aligned}
\lim_{N\to\infty}D(k) &= \lim_{N\to\infty}\sum_{p=0}^\infty(1/2)^{k+2p}\binom{k+2p}{k+p}P_N(\uparrow g+\downarrow g=k+2p)\\
&= \sum_{p=0}^\infty(1/2)^{k+2p}\binom{k+2p}{k+p}\frac{1}{e(k+2p)!}\\
&= \frac{1}{e}\sum_{p=0}^\infty\frac{(1/2)^{k+2p}}{(k+p)!p!}
\end{aligned}$$

The general case $t\in(0,1]$ follows by performing some modifications in the above computation. The asymptotic density is computed as follows:

$$\begin{aligned}
\lim_{N\to\infty}D(k) &= \lim_{N\to\infty}\sum_{p=0}^\infty(1/2)^{k+2p}\binom{k+2p}{k+p}P_N(\uparrow g+\downarrow g=k+2p)\\
&= \sum_{p=0}^\infty(1/2)^{k+2p}\binom{k+2p}{k+p}\frac{t^{k+2p}}{e^t(k+2p)!}\\
&= e^{-t}\sum_{p=0}^\infty\frac{(t/2)^{k+2p}}{(k+p)!p!}
\end{aligned}$$

Together with $D(-k)=D(k)$, this gives the formula in the statement. $\square$

Observe that the measure found above is $b_t=e^{-t}\sum_{k=-\infty}^\infty\delta_k\,f_k(t/2)$, where f_k is the Bessel function of the first kind:

$$f_k(t)=\sum_{p=0}^\infty\frac{t^{|k|+2p}}{(|k|+p)!p!}$$

Next, we have the following result, once again from [14]:

Theorem 8.20. *The Bessel laws b_t have the additivity property*

$$b_s*b_t=b_{s+t}$$

so they form a truncated one-parameter semigroup with respect to convolution.

Proof. The Fourier transform of b_t is given by:

$$Fb_t(y) = e^{-t} \sum_{k=-\infty}^{\infty} e^{ky} f_k(t/2)$$

We compute now the derivative with respect to t:

$$Fb_t(y)' = -Fb_t(y) + \frac{e^{-t}}{2} \sum_{k=-\infty}^{\infty} e^{ky} f_k'(t/2)$$

On the other hand, the derivative of f_k with $k \geq 1$ is given by:

$$\begin{aligned} f_k'(t) &= \sum_{p=0}^{\infty} \frac{(k+2p)t^{k+2p-1}}{(k+p)!p!} \\ &= \sum_{p=0}^{\infty} \frac{(k+p)t^{k+2p-1}}{(k+p)!p!} + \sum_{p=0}^{\infty} \frac{p\, t^{k+2p-1}}{(k+p)!p!} \\ &= \sum_{p=0}^{\infty} \frac{t^{k+2p-1}}{(k+p-1)!p!} + \sum_{p=1}^{\infty} \frac{t^{k+2p-1}}{(k+p)!(p-1)!} \\ &= \sum_{p=0}^{\infty} \frac{t^{(k-1)+2p}}{((k-1)+p)!p!} + \sum_{p=1}^{\infty} \frac{t^{(k+1)+2(p-1)}}{((k+1)+(p-1))!(p-1)!} \\ &= f_{k-1}(t) + f_{k+1}(t) \end{aligned}$$

This computation works in fact for any k, so we get:

$$\begin{aligned} Fb_t(y)' &= -Fb_t(y) + \frac{e^{-t}}{2} \sum_{k=-\infty}^{\infty} e^{ky}(f_{k-1}(t/2) + f_{k+1}(t/2)) \\ &= -Fb_t(y) + \frac{e^{-t}}{2} \sum_{k=-\infty}^{\infty} e^{(k+1)y} f_k(t/2) + e^{(k-1)y} f_k(t/2) \\ &= -Fb_t(y) + \frac{e^y + e^{-y}}{2} Fb_t(y) \\ &= \left(\frac{e^y + e^{-y}}{2} - 1\right) Fb_t(y) \end{aligned}$$

Thus the log of the Fourier transform is linear in t, and we get the assertion. □

In order to discuss now the free analogue β_t of the above measure b_t, as well as the s-analogues b_t^s, β_t^s of the measures b_t, β_t, we need some free probability.

We have the following notion, extending the Poisson limit theory from section 5:

Definition 8.21. *Associated to any compactly supported positive measure ρ on $\mathbb{R}$ are the probability measures*

$$p_\rho=\lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0+\frac{1}{n}\rho\right)^{*n}\quad,\quad \pi_\rho=\lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0+\frac{1}{n}\rho\right)^{\boxplus n}$$

where $c=mass(\rho)$, called compound Poisson and compound free Poisson laws.

In what follows we will be interested in the case where ρ is discrete, as is for instance the case for $\rho=\delta_t$ with $t>0$, which produces the Poisson and free Poisson laws.

The following result allows one to detect compound Poisson/free Poisson laws:

Proposition 8.22. *For $\rho=\sum_{i=1}^s c_i\delta_{z_i}$ with $c_i>0$ and $z_i\in\mathbb{R}$, we have*

$$F_{p_\rho}(y)=\exp\left(\sum_{i=1}^s c_i(e^{iyz_i}-1)\right)\quad,\quad R_{\pi_\rho}(y)=\sum_{i=1}^s\frac{c_iz_i}{1-yz_i}$$

where F,R denote respectively the Fourier transform, and Voiculescu's R-transform.

Proof. Let μ_n be the measure appearing in Definition 8.21, under the convolution signs. In the classical case, we have the following computation:

$$\begin{aligned}
&F_{\mu_n}(y)=\left(1-\frac{c}{n}\right)+\frac{1}{n}\sum_{i=1}^s c_ie^{iyz_i}\\
\implies\quad &F_{\mu_n^{*n}}(y)=\left(\left(1-\frac{c}{n}\right)+\frac{1}{n}\sum_{i=1}^s c_ie^{iyz_i}\right)^n\\
\implies\quad &F_{p_\rho}(y)=\exp\left(\sum_{i=1}^s c_i(e^{iyz_i}-1)\right)
\end{aligned}$$

In the free case now, we use a similar method. The Cauchy transform of μ_n is:

$$G_{\mu_n}(\xi)=\left(1-\frac{c}{n}\right)\frac{1}{\xi}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{\xi-z_i}$$

Consider now the R-transform of the measure $\mu_n^{\boxplus n}$, which is given by:

$$R_{\mu_n^{\boxplus n}}(y)=nR_{\mu_n}(y)$$

The above formula of G_{μ_n} shows that the equation for $R=R_{\mu_n^{\boxplus n}}$ is as follows:

$$\begin{aligned}
&\left(1-\frac{c}{n}\right)\frac{1}{y^{-1}+R/n}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{y^{-1}+R/n-z_i}=y\\
\implies\quad &\left(1-\frac{c}{n}\right)\frac{1}{1+yR/n}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{1+yR/n-yz_i}=1
\end{aligned}$$

Now by multiplying by n, rearranging the terms, and letting $n \to \infty$, we get:

$$\begin{aligned}
\frac{c+yR}{1+yR/n} &= \sum_{i=1}^s \frac{c_i}{1+yR/n-yz_i}\\
\implies \quad c+yR_{\pi_\rho}(y) &= \sum_{i=1}^s \frac{c_i}{1-yz_i}\\
\implies \quad R_{\pi_\rho}(y) &= \sum_{i=1}^s \frac{c_iz_i}{1-yz_i}
\end{aligned}$$

This finishes the proof in the free case, and we are done. □

We have as well the following result, providing an alternative to Definition 8.21:

Theorem 8.23. *For $\rho = \sum_{i=1}^s c_i\delta_{z_i}$ with $c_i > 0$ and $z_i \in \mathbb{R}$, we have*

$$p_\rho/\pi_\rho = \mathrm{law}\left(\sum_{i=1}^s z_i\alpha_i\right)$$

where the variables α_i are Poisson/free Poisson(c_i), independent/free.

Proof. Let α be the sum of Poisson/free Poisson variables in the statement. We will show that the Fourier/R-transform of α is given by the formulae in Proposition 8.22.

Indeed, by using some well-known Fourier transform formulae, we have:

$$\begin{aligned}
F_{\alpha_i}(y) = \exp(c_i(e^{iy}-1)) \quad &\implies \quad F_{z_i\alpha_i}(y) = \exp(c_i(e^{iyz_i}-1))\\
&\implies \quad F_\alpha(y) = \exp\left(\sum_{i=1}^s c_i(e^{iyz_i}-1)\right)
\end{aligned}$$

Also, by using some well-known R-transform formulae, we have:

$$\begin{aligned}
R_{\alpha_i}(y) = \frac{c_i}{1-y} \quad &\implies \quad R_{z_i\alpha_i}(y) = \frac{c_iz_i}{1-yz_i}\\
&\implies \quad R_\alpha(y) = \sum_{i=1}^s \frac{c_iz_i}{1-yz_i}
\end{aligned}$$

Thus we have indeed the same formulae as those in Proposition 8.22. □

We can go back now to quantum reflection groups, and we have:

Theorem 8.24. *The asymptotic laws of truncated characters are as follows, where ε_s with $s \in \{1, 2, \ldots, \infty\}$ is the uniform measure on the s-th roots of unity:*

1. *For H_N^s we obtain the compound Poisson law $b_t^s = p_{t\varepsilon_s}$.*
2. *For H_N^{s+} we obtain the compound free Poisson law $\beta_t^s = \pi_{t\varepsilon_s}$.*

These measures are in Bercovici-Pata bijection.

Proof. This follows from easiness, and from the Weingarten formula. To be more precise, at $t=1$ this follows by counting the partitions, and at $t\in(0,1]$ general, this follows in the usual way, for instance by using cumulants. For details here, we refer to [6]. $\square$

The above measures are called Bessel and free Bessel laws. This is because at $s=2$ we have $b_t^2=e^{-t}\sum_{k=-\infty}^\infty f_k(t/2)\delta_k$, with f_k being the Bessel function of the first kind:

$$f_k(t)=\sum_{p=0}^\infty\frac{t^{|k|+2p}}{(|k|+p)!p!}$$

The Bessel and free Bessel laws have particularly interesting properties at the parameter values $s=2,\infty$. So, let us record the precise statement here:

Theorem 8.25. *The asymptotic laws of truncated characters are as follows:*

(1) *For H_N we obtain the real Bessel law $b_t=p_{t\varepsilon_2}$.*
(2) *For K_N we obtain the complex Bessel law $B_t=p_{t\varepsilon_\infty}$.*
(3) *For H_N^+ we obtain the free real Bessel law $\beta_t=\pi_{t\varepsilon_2}$.*
(4) *For K_N^+ we obtain the free complex Bessel law $\mathfrak{B}_t=\pi_{t\varepsilon_\infty}$.*

Proof. This follows indeed from Theorem 8.24 above, at $s=2,\infty$. $\square$

In addition to what has been said above, there are as well some interesting results about the Bessel and free Bessel laws involving the multiplicative convolution $\times$, and the multiplicative free convolution $\boxtimes$ from [88]. For details, we refer here to [6].

9. Complex reflections

We have seen in the previous section that the basic reflection groups $H_N^s = \mathbb{Z}_s \wr S_N$ have free analogues $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$, and that the theory of these quantum groups, both classical and free, is very interesting, algebrically and analytically speaking.

In this section we explore more general classes of quantum reflection groups. As we will see, the subject is extremely interesting, and there are many open questions, currently under investigation. In fact, this is one of the main "hot areas" in quantum groups.

In order to get started, let us recall the theory from the classical case. The theorem here, a celebrated result by Shephard and Todd, from the 50s, is as follows:

Theorem 9.1. *The irreducible complex reflection groups are*

$$H_N^{sd} = \left\{ U \in H_N^s \Big| (\det U)^d = 1 \right\}$$

along with 34 *exceptional examples.*

Proof. This is something quite advanced, and we refer here to the paper of Shephard and Todd [81], and to the subsequent literature on the subject. □

In the general quantum case now, the axiomatization and classification of the quantum reflection groups is an interesting question, which is not understood yet.

We will be interested in what follows in the "twistable" case, where the theory is more advanced than in the general case. Let us start with the following definition:

Definition 9.2. *A closed subgroup $G \subset U_N^+$ is called:*

(1) *Half-homogeneous, when it contains the alternating group, $A_N \subset G$.*
(2) *Homogeneous, when it contains the symmetric group, $S_N \subset G$.*
(3) *Twistable, when it contains the hyperoctahedral group, $H_N \subset G$.*

Observe that in the classical case, the complex reflection groups H_N^{sd} are all half-homogeneous, but only some of them are homogeneous, or twistable.

In general, the above notions are mostly motivated by the easy case. Here we have by definition $S_N \subset G \subset U_N^+$, and so our quantum group is automatically homogeneous. The point now is that the twistability assumption corresponds to the following condition, at the level of the associated category of partitions $D \subset P$:

$$D \subset P_{even}$$

We recognize here the condition which is needed for performing the Schur-Weyl twisting operation, explained in section 7 above, and based on the signature map:

$$\varepsilon : P_{even} \to \{\pm 1\}$$

As a conclusion, in the easy case our notion of twistability is the correct one. In general, there are of course more general twisting methods, usually requiring $\mathbb{Z}_2^N \subset G$ only. But in the half-homogeneous case, the condition $\mathbb{Z}_2^N \subset G$ is equivalent to $H_N \subset G$.

With this discussion done, let us formulate now the following definition:

Definition 9.3. *A twistable quantum reflection group is an intermediate subgroup*

$$H_N\subset K\subset K_N^+$$

between the group $H_N=\mathbb Z_2\wr S_N$, and the quantum group $K_N^+=\mathbb T\wr_*S_N^+$.

As already mentioned, this is something quite restrictive, mainly motivated by the easy quantum group case, and in the classical case, not all the complex reflection groups H_N^{sd} from Theorem 9.1 fit into this scheme. However, the theory is quite advanced under the twistability assumption, and explaining it will be our main purpose in this section.

Here is now another definition, which is important for general compact quantum group purposes, and which provides motivations for our formalism from Definition 9.3:

Definition 9.4. *Given a closed subgroup $G\subset U_N^+$ which is twistable, in the sense that we have $H_N\subset G$, we define its associated reflection subgroup to be*

$$K=G\cap K_N^+$$

with the intersection taken inside U_N^+. We say that G appears as a soft liberation of its classical version $G_{class}=G\cap U_N$ when $G=<G_{class},K>$.

These notions are important in the classification theory of compact quantum groups, and in connection with certain noncommutative geometry questions as well. As a first observation, with K being as above, we have an intersection diagram, as follows:

$$\begin{tikzcd}
K \arrow[r] & G \\
K_{class} \arrow[u] \arrow[r] & G_{class} \arrow[u]
\end{tikzcd}$$

The soft liberation condition states that this diagram must be a generation diagram. We will be back to this in a moment, with some further theoretical comments.

Let us work out some examples. As a basic result, we have:

Theorem 9.5. *The reflection subgroups of the basic unitary quantum groups are*

$$\begin{tikzcd}
U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\
O_N \arrow[u] \arrow[r] & O_N^* \arrow[u] \arrow[r] & O_N^+ \arrow[u]
\end{tikzcd}
\quad\to\quad
\begin{tikzcd}
K_N \arrow[r] & K_N^* \arrow[r] & K_N^+ \\
H_N \arrow[u] \arrow[r] & H_N^* \arrow[u] \arrow[r] & H_N^+ \arrow[u]
\end{tikzcd}$$

and these unitary quantum groups all appear via soft liberation.

Proof. The fact that the reflection subgroups of the quantum groups on the left are those on the right is clear in all cases, with the middle objects being by definition:

$$H_N^* = H_N \cap O_N^* \quad , \quad K_N^* = K_N \cap U_N^*$$

Regarding the second assertion, things are quite tricky here, as follows:

(1) In the classical case there is nothing to prove, because any classical group is by definition a soft liberation of itself.

(2) In the half-classical case the results are non-trivial, but can be proved by using the technology developed by Bichon and Dubois-Violette.

(3) In the free case the results are highly non-trivial, and the only known proof so far uses the recurrence methods developed by Chirvasitu in [42]. □

Summarizing, we are here into recent and interesting quantum group theory. We will discuss a bit later the concrete applications of Theorem 9.5.

There is a connection here as well with the notion of diagonal torus, introduced in section 1 above. We can indeed refine Definition 9.4, in the following way:

Definition 9.6. *Given $H_N \subset G \subset U_N^+$, the diagonal tori $T = G \cap \mathbb{T}_N^+$ and reflection subgroups $K = G \cap K_N^+$ for G and for $G_{class} = G \cap U_N$ form a diagram as follows:*

$$\begin{array}{ccccc} T & \longrightarrow & K & \longrightarrow & G \\ \uparrow & & \uparrow & & \uparrow \\ T_{class} & \longrightarrow & K_{class} & \longrightarrow & G_{class} \end{array}$$

We say that G appears as a soft/hard liberation when it is generated by G_{class} and by K/T, which means that the right square/whole rectangle should be generation diagrams.

It is in fact possible to further complicate the picture, by adding free versions as well, with these free versions being by definition given by the following formula:

$$G_{free} = < G, S_N^+ >$$

Importantly, we can equally add the parameter $N \in \mathbb{N}$ to the picture, the idea being that we have a kind of "ladder", whose steps are the diagrams in Definition 9.6, perhaps extended with the free versions too, at fixed values of $N \in \mathbb{N}$.

The various generation and intersection properties of this ladder are important properties of $G = (G_N)$ itself, with subtle relations between them. In fact, as already mentioned in the proof of Theorem 9.5 above, the proof of the soft generation property for O_N^+, U_N^+ uses in fact this ladder, via the recurrence methods developed in [42].

All this is quite technical, so as a concrete result in connection with the above hard liberation notion, we have the following statement, improving Theorem 9.5:

Theorem 9.7. *The diagonal tori of the basic unitary quantum groups are*

$$\begin{matrix}\begin{tikzcd} U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\ O_N \arrow[u] \arrow[r] & O_N^* \arrow[u] \arrow[r] & O_N^+ \arrow[u] \end{tikzcd} & \to & \begin{tikzcd} \mathbb{T}_N \arrow[r] & \mathbb{T}_N^* \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[u] \arrow[r] & T_N^* \arrow[u] \arrow[r] & T_N^+ \arrow[u] \end{tikzcd}\end{matrix}$$

and these unitary quantum groups all appear via hard liberation.

Proof. The first assertion is something that we already know, from section 1 above, with the various tori appearing there being by definition the following group duals:

$$\begin{tikzcd} \widehat{\mathbb{Z}^N} \arrow[r] & \widehat{\mathbb{Z}^{\circ N}} \arrow[r] & \widehat{\mathbb{Z}^{*N}} \\ \widehat{\mathbb{Z}_2^N} \arrow[u] \arrow[r] & \widehat{\mathbb{Z}_2^{\circ N}} \arrow[u] \arrow[r] & \widehat{\mathbb{Z}_2^{*N}} \arrow[u] \end{tikzcd}$$

As for the second assertion, this can be proved by carefully examining the proof of Theorem 9.5, and performing some suitable modifications, where needed. $\square$

As an interesting remark, some subtleties appear in the following way:

Proposition 9.8. *The diagonal tori of the basic quantum reflection groups are*

$$\begin{matrix}\begin{tikzcd} K_N \arrow[r] & K_N^* \arrow[r] & K_N^+ \\ H_N \arrow[u] \arrow[r] & H_N^* \arrow[u] \arrow[r] & H_N^+ \arrow[u] \end{tikzcd} & \to & \begin{tikzcd} \mathbb{T}_N \arrow[r] & \mathbb{T}_N^* \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[u] \arrow[r] & T_N^* \arrow[u] \arrow[r] & T_N^+ \arrow[u] \end{tikzcd}\end{matrix}$$

and these quantum reflection groups do not all appear via hard liberation.

Proof. The first assertion is clear, for instance as a consequence of Theorem 9.7, because the diagonal torus is the same for a quantum group, and for its reflection subgroup:

$$G \cap \mathbb{T}_N^+ = (G \cap K_N^+) \cap \mathbb{T}_N^+$$

Regarding the second assertion, things are quite tricky here, as follows:

(1) In the classical case the hard liberation property definitely holds, because any classical group is by definition a hard liberation of itself.

(2) In the half-classical case the answer is again positive, and this is non-trivial, but can be proved by using the technology developed by Bichon and Dubois-Violette.

(3) In the free case the hard liberation property fails, due to some intermediate quantum groups $H_N^{[\infty]}$, $K_N^{[\infty]}$, where "hard liberation stops". We will be back to this. $\square$

As a conjectural solution to these latter difficulties, coming from Proposition 9.8, we have the notion of Fourier liberation, to be discussed later on, in section 9 below.

Finally, as a last piece of general theory, let us mention that all this is interesting in connection with noncommutative geometry. From a very naive viewpoint, which is actually a very good viewpoint, an abstract noncommutative geometry theory should include at least a noncommutative sphere S, a noncommutative torus T, a unitary quantum group U, and a quantum reflection group K, with connections between them, as follows:

$$\begin{tikzcd} S \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ U \arrow[r, leftrightarrow] & K \end{tikzcd}$$

The point now is that when trying to axiomatize such quadruplets (S, T, U, K), we are led in particular to the study of the relations between T, U, K, and so to the above soft and hard liberation questions. We refer here to the literature on the subject.

With this done, let us go back now to Definition 9.3, and try to do some classification work there. We are interested in intermediate quantum groups, as follows:

$$H_N \subset K \subset K_N^+$$

We have already met a few examples of such quantum groups, in the various results above, and their proofs. We can put these examples together, as follows:

Proposition 9.9. *We have quantum reflection groups as follows, with $H_N^{[\infty]}, K_N^{[\infty]}$ being obtained via the relations $abc = 0$, for any $a \neq c$ on the same row or column of $u, \bar{u}$:*

$$\begin{tikzcd} K_N \arrow[r] & K_N^* \arrow[r] & K_N^{[\infty]} \arrow[r] & K_N^+ \\ H_N \arrow[r] \arrow[u] & H_N^* \arrow[r] \arrow[u] & H_N^{[\infty]} \arrow[r] \arrow[u] & H_N^+ \arrow[u] \end{tikzcd}$$

In addition, we can intersect these quantum groups with H_N^{s+} with $s \in 2\mathbb{N} \cup \{\infty\}$, with the $s = 2, \infty$ intersections corresponding to the lower and upper row.

Proof. These are quantum groups that we already know, with the exception of $H_N^{[\infty]},K_N^{[\infty]}$, that only appeared briefly, in the proof of Proposition 9.8 above.

But these latter quantum groups can be indeed be defined via the relations $abc=0$ in the statement, with the construction of Δ,ε,S being standard, by universality, and with the inclusions $H_N^*\subset H_N^{[\infty]}$ and $K_N^*\subset K_N^{[\infty]}$ being clear as well. $\square$

Remarkably, all the above quantum groups are easy, as follows:

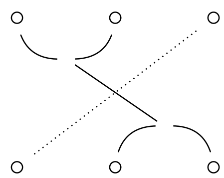
Theorem 9.10. *The above quantum groups are all easy, the categories of partitions being as follows, with $P_{even}^{[\infty]}=<\eta>$ with $\eta=\ker\binom{xxy}{yxx}$, and with $\mathcal{P}_{even}^{[\infty]}=P_{even}^{[\infty]}\cap\mathcal{P}_{even}$:*

$$\begin{tikzcd}
\mathcal{P}_{even} \arrow[d] & \mathcal{P}_{even}^* \arrow[l] \arrow[d] & \mathcal{P}_{even}^{[\infty]} \arrow[l] \arrow[d] & \mathcal{NC}_{even} \arrow[l] \arrow[d] \\
P_{even} & P_{even}^* \arrow[l] & P_{even}^{[\infty]} \arrow[l] & NC_{even} \arrow[l]
\end{tikzcd}$$

When intersecting with H_N^{s+} with $s\in 2\mathbb{N}\cup\{\infty\}$, the quantum groups that we obtain are easy as well, the corresponding categories being $D_{G\cap H_N^{s+}}=<D_G,P^s>$.

Proof. These are basically results that we already know, with the exception of the results for $H_N^{[\infty]},K_N^{[\infty]}$. In order to prove easiness here, consider the following partition:

η =



It is routine to check that this partition implements the relations $abc=0$ defining the quantum group $H_N^{[\infty]}$, and we conclude that this quantum group is indeed easy, the corresponding category being $P_{even}^{[\infty]}=<\eta>$. The proof for $K_N^{[\infty]}$ is similar. $\square$

The above quantum groups $H_N^*,H_N^{[\infty]}$ are quite interesting objects. Here are some further results regarding the associated categories $P_{even}^{[\infty]},P_{even}^*$, that we will need later:

Proposition 9.11. *We have the following formulae,*

$$\begin{aligned}
P_{even}^{[\infty]} &= \left\{\pi\in P_{even}\Big|\varepsilon(\tau)=1,\forall\tau\leq\pi\right\}\\
P_{even}^* &= \left\{\pi\in P_{even}\Big|\varepsilon(\tau)=1,\forall\tau\leq\pi,|\tau|=2\right\}
\end{aligned}$$

where $|.|$ denotes the number of blocks.

Proof. We first prove the second equality. Given $\pi \in P_{even}$, we have $\tau \leq \pi, |\tau| = 2$ precisely when $\tau = \pi^\beta$ is the partition obtained from π by merging all the legs of a certain subpartition $\beta \subset \pi$, and by merging as well all the other blocks. Now observe that π^β does not depend on π, but only on β, and that the number of switches required for making π^β noncrossing is $c = N_\bullet - N_\circ$ modulo 2, where $N_\bullet/N_\circ$ is the number of black/white legs of β, when labelling the legs of π counterclockwise $\circ\bullet\circ\bullet\ldots$ Thus $\varepsilon(\pi^\beta) = 1$ holds precisely when $\beta \in \pi$ has the same number of black and white legs, and this gives the result.

We prove now the first equality. We recall that we have:

$$P_{even}^{[\infty]}(k,l) = \left\{ \ker \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} \Big| g_{i_1} \ldots g_{i_k} = g_{j_1} \ldots g_{j_l} \text{ inside } \mathbb{Z}_2^{*N} \right\}$$

In other words, the partitions in $P_{even}^{[\infty]}$ are those describing the relations between free variables, subject to the conditions $g_i^2 = 1$. We conclude that $P_{even}^{[\infty]}$ appears from NC_{even} by "inflating blocks", in the sense that each $\pi \in P_{even}^{[\infty]}$ can be transformed into a partition $\pi' \in NC_{even}$ by deleting pairs of consecutive legs, belonging to the same block.

Now since this inflation operation leaves invariant modulo 2 the number $c \in \mathbb{N}$ of switches in the definition of the signature, it leaves invariant the signature $\varepsilon = (-1)^c$ itself, and we obtain in this way the inclusion "$\subset$" in the statement.

Conversely, given $\pi \in P_{even}$ satisfying $\varepsilon(\tau) = 1$, $\forall \tau \leq \pi$, our claim is that:

$$\rho \leq \sigma \subset \pi, |\rho| = 2 \implies \varepsilon(\rho) = 1$$

Indeed, let us denote by α, β the two blocks of ρ, and by γ the remaining blocks of π, merged altogether. We know that the partitions $\tau_1 = (\alpha \wedge \gamma, \beta)$, $\tau_2 = (\beta \wedge \gamma, \alpha)$, $\tau_3 = (\alpha, \beta, \gamma)$ are all even. On the other hand, putting these partitions in noncrossing form requires respectively $s+t, s'+t, s+s'+t$ switches, where t is the number of switches needed for putting $\rho = (\alpha, \beta)$ in noncrossing form. Thus t is even, and we are done.

With the above claim in hand, we conclude, by using the second equality in the statement, that we have $\sigma \in P_{even}^*$. Thus we have $\pi \in P_{even}^{[\infty]}$, which ends the proof of "$\supset$". □

Summarizing, the quantum groups $H_N^*, H_N^{[\infty]}$ are quite interesting objects, and we have a variety of combinatorial methods for investigating them, via partitions.

Following now [78], let us discuss the classification of the easy reflection groups in the real case, $H_N \subset G \subset H_N^+$. For various reasons, including those coming from the general results presented above, providing motivation for the subject, it is convenient to include the unitary quantum groups as well. We first have the following result:

Proposition 9.12. *The easy quantum groups $H_N\subset G\subset O_N^+$ are as follows,*

$$\begin{tikzcd}
O_N \arrow[r] & O_N^* \arrow[r] & O_N^+ \\
H_N \arrow[u] \arrow[r, dotted] & H_N^{[\infty]} \arrow[r, dotted] & H_N^+ \arrow[u]
\end{tikzcd}$$

with the dotted arrows indicating that we have intermediate quantum groups there.

Proof. This is a key result in the classification of easy quantum groups:

(1) The first dichotomy, $O_N\subset G\subset O_N^+$ vs. $H_N\subset G\subset H_N^+$, comes from the early classification results, from [17], [32], [33]. In addition, these results solve as well the first problem, $O_N\subset G\subset O_N^+$, with $G=O_N^*$ being the unique non-trivial solution.

(2) The second dichotomy, $H_N\subset G\subset H_N^{[\infty]}$ vs. $H_N^{[\infty]}\subset G\subset H_N^+$, comes from various papers, and more specifically from the final classification paper [78], where the quantum groups $S_N\subset G\subset H_N^+$ with $G\not\subset H_N^{[\infty]}$ were classified, and shown to contain $H_N^{[\infty]}$. $\square$

Regarding now the case $H_N^{[\infty]}\subset G\subset H_N^+$, the precise result here, from [78], is:

Proposition 9.13. *Let $H_N^{[r]}\subset H_N^+$ be the easy quantum group coming from:*

$$\pi_r=\ker\begin{pmatrix}1&\ldots&r&r&\ldots&1\\1&\ldots&r&r&\ldots&1\end{pmatrix}$$

We have then inclusions of quantum groups as follows,

$$H_N^+=H_N^{[1]}\supset H_N^{[2]}\supset H_N^{[3]}\supset\ldots\ldots\supset H_N^{[\infty]}$$

and we obtain in this way all the intermediate easy quantum groups

$$H_N^{[\infty]}\subset G\subset H_N^+$$

satisfying the assumption $G\neq H_N^{[\infty]}$.

Proof. Once again, this is something technical, and we refer here to [78]. $\square$

It remains to discuss the easy quantum groups $H_N\subset G\subset H_N^{[\infty]}$, with the endpoints $G=H_N,H_N^{[\infty]}$ included. Once again, we follow here [78]. First, we have:

Definition 9.14. *A discrete group generated by real reflections, $g_i^2=1$,*

$$\Gamma=<g_1,\ldots,g_N>$$

is called uniform if each $\sigma\in S_N$ produces a group automorphism, $g_i\to g_{\sigma(i)}$.

Given a uniform reflection group $\mathbb{Z}_2^{*N} \to \Gamma \to \mathbb{Z}_2^N$, we can associate to it a family of subsets $D(k,l) \subset P(k,l)$, which form a category of partitions, as follows:

$$D(k,l) = \left\{ \pi \in P(k,l) \middle| \ker\binom{i}{j} \leq \pi \implies g_{i_1} \ldots g_{i_k} = g_{j_1} \ldots g_{j_l} \right\}$$

Observe that we have $P_{even}^{[\infty]} \subset D \subset P_{even}$, with the inclusions coming respectively from $\eta \in D$, and from $\Gamma \to \mathbb{Z}_2^N$. Conversely, given a category of partitions $P_{even}^{[\infty]} \subset D \subset P_{even}$, we can associate to it a uniform reflection group $\mathbb{Z}_2^{*N} \to \Gamma \to \mathbb{Z}_2^N$, as follows:

$$\Gamma = \left\langle g_1, \ldots g_N \middle| g_{i_1} \ldots g_{i_k} = g_{j_1} \ldots g_{j_l}, \forall i,j,k,l, \ker\binom{i}{j} \in D(k,l) \right\rangle$$

As explained in [78], the correspondences $\Gamma \to D$ and $D \to \Gamma$ are bijective, and inverse to each other, at $N = \infty$. We have in fact the following result, from [78]:

Proposition 9.15. *We have correspondences between:*

1. *Uniform reflection groups $\mathbb{Z}_2^{*\infty} \to \Gamma \to \mathbb{Z}_2^\infty$.*
2. *Categories of partitions $P_{even}^{[\infty]} \subset D \subset P_{even}$.*
3. *Easy quantum groups $G = (G_N)$, with $H_N^{[\infty]} \supset G_N \supset H_N$.*

Proof. As an illustration, if we denote by $\mathbb{Z}_2^{\circ N}$ the quotient of $\mathbb{Z}_2^{*N}$ by the relations of type $abc = cba$ between the generators, we have the following correspondences:

$$\begin{tikzcd}
\mathbb{Z}_2^N \arrow[d, leftrightsquigarrow] & \mathbb{Z}_2^{\circ N} \arrow[l] \arrow[d, leftrightsquigarrow] & \mathbb{Z}_2^{*N} \arrow[l] \arrow[d, leftrightsquigarrow] \\
H_N \arrow[r] & H_N^* \arrow[r] & H_N^{[\infty]}
\end{tikzcd}$$

More generally, for any $s \in \{2, 4, \ldots, \infty\}$, the quantum groups $H_N^{(s)} \subset H_N^{[s]}$ constructed in [17] come from the quotients of $\mathbb{Z}_2^{\circ N} \leftarrow \mathbb{Z}_2^{*N}$ by the relations $(ab)^s = 1$. See [78]. $\square$

We can now formulate a final classification result, as follows:

Theorem 9.16. *The easy quantum groups $H_N \subset G \subset O_N^+$ are as follows,*

$$\begin{tikzcd}
O_N \arrow[rr] & & O_N^* \arrow[rr] & & O_N^+ \\
H_N \arrow[u] \arrow[r] & H_N^\Gamma \arrow[r] & H_N^{[\infty]} \arrow[r] & H_N^{[r]} \arrow[r] & H_N^+ \arrow[u]
\end{tikzcd}$$

with the family H_N^Γ covering $H_N, H_N^{[\infty]}$, and with the series $H_N^{[r]}$ covering H_N^+.

Proof. This follows indeed from Proposition 9.12, Proposition 9.13 and Proposition 9.15 above. For further details, we refer to the paper of Raum and Weber [78]. □

As an application of the above results, we can fully classify the Schur-Weyl twists as well, in the orthogonal case. Here are some basic examples of such twists:

Proposition 9.17. *$\bar{O}_N, \bar{O}_N^* \subset O_N^+$ are obtained respectively by imposing the relations*

$$ab = \begin{cases} -ba & \text{for } a \neq b \text{ on the same row or column of } u \\ ba & \text{otherwise} \end{cases}$$

$$abc = \begin{cases} -cba & \text{for } r \leq 2, s = 3 \text{ or } r = 3, s \leq 2 \\ cba & \text{for } r \leq 2, s \leq 2 \text{ or } r = s = 3 \end{cases}$$

where $r, s \in \{1,2,3\}$ are the number of rows/columns of u spanned by $a, b, c \in \{u_{ij}\}$.

Proof. Assume that a compact quantum group $G \subset O_N^+$ appears via a relation of the following type, for a certain partition $\pi \in P(k,l)$:

$$T_\pi \in Hom(u^{\otimes k}, u^{\otimes l})$$

According to our general theory of Schur-Weyl twisting, from section 7 above, its twist $\bar{G} \subset O_N^+$ must appear then via the following relation:

$$\bar{T}_\pi \in Hom(u^{\otimes k}, u^{\otimes l})$$

We conclude that the quantum groups $\bar{O}_N, \bar{O}_N^*$ appear respectively via the relations $\bar{T}_{\slash\!\!\!\backslash} \in End(u^{\otimes 2})$, $\bar{T}_{\slash\!\!\!\backslash\!\!|} \in End(u^{\otimes 3})$, and the result follows. □

We will show that $\bar{O}_N, \bar{O}_N^*$ are in fact the only possible twists. First, we have:

Proposition 9.18. *The basic quantum groups $H_N \subset G \subset H_N^+$, namely*

$$H_N \subset H_N^* \subset H_N^{[\infty]} \subset H_N^+$$

are equal to their own twists.

Proof. We know that the corresponding categories of partitions are:

$$P_{even} \supset P_{even}^* \supset P_{even}^{[\infty]} \supset NC_{even}$$

With this observation in hand, the proof goes as follows:

(1) H_N^+. We know that for $\pi \in NC_{even}$ we have $\bar{T}_\pi = T_\pi$, and since we are in the situation $D \subset NC_{even}$, the definitions of $G, \bar{G}$ coincide.

(2) $H_N^{[\infty]}$. Here we can use the same argument as in (1), based this time on the description of $P_{even}^{[\infty]}$ found in Proposition 9.11 above.

(3) H_N^*. We have $H_N^* = H_N^{[\infty]} \cap O_N^*$, so $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the defining relations for $\bar{O}_N^*$. But all the $abc = -cba$ relations defining $\bar{H}_N^*$ are automatic,

of type $0 = 0$, and it follows that $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the relations $abc = cba$, for any $a, b, c \in \{u_{ij}\}$. Thus we have $\bar{H}_N^* = H_N^{[\infty]} \cap O_N^* = H_N^*$, as claimed.

(4) H_N. We have $H_N = H_N^* \cap O_N$, and by functoriality, $\bar{H}_N = \bar{H}_N^* \cap \bar{O}_N = H_N^* \cap \bar{O}_N$. But this latter intersection is equal to H_N, as claimed. □

In order to investigate now the general case, we need to establish the precise relation between the maps $T_\pi, \bar{T}_\pi$. As an example here, we have:

$$\bar{T}_{\times} = -T_{\times} + 2T_{\ker\binom{aa}{aa}}$$

$$\bar{T}_{\times\!\!\!\!|} = -\bar{T}_{\times\!\!\!\!|} + 2T_{\ker\binom{aab}{baa}} + 2T_{\ker\binom{aba}{aba}} + 2T_{\ker\binom{baa}{aab}} - 4T_{\ker\binom{aaa}{aaa}}$$

In general, the answer comes from the Möbius inversion formula. We recall that the Möbius function of any lattice, and in particular of P_{even}, is given by:

$$\mu(\sigma, \pi) = \begin{cases} 1 & \text{if } \sigma = \pi \\ -\sum_{\sigma \leq \tau < \pi} \mu(\sigma, \tau) & \text{if } \sigma < \pi \\ 0 & \text{if } \sigma \not\leq \pi \end{cases}$$

With this notation, we have the following result:

Proposition 9.19. *For any partition $\pi \in P_{even}$ we have the formula*

$$\bar{T}_\pi = \sum_{\tau \leq \pi} \alpha_\tau T_\tau$$

where $\alpha_\sigma = \sum_{\sigma \leq \tau \leq \pi} \varepsilon(\tau)\mu(\sigma, \tau)$, with μ being the Möbius function of P_{even}.

Proof. The linear combinations $T = \sum_{\tau \leq \pi} \alpha_\tau T_\tau$ acts on tensors as follows:

$$\begin{aligned} T(e_{i_1} \otimes \ldots \otimes e_{i_k}) &= \sum_{\tau \leq \pi} \alpha_\tau T_\tau(e_{i_1} \otimes \ldots \otimes e_{i_k}) \\ &= \sum_{\tau \leq \pi} \alpha_\tau \sum_{\sigma \leq \tau} \sum_{j : \ker\binom{i}{j} = \sigma} e_{j_1} \otimes \ldots \otimes e_{j_l} \\ &= \sum_{\sigma \leq \pi} \left(\sum_{\sigma \leq \tau \leq \pi} \alpha_\tau \right) \sum_{j : \ker\binom{i}{j} = \sigma} e_{j_1} \otimes \ldots \otimes e_{j_l} \end{aligned}$$

Thus, in order to have $\bar{T}_\pi = \sum_{\tau \leq \pi} \alpha_\tau T_\tau$, we must have, for any $\sigma \leq \pi$:

$$\varepsilon(\sigma) = \sum_{\sigma \leq \tau \leq \pi} \alpha_\tau$$

But this problem can be solved by using the Möbius inversion formula, and we obtain the numbers $\alpha_\sigma = \sum_{\sigma \leq \tau \leq \pi} \varepsilon(\tau)\mu(\sigma, \tau)$ in the statement. □

Observe that the above formula fits with the examples given above.

Now back to the general twisting problem, the answer here is:

Proposition 9.20. *The twists of the easy quantum groups $H_N \subset G \subset O_N^+$ are:*

1. *For $G = O_N, O_N^*$ we obtain $\bar{G} = \bar{O}_N, \bar{O}_N^*$.*
2. *For $G \neq O_N, O_N^*$ we have $G = \bar{G}$.*

Proof. We use the classification result in Theorem 9.16 above. We have to examine the 3 cases left, namely $G = O_N^+, H_N^{[r]}, H_N^\Gamma$, and the proof goes as follows:

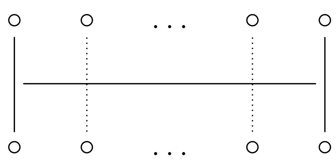
(1) Let $G = O_N^+$. We know that for $\pi \in NC_{even}$ we have $\bar{T}_\pi = T_\pi$, and since we are in the situation $D \subset NC_{even}$, the definitions of $G, \bar{G}$ coincide.

(2) Let $G = H_N^{[r]}$. We know that the generating partition is:

$$\pi_r = \ker \begin{pmatrix} 1 & \dots & r & r & \dots & 1 \\ 1 & \dots & r & r & \dots & 1 \end{pmatrix}$$

By symmetry, putting this partition in noncrossing form requires the same number of upper switches and lower switches, and so requires an even number of total switches. Thus π_r is even, and the same argument shows in fact that all its subpartitions are even as well. It follows that we have $T_{\pi_k} = \bar{T}_{\pi_k}$, and this gives the result.

(3) Let $G = H_N^\Gamma$. We denote by $P_{even}^{[\infty]} \subset D \subset P_{even}$ the corresponding category of partitions. According to the description of $P_{even}^{[\infty]}$ worked out above, this category contains the following type of partition:



The point now is that, by "capping" with such partitions, we can merge any pair of blocks of $\pi \in D$, by staying inside D. Thus, D has the following property:

$$\tau \leq \pi \in D \implies \tau \in D$$

We deduce from this that $\bar{T}_\pi$ is an intertwiner for G, and so $G \subset \bar{G}$. By symmetry we must have $\bar{G} \subset G$ as well, and this finishes the proof. $\square$

We can now formulate a final result in the orthogonal case, regarding the easy quantum groups, their twists, and the intersections as well, as follows:

Theorem 9.21. *The easy quantum groups $H_N \subset G \subset O_N^+$ and their twists are*

$$\begin{tikzcd}
 & O_N \arrow[r] & O_N^* \arrow[dr] & \\
H_N \arrow[ur] \arrow[r] \arrow[dr] & H_N^\Gamma \arrow[r] & H_N^{\diamond k} \arrow[r] & O_N^+ \\
 & \bar{O}_N \arrow[r] & \bar{O}_N^* \arrow[ur] &
\end{tikzcd}$$

and the set formed by these quantum groups is stable by intersections.

Proof. The first assertion follows from the above results. Regarding now the intersection assertion, the point is that we have the following intersection diagram:

$$\begin{tikzcd}
 & O_N \arrow[r] & O_N^* \arrow[dr] & \\
H_N \arrow[ur] \arrow[r] \arrow[dr] & H_N^* \arrow[ur] \arrow[r] \arrow[dr] & H_N^+ \arrow[r] & O_N^+ \\
 & \bar{O}_N \arrow[r] & \bar{O}_N^* \arrow[ur] &
\end{tikzcd}$$

More precisely, this diagram has the property that any intersection $G \cap H$ appears on the diagram, as the biggest quantum group contained in both G, H.

With this diagram in hand, the assertion follows. Indeed, the intersections between the quantum groups $O_N^\times$ are their twists are all on this diagram, and hence on the diagram in the statement as well. Regarding now the intersections of an easy quantum group $H_N \subset G \subset H_N^+$ with the twists $\bar{O}_N, \bar{O}_N^*$, we can use again the above diagram. Indeed, from $H_N^+ \cap \bar{O}_N^* = H_N^*$ we deduce that both $K = G \cap \bar{O}_N, K' = G \cap \bar{O}_N^*$ appear as intermediate easy quantum groups $H_N \subset K^\times \subset H_N^*$, and we are done. $\square$

Getting back now to the reflection groups, we have the following result, from [78]:

Proposition 9.22. *The easy quantum groups $H_N \subset G_N \subset H_N^+$, and the corresponding diagonal tori, are as follows,*

$$\begin{tikzcd}
H_N \arrow[r] & H_N^\Gamma \arrow[r] & H_N^{[\infty]} \arrow[r] & H_N^{[r]} \arrow[r] & H_N^+ \\
T_N \arrow[u] \arrow[r] & \widehat{\Gamma} \arrow[u] \arrow[r] & T_N^+ \arrow[u] \arrow[r] & T_N^+ \arrow[u] \arrow[r] & T_N^+ \arrow[u]
\end{tikzcd}$$

with the family H_N^Γ and the series $H_N^{[r]}$ being constructed as above.

Proof. The classification result follows by combining the above results, and the assertion about the diagonal tori is clear from definitions. See [78]. □

With these results in hand, we can go back now to our hard liberation questions. Regarding the quantum groups of type H_N^Γ, we have here the following result:

Theorem 9.23. *The quantum groups H_N^Γ appear via hard liberation, as follows:*

$$H_N^\Gamma =< H_N, \widehat{\Gamma} >$$

In particular, we have the "master formula" $H_N^{[\infty]} =< H_N, T_N^+ >$.

Proof. We use the basic fact, from [78], and which is complementary to the easiness considerations above, that we have a crossed product decomposition as follows:

$$H_N^\Gamma = \widehat{\Gamma} \rtimes S_N$$

With this result in hand, we obtain that we have the missing inclusion, namely:

$$H_N^\Gamma \ =< S_N, \widehat{\Gamma} > \subset < H_N, \widehat{\Gamma} >$$

Finally, the last assertion is clear, by taking $\Gamma = \mathbb{Z}_2^{*N}$. Indeed, this group produces $H_N^{[\infty]}$, and the corresponding group dual is the free real torus T_N^+. □

Let us discuss now the general complex case. We first have:

Proposition 9.24. *We have easy quantum groups $K_N^\times$ as follows,*

$$\begin{array}{ccccccccc}
K_N & \rightarrow & K_N^\Gamma & \rightarrow & K_N^{[\infty]} & \rightarrow & K_N^{[r]} & \rightarrow & K_N^+\\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow\\
H_N & \rightarrow & H_N^\Gamma & \rightarrow & H_N^{[\infty]} & \rightarrow & H_N^{[r]} & \rightarrow & H_N^+
\end{array}$$

obtained by soft intermediate liberation, $K_N^\times =< K_N, H_N^\times >$.

Proof. This is more of an empty statement, with perhaps the only thing to be justified being the fact that $K_N, K_N^{[\infty]}, K_N^+$, which are already known, appear indeed via soft liberation. But this latter fact follows by interesting categories, with input from [1], [85]. □

In relation now with our hard liberation questions, we first have:

Proposition 9.25. *The diagonal tori of the quantum groups $K_N^\times$ are as follows,*

$$\begin{tikzcd}
K_N \arrow[r] & K_N^\Gamma \arrow[r] & K_N^{[\infty]} \arrow[r] & K_N^{[r]} \arrow[r] & K_N^+ \\
\mathbb{T}_N \arrow[r] \arrow[u] & \widehat{\Gamma}_c \arrow[r] \arrow[u] & \mathbb{T}_N^+ \arrow[r] \arrow[u] & \mathbb{T}_N^+ \arrow[r] \arrow[u] & \mathbb{T}_N^+ \arrow[u]
\end{tikzcd}$$

with $\Gamma \to \Gamma_c$ being a certain complexification operation, satisfying $< \mathbb{T}_N, \widehat{\Gamma} > \subset \widehat{\Gamma}_c$.

Proof. As a first observation, the results are clear and well-known for the endpoints K_N, K_N^+ and for the middle point $K_N^{[\infty]}$ as well. Indeed, these are known quantum groups.

By functoriality it follows that the diagonal torus of $K_N^{[r]}$ must be the free complex torus $\mathbb{T}_N^+$, for any $r \in \mathbb{N}$, so we are done with the right part of the diagram.

Regarding now the left part of the diagram, concerning the quantum groups K_N^Γ, if we denote by $T_1(.)$ the diagonal torus, we have:

$$\begin{aligned}
T_1(K_N^\Gamma) &= T_1(< K_N, H_N^\Gamma >) \\
&\supset < T_1(K_N), T_1(H_N^\Gamma) > \\
&= < \mathbb{T}_N, \widehat{\Gamma} >
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Observe that the above inclusion $< \mathbb{T}_N, \widehat{\Gamma} > \subset \widehat{\Gamma}_c$ fails to be an isomorphism, and this for instance for $\Gamma = \mathbb{Z}_2^{*N}$. However, the construction $\Gamma \to \Gamma_c$ can be in principle explicitely computed, for instance by using Tannakian methods. Indeed, our soft liberation formula $K_N^\Gamma =< K_N, H_N^\Gamma >$ translates into a Tannakian formula, as follows:

$$\mathcal{P}_{even}^\Gamma = \mathcal{P}_{even} \cap P_{even}^\Gamma$$

The problem is that of explicitely computing the category on the left, corresponding to K_N^Γ, and then of deducing from this a presentation formula for the associated diagonal torus $\widehat{\Gamma}_c$. Now back to the hard liberation question, we have the following result:

Theorem 9.26. *The quantum groups K_N^Γ appear via hard liberation, and this even in a stronger form, as follows:*

$$K_N^\Gamma =< K_N, \widehat{\Gamma} >$$

In particular, we have the formula $K_N^{[\infty]} =< K_N, T_N^+ >$.

Proof. This follows from the above results. Indeed, we have:

$$\begin{aligned} K_N^\Gamma &= < K_N, H_N^\Gamma > \\ &= < K_N, H_N, \widehat{\Gamma} > \\ &= < K_N, \widehat{\Gamma} > \end{aligned}$$

Thus we have the formula in the statement, and the fact that this implies the fact that K_N^Γ appears indeed via hard liberation follows from the above results.

Finally, with $\Gamma = \mathbb{Z}_2^{*N}$ we obtain from this the formula $K_N^{[\infty]} =< K_N, T_N^+ >$. □

Regarding the hard liberation question for the quantum groups $K_N^{[r]}$, the problem here is open, the difficulties being similar to those for the quantum groups $H_N^{[r]}$.

10. Orbits, orbitals

We have seen so far that the quantum permutation groups S_N^+, as well as some of their subgroups, such as the quantum reflection ones H_N^{s+}, can be quite well understood.

In this section and in the next two ones we discuss the structure of the arbitrary subgroups $G\subset S_N^+$, with a number of general results on the subject.

We will first discuss, here in this section, the notions of orbits and orbitals, with some general results, and then with applications to toral subgroup questions.

The notions of orbits, and of transitivity, for the subgroups $G\subset S_N^+$ go back to Bichon's paper [37]. Bichon constructed there the orbits, and used them for classifying the group dual subgroups $\widehat{\Gamma}\subset S_N^+$. We will explain here this material. Let us start with:

Theorem 10.1. *Given a closed subgroup $G\subset S_N^+$, with standard coordinates denoted $u_{ij}\in C(G)$, the following defines an equivalence relation on $\{1,\ldots,N\}$,*

$$i\sim j\iff u_{ij}\neq 0$$

that we call orbit decomposition associated to the action $G\curvearrowright\{1,\ldots,N\}$. In the classical case, $G\subset S_N$, this is the usual orbit equivalence coming from the action of G.

Proof. We first check the fact that we have indeed an equivalence relation:

(1) The reflexivity axiom $i\sim i$ follows by using the counit, as follows:

$$\begin{aligned}\varepsilon(u_{ij})=\delta_{ij}&\implies\varepsilon(u_{ii})=1\\&\implies u_{ii}\neq 0\end{aligned}$$

(2) The symmetry axiom $i\sim j\implies j\sim i$ follows by using the antipode:

$$S(u_{ij})=u_{ji}\implies[u_{ij}\neq 0\implies u_{ji}\neq 0]$$

(3) As for the transitivity axiom $i\sim j,j\sim k\implies i\sim k$, this follows by using the comultiplication. Consider indeed the following formula:

$$\Delta(u_{ik})=\sum_j u_{ij}\otimes u_{jk}$$

On the right we have a sum of projections, and we obtain from this:

$$\begin{aligned}u_{ij}\neq 0,u_{jk}\neq 0&\implies u_{ij}\otimes u_{jk}>0\\&\implies\Delta(u_{ik})>0\\&\implies u_{ik}\neq 0\end{aligned}$$

Finally, in the classical case, where $G\subset S_N$, the standard coordinates are the characteristic functions $u_{ij}=\chi(\sigma\in G|\sigma(j)=i)$. Thus the condition $u_{ij}\neq 0$ is equivalent to the existence of an element $\sigma\in G$ such that $\sigma(j)=i$, and this means precisely that i,j must be in the same orbit under the action of G, as claimed. $\square$

Generally speaking, the theory from the classical case extends well to the quantum group setting, and we have in particular the following result, also from [37]:

Theorem 10.2. *Given a closed subgroup $G\subset S_N^+$, consider the associated coaction*

$$\Phi:C(X)\to C(X)\otimes C(G)\quad,\quad \Phi(e_i)=\sum_j e_j\otimes u_{ji}$$

where $X=\{1,\ldots,N\}$. The fixed point algebra of this coaction is then given by

$$Fix(\Phi)=\left\{\xi\in C(X)\Big|i\sim j\implies \xi(i)=\xi(j)\right\}$$

where $\sim$ is the orbit decomposition constructed in Theorem 10.1.

Proof. Consider the fixed point algebra of the coaction in the statement:

$$Fix(\Phi)=\left\{\xi\in\mathbb C^N\Big|\Phi(\xi)=\xi\otimes 1\right\}$$

By doing a number of manipulations, we obtain the result. See [37]. □

We can derive some explicit consequences of the above result, of representation theory flavor, by using the standard fact that the fixed point space of a corepresentation coincides with the fixed point space of the associated coaction. Indeed, this gives:

$$Fix(\Phi)=Fix(u)$$

As a first consequence, we have an algebraic result, as follows:

Proposition 10.3. *Given a closed subgroup $G\subset S_N^+$, we have*

$$Fix(u)=\left\{\xi\in\mathbb C^N\Big|i\sim j\implies \xi_i=\xi_j\right\}$$

and in particular, with $F=Fix(u)$, the following happen:

1. $\dim(F)=\sum N_i^2$*, where* $N=\sum N_i$ *is the orbit decomposition.*
2. $\dim(Z(F))$ *is the number of orbits of the action* $G\curvearrowright\{1,\ldots,N\}$*.*

Proof. This follows indeed from Theorem 10.2 above, by using the above-mentioned identification $Fix(\Phi)=Fix(u)$, relating fixed points of actions and representations. □

We have as well a useful analytic result, as follows:

Theorem 10.4. *Given a closed subgroup $G\subset S_N^+$, consider the following matrix:*

$$P_{ij}=\int_G u_{ij}$$

Then P is the orthogonal projection onto the linear space

$$F=\left\{\xi\in\mathbb C^N\Big|i\sim j\implies \xi_i=\xi_j\right\}$$

and so the orbits and their sizes can be deduced from the knowledge of P.

Proof. This follows from the above results, and from the standard fact, coming from the Peter-Weyl theory, that P is the orthogonal projection onto $Fix(u)$. □

There are of course many explicit formulae that can be deduced from Theorem 10.4, and we will work out some of them in the next section, in connection with the transitive case, the idea being that $G \subset S_N^+$ is transitive precisely when the following happens:

$$\int_G u_{ij} = \frac{1}{N}$$

As another comment, the result in Theorem 10.4 makes it clear that the various notions in relation with the orbit decomposition, coming from Theorem 10.1, in the quantum permutation group case, $G \subset S_N^+$, can be normally extended, for instance by using an analytic approach, to the general quantum symmetry group case:

$$G \subset S_X^+$$

There is quite some work to be done here, but instead of getting into this subject, which is quite technical, let us stay with the usual quantum permutation groups, $G \subset S_N^+$, and try to understand if a theory of higher orbitals for them can be developed.

Let us start with the following standard definition, from the classical case:

Proposition 10.5. *Given a subgroup $G \subset S_N$, consider its magic unitary $u = (u_{ij})$, given by $u_{ij} = \chi\{\sigma \in G|\sigma(j) = i\}$. The following conditions are then equivalent:*

(1) $\sigma(i_1) = j_1, \ldots, \sigma(i_k) = j_k$, *for some* $\sigma \in G$.
(2) $u_{i_1j_1} \ldots u_{i_kj_k} \neq 0$.

These conditions produce an equivalence relation $(i_1, \ldots, i_k) \sim (j_1, \ldots, j_k)$, and the corresponding equivalence classes are the k-orbitals of G.

Proof. The fact that we have indeed an equivalence as in the statement, which produces an equivalence relation, is indeed clear from definitions. □

In the quantum case, the situation is more complicated. We follow the approach to the orbits and orbitals developed in [37], [67], and in [74] as well. We first have:

Proposition 10.6. *Let $G \subset S_N^+$ be a closed subgroup, with magic unitary $u = (u_{ij})$, and let $k \in \mathbb{N}$. The relation $(i_1, \ldots, i_k) \sim (j_1, \ldots, j_k)$ when $u_{i_1j_1} \ldots u_{i_kj_k} \neq 0$ is:*

(1) *Reflexive.*
(2) *Symmetric.*
(3) *Transitive at $k = 1, 2$.*

Proof. This is basically known from [37], [67], [74], the proof being as follows:

(1) This simply follows by using the counit:

$$\begin{aligned}\varepsilon(u_{i_ri_r})=1,\forall r &\implies \varepsilon(u_{i_1i_1}\ldots u_{i_ki_k})=1\\ &\implies u_{i_1i_1}\ldots u_{i_ki_k}\neq 0\\ &\implies (i_1,\ldots,i_k)\sim(i_1,\ldots,i_k)\end{aligned}$$

(2) This follows by applying the antipode, and then the involution:

$$\begin{aligned}(i_1,\ldots,i_k)\sim(j_1,\ldots,j_k) &\implies u_{i_1j_1}\ldots u_{i_kj_k}\neq 0\\ &\implies u_{j_ki_k}\ldots u_{j_1i_1}\neq 0\\ &\implies u_{j_1i_1}\ldots u_{j_ki_k}\neq 0\\ &\implies (j_1,\ldots,j_k)\sim(i_1,\ldots,i_k)\end{aligned}$$

(3) This is something more tricky. We need to prove that we have:

$$u_{i_1j_1}\ldots u_{i_kj_k}\neq 0\ ,\ u_{j_1l_1}\ldots u_{j_kl_k}\neq 0\implies u_{i_1l_1}\ldots u_{i_kl_k}\neq 0$$

In order to do so, we use the following formula:

$$\Delta(u_{i_1l_1}\ldots u_{i_kl_k})=\sum_{s_1\ldots s_k}u_{i_1s_1}\ldots u_{i_ks_k}\otimes u_{s_1l_1}\ldots u_{s_kl_k}$$

At $k=1$ the result is clear, because on the right we have a sum of projections, which is therefore strictly positive when one of these projections is nonzero.

At $k=2$ now, the result follows from the following trick, from [67]:

$$\begin{aligned}&(u_{i_1j_1}\otimes u_{j_1l_1})\Delta(u_{i_1l_1}u_{i_2l_2})(u_{i_2j_2}\otimes u_{j_2l_2})\\ =&\sum_{s_1s_2}u_{i_1j_1}u_{i_1s_1}u_{i_2s_2}u_{i_2j_2}\otimes u_{j_1l_1}u_{s_1l_1}u_{s_2l_2}u_{j_2l_2}\\ =&\ u_{i_1j_1}u_{i_2j_2}\otimes u_{j_1l_1}u_{j_2l_2}\end{aligned}$$

Indeed, we obtain from this that we have $u_{i_1l_1}u_{i_2l_2}\neq 0$, as desired. □

In view of the results that we have so far, we can formulate:

Definition 10.7. *Given a closed subgroup $G\subset S_N^+$, consider the relation $\sim_k$ defined by $(i_1,\ldots,i_k)\sim(j_1,\ldots,j_k)$ when $u_{i_1j_1}\ldots u_{i_kj_k}\neq 0$.*

1. *The equivalence classes with respect to $\sim_1$ are called orbits of G.*
2. *The equivalence classes with respect to $\sim_2$ are called orbitals of G.*

In the case where $\sim_k$ with $k\geq 3$ happens to be transitive, and so is an equivalence relation, we call its equivalence classes the algebraic k-orbitals of G.

Generally speaking, examples and counterexamples for all this can be found by using group duals. Regarding now the quantum permutation group S_N^+ itself, we have here:

Theorem 10.8. *For the quantum permutation group S_N^+, with $N\geq 4$, we have*

$$(i_1,\ldots,i_k)\sim(j_1,\ldots,j_k)\iff\begin{cases}i_1=i_2\iff j_1=j_2\\ i_2=i_3\iff j_2=j_3\\ \ldots\\ i_{k-1}=i_k\iff j_{k-1}=j_k\end{cases}$$

and so $\sim$ is an equivalence relation, at any $k\in\mathbb{N}$. The number of orbits is 2^{k-1}.

Proof. The implication $\implies$ is clear, because if one of the conditions on the right does not hold, we have $u_{i_1j_1}\ldots u_{i_kj_k}=0$, due to a cancellation between consecutive terms.

Conversely now, we have to show that a vanishing formula of type $u_{i_1j_1}\ldots u_{i_kj_k}=0$ can only come from "trivial reasons", as in the statement. But this follows by using group duals, and more specifically by using an embedding as follows:

$$\widehat{\mathbb{Z}_2*\mathbb{Z}_2}\subset S_4^+\subset S_N^+$$

Finally, the last assertion is clear, because when counting the orbits for $\sim$, at the level of the pairs (i_1i_2) we have one binary choice to be made, namely $i_1=i_2$ vs. $i_2\neq i_2$, then for the pairs (i_2i_3) we have another binary choice, and so on up to a final binary choice, for $(i_{k-1}i_k)$. Thus, we have $k-1$ binary choices, and so 2^{k-1} orbits. $\square$

As an interesting consequence, the algebraic 3-orbitals differ for S_N and S_N^+:

Proposition 10.9. *The algebraic 3-orbitals for S_N and S_N^+ are as follows:*

(1) *For S_N we have 5 such orbitals, corresponding to* ⊓⊓, ⊓|, |⊓, ⊓̣, |||.
(2) *For S_N^+ we have 4 such orbitals, corresponding to* ⊓⊓, ⊓|, |⊓, ⊓̣|||.

Proof. For the symmetric group S_N, it follows from definitions that the k-orbitals are indexed by the partitions $\pi\in P(k)$, as follows:

$$C_\pi=\left\{(i_1,\ldots,i_k)\Big|\ker i=\pi\right\}$$

Regarding now S_N^+, the k-orbitals are those computed above, and at $k=3$ they can be naturally indexed by the above diagrams, with the last one standing for the fact that the corresponding 3-orbital merges the ⊓̣ and ||| 3-orbitals from the classical case. $\square$

Let us discuss now an analytic approach to all this. We first have:

Proposition 10.10. *For a subgroup $G\subset S_N$, which fundamental corepresentation denoted $u=(u_{ij})$, the following numbers are equal:*

(1) *The number of k-orbitals.*
(2) *The dimension of space $Fix(u^{\otimes k})$.*
(3) *The number $\int_G\chi^k$, where $\chi=\sum_i u_{ii}$.*

Proof. This is well-known, the proof being as follows:

(1) = (2) Consider an element $\sigma\in G$, and a vector, as follows:

$$\xi=\sum_{i_1\ldots i_k}\alpha_{i_1\ldots i_k}e_{i_1}\otimes\ldots\otimes e_{i_k}$$

We have then the following formulae:

$$\begin{aligned}\sigma^{\otimes k}\xi&=\sum_{i_1\ldots i_k}\alpha_{i_1\ldots i_k}e_{\sigma(i_1)}\otimes\ldots\otimes e_{\sigma(i_k)}\\ \xi&=\sum_{i_1\ldots i_k}\alpha_{\sigma(i_1)\ldots\sigma(i_k)}e_{\sigma(i_1)}\otimes\ldots\otimes e_{\sigma(i_k)}\end{aligned}$$

Thus $\sigma^{\otimes k}\xi=\xi$ holds for any $\sigma\in G$ precisely when α is constant on the k-orbitals of G, and this gives the equality between the numbers in (1) and (2).

(2) = (3) This follows from the Peter-Weyl theory, because $\chi=\sum_i u_{ii}$ is the character of the fundamental corepresentation u. □

In the general case now, $G\subset S_N^+$, by the general Peter-Weyl type results established by Woronowicz in [99], we still have the following formula:

$$\dim(Fix(u^{\otimes k}))=\int_G\chi^k$$

The problem is that of understanding the k-orbital interpretation of this number. We first have the following result, basically coming from [37], [67]:

Proposition 10.11. *Given a closed subgroup $G\subset S_N^+$, and a number $k\in\mathbb N$, consider the following linear space:*

$$F_k=\left\{\xi\in(\mathbb C^N)^{\otimes k}\Big|\xi_{i_1\ldots i_k}=\xi_{j_1\ldots j_k},\forall(i_1,\ldots,i_k)\sim(j_1,\ldots,j_k)\right\}$$

1. *We have $F_k\subset Fix(u^{\otimes k})$.*
2. *At $k=1,2$ we have $F_k=Fix(u^{\otimes k})$.*
3. *In the classical case, we have $F_k=Fix(u^{\otimes k})$.*
4. *For $G=S_N^+$ with $N\geq 4$ we have $F_3\neq Fix(u^{\otimes 3})$.*

Proof. The tensor power $u^{\otimes k}$ being the corepresentation $(u_{i_1,\ldots i_k,j_1\ldots j_k})_{i_1\ldots i_k,j_1\ldots j_k}$, the corresponding fixed point space $Fix(u^{\otimes k})$ consists of the vectors ξ satisfying:

$$\sum_{j_1\ldots j_k}u_{i_1j_1}\ldots u_{i_kj_k}\xi_{j_1\ldots j_k}=\xi_{i_1\ldots i_k}\quad,\quad\forall i_1,\ldots,i_k$$

With this formula in hand, the proof goes as follows:

(1) Assuming $\xi\in F_k$, the above fixed point formula holds indeed, because:

$$\sum_{j_1\ldots j_k}u_{i_1j_1}\ldots u_{i_kj_k}\xi_{j_1\ldots j_k}=\sum_{j_1\ldots j_k}u_{i_1j_1}\ldots u_{i_kj_k}\xi_{i_1\ldots i_k}=\xi_{i_1\ldots i_k}$$

(2) This is something more tricky, coming from the following formulae:

$$u_{ik}\left(\sum_j u_{ij}\xi_j-\xi_i\right)=u_{ik}(\xi_k-\xi_i)$$

$$u_{i_1k_1}\left(\sum_{j_1j_2}u_{i_1j_1}u_{i_2j_2}\xi_{j_1j_2}-\xi_{i_1i_2}\right)u_{i_2k_2}=u_{i_1k_1}u_{i_2k_2}(\xi_{k_1k_2}-\xi_{i_1i_2})$$

(3) This follows indeed from the results above.

(4) This follows from the results above, and from the representation theory of S_N^+ with $N\geq 4$, the dimensions of the two spaces involved being $4<5$. $\square$

The above considerations suggest formulating the following definition:

Definition 10.12. *Given a closed subgroup $G\subset U_N^+$, the integer*

$$\dim(Fix(u^{\otimes k}))=\int_G\chi^k$$

is called number of analytic k-orbitals.

We have the following result, which brings more support for our definition:

Proposition 10.13. *For $G\subset S_N^+$, and $k\leq 3$, the following are equivalent:*

(1) *G is k-transitive, in the sense that $Fix(u^{\otimes k})$ has dimension* $1,2,5$.

(2) *The k-th moment of the main character is $\int_G\chi^k=1,2,5$.*

(3) *$\int_G u_{i_1j_1}\ldots u_{i_kj_k}=\frac{(N-k)!}{N!}$ for distinct indices i_r and distinct indices j_r.*

(4) *$\int_G u_{i_1j_1}\ldots u_{i_kj_k}$ equals $\frac{(N-|\ker i|)!}{N!}$ when $\ker i=\ker j$, and equals 0, otherwise.*

Proof. Most of these implications are known since [4], the idea being as follows:

(1) $\iff$ (2) This follows from the Peter-Weyl type theory from [99], because the k-th moment of the character counts the number of fixed points of $u^{\otimes k}$.

(2) $\iff$ (3) This follows from the Schur-Weyl duality results for S_N,S_N^+ and from $P(k)=NC(k)$ at $k\leq 3$, as explained in [4].

(3) $\iff$ (4) Once again this follows from $P(k)=NC(k)$ at $k\leq 3$, and from a standard integration result for S_N, as explained in [4]. $\square$

As a conclusion to all these considerations, we have:

Theorem 10.14. *For a closed subgroup $G\subset S_N^+$, and an integer $k\in\mathbb N$, the number $\dim(Fix(u^{\otimes k}))=\int_G\chi^k$ of "analytic k-orbitals" has the following properties:*

(1) *In the classical case, this is the number of k-orbitals.*

(2) *In general, at $k=1,2$, this is the number of k-orbitals.*

(3) *At $k=3$, when this number is minimal, G is 3-transitive in the above sense.*

Proof. This follows indeed from the above considerations. $\square$

Let us discuss now some applications of all this material, following [31], [37], to general structure and classification questions for the compact quantum groups.

As a starting point, we have the following basic statement:

Proposition 10.15. *Let $G \subset U_N^+$ be a compact quantum group, and consider the group dual subgroups $\widehat{\Lambda} \subset G$, also called toral subgroups, or simply "tori".*

1. *In the classical case, where $G \subset U_N$ is a compact Lie group, these are the usual tori, where by torus we mean here closed abelian subgroup.*
2. *In the group dual case, $G = \widehat{\Gamma}$ with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group, these are the duals of the various quotients $\Gamma \to \Lambda$.*

Proof. Both these assertions are elementary, as follows:

(1) This follows indeed from the fact that a closed subgroup $H \subset U_N^+$ is at the same time classical, and a group dual, precisely when it is classical and abelian.

(2) This follows from the general propreties of the Pontrjagin duality, and more precisely from the fact that the subgroups $\widehat{\Lambda} \subset \widehat{\Gamma}$ correspond to the quotients $\Gamma \to \Lambda$. □

At a more concrete level now, most of the tori that we met appear as diagonal tori. Let us first review this material. We first have:

Proposition 10.16. *The diagonal torus $T \subset G$, which appears via the formula*

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

can be defined as well via the following intersection formula, inside U_N^+,

$$T = G \cap \mathbb{T}_N^+$$

where $\mathbb{T}_N^+ \subset U_N^+$ is the dual of the free group $F_N =< g_1, \ldots, g_N >$, with $u = diag(g_i)$.

Proof. According to our results above, the free torus $\mathbb{T}_N^+$ appears as follows:

$$C(\mathbb{T}_N^+) = C(U_N^+) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

Thus, by intersecting with G we obtain the diagonal torus of G. □

Most of our computations so far of diagonal tori concern various classes of easy quantum groups. In the general easy case, we have the following result:

Proposition 10.17. *For an easy quantum group $G \subset U_N^+$, coming from a category of partitions $D \subset P$, the associated diagonal torus is $T = \widehat{\Gamma}$, with:*

$$\Gamma = F_N \Big/ \left\langle g_{i_1} \ldots g_{i_k} = g_{j_1} \ldots g_{j_l} \Big| \forall i, j, k, l, \exists \pi \in D(k,l), \delta_\pi \begin{pmatrix} i \\ j \end{pmatrix} \neq 0 \right\rangle$$

Moreover, we can just use partitions π which generate the category D.

Proof. If we denote by $g_i = u_{ii}$ the standard coordinates on the associated diagonal torus T, then we have, with $g = diag(g_1, \ldots, g_N)$:

$$\begin{aligned}
C(T) &= \left[C(U_N^+)\Big/\left\langle T_\pi \in Hom(u^{\otimes k}, u^{\otimes l})\Big|\forall \pi \in D\right\rangle\right] \Big/ \left\langle u_{ij} = 0\Big|\forall i \neq j\right\rangle \\
&= \left[C(U_N^+)\Big/\left\langle u_{ij} = 0\Big|\forall i \neq j\right\rangle\right] \Big/ \left\langle T_\pi \in Hom(u^{\otimes k}, u^{\otimes l})\Big|\forall \pi \in D\right\rangle \\
&= C^*(F_N)\Big/\left\langle T_\pi \in Hom(g^{\otimes k}, g^{\otimes l})\Big|\forall \pi \in D\right\rangle
\end{aligned}$$

Thus, we obtain the formula in the statement. Finally, the last assertion follows from Tannakian duality, because we can replace everywhere D by a generating subset. □

In practice now, in the continuous case we have the following result:

Theorem 10.18. *The diagonal tori of the basic unitary quantum groups, namely*

$$\begin{array}{ccccc}
U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+ \\
\uparrow & & \uparrow & & \uparrow \\
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+
\end{array}$$

and of their $q = -1$ twists as well, are $T_N = \mathbb{Z}_2^N, \mathbb{T}_N = \mathbb{T}^N$ and their liberations:

$$\begin{array}{ccccc}
\mathbb{T}_N & \longrightarrow & \mathbb{T}_N^* & \longrightarrow & \mathbb{T}_N^+ \\
\uparrow & & \uparrow & & \uparrow \\
T_N & \longrightarrow & T_N^* & \longrightarrow & T_N^+
\end{array}$$

Also, for the quantum groups B_N, B_N^+, C_N, C_N^+, the diagonal torus collapses to $\{1\}$.

Proof. The main assertion, regarding the basic unitary quantum groups, is something that we already know, from section 1 above, with the various liberations $T_N^\times, \mathbb{T}_N^\times$ of the basic tori $T_N, \mathbb{T}_N$ in the statement being by definition those appearing there.

Regarding the invariance under twisting, this is best seen by using Proposition 10.17. Indeed, the computation in the proof there applies in the same way to the general quizzy case, and shows that the diagonal torus is invariant under twisting.

Finally, in the bistochastic case the fundamental corepresentation $g = diag(g_1, \ldots, g_N)$ of the diagonal torus must be bistochastic, and so $g_1 = \ldots = g_N = 1$, as claimed. □

Regarding now the discrete case, the result is as follows:

Theorem 10.19. *The diagonal tori of the basic quantum reflection groups, namely*

$$\begin{tikzcd} K_N \arrow[r] & K_N^* \arrow[r] & K_N^+ \\ H_N \arrow[r] \arrow[u] & H_N^* \arrow[r] \arrow[u] & H_N^+ \arrow[u] \end{tikzcd}$$

are the same as those for $O_N^\times, U_N^\times$, given above. Also, for S_N, S_N^+ we have $T = \{1\}$.

Proof. The first assertion follows from the general fact that the diagonal torus of $G_N \subset U_N^+$ equals the diagonal torus of the discrete version $G_N^d = G_N \cap K_N^+$, which follows from definitions. As for the second assertion, this follows from $S_N \subset B_N$, $S_N^+ \subset B_N^+$. □

As a conclusion, the diagonal torus $T \subset G$ is usually a quite interesting object, but for certain quantum groups like the bistochastic ones, or the quantum permutation group ones, this torus collapses to $\{1\}$, and so it cannot be of use in the study of G.

In order to deal with this issue, the idea, from [31], will be that of using:

Proposition 10.20. *Given a closed subgroup $G \subset U_N^+$ and a matrix $Q \in U_N$, we let $T_Q \subset G$ be the diagonal torus of G, with fundamental representation spinned by Q:*

$$C(T_Q) = C(G) \Big/ \left\langle (QuQ^*)_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T_Q = \widehat{\Lambda}_Q$, where $\Lambda_Q =< g_1, \ldots, g_N >$ is the discrete group generated by the elements $g_i = (QuQ^*)_{ii}$, which are unitaries inside $C(T_Q)$.

Proof. This follows indeed from our results, because, as said in the statement, T_Q is by definition a diagonal torus. Equivalently, since $v = QuQ^*$ is a unitary corepresentation, its diagonal entries $g_i = v_{ii}$, when regarded inside $C(T_Q)$, are unitaries, and satisfy:

$$\Delta(g_i) = g_i \otimes g_i$$

Thus $C(T_Q)$ is a group algebra, and more specifically we have $C(T_Q) = C^*(\Lambda_Q)$, where $\Lambda_Q =< g_1, \ldots, g_N >$ is the group in the statement, and this gives the result. □

Summarizing, associated to any closed subgroup $G \subset U_N^+$ is a whole family of tori, indexed by the unitaries $U \in U_N$. We use the following terminology:

Definition 10.21. *Let $G \subset U_N^+$ be a closed subgroup.*

(1) *The tori $T_Q \subset G$ constructed above are called standard tori of G.*
(2) *The collection of tori $T = \left\{ T_Q \subset G \middle| Q \in U_N \right\}$ is called skeleton of G.*

This might seem a bit awkward, but in view of various results, examples and counterexamples, to be presented below, this is perhaps the best terminology.

As a first general result regarding these tori, we have:

Theorem 10.22. *Any torus $T \subset G$ appears as follows, for a certain $Q \in U_N$:*

$$T \subset T_Q \subset G$$

In other words, any torus appears inside a standard torus.

Proof. Given a torus $T \subset G$, we have an inclusion $T \subset G \subset U_N^+$. On the other hand, we know that each torus $T = \widehat{\Lambda} \subset U_N^+$, coming from a discrete group $\Lambda =< g_1, \ldots, g_N >$, has a fundamental corepresentation as follows, with $Q \in U_N$:

$$u = Qdiag(g_1, \ldots, g_N)Q^*$$

But this shows that we have $T \subset T_Q$, and this gives the result. □

Let us do now some computations. In the classical case, the result is as follows:

Proposition 10.23. *For a closed subgroup $G \subset U_N$ we have*

$$T_Q = G \cap (Q^* \mathbb{T}^N Q)$$

where $\mathbb{T}^N \subset U_N$ is the group of diagonal unitary matrices.

Proof. This is indeed clear at $Q = 1$, where Γ_1 appears by definition as the dual of the compact abelian group $G \cap \mathbb{T}^N$. In general, this follows by conjugating by Q. □

In the group dual case now, we have the following result:

Proposition 10.24. *Given a discrete group $\Gamma =< g_1, \ldots, g_N >$, consider its dual compact quantum group $G = \widehat{\Gamma}$, diagonally embedded into U_N^+. We have then*

$$\Lambda_Q = \Gamma / < g_i = g_j | \exists k, Q_{ki} \neq 0, Q_{kj} \neq 0 >$$

with the embedding $T_Q \subset G = \widehat{\Gamma}$ coming from the quotient map $\Gamma \to \Lambda_Q$.

Proof. Assume indeed that $\Gamma =< g_1, \ldots, g_N >$ is a discrete group, with $\widehat{\Gamma} \subset U_N^+$ coming via $u = diag(g_1, \ldots, g_N)$. With $v = QuQ^*$, we have:

$$\begin{aligned}
\sum_s \bar{Q}_{si} v_{sk} &= \sum_{st} \bar{Q}_{si} Q_{st} \bar{Q}_{kt} g_t \\
&= \sum_t \delta_{it} \bar{Q}_{kt} g_t \\
&= \bar{Q}_{ki} g_i
\end{aligned}$$

Thus $v_{ij} = 0$ for $i \neq j$ gives $\bar{Q}_{ki} v_{kk} = \bar{Q}_{ki} g_i$, which is the same as saying that $Q_{ki} \neq 0$ implies $g_i = v_{kk}$. But this latter equality reads:

$$g_i = \sum_j |Q_{kj}|^2 g_j$$

We conclude from this that $Q_{ki} \neq 0, Q_{kj} \neq 0$ implies $g_i = g_j$, as desired. As for the converse, this is elementary to establish as well. □

According to the above results, we can expect the skeleton T to encode various algebraic and analytic properties of G. We first have the following result:

Theorem 10.25. *The following results hold, both over the category of compact Lie groups, and over the category of duals of finitely generated discrete groups:*

1. *Injectivity: the construction $G \to T$ is injective, in the sense that $G \neq H$ implies $T_Q(G) \neq T_Q(H)$, for some $Q \in U_N$.*
2. *Monotony: the construction $G \to T$ is increasing, in the sense that passing to a subgroup $H \subset G$ decreases at least one of the tori T_Q.*
3. *Generation: any closed quantum subgroup $G \subset U_N^+$ has the generation property $G =< T_Q | Q \in U_N >$. In other words, G is generated by its tori.*

Proof. In the classical case, where $G \subset U_N$, the proof is elementary, based on standard facts from linear algebra, and goes as follows:

(1) Injectivity. This follows from the generation statement, explained below.

(2) Monotony. Once again, this follows from the generation statement.

(3) Generation. We use the following formula, established above:

$$T_Q = G \cap Q^* \mathbb{T}^N Q$$

Since any group element $U \in G$ is diagonalizable, $U = Q^* D Q$ with $Q \in U_N, D \in \mathbb{T}^N$, we have $U \in T_Q$ for this value of $Q \in U_N$, and this gives the result.

Regarding now the group duals, here everything is trivial. Indeed, when the group duals are diagonally embedded we can take $Q = 1$, and when the group duals are embedded by using a spinning matrix $Q \in U_N$, we can use precisely this matrix Q. □

We have as well the following result, also from [31]:

Theorem 10.26. *The following results hold, both over the category of compact Lie groups, and over the category of duals of finitely generated discrete groups:*

1. *Characters: if G is connected, for any nonzero $P \in C(G)_{central}$ there exists $Q \in U_N$ such that P becomes nonzero, when mapped into $C(T_Q)$.*
2. *Amenability: a closed subgroup $G \subset U_N^+$ is coamenable if and only if each of the tori T_Q is coamenable, in the usual discrete group sense.*
3. *Growth: assuming $G \subset U_N^+$, the discrete quantum group $\widehat{G}$ has polynomial growth if and only if each the discrete groups $\widehat{T_Q}$ has polynomial growth.*

Proof. In the classical case, where $G \subset U_N$, the proof goes as follows:

(1) Characters. We can take here $Q \in U_N$ to be such that $QTQ^* \subset \mathbb{T}^N$, where $T \subset U_N$ is a maximal torus for G, and this gives the result.

(2) Amenability. This conjecture holds trivially in the classical case, $G \subset U_N$, due to the fact that these latter quantum groups are all coamenable.

(3) Growth. This is something nontrivial, well-known from the theory of compact Lie groups, and we refer here for instance to the literature.

Regarding now the group duals, here everything is trivial. Indeed, when the group duals are diagonally embedded we can take $Q = 1$, and when the group duals are embedded by using a spinning matrix $Q \in U_N$, we can use precisely this matrix Q. □

The various statements in Theorem 10.25 and Theorem 10.26 are conjectured to hold for any compact quantum group. We refer to [31] and to subsequent papers for a number of verifications, notably covering many basic examples of easy quantum groups.

Let us focus now on the generation property, from Theorem 10.25 (3), which is perhaps the most important. In order to discuss the general case, we will need:

Proposition 10.27. *Given a closed subgroup $G \subset U_N^+$ and a matrix $Q \in U_N$, the corresponding standard torus and its Tannakian category are given by*

$$T_Q = G \cap \mathbb{T}_Q \quad , \quad C_{T_Q} =< C_G, C_{\mathbb{T}_Q} >$$

where $\mathbb{T}_Q \subset U_N^+$ is the dual of the free group $F_N =< g_1, \ldots, g_N >$, with the fundamental corepresentation of $C(\mathbb{T}_Q)$ being the matrix $u = Qdiag(g_1, \ldots, g_N)Q^*$.

Proof. The first assertion comes from the well-known fact that given two closed subgroups $G, H \subset U_N^+$, the corresponding quotient algebra $C(U_N^+) \to C(G \cap H)$ appears by dividing by the kernels of both the quotient maps $C(U_N^+) \to C(G)$ and $C(U_N^+) \to C(H)$.

Indeed, the construction of T_Q amounts precisely in performing this operation, with $H = \mathbb{T}_Q$, and so we obtain $T_Q = G \cap \mathbb{T}_Q$, as claimed.

As for the Tannakian category formula, this follows from this, and from the general duality formula $C_{G\cap H} =< C_G, C_H >$. □

We have the following Tannakian reformulation of the toral generation property:

Theorem 10.28. *Given a closed subgroup $G \subset U_N^+$, the subgroup $G' =< T_Q | Q \in U_N >$ generated by its standard tori has the following Tannakian category:*

$$C_{G'} = \bigcap_{Q \in U_N} < C_G, C_{\mathbb{T}_Q} >$$

In particular we have $G = G'$ when this intersection reduces to C_G.

Proof. Consider indeed the subgroup $G' \subset G$ constructed in the statement. We have:

$$C_{G'} = \bigcap_{Q \in U_N} C_{T_Q}$$

Together with the formula in Proposition 10.27, this gives the result. □

The above result can be used for investigating the toral generation conjecture, but the combinatorics is quite difficult, and there are no results yet, along these lines.

Now back to the tori, we have the following key result, from [37]:

Theorem 10.29. *Consider a quotient group as follows, with $N = N_1 + \ldots + N_k$:*

$$\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$$

We have then $\widehat{\Gamma} \subset S_N^+$, and any group dual subgroup of S_N^+ appears in this way.

Proof. The fact that we have a subgroup as in the statement follows from:

$$\begin{aligned}\widehat{\Gamma} &\subset \widehat{\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}} = \widehat{\mathbb{Z}_{N_1}} \,\hat{*}\, \ldots \,\hat{*}\, \widehat{\mathbb{Z}_{N_k}}\\ &\simeq \mathbb{Z}_{N_1} \,\hat{*}\, \ldots \,\hat{*}\, \mathbb{Z}_{N_k} \subset S_{N_1} \,\hat{*}\, \ldots \,\hat{*}\, S_{N_k}\\ &\subset S_{N_1}^+ \,\hat{*}\, \ldots \,\hat{*}\, S_{N_k}^+ \subset S_N^+\end{aligned}$$

Conversely, assume that we have a group dual subgroup $\widehat{\Gamma} \subset S_N^+$. By Theorem 10.22, the corresponding magic unitary must be of the following form, with $U \in U_N$:

$$u = Udiag(g_1, \ldots, g_N)U^*$$

Now if we denote by $N = N_1 + \ldots + N_k$ the orbit decomposition for $\widehat{\Gamma} \subset S_N^+$, coming from Theorem 10.1, we conclude that u has a $N = N_1 + \ldots + N_k$ block-diagonal pattern, and so that U has as well this $N = N_1 + \ldots + N_k$ block-diagonal pattern.

But this discussion reduces our problem to its $k = 1$ particular case, with the statement here being that the cyclic group $\mathbb{Z}_N$ is the only transitive group dual $\widehat{\Gamma} \subset S_N^+$. The proof of this latter fact being elementary, we obtain the result. See [37]. $\square$

Here is a related result, which is useful for our purposes:

Theorem 10.30. *For the quantum permutation group S_N^+, we have:*

(1) *Given $Q \in U_N$, the quotient $F_N \to \Lambda_Q$ comes from the following relations:*

$$\begin{cases}g_i = 1 & \text{if } \sum_l Q_{il} \neq 0\\ g_ig_j = 1 & \text{if } \sum_l Q_{il}Q_{jl} \neq 0\\ g_ig_jg_k = 1 & \text{if } \sum_l Q_{il}Q_{jl}Q_{kl} \neq 0\end{cases}$$

(2) *Given a decomposition $N = N_1 + \ldots + N_k$, for the matrix $Q = diag(F_{N_1}, \ldots, F_{N_k})$, where $F_N = \frac{1}{\sqrt{N}}(\xi^{ij})_{ij}$ with $\xi = e^{2\pi i/N}$ is the Fourier matrix, we obtain:*

$$\Lambda_Q = \mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}$$

(3) *Given an arbitrary matrix $Q \in U_N$, there exists a decomposition $N = N_1 + \ldots + N_k$, such that Λ_Q appears as quotient of $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}$.*

Proof. This is more or less equivalent to the classification of the group dual subgroups $\widehat{\Gamma} \subset S_N^+$ from Theorem 10.29, but with the result formulated in an alternative way, and the proof can be deduced either from it, or from some direct computations. $\square$

Summarizing, in the quantum permutation group case, the standard tori parametrized by Fourier matrices play a special role. All this discussion suggests formulating:

Definition 10.31. *Consider a closed subgroup $G \subset U_N^+$.*

(1) *Its standard tori T_F, with $F = F_{N_1} \otimes \ldots \otimes F_{N_k}$, and $N = N_1 + \ldots + N_k$ being regarded as a partition, are called Fourier tori.*
(2) *In the case where we have $G_N =< G_N^c, (T_F)_F >$, we say that G_N appears as a Fourier liberation of its classical version G_N^c.*

The conjecture is that all the easy quantum groups should appear as Fourier liberations. The situation in the free case is as follows:

(1) O_N^+, U_N^+ are diagonal liberations, so they are Fourier liberations as well.
(2) B_N^+, C_N^+ are Fourier liberations too, by using a Fourier transform.
(3) S_N^+ is a Fourier liberation too, being generated by its tori [39], [43].
(4) H_N^+, K_N^+ remain to be investigated, by using the general theory in [78].

As an application of all this, let us go back to quantum permutation groups. One interesting question is whether $G^+(X)$ appears as a Fourier liberation of $G(X)$.

Generally speaking, this is something quite difficult, because for the empty graph itself we are in need of the above-mentioned technical results from [39], [43].

In order to discuss however this question, let us begin with:

Proposition 10.32. *The Fourier tori of $G^+(X)$ are the biggest quotients*

$$\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$$

whose duals act on the graph, $\widehat{\Gamma} \curvearrowright X$.

Proof. We have indeed the following computation, at $F = 1$:

$$\begin{aligned} C(T_1(G^+(X))) &= C(G^+(X))/ < u_{ij} = 0, \forall i \neq j > \\ &= [C(S_N^+)/ < [d,u] = 0 >]/ < u_{ij} = 0, \forall i \neq j > \\ &= [C(S_N^+)/ < u_{ij} = 0, \forall i \neq j >]/ < [d,u] = 0 > \\ &= C(T_1(S_N^+))// < [d,u] = 0 > \end{aligned}$$

Thus, we obtain the result, with the remark that the quotient that we are interested in appears via relations of type $d_{ij} = 1 \implies g_i = g_j$. The proof in general is similar. □

An interesting question is whether the "non quantum symmetry" property can be seen at the level of Fourier tori. In order to comment on this, let us start with:

Proposition 10.33. *Consider the following conditions:*

(1) *We have $G(X) = G^+(X)$.*
(2) *$G(X) \subset G^+(X)$ is a Fourier liberation.*
(3) *$\widehat{\Gamma} \curvearrowright X$ implies that Γ is abelian.*

We have then $(1) \iff (2) + (3)$.

Proof. This is something elementary, the proof being as follows:

(1) $\implies$ (2, 3) Here both the implications are trivial.

(2, 3) $\implies$ (1) Assuming $G(X) \neq G^+(X)$, from (2) we know that $G^+(X)$ has at least one non-classical Fourier torus, and this contradicts (3). □

With this observation in hand, our question is whether (3) $\implies$ (1) holds.

In other words, our conjecture would be that a graph X has no quantum symmetry if and only if any action $\widehat{\Gamma} \curvearrowright X$ of a quotient $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$ must come from an abelian quotient $\mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k} \to \Gamma$. This would be of course something very useful.

We have the following result, regarding the torus coactions on finite graphs:

Proposition 10.34. *For a quotient group $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, and a graph X having $N = N_1 + \ldots + N_k$ vertices, the condition $\widehat{\Gamma} \curvearrowright X$ is equivalent to*

$$(F^*dF)_{ij} \neq 0 \implies I_i = I_j$$

where $F = diag(F_{N_1}, \ldots, F_{N_k})$, and where $I = diag(I_1, \ldots, I_k)$ is the diagonal matrix formed by the elements of the images of $\mathbb{Z}_{N_1}, \ldots, \mathbb{Z}_{N_k}$.

Proof. We know that with F, I being as in the statement, we have $u = FIF^*$. Now with this formula in hand, we have the following equivalences:

$$\begin{aligned}
\widehat{\Gamma} \curvearrowright X &\iff du = ud \\
&\iff dFIF^* = FIF^*d \\
&\iff [F^*dF, I] = 0
\end{aligned}$$

Also, since the matrix I is diagonal, with $M = F^*dF$ have:

$$\begin{aligned}
MI = IM &\iff (MI)_{ij} = (IM)_{ij} \\
&\iff M_{ij}I_j = I_iM_{ij} \\
&\iff [M_{ij} \neq 0 \implies I_i = I_j]
\end{aligned}$$

Thus, we obtain the condition in the statement. □

Observe now that in the cyclic case, where $F = F_N$ is a usual Fourier matrix, associated to a cyclic group $\mathbb{Z}_N$, we have the following formula, with $w = e^{2\pi i/N}$:

$$(F^*dF)_{ij} = \sum_{kl} (F^*)_{ik} d_{kl} F_{lj} = \sum_{kl} w^{lj-ik} d_{kl} = \sum_{k \sim l} w^{lj-ik}$$

All this suggests that the random graphs should be "weakly rigid", in the sense that there are no group dual actions on them. Indeed, this should follow in principle from the observation that if $d \in M_N(0, 1)$ is random, then we will have $(F^*dF)_{ij} \neq 0$ almost everywhere, and so we will obtain $I_i = I_j$ almost everywhere, and so abelianity.

11. Transitive groups

We have seen in the previous section that a theory of orbits and orbitals can be developed for the closed subgroups $G\subset S_N^+$, and that all this is particularly interesting in connection with tori. In this section we restrict the attention to the transitive case.

Let us first review the basic theory, that we will need in what follows. The notion of transitivity, which goes back to Bichon's paper [37], can be introduced as follows:

Definition 11.1. *Let $G\subset S_N^+$ be a closed subgroup, with magic unitary $u=(u_{ij})$, and consider the equivalence relation on $\{1,\ldots,N\}$ given by $i\sim j\iff u_{ij}\neq 0$.*

1. *The equivalence classes under $\sim$ are called orbits of G.*
2. *G is called transitive when the action has a single orbit.*

In other words, we call a subgroup $G\subset S_N^+$ transitive when $u_{ij}\neq 0$, for any i,j.

This transitivity notion is standard, coming in a straightforward way from Theorem 10.1. In the classical case, we obtain of course the usual notion of transitivity.

We will need as well the following result, once again coming from [37]:

Theorem 11.2. *For a closed subgroup $G\subset S_N^+$, the following are equivalent:*

1. *G is transitive.*
2. *$Fix(u)=\mathbb C\xi$, where ξ is the all-one vector.*
3. *$\int_Gu_{ij}=\frac{1}{N}$, for any i,j.*

Proof. This is well-known in the classical case. In general, the proof is as follows:

(1) $\iff$ (2) We use the standard fact that the fixed point space of a corepresentation coincides with the fixed point space of the associated coaction:

$$Fix(u)=Fix(\Phi)$$

Thus, Theorem 10.2 above, also from [37], tells us that the fixed point space of the magic corepresentation $u=(u_{ij})$ has the following interpretation, in terms of orbits:

$$Fix(u)=\left\{\xi\in C(X)\Big|i\sim j\implies \xi(i)=\xi(j)\right\}$$

In particular, the transitivity condition corresponds to $Fix(u)=\mathbb C\xi$, as stated.

(2) $\iff$ (3) This is clear from the general properties of the Haar integration, and more precisely from the fact that $(\int_Gu_{ij})_{ij}$ is the projection onto $Fix(u)$. □

Let us recall now that in the classical case, in the situation where we have a transitive subgroup $G\subset S_N$, by setting $H=\{\sigma\in G|\sigma(1)=1\}$ we have:

$$G/H=\{1,\ldots,N\}$$

Conversely, any subgroup $H\subset G$ produces an action $G\curvearrowright G/H$, given by $g(hH)=(gh)H$, and so a morphism $G\to S_N$, where $N=[G:H]$.

This latter morphism is injective when the following condition is satisfied:

$$hgh^{-1} \in H, \forall h \in G \implies g = 1$$

In the quantum case now, it is very unclear how to generalize this structure result. To be more precise, the various examples from [7] show that we cannot expect to have an elementary generalization of the above $G/H = \{1, \ldots, N\}$ isomorphism.

However, we can at least try to extend the obvious fact that $G = N|H|$ must be a multiple of N. And here, we have the following result, from [19]:

Theorem 11.3. *If $G \subset S_N^+$ is finite and transitive, then N divides $|G|$. Moreover:*

(1) *The case $|G| = N$ comes from the classical finite groups, of order N, acting on themselves.*
(2) *The case $|G| = 2N$ is possible, in the non-classical setting, an example here being the Kac-Paljutkin quantum group, at $N = 4$.*

Proof. In order to prove the first assertion, we use the coaction of $C(G)$ on the algebra $\mathbb{C}^N = C(1, \ldots, N)$. In terms of the standard coordinates u_{ij}, the formula is:

$$\Phi : \mathbb{C}^N \to \mathbb{C}^N \otimes C(G) \quad , \quad e_i \to \sum_j e_j \otimes u_{ji}$$

For $a \in \{1, \ldots, N\}$ consider the evaluation map $ev_a : \mathbb{C}^N \to \mathbb{C}$ at a. By composing Φ with $ev_a \otimes id$ we obtain a $C(G)$-comodule map, as follows:

$$I_a : \mathbb{C}^N \to C(G) \quad , \quad e_i \to u_{ia}$$

Our transitivity assumption on G ensures that this map I_a is injective. In other words, we have realized $\mathbb{C}^N$ as a coideal subalgebra of $C(G)$.

We recall now that a finite dimensional Hopf algebra is free as a module over a coideal subalgebra A provided that the latter is Frobenius, in the sense that there exists a non-degenerate bilinear form $b : A \otimes A \to \mathbb{C}$ satisfying $b(xy, z) = b(x, yz)$.

We can apply this result to the coideal subalgebra $I_a(\mathbb{C}^N) \subset C(G)$, with the remark that $\mathbb{C}^N$ is indeed Frobenius, with bilinear form as follows:

$$b(fg) = \frac{1}{N} \sum_{i=1}^N f(i)g(i)$$

Thus $C(G)$ is a free module over the N-dimensional algebra $\mathbb{C}^N$, and this gives the result. Regarding now the remaining assertions, the proof here goes as follows:

(1) Since $C(G) =< u_{ij} >$ is of dimension N, and its commutative subalgebra $< u_{i1} >$ is of dimension N already, $C(G)$ must be commutative. Thus G must be classical, and by transitivity, the inclusion $G \subset S_N$ must come from the action of G on itself.

(2) The closed subgroups $G \subset S_4^+$ are fully classified, and among them we have indeed the Kac-Paljutkin quantum group, which satisfies $|G| = 8$, and is transitive. □

Here is now a list of examples of transitive quantum groups, coming from the various constructions from the previous sections:

Theorem 11.4. *The following are transitive subgroups $G \subset S_N^+$:*

(1) *The quantum permutation group S_N^+ itself.*
(2) *The transitive subgroups $G \subset S_N$. These are the classical examples.*
(3) *The subgroups $\widehat{G} \subset S_{|G|}$, with G abelian. These are the group dual examples.*
(4) *The quantum groups $F \subset S_N^+$ which are finite, $|F| < \infty$, and transitive.*
(5) *The quantum automorphism groups of transitive graphs $G^+(X)$, with $|X| = N$.*
(6) *In particular, we have the hyperoctahedral quantum group $H_n^+ \subset S_N^+$, with $N = 2n$.*
(7) *We have as well the twisted orthogonal group $O_n^{-1} \subset S_N^+$, with $N = 2^n$.*

In addition, the class of transitive quantum permutation groups $\{G \subset S_N^+ | N \in \mathbb{N}\}$ is stable under direct products $\times$, wreath products $\wr$ and free wreath products $\wr_*$.

Proof. All these assertions are well-known. In what follows we briefly describe the idea of each proof, and indicate a reference. We will be back to all these examples, gradually, in the context of certain matrix modelling questions, to be formulated later on.

(1) This comes from the fact that we have an inclusion $S_N \subset S_N^+$. Indeed, since S_N is transitive, so must be S_N^+, because its coordinates u_{ij} map to those of S_N. See [29].

(2) This is again trivial. Indeed, for a classical group $G \subset S_N$, the variables $u_{ij} = \chi(\sigma \in S_N | \sigma(j) = i)$ are all nonzero precisely when G is transitive. See [29].

(3) This follows from the general results of Bichon in [37], who classified there all the group dual subgroups $\widehat{\Gamma} \subset S_N^+$. For a discussion here, we refer to [29].

(4) Here we use the convention $|F| = \dim_{\mathbb{C}} C(F)$, and the statement itself is empty, and is there just for reminding us that these examples are to be investigated.

(5) This is trivial, because X being transitive means that $G(X) \curvearrowright X$ is transitive, and by definition of $G^+(X)$, we have $G(X) \subset G^+(X)$. See [1].

(6) This comes from a result from [14], stating that we have $H_n^+ = G^+(I_n)$, where I_n is the graph formed by n segments, having $N = 2n$ vertices.

(7) Once again this comes from a result from [14], stating that we have $O_n^{-1} = G^+(K_n)$, where K_n is the n-dimensional hypercube, having $N = 2^n$ vertices.

Finally, the stability assertion is clear from the definition of the various products involved, from [36], [92]. This is well-known, and we will be back later on to this. $\square$

Summarizing, we have a substantial list of examples. We will see in the next section that there are several other interesting examples, coming from the matrix models.

We will be back with more general theory at the end of this section.

Let us discuss now classification results at small values of N. We first have:

Theorem 11.5. *The closed subgroups of $S_4^+ = SO_3^{-1}$ are as follows:*

(1) *Infinite quantum groups: S_4^+, O_2^{-1}, $\widehat{D}_\infty$.*
(2) *Finite groups: S_4, and its subgroups.*
(3) *Finite group twists: S_4^{-1}, A_5^{-1}.*
(4) *Series of twists: D_{2n}^{-1} $(n \geq 3)$, DC_n^{-1} $(n \geq 2)$.*
(5) *A group dual series: $\widehat{D}_n$, with $n \geq 3$.*

Moreover, these quantum groups are subject to an ADE classification result.

Proof. The idea here is that the classification result can be obtained by taking some inspiration from the McKay classification of the subgroups of SO_3. See [9]. □

By restricting the attention to the transitive case, we obtain:

Theorem 11.6. *The small order transitive quantum groups are as follows:*

(1) *At $N = 1, 2, 3$ we have $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, S_3.*
(2) *At $N = 4$ we have $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, D_4, A_4, S_4, O_2^{-1}, S_4^+$ and S_4^{-1}, A_5^{-1}.*

Proof. This follows from the above result, the idea being as follows:
(1) This follows from the fact that we have $S_N = S_N^+$ at $N \leq 3$, from [93].
(2) This follows from the ADE classification of the subgroups $G \subset S_4^+$, from [7], with all the twists appearing in the statement being standard twists. See [7]. □

As an interesting consequence of the above result, we have:

Proposition 11.7. *The inclusion of compact quantum groups*

$$S_4 \subset S_4^+$$

is maximal, in the sense that there is no quantum group in between.

Proof. This follows indeed from the above classification result. See [9]. □

Let us study now the quantum subgroups $G \subset S_5^+$. We first have the following elementary observations, regarding such subgroups:

Proposition 11.8. *We have the following examples of subgroups $G \subset S_5^+$:*

(1) *The classical subgroups, $G \subset S_5$. There are 16 such subgroups, having order $1, 2, 3, 4, 4, 5, 6, 6, 8, 10, 12, 12, 20, 24, 60, 120$.*
(2) *The group duals, $G = \widehat{\Gamma} \subset S_5^+$. These appear, via a Fourier transform construction, from the various quotients Γ of the groups $\mathbb{Z}_4, \mathbb{Z}_2 * \mathbb{Z}_2, \mathbb{Z}_2 * \mathbb{Z}_3$.*

In addition, we have as well all the ADE quantum groups $G \subset S_4^+ \subset S_5^+$ from Theorem 11.5 above, embedded via the 5 standard embeddings $S_4^+ \subset S_5^+$.

Proof. These results are well-known, the proof being as follows:

(1) This is a classical result, with the groups which appear being respectively the cyclic groups $\{1\}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathbb{Z}_5, \mathbb{Z}_6, S_3, D_4, D_5, A_4$, then a copy of $S_3 \rtimes \mathbb{Z}_2$, the general affine group $GA_1(5) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$, and finally S_4, A_5, S_5.

(2) This follows from Bichon's result in [36], stating that the group dual subgroups $G = \widehat{\Gamma} \subset S_N^+$ appear from the various quotients $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, with $N_1 + \ldots + N_k = N$. At $N = 5$ the partitions are $5 = 1 + 4, 1 + 2 + 2, 2 + 3$, and this gives the result. □

Summarizing, the classification of the subgroups $G \subset S_5^+$ is a particularly difficult task, the situation here being definitely much more complicated than at $N = 4$.

Consider now an intermediate compact quantum group, as follows:

$$S_N \subset G \subset S_N^+$$

Then G must be transitive. Thus, we can restrict the attention to such quantum groups. Regarding such quantum groups, we first have the following elementary result:

Proposition 11.9. *We have the following examples of transitive subgroups $G \subset S_5^+$:*

1. *The classical transitive subgroups $G \subset S_5$. There are only 5 such subgroups, namely $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5$.*
2. *The transitive group duals, $G = \widehat{\Gamma} \subset S_5^+$. There is only one example here, namely the dual of $\Gamma = \mathbb{Z}_5$, which is $\mathbb{Z}_5$, already appearing above.*

In addition, all the ADE quantum groups $G \subset S_4^+ \subset S_5^+$ are not transitive.

Proof. This follows indeed by examining the lists in Proposition 11.8:

(1) The result here is well-known, and elementary. Observe that $GA_1(5) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$, which is by definition the general affine group of $\mathbb{F}_5$, is indeed transitive.

(2) This follows from the results in [36], because with $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$ as in the proof of Proposition 11.8 (2), the orbit decomposition is precisely $N = N_1 + \ldots + N_k$.

Finally, the last assertion is clear, because the embedding $S_4^+ \subset S_5^+$ is obtained precisely by fixing a point. Thus S_4^+ and its subgroups are not transitive, as claimed. □

In order to prove the uniqueness result, we will use the recent progress in subfactor theory [60], concerning the classification of the small index subfactors.

For our purposes, the most convenient formulation of the result in [60] is:

Theorem 11.10. *The principal graphs of the irreducible index 5 subfactors are:*

1. *A_∞, and a non-extremal perturbation of $A_\infty^{(1)}$.*
2. *The McKay graphs of $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5$.*
3. *The twists of the McKay graphs of A_5, S_5.*

Proof. This is a heavy result, and we refer to [60] for the whole story. The above formulation is the one from [60], with the subgroup subfactors there replaced by fixed point subfactors [3], and with the cyclic groups denoted as usual by $\mathbb{Z}_N$. □

In the quantum permutation group setting, this result becomes:

Theorem 11.11. *The set of principal graphs of the transitive subgroups $G \subset S_5^+$ coincide with the set of principal graphs of the subgroups $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5, S_5^+$.*

Proof. We must take the list of graphs in Theorem 11.10, and exclude some of the graphs, on the grounds that the graph cannot be realized by a transitive subgroup $G \subset S_5^+$.

We have 3 cases here to be studied, as follows:

(1) The graph A_∞ corresponds to S_5^+ itself. As for the perturbation of $A_\infty^{(1)}$, this dissapears, because our notion of transitivity requires the subfactor extremality.

(2) For the McKay graphs of $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5$ there is nothing to be done, all these graphs being solutions to our problem.

(3) The possible twists of A_5, S_5, coming from the graphs in Theorem 11.10 (3) above, cannot contain S_5, because their cardinalities are smaller or equal than $|S_5| = 120$. $\square$

In connection now with our maximality questions, we have:

Theorem 11.12. *The inclusion $S_5 \subset S_5^+$ is maximal.*

Proof. This follows indeed from Theorem 11.11, with the remark that S_5 being transitive, so must be any intermediate subgroup $S_5 \subset G \subset S_5^+$. $\square$

With a little more work, the above considerations can give the full list of transitive subgroups $G \subset S_5^+$. To be more precise, we have here the various subgroups appearing in Theorem 11.11, plus some possible twists of A_5, S_5, which remain to be investigated.

In general, the maximality of $S_N \subset S_N^+$ is a difficult question. The only known general result here is in the easy case, as follows:

Theorem 11.13. *There is no intermediate easy quantum group $S_N \subset G \subset S_N^+$.*

Proof. This follows by doing some combinatorics. To be more precise, the idea is to show that any $\pi \in P - NC$ has the property $< \pi >= P$. And, in order to establish this formula, the idea is to cap π with semicircles, as to preserve one crossing, chosen in advance, and to end up, by a recurrence procedure, with the standard crossing. $\square$

The corresponding orthogonal quantum group questions are somehow easier, and our purpose in what follows will be that of discussing all this. We first have:

Theorem 11.14. *There is only one proper intermediate easy quantum group*

$$O_N \subset G \subset O_N^+$$

namely the half-classical orthogonal group O_N^*.

Proof. We must compute here the categories of pairings $NC_2 \subset D \subset P_2$, and this can be done via some standard combinatorics, in three steps, as follows:

(1) Let $\pi \in P_2 - NC_2$, having $s \geq 4$ strings. Our claim is that:

– If $\pi\in P_2-P_2^*$, there exists a semicircle capping $\pi'\in P_2-P_2^*$.
– If $\pi\in P_2^*-NC_2$, there exists a semicircle capping $\pi'\in P_2^*-NC_2$.
Indeed, both these assertions can be easily proved, by drawing pictures.

(2) Consider now a partition $\pi\in P_2(k,l)-NC_2(k,l)$. Our claim is that:
– If $\pi\in P_2(k,l)-P_2^*(k,l)$ then $<\pi>=P_2$.
– If $\pi\in P_2^*(k,l)-NC_2(k,l)$ then $<\pi>=P_2^*$.
This can be indeed proved by recurrence on the number of strings, $s=(k+l)/2$, by using (1), which provides us with a descent procedure $s\to s-1$, at any $s\geq 4$.

(3) Finally, assume that we are given an easy quantum group $O_N\subset G\subset O_N^+$, coming from certain sets of pairings $D(k,l)\subset P_2(k,l)$. We have three cases:
– If $D\not\subset P_2^*$, we obtain $G=O_N$.
– If $D\subset P_2,D\not\subset NC_2$, we obtain $G=O_N^*$.
– If $D\subset NC_2$, we obtain $G=O_N^+$.
Thus, we are led to the conclusion in the statement. □

We have as well the following result, going beyond easiness:

Theorem 11.15. *The following inclusions are maximal:*

(1) $\mathbb{T}O_N\subset U_N$.
(2) $PO_N\subset PU_N$.

Proof. In order to prove these results, consider as well the group $\mathbb{T}SO_N$. Observe that we have $\mathbb{T}SO_N=\mathbb{T}O_N$ if N is odd. If N is even the group $\mathbb{T}O_N$ has two connected components, with $\mathbb{T}SO_N$ being the component containing the identity.

Let us denote by $\mathfrak{so}_N,\mathfrak{u}_N$ the Lie algebras of SO_N,U_N. It is well-known that $\mathfrak{u}_N$ consists of the matrices $M\in M_N(\mathbb{C})$ satisfying $M^*=-M$, and that $\mathfrak{so}_N=\mathfrak{u}_N\cap M_N(\mathbb{R})$. Also, it is easy to see that the Lie algebra of $\mathbb{T}SO_N$ is $\mathfrak{so}_N\oplus i\mathbb{R}$.

Step 1. Our first claim is that if $N\geq 2$, the adjoint representation of SO_N on the space of real symmetric matrices of trace zero is irreducible.

Let indeed $X\in M_N(\mathbb{R})$ be symmetric with trace zero. We must prove that the following space consists of all the real symmetric matrices of trace zero:

$$V=span\left\{UXU^t\Big|U\in SO_N\right\}$$

We first prove that V contains all the diagonal matrices of trace zero. Since we may diagonalize X by conjugating with an element of SO_N, our space V contains a nonzero diagonal matrix of trace zero. Consider such a matrix:

$$D=diag(d_1,d_2,\ldots,d_N)$$

We can conjugate this matrix by the following matrix:

$$\begin{pmatrix}0&-1&0\\1&0&0\\0&0&I_{N-2}\end{pmatrix}\in SO_N$$

We conclude that our space V contains as well the following matrix:

$$D'=diag(d_2,d_1,d_3,\ldots,d_N)$$

More generally, we see that for any $1\leq i,j\leq N$ the diagonal matrix obtained from D by interchanging d_i and d_j lies in V. Now since S_N is generated by transpositions, it follows that V contains any diagonal matrix obtained by permuting the entries of D. But it is well-known that this representation of S_N on the diagonal matrices of trace zero is irreducible, and hence V contains all such diagonal matrices, as claimed.

In order to conclude now, assume that Y is an arbitrary real symmetric matrix of trace zero. We can find then an element $U\in SO_N$ such that UYU^t is a diagonal matrix of trace zero. But we then have $UYU^t\in V$, and hence also $Y\in V$, as desired.

Step 2. Our claim is that the inclusion $\mathbb TSO_N\subset U_N$ is maximal in the category of connected compact groups.

Let indeed G be a connected compact group satisfying $\mathbb TSO_N\subset G\subset U_N$. Then G is a Lie group. Let $\mathfrak g$ denote its Lie algebra, which satisfies:

$$\mathfrak{so}_N\oplus i\mathbb R\subset\mathfrak g\subset\mathfrak u_N$$

Let ad_G be the action of G on $\mathfrak g$ obtained by differentiating the adjoint action of G on itself. This action turns $\mathfrak g$ into a G-module. Since $SO_N\subset G$, $\mathfrak g$ is also a SO_N-module.

Now if $G\neq\mathbb TSO_N$, then since G is connected we must have $\mathfrak{so}_N\oplus i\mathbb R\neq\mathfrak g$. It follows from the real vector space structure of the Lie algebras $\mathfrak u_N$ and $\mathfrak{so}_N$ that there exists a nonzero symmetric real matrix of trace zero X such that:

$$iX\in\mathfrak g$$

We know that the space of symmetric real matrices of trace zero is an irreducible representation of SO_N under the adjoint action. Thus $\mathfrak g$ must contain all such X, and hence $\mathfrak g=\mathfrak u_N$. But since U_N is connected, it follows that $G=U_N$.

Step 3. Our claim is that the commutant of SO_N in $M_N(\mathbb C)$ is as follows:

(1) $SO_2'=\left\{\begin{pmatrix}\alpha&\beta\\-\beta&\alpha\end{pmatrix}\Big|\alpha,\beta\in\mathbb C\right\}$.
(2) If $N\geq3$, $SO_N'=\{\alpha I_N|\alpha\in\mathbb C\}$.

Indeed, at $N=2$ this is a direct computation. At $N\geq3$, an element in $X\in SO_N'$ commutes with any diagonal matrix having exactly $N-2$ entries equal to 1 and two entries equal to -1. Hence X is a diagonal matrix. Now since X commutes with any even permutation matrix and $N\geq3$, it commutes in particular with the permutation matrix associated with the cycle (i,j,k) for any $1<i<j<k$, and hence all the entries of X are the same. We conclude that X is a scalar matrix, as claimed.

Step 4. Our claim is that the set of matrices with nonzero trace is dense in SO_N.

At $N=2$ this is clear, since the set of elements in SO_2 having a given trace is finite. So assume $N>2$, and let $T\in SO_N\simeq SO(\mathbb{R}^N)$ with $Tr(T)=0$. Let $E\subset\mathbb{R}^N$ be a 2-dimensional subspace preserved by T, such that $T_{|E}\in SO(E)$.

Let $\varepsilon>0$ and let $S_\varepsilon\in SO(E)$ with $||T_{|E}-S_\varepsilon||<\varepsilon$, and with $Tr(T_{|E})\neq Tr(S_\varepsilon)$, in the $N=2$ case. Now define $T_\varepsilon\in SO(\mathbb{R}^N)=SO_N$ by:

$$T_{\varepsilon|E}=S_\varepsilon\quad,\quad T_{\varepsilon|E^\perp}=T_{|E^\perp}$$

It is clear that $||T-T_\varepsilon||\leq||T_{|E}-S_\varepsilon||<\varepsilon$ and that:

$$Tr(T_\varepsilon)=Tr(S_\varepsilon)+Tr(T_{|E^\perp})\neq 0$$

Thus, we have proved our claim.

Step 5. Our claim is that $\mathbb{T}O_N$ is the normalizer of $\mathbb{T}SO_N$ in U_N, i.e. is the subgroup of U_N consisting of the unitaries U for which $U^{-1}XU\in\mathbb{T}SO_N$ for all $X\in\mathbb{T}SO_N$.

It is clear that the group $\mathbb{T}O_N$ normalizes $\mathbb{T}SO_N$, so in order to prove the result, we must show that if $U\in U_N$ normalizes $\mathbb{T}SO_N$ then $U\in\mathbb{T}O_N$.

First note that U normalizes SO_N. Indeed if $X\in SO_N$ then $U^{-1}XU\in\mathbb{T}SO_N$, so $U^{-1}XU=\lambda Y$ for some $\lambda\in\mathbb{T}$ and $Y\in SO_N$. If $Tr(X)\neq0$, we have $\lambda\in\mathbb{R}$ and hence:

$$\lambda Y=U^{-1}XU\in SO_N$$

The set of matrices having nonzero trace being dense in SO_N, we conclude that $U^{-1}XU\in SO_N$ for all $X\in SO_N$. Thus, we have:

$$\begin{aligned}X\in SO_N&\implies (UXU^{-1})^t(UXU^{-1})=I_N\\&\implies X^tU^tUX=U^tU\\&\implies U^tU\in SO_N'\end{aligned}$$

It follows that at $N\geq3$ we have $U^tU=\alpha I_N$, with $\alpha\in\mathbb{T}$, since U is unitary. Hence we have $U=\alpha^{1/2}(\alpha^{-1/2}U)$ with $\alpha^{-1/2}U\in O_N$, and $U\in\mathbb{T}O_N$. If $N=2$, $(U^tU)^t=U^tU$ gives again that $U^tU=\alpha I_2$, and we conclude as in the previous case.

Step 6. Our claim is that the inclusion $\mathbb{T}O_N\subset U_N$ is maximal in the category of compact groups.

Suppose indeed that $\mathbb{T}O_N\subset G\subset U_N$ is a compact group such that $G\neq U_N$. It is a well-known fact that the connected component of the identity in G is a normal subgroup, denoted G_0. Since we have $\mathbb{T}SO_N\subset G_0\subset U_N$, we must have $G_0=\mathbb{T}SO_N$. But since G_0 is normal in G, the group G normalizes $\mathbb{T}SO_N$, and hence $G\subset\mathbb{T}O_N$.

Step 7. Our claim is that the inclusion $PO_N\subset PU_N$ is maximal in the category of compact groups.

This follows from the above result. Indeed, if $PO_N \subset G \subset PU_N$ is a proper intermediate subgroup, then its preimage under the quotient map $U_N \to PU_N$ would be a proper intermediate subgroup of $\mathbb{T}O_N \subset U_N$, which is a contradiction. $\square$

Summarizing, we have many interesting questions here, and both the maximality of $S_N \subset S_N^+$ and of $O_N^* \subset O_N^+$ are main problems in the area.

Let us go back now to the theoretical questions, in relation with the notion of transitivity. It is convenient to introduce a few more related objects, as follows:

Definition 11.16. *Associated to a quantum group $G \subset S_N^+$, producing the equivalence relation on $\{1,\ldots,N\}$ given by $i \sim j$ when $u_{ij} \neq 0$, are as well:*

1. *The partition $\pi \in P(N)$ having as blocks the equivalence classes under $\sim$.*
2. *The binary matrix $\varepsilon \in M_N(0,1)$ given by $\varepsilon_{ij} = \delta_{u_{ij},0}$.*

Observe that each of the objects $\sim, \pi, \varepsilon$ determines the other two ones.

We will often assume, without mentioning it, that the orbits of $G \subset S_N^+$ come in increasing order, in the sense that the corresponding partition is as follows:

$$\pi = \{1,\ldots,K_1\},\ldots,\{K_1+\ldots+K_{M-1}+1,\ldots,K_1+\ldots+K_M\}$$

Indeed, at least for the questions that we are interested in here, we can always assume that it is so, simply by conjugating everything by a suitable permutation $\sigma \in S_N$.

In terms of these objects, the notion of transitivity reformulates as follows:

Definition 11.17. *We call $G \subset S_N^+$ transitive when $u_{ij} \neq 0$ for any i,j. Equivalently:*

1. *$\sim$ must be trivial, $i \sim j$ for any i,j.*
2. *π must be the 1-block partition.*
3. *ε must be the all-1 matrix.*

Let us discuss now the quantum analogue of the fact that given a subgroup $G \subset S_N$, with orbits of lenghts $K_1,\ldots,K_M$, we have an inclusion as follows:

$$G \subset S_{K_1} \times \ldots \times S_{K_M}$$

Given two quantum permutation groups $G \subset S_K^+, H \subset S_L^+$, with magic corepresentations denoted u, v, we can consider the following algebra, and matrix:

$$A = C(G) * C(H) \quad , \quad w = diag(u,v)$$

The pair (A, w) satisfies Woronowicz's axioms, and since w is magic, we therefore obtain a quantum permutation group, denoted $G \,\hat{*}\, H \subset S_{K+L}^+$. See [92].

With this notion in hand, we have the following result:

Proposition 11.18. *Given a quantum group $G \subset S_N^+$, with associated orbit decomposition partition $\pi \in P(N)$, having blocks of length $K_1, \ldots, K_M$, we have an inclusion*

$$G \subset S_{K_1}^+ \,\hat{*}\, \ldots \,\hat{*}\, S_{K_M}^+$$

where the product on the right is constructed with respect to the blocks of π. In the classical case, $G \subset S_N$, we obtain in this way the usual inclusion $G \subset S_{K_1} \times \ldots \times S_{K_M}$.

Proof. Since the standard coordinates $u_{ij} \in C(G)$ satisfy $u_{ij} = 0$ for $i \not\sim j$, the algebra $C(G)$ appears as quotient of the following algebra:

$$\begin{aligned} C(S_N^+)\Big/\langle u_{ij} = 0, \forall i \not\sim j\rangle &= C(S_{K_1}^+) * \ldots * C(S_{K_M}^+) \\ &= C(S_{K_1}^+ \,\hat{*}\, \ldots \,\hat{*}\, S_{K_M}^+) \end{aligned}$$

Thus, we have an inclusion of quantum groups, as in the statement. Finally, observe that the classical version of the quantum group $S_{K_1}^+ \,\hat{*}\, \ldots \,\hat{*}\, S_{K_M}^+$ is given by:

$$\begin{aligned} (S_{K_1}^+ \,\hat{*}\, \ldots \,\hat{*}\, S_{K_M}^+)_{class} &= (S_{K_1} \times \ldots \times S_{K_M})_{class} \\ &= S_{K_1} \times \ldots \times S_{K_M} \end{aligned}$$

Thus in the classical case we obtain $G \subset S_{K_1} \times \ldots \times S_{K_M}$, as claimed. $\square$

Let us discuss now what happens in the group dual case, where the situation is non-trivial. Following the work of Bichon in [37], we have the following result:

Proposition 11.19. *Given a decomposition $N = K_1 + \ldots + K_M$, and a quotient group $\mathbb{Z}_{K_1} * \ldots * \mathbb{Z}_{K_M} \to \Gamma$, we have an embedding, as follows:*

$$\widehat{\Gamma} \subset \mathbb{Z}_{K_1} \,\hat{*}\, \ldots \,\hat{*}\, \mathbb{Z}_{K_M} \subset S_{K_1}^+ \,\hat{*}\, \ldots \,\hat{*}\, S_{K_M}^+ \subset S_N^+$$

Moreover, modulo the action of $S_N \times S_N$ on the magic unitaries, obtained by permuting the rows and columns, we obtain in this way all the group dual subgroups $\widehat{\Gamma} \subset S_N^+$.

Proof. Given a quotient group Γ as in the statement, by composing a number of standard embeddings and identifications, we obtain indeed an embedding, as follows:

$$\begin{aligned} \widehat{\Gamma} &\subset \widehat{\mathbb{Z}_{K_1} * \ldots * \mathbb{Z}_{K_M}} \\ &= \widehat{\mathbb{Z}}_{K_1} \,\hat{*}\, \ldots \,\hat{*}\, \widehat{\mathbb{Z}}_{K_M} \\ &\simeq \mathbb{Z}_{K_1} \,\hat{*}\, \ldots \,\hat{*}\, \mathbb{Z}_{K_M} \\ &\subset S_{K_1} \hat{*} \ldots \,\hat{*}\, S_{K_M} \\ &\subset S_{K_1}^+ \hat{*} \ldots \,\hat{*}\, S_{K_M}^+ \\ &\subset S_{K_1 + \ldots + K_M}^+ \end{aligned}$$

Regarding now the last assertion, this basically follows by letting $N = K_1 + \ldots + K_M$ be the decomposition coming from the orbit structure of $\widehat{\Gamma} \subset S_N^+$. See [37]. $\square$

Let us now consider the case where the decomposition $N = K_1 + \ldots + K_M$ is "minimal", in the sense that the quotient map $\mathbb{Z}_{K_1} * \ldots * \mathbb{Z}_{K_M} \to \Gamma$ is faithful on each $\mathbb{Z}_{K_i}$.

With this assumption made, we have the following result:

Theorem 11.20. *Assume that $\widehat{\Gamma} \subset S_N^+$ comes from a quotient group $\mathbb{Z}_{K_1} * \ldots * \mathbb{Z}_{K_M} \to \Gamma$ with $K_1 + \ldots + K_M = N$, such that the quotient map is faithful on each $\mathbb{Z}_{K_i}$.*

(1) *The associated orbit decomposition is $N = K_1 + \ldots + K_M$.*

(2) *The inclusions $\widehat{\Gamma} \subset S_{K_1}^+ \,\hat{*}\, \ldots \,\hat{*}\, S_{K_M}^+$ from Propositions 11.18 and 11.19 coincide.*

Proof. We recall from Proposition 11.18 that the subgroup $\widehat{\Gamma} \subset S_N^+$ appears as follows:

$$\begin{aligned}\widehat{\Gamma} &\subset \mathbb{Z}_{K_1} \,\hat{*}\, \ldots \,\hat{*}\, \mathbb{Z}_{K_M}\\ &\subset S_{K_1}^+ \,\hat{*}\, \ldots \,\hat{*}\, S_{K_M}^+\\ &\subset S_N^+\end{aligned}$$

(1) By construction of $\widehat{\Gamma} \subset S_N^+$, the orbit decomposition for this quantum group must appear via a refinement of the decomposition $N = K_1 + \ldots + K_M$.

On the other hand, consider the following elements:

$$(K_1 + \ldots + K_{i-1}) + 1, \ldots\ldots, (K_1 + \ldots + K_{i-1}) + K_i$$

Since the quotient map $\mathbb{Z}_{K_1} * \ldots * \mathbb{Z}_{K_M} \to \Gamma$ is faithful on each $\mathbb{Z}_{K_i}$, these elements must belong to the same orbit under the action of $\widehat{\Gamma}$, and we are done.

(2) This is just an observation, which is clear from (1) above. $\square$

Let us discuss now an extension of the notion of transitivity, as follows:

Definition 11.21. *A quantum permutation group $G \subset S_N^+$ is called quasi-transitive when all its orbits have the same size. Equivalently:*

(1) *$\sim$ has equivalence classes of same size.*

(2) *π has all the blocks of equal length.*

(3) *ε is block-diagonal with blocks the flat matrix of size K.*

As a first example, if G is transitive then it is quasi-transitive. In general now, if we denote by $K \in \mathbb{N}$ the common size of the blocks, and by $M \in \mathbb{N}$ their multiplicity, then we must have $N = KM$. We have the following result:

Proposition 11.22. *Assuming that $G \subset S_N^+$ is quasi-transitive, we must have*

$$G \subset \underbrace{S_K^+ \,\hat{*}\, \ldots \,\hat{*}\, S_K^+}_{M\ terms}$$

where $K \in \mathbb{N}$ is the common size of the orbits, and $M \in \mathbb{N}$ is their number.

Proof. This simply follows from the above results, because, with the previous notations, in the quasi-transitive case we must have $K_1 = \ldots = K_M = K$. $\square$

Observe that in the classical case, we obtain in this way the usual embedding:

$$G \subset \underbrace{S_K \times \ldots \times S_K}_{M\ terms}$$

Let us discuss now the examples. Assume that we are in the following situation:

$$G \subset S_K^+ \,\hat{*}\, \ldots \,\hat{*}\, S_K^+$$

If u, v are the fundamental corepresentations of $C(S_N^+), C(S_K^+)$, consider the quotient map $\pi_i : C(S_N^+) \to C(S_K^+)$ constructed as follows:

$$u \to diag(1_K, \ldots, 1_K, \underbrace{v}_{i-th\ term}, 1_K, \ldots, 1_K)$$

We can then set $C(G_i) = \pi_i(C(G))$, and we have the following result:

Proposition 11.23. *If G_i is transitive for all i, then G is quasi-transitive.*

Proof. We know that we have embeddings as follows:

$$G_1 \times \ldots \times G_M \subset G \subset \underbrace{S_K^+ \,\hat{*}\, \ldots \,\hat{*}\, S_K^+}_{M\ terms}$$

It follows that the size of any orbit of G is at least K, because it contains $G_1 \times \ldots \times G_M$, and at most K, because it is contained in $S_K^+ \,\hat{*}\, \ldots \,\hat{*}\, S_K^+$. Thus, G is quasi-transitive. □

We call the quasi-transitive subgroups appearing as above "of product type". Observe that there are quasi-transitive groups which are not of product type, as for instance the group $G = S_2 \subset S_2 \times S_2 \subset S_4$ obtained by using the embedding $\sigma \to (\sigma, \sigma)$. Indeed, the quasi-transitivity is clear, say by letting G act on the vertices of a square. On the other hand, since we have $G_1 = G_2 = \{1\}$, this group is not of product type.

In general, we can construct examples by using various product operations:

Proposition 11.24. *Given transitive subgroups $G_1, \ldots, G_M \subset S_K^+$, the following constructions produce quasi-transitive subgroups $G \subset \underbrace{S_K^+ \,\hat{*}\, \ldots \,\hat{*}\, S_K^+}_{M\ terms}$, of product type:*

1. *The usual product: $G = G_1 \times \ldots \times G_M$.*
2. *The dual free product: $G = G_1 \,\hat{*}\, \ldots \,\hat{*}\, G_M$.*

Proof. All these assertions are clear from definitions, because in each case, the quantum groups $G_i \subset S_K^+$ constructed before are those in the statement. □

In the group dual case, we have the following result:

Proposition 11.25. *The group duals* $\widehat{\Gamma}\subset\underbrace{S_K^+\,\hat{*}\,\ldots\,\hat{*}\,S_K^+}_{M\ terms}$ *which are of product type are precisely those appearing from intermediate groups of the following type:*

$$\underbrace{\mathbb{Z}_K*\ldots*\mathbb{Z}_K}_{M\ terms}\to\Gamma\to\underbrace{\mathbb{Z}_K\times\ldots\times\mathbb{Z}_K}_{M\ terms}$$

Proof. It is clear that any intermediate quotient Γ as in the statement produces a quantum permutation group $\widehat{\Gamma}\subset S_N^+$ which is of product type.

Conversely, given a group dual $\widehat{\Gamma}\subset S_N^+$, coming from a quotient group $\mathbb{Z}_K^{*M}\to\Gamma$, the subgroups $G_i\subset\widehat{\Gamma}$ constructed above must be group duals as well, $G_i=\widehat{\Gamma}_i$, for certain quotient groups $\Gamma\to\Gamma_i$.

Now if $\widehat{\Gamma}$ is of product type, $\widehat{\Gamma}_i\subset S_K^+$ must be transitive, and hence equal to $\widehat{\mathbb{Z}}_K$. We then conclude that we have $\widehat{\mathbb{Z}_K^M}\subset\widehat{\Gamma}$, and so $\Gamma\to\mathbb{Z}_K^M$. □

In order to construct now some other classes of examples, we use the notion of normality for compact quantum groups. This notion is introduced as follows:

Definition 11.26. *Given a quantum subgroup $H\subset G$, coming from a quotient map $\pi:C(G)\to C(H)$, the following are equivalent:*

1. $A=\{a\in C(G)|(id\otimes\pi)\Delta(a)=a\otimes1\}$ *satisfies* $\Delta(A)\subset A\otimes A$.
2. $B=\{a\in C(G)|(\pi\otimes id)\Delta(a)=1\otimes a\}$ *satisfies* $\Delta(B)\subset B\otimes B$.
3. *We have $A=B$, as subalgebras of $C(G)$.*

If these conditions are satisfied, we say that $H\subset G$ is a normal subgroup.

In the classical case we obtain the usual normality notion for the subgroups. Also, in the group dual case the normality of any subgroup, which must be a group dual subgroup, is automatic. Now with this notion in hand, we have:

Theorem 11.27. *Assuming that $G\subset S_N^+$ is transitive, and that $H\subset G$ is normal, $H\subset S_N^+$ follows to be quasi-transitive.*

Proof. Consider the quotient map $\pi:C(G)\to C(H)$, given at the level of standard coordinates by $u_{ij}\to v_{ij}$. Consider two orbits O_1,O_2 of H and set:

$$x_i=\sum_{j\in O_1}u_{ij}\quad,\quad y_i=\sum_{j\in O_2}u_{ij}$$

These two elements are orthogonal projections in $C(G)$ and they are nonzero, because they are sums of nonzero projections by transitivity of G. We have:

$$\begin{aligned}(id\otimes\pi)\Delta(x_i) &= \sum_k\sum_{j\in O_1}u_{ik}\otimes v_{kj}\\ &= \sum_{k\in O_1}\sum_{j\in O_1}u_{ik}\otimes v_{kj}\\ &= \sum_{k\in O_1}u_{ik}\otimes 1\\ &= x_i\otimes 1\end{aligned}$$

Thus by normality of H we have the following formula:

$$(\pi\otimes id)\Delta(x_i)=1\otimes x_i$$

On the other hand, assuming that we have $i\in O_2$, we obtain:

$$\begin{aligned}(\pi\otimes id)\Delta(x_i) &= \sum_k\sum_{j\in O_1}v_{ik}\otimes u_{kj}\\ &= \sum_{k\in O_2}v_{ik}\otimes x_k\end{aligned}$$

Multiplying this by $v_{ik}\otimes 1$ with $k\in O_2$ yields $v_{ik}\otimes x_k=v_{ik}\otimes x_i$, that is to say $x_k=x_i$. In other words, x_i only depends on the orbit of i. The same is of course true for y_i.

By using this observation, we can compute the following element:

$$\begin{aligned}z &= \sum_{k\in O_2}\sum_{j\in O_1}u_{kj}\\ &= \sum_{k\in O_2}x_k\\ &= |O_2|x_i\end{aligned}$$

On the other hand, by applying the antipode, we have as well:

$$\begin{aligned}S(z) &= \sum_{k\in O_2}\sum_{j\in O_1}u_{jk}\\ &= \sum_{j\in O_1}y_j\\ &= |O_1|y_j\end{aligned}$$

We therefore obtain the following formula:

$$S(x_i)=\frac{|O_1|}{|O_2|}y_j$$

Now since both x_i and y_j have norm one, we conclude that the two orbits have the same size, and this finishes the proof. $\square$

Some additional interesting transitivity questions appear in the graph context.

12. Matrix models

One interesting method for the study of the subgroups $G\subset S_N^+$, that we have not tried yet, consists in modelling the coordinates $u_{ij}\in C(G)$ by concrete variables $U_{ij}\in B$.

Indeed, assuming that the model is faithful in some suitable sense, that the algebra B is something quite familiar, and that the variables U_{ij} are not too complicated, all questions about G would correspond in this way to routine questions inside B.

We discuss here these questions, first for the arbitrary quantum groups $G\subset U_N^+$, and then for the quantum permutation groups $G\subset S_N^+$.

Regarding the choice of the target algebra B, some very convenient algebras are the random matrix ones, $B=M_K(C(T))$, with $K\in\mathbb N$, and T being a compact space.

These algebras generalize indeed the most familiar algebras that we know, namely the matrix ones $M_K(\mathbb C)$, and the commutative ones $C(T)$.

We are led in this way to the following general definition:

Definition 12.1. *A matrix model for $G\subset U_N^+$ is a morphism of C^*-algebras*
$$\pi:C(G)\to M_K(C(T))$$
where T is a compact space, and $K\geq 1$ is an integer.

There are many examples of such models, and will discuss them later on. For the moment, let us develop some general theory. The question to be solved is that of understanding the suitable faithfulness assumptions needed on π, as for the model to "remind" the quantum group. As we will see, this is something quite tricky.

The simplest situation is when π is faithful in the usual sense. This is of course something quite restrictive, because the algebra $C(G)$ must be of type I in this case.

However, there are many interesting examples here, and all this is worth a detailed look. Let us introduce the following notion, which is related to faithfulness:

Definition 12.2. *A matrix model $\pi:C(G)\to M_K(C(T))$ is called stationary when*
$$\int_G=\left(tr\otimes\int_T\right)\pi$$
where $\int_T$ is the integration with respect to a given probability measure on T.

Here the term "stationary" comes from a functional analytic interpretation of all this, with a certain Cesàro limit being needed to be stationary, and this will be explained later. Yet another explanation comes from a certain relation with the lattice models, but this relation is rather something folklore, not axiomatized yet. We will be back to this.

As a first result now, which is something which is not exactly trivial, and whose proof requires some functional analysis, the stationarity property implies the faithfulness:

Theorem 12.3. *Assuming that subgroup $G\subset U_N^+$ has a stationary model,*

$$\pi:C(G)\to M_K(C(T))\quad,\quad \int_G=\left(tr\otimes\int_T\right)\pi$$

it follows that G must be coamenable, and that the model is faithful.

Proof. Assume that we have a stationary model, as in the statement. By performing the GNS construction with respect to $\int_G$, we obtain a factorization as follows, which commutes with the respective canonical integration functionals:

$$\pi:C(G)\to C(G)_{red}\subset M_K(C(T))$$

Thus, in what regards the coamenability question, we can assume that π is faithful. With this assumption made, observe that we have embeddings as follows:

$$C^\infty(G)\subset C(G)\subset M_K(C(T))$$

Now observe that the GNS construction gives a better embedding, as follows:

$$L^\infty(G)\subset M_K(L^\infty(T))$$

Now since the von Neumann algebra on the right is of type I, so must be its subalgebra $A=L^\infty(G)$. This means that, when writing the center of this latter algebra as $Z(A)=L^\infty(X)$, the whole algebra decomposes over X, as an integral of type I factors:

$$L^\infty(G)=\int_X M_{K_x}(\mathbb C)\,dx$$

In particular, we can see from this that $C^\infty(G)\subset L^\infty(G)$ has a unique C^*-norm, and so G is coamenable. Thus we have proved our first assertion, and the second assertion follows as well, because our factorization of π consists of the identity, and of an inclusion. $\square$

Summarizing, what we have so far is a slight strengthening of the notion of faithfulness. We will see later on that are many interesting examples of such models.

Let us discuss now the general, non-coamenable case, with the aim of finding a weaker notion of faithfulness, which still does the job, of "reminding" the quantum group.

The idea comes by looking at the group duals $G=\widehat{\Gamma}$. Consider indeed a general model for the associated algebra, which can be written as follows:

$$\pi:C^*(\Gamma)\to M_K(C(T))$$

The point now is that such a representation of the group algebra must come by linearization from a unitary group representation, as follows:

$$\rho:\Gamma\to C(T,U_K)$$

Now observe that when ρ is faithful, the representation π is in general not faithful, for instance because when $T=\{.\}$ its target algebra is finite dimensional. On the other hand, this representation "reminds" Γ, so can be used in order to fully understand Γ.

Summarizing, we have a new idea here, basically saying that, for practical purposes, the faithfuless property can be replaced with something much weaker. This weaker notion is called "inner faithfulness", and the theory here, from [9], is as follows:

Definition 12.4. *Let $\pi : C(G) \to M_K(C(T))$ be a matrix model.*

(1) *The Hopf image of π is the smallest quotient Hopf C^*-algebra $C(G) \to C(H)$ producing a factorization of type $\pi : C(G) \to C(H) \to M_K(C(T))$.*
(2) *When the inclusion $H \subset G$ is an isomorphism, i.e. when there is no non-trivial factorization as above, we say that π is inner faithful.*

These constructions work in fact for any C^*-algebra representation $\pi : C(G) \to B$, but here we will be only interested in the random matrix case, $B = M_K(C(T))$.

In the case where $G = \widehat{\Gamma}$ is a group dual, π must come from a group representation $\rho : \Gamma \to C(T, U_K)$, and the above factorization is simply the one obtained by taking the image, $\rho : \Gamma \to \Lambda \subset C(T, U_K)$. Thus π is inner faithful when $\Gamma \subset C(T, U_K)$.

Also, given a compact group G, and elements $g_1, \ldots, g_K \in G$, we have a representation $\pi : C(G) \to \mathbb{C}^K$, given by $f \to (f(g_1), \ldots, f(g_K))$. The minimal factorization of π is then via $C(H)$, with $H = \overline{< g_1, \ldots, g_K >}$, and π is inner faithful when $G = H$.

In general, the existence and uniqueness of the Hopf image comes from dividing $C(G)$ by a suitable ideal, as explained in [9]. In Tannakian terms, we have:

Theorem 12.5. *Assuming $G \subset U_N^+$, with fundamental corepresentation $u = (u_{ij})$, the Hopf image of $\pi : C(G) \to M_K(C(T))$ comes from the Tannakian category*

$$C_{kl} = Hom(U^{\otimes k}, U^{\otimes l})$$

where $U_{ij} = \pi(u_{ij})$, and where the spaces on the right are taken in a formal sense.

Proof. Since the morphisms increase the intertwining spaces, when defined either in a representation theory sense, or just formally, we have inclusions as follows:

$$Hom(u^{\otimes k}, u^{\otimes l}) \subset Hom(U^{\otimes k}, U^{\otimes l})$$

More generally, we have such inclusions when replacing (G, u) with any pair producing a factorization of π. Thus, by Tannakian duality, the Hopf image must be given by the fact that the intertwining spaces must be the biggest, subject to the above inclusions.

On the other hand, since u is biunitary, so is U, and it follows that the spaces on the right form a Tannakian category. Thus, we have a quantum group (H, v) given by:

$$Hom(v^{\otimes k}, v^{\otimes l}) = Hom(U^{\otimes k}, U^{\otimes l})$$

By the above discussion, $C(H)$ follows to be the Hopf image of π, as claimed. □

Regarding now the study of the inner faithful models, a key problem is that of computing the Haar integration functional. The result here, from [94], is as follows:

Theorem 12.6. *Given an inner faithful model $\pi : C(G) \to M_K(C(T))$, we have*

$$\int_G = \lim_{k\to\infty} \frac{1}{k} \sum_{r=1}^k \int_G^r$$

where $\int_G^r = (\varphi \circ \pi)^{*r}$, *with* $\varphi = tr \otimes \int_T$ *being the random matrix trace.*

Proof. As a first observation, there is an obvious similarity here with the Woronowicz construction of the Haar measure, explained in section 1 above. In fact, the above result holds more generally for any model $\pi : C(G) \to B$, with $\varphi \in B^*$ being a faithful trace. With this picture in hand, the Woronowicz construction simply corresponds to the case $\pi = id$, and the result itself is therefore a generalization of Woronowicz's result.

In order to prove now the result, we can proceed as in section 1. If we denote by $\int_G'$ the limit in the statement, we must prove that this limit converges, and that we have $\int_G' = \int_G$. It is enough to check this on the coefficients of corepresentations, and if we let $v = u^{\otimes k}$ be one of the Peter-Weyl corepresentations, we must prove that we have:

$$\left(id \otimes \int_G'\right) v = \left(id \otimes \int_G\right) v$$

We already know, from section 1 above, that the matrix on the right is the orthogonal projection onto $Fix(v)$. Regarding now the matrix on the left, the trick in [99] applied to the linear form $\varphi\pi$ tells us that this is the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi\pi)v$. Now observe that, if we set $V_{ij} = \pi(v_{ij})$, we have:

$$(id \otimes \varphi\pi)v = (id \otimes \varphi)V$$

Thus, we can apply the trick in [99], or rather use the same computation as there, which is only based on the biunitarity condition, and we conclude that the 1-eigenspace that we are interested in equals $Fix(V)$. But, according to Theorem 12.5, we have:

$$Fix(V) = Fix(v)$$

Thus, we have proved that we have $\int_G' = \int_G$, as desired. □

In order to detect the stationary models, we have the following criterion:

Theorem 12.7. *For $\pi : C(G) \to M_K(C(T))$, the following are equivalent:*

(1) *$Im(\pi)$ is a Hopf algebra, and $(tr \otimes \int_T)\pi$ is the Haar integration on it.*
(2) *$\psi = (tr \otimes \int_X)\pi$ satisfies the idempotent state property $\psi * \psi = \psi$.*
(3) *$T_e^2 = T_e$, $\forall p \in \mathbb{N}$, $\forall e \in \{1, *\}^p$, where $(T_e)_{i_1\ldots i_p, j_1\ldots j_p} = (tr \otimes \int_T)(U_{i_1j_1}^{e_1} \ldots U_{i_pj_p}^{e_p})$.*

If these conditions are satisfied, we say that π is stationary on its image.

Proof. Given a matrix model $\pi : C(G) \to M_K(C(T))$ as in the statement, we can factorize it via its Hopf image, as in Definition 12.4 above:

$$\pi : C(G) \to C(H) \to M_K(C(T))$$

Now observe that the conditions (1,2,3) in the statement depend only on the factorized representation $\nu : C(H) \to M_K(C(T))$. Thus, we can assume in practice that we have $G = H$, which means that we can assume that π is inner faithful.

With this assumption made, the general integration formula from Theorem 12.6 applies to our situation, and the proof of the equivalences goes as follows:

(1) $\implies$ (2) This is clear from definitions, because the Haar integration on any compact quantum group satisfies the idempotent equation $\psi * \psi = \psi$.

(2) $\implies$ (1) Assuming $\psi * \psi = \psi$, we have $\psi^{*r} = \psi$ for any $r \in \mathbb{N}$, and Theorem 12.6 gives $\int_G = \psi$. By using now Theorem 12.3, we obtain the result.

In order to establish now (2) $\iff$ (3), we use the following elementary formula, which comes from the definition of the convolution operation:

$$\psi^{*r}(u_{i_1j_1}^{e_1} \ldots u_{i_pj_p}^{e_p}) = (T_e^r)_{i_1\ldots i_p,j_1\ldots j_p}$$

(2) $\implies$ (3) Assuming $\psi * \psi = \psi$, by using the above formula at $r = 1, 2$ we obtain that the matrices T_e and T_e^2 have the same coefficients, and so they are equal.

(3) $\implies$ (2) Assuming $T_e^2 = T_e$, by using the above formula at $r = 1, 2$ we obtain that the linear forms ψ and $\psi * \psi$ coincide on any product of coefficients $u_{i_1j_1}^{e_1} \ldots u_{i_pj_p}^{e_p}$. Now since these coefficients span a dense subalgebra of $C(G)$, this gives the result. □

Let us get now into the quantum permutation group case. With the convention that we identify the rank one projections in $M_K(\mathbb{C})$ with the corresponding elements of the complex projective space $P_{\mathbb{C}}^{K-1}$, we have the following result, from [29]:

Theorem 12.8. *Given a quasi-transitive group $G \subset S_N$, with orbits having size K, the associated universal quasi-flat model space is $X_G = E_K \times L_{N,K}^G$, where:*

$$E_K = \left\{ P_1, \ldots, P_K \in P_{\mathbb{C}}^{K-1} \Big| P_i \perp P_j, \forall i, j \right\}$$

$$L_{N,K}^G = \left\{ \sigma_1, \ldots, \sigma_K \in G \Big| \sigma_1(i), \ldots, \sigma_K(i) \text{ distinct}, \forall i \in \{1, \ldots, N\} \right\}$$

In addition, assuming that we have $L_{N,K}^G \neq \emptyset$, the universal quasi-flat model is stationary, with respect to the Haar measure on E_K times the discrete measure on $L_{N,K}^G$.

Proof. This result is from [29], the idea being as follows:

(1) Let us call "sparse Latin square" any matrix $L \in M_N(*, 1, \ldots, K)$ whose rows and columns consists of a permutation of the numbers $1, \ldots, K$, completed with $*$ entries.

(2) Our claim is that the quasi-flat representations $\pi : C(S_N) \to M_K(\mathbb{C})$ appear as follows, where $P_1, \ldots, P_K \in M_K(\mathbb{C})$ are rank 1 projections, summing up to 1, and where $L \in M_N(*, 1, \ldots, K)$ is a sparse Latin square, with the convention $P_* = 0$:

$$u_{ij} \mapsto P_{L_{ij}}$$

Indeed, assuming that $\pi : C(S_N) \to M_K(\mathbb{C})$ is quasi-flat, the elements $P_{ij} = \pi(u_{ij})$ are projections of rank ≤ 1, which pairwise commute, and form a magic unitary.

Let $P_1, \ldots, P_K \in M_K(\mathbb{C})$ be the rank one projections appearing in the first row of $P = (P_{ij})$. Since these projections form a partition of unity with rank one projections, any rank one projection $Q \in M_K(\mathbb{C})$ commuting with all of them satisfies $Q \in \{P_1, \ldots, P_K\}$. In particular we have $P_{ij} \in \{P_1, \ldots, P_K\}$ for any i, j such that $P_{ij} \neq 0$. Thus we can write $u_{ij} \mapsto P_{L_{ij}}$, for a certain matrix $L \in M_N(*, 1, \ldots, K)$, with the convention $P_* = 0$.

In order to finish, the remark is that $u_{ij} \mapsto P_{L_{ij}}$ defines a representation $\pi : C(S_N) \to M_K(\mathbb{C})$ precisely when the matrix $P = (P_{L_{ij}})_{ij}$ is magic. But this condition tells us precisely that L must be a sparse Latin square, as desired.

(3) In order to finish, we must compute the Hopf image. Given a sparse Latin square $L \in M_N(*, 1, \ldots, K)$, consider the permutations $\sigma_1, \ldots, \sigma_K \in S_N$ given by:

$$\sigma_x(j) = i \iff L_{ij} = x$$

Our claim is that the Hopf image associated to a representation $\pi : C(S_N) \to M_K(\mathbb{C})$, $u_{ij} \mapsto P_{L_{ij}}$ as above is then the algebra $C(G_L)$, where $G_L =< \sigma_1, \ldots, \sigma_K >\subset S_N$.

Indeed, the image of π being generated by $P_1, \ldots, P_K$, we have an isomorphism of algebras $\alpha : Im(\pi) \simeq C(1, \ldots, K)$ given by $P_i \mapsto \delta_i$. Consider the following diagram:

$$\begin{tikzcd}
C(S_N) \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & Im(\pi) \arrow[r] \arrow[d, "\alpha"] & M_K(\mathbb{C}) \\
& C(1, \ldots, K) &
\end{tikzcd}$$

Here the map on the right is the canonical inclusion and $\varphi = \alpha\pi$. Since the Hopf image of π coincides with the one of φ, it is enough to compute the latter. We know that φ is given by $\varphi(u_{ij}) = \delta_{L_{ij}}$, with the convention $\delta_* = 0$. By Gelfand duality, φ must come from a certain map $\sigma : \{1, \ldots, K\} \to S_N$, via the transposition formula $\varphi(f)(x) = f(\sigma_x)$.

With the choice $f = u_{ij}$, we obtain $\delta_{L_{ij}}(x) = u_{ij}(\sigma_x)$. Now observe that:

$$\delta_{L_{ij}}(x) = \begin{cases} 1 & \text{if } L_{ij} = x \\ 0 & \text{otherwise} \end{cases} \quad , \quad u_{ij}(\sigma_x) = \begin{cases} 1 & \text{if } \sigma_x(j) = i \\ 0 & \text{otherwise} \end{cases}$$

We conclude that σ_x is the permutation in the statement. Summarizing, we have shown that φ comes by transposing the map $x \to \sigma_x$, with σ_x being as in the statement. Thus the Hopf image of φ is the algebra $C(G_L)$, with $G_L =< \sigma_1, \ldots, \sigma_K >$, as desired. $\square$

We have the following result:

Proposition 12.9. *Assuming that $\pi : C(G) \to M_K(C(X))$ is inner faithful and quasi-flat, mapping $u_{ij} \to Proj(\xi^x_{ij})$, with $||\xi^x_{ij}|| \in \{0,1\}$, the above matrices T_p are given by*

$$T_p = \int_X T_p(\xi^x)dx$$

where the matrix $T_p(\xi) \in M_{N^p}(\mathbb{C})$, associated to an array $\xi \in M_N(\mathbb{C}^K)$ is given by

$$T_p(\xi)_{i_1\ldots i_p,j_1\ldots j_p} = \frac{1}{K} < \xi_{i_1j_1}, \xi_{i_2j_2} >< \xi_{i_2j_2}, \xi_{i_3j_3} > \ldots\ldots < \xi_{i_pj_p}, \xi_{i_1j_1} >$$

with the scalar product being the usual one on $\mathbb{C}^K$, taken linear at right.

Proof. We have the following well-known computation, valid for any vectors $\xi_1, \ldots, \xi_p$ having norms $||\xi_i|| \in \{0,1\}$, with the scalar product being linear at right:

$$\begin{aligned} & Proj(\xi_i)x =< \xi_i, x > \xi_i, \forall i \\ \implies & Proj(\xi_1)\ldots Proj(\xi_p)(x) =< \xi_1, \xi_2 > \ldots\ldots < \xi_{p-1}, \xi_p >< \xi_p, x > \xi_1 \\ \implies & Tr(Proj(\xi_1)\ldots Proj(\xi_p)) =< \xi_1, \xi_2 > \ldots\ldots < \xi_{p-1}, \xi_p >< \xi_p, \xi_1 > \end{aligned}$$

Thus, the matrices T_p from Theorem 12.7 can be computed as follows:

$$\begin{aligned} (T_p)_{i_1\ldots i_p,j_1\ldots j_p} &= \int_X tr\left(Proj(\xi^x_{i_1j_1})Proj(\xi^x_{i_2j_2})\ldots Proj(\xi^x_{i_pj_p})\right)dx \\ &= \frac{1}{K}\int_X < \xi^x_{i_1j_1}, \xi^x_{i_2j_2} >< \xi^x_{i_2j_2}, \xi^x_{i_3j_3} > \ldots\ldots < \xi^x_{i_pj_p}, \xi^x_{i_1j_1} > dx \\ &= \int_X (T_p(\xi^x))_{i_1\ldots i_p,j_1\ldots j_p}dx \end{aligned}$$

We therefore obtain the formula in the statement. See [8], [30]. □

One can do even better than this, with a more conceptual result, as follows:

Theorem 12.10. *Given an inner faithful quasi-flat model*

$$\pi : C(G) \to M_K(C(X)) \quad , \quad u_{ij} \to Proj(\xi^x_{ij})$$

with $||\xi^x_{ij}|| \in \{0,1\}$, the law of the normalized character χ/K with respect to the truncated integral $\int_G^r$ coincides with that of the Gram matrix of the vectors

$$\xi^x_{i_1\ldots i_r} = \frac{1}{\sqrt{K}} \cdot \xi^{x_1}_{i_1i_2} \otimes \xi^{x_2}_{i_2i_3} \otimes \ldots \otimes \xi^{x_r}_{i_ri_1}$$

with respect to the normalized matrix trace, and to the integration functional on X^r.

Proof. This was proved in [8], [30] under various supplementary assumptions on the model, which are actually not needed. First of all, by using the above results, the moments C_p

of the measure that we are interested in are given by:

$$C_p=\frac{1}{K^p}\int_G^r\left(\sum_i u_{ii}\right)^p=\frac{1}{K^p}\sum_{i_1\ldots i_p}(T_p^r)_{i_1\ldots i_p,i_1\ldots i_p}=\frac{1}{K^p}\cdot Tr(T_p^r)$$

The trace on the right is given by the following formula:

$$Tr(T_p^r)=\sum_{i_1^1\ldots i_p^r}(T_p)_{i_1^1\ldots i_p^1,i_1^2\ldots i_p^2}\ldots\ldots(T_p)_{i_1^r\ldots i_p^r,i_1^1\ldots i_p^1}$$

In view of the formula in Proposition 12.9, this quantity will expand in terms of the matrices $T_p(\xi)$ constructed there. To be more precise, we have:

$$Tr(T_p^r)=\int_{X^r}\sum_{i_1^1\ldots i_p^r}T_p(\xi^{x_1})_{i_1^1\ldots i_p^1,i_1^2\ldots i_p^2}\ldots\ldots T_p(\xi^{x_r})_{i_1^r\ldots i_p^r,i_1^1\ldots i_p^1}\,dx$$

By using now the explicit formula of each $T_p(\xi)$, from Proposition 12.9, we have:

$$\begin{aligned}Tr(T_p^r)\ &=\ \frac{1}{K^r}\int_{X^r}\sum_{i_1^1\ldots i_p^r}<\xi^{x_1}_{i_1^1i_1^2},\xi^{x_1}_{i_2^1i_2^2}>\ldots\ldots<\xi^{x_1}_{i_p^1i_p^2},\xi^{x_1}_{i_1^1i_1^2}>\\&\qquad\qquad\qquad\qquad\qquad\qquad\ldots\\&\qquad\qquad\qquad<\xi^{x_r}_{i_1^ri_1^1},\xi^{x_r}_{i_2^ri_2^1}>\ldots\ldots<\xi^{x_r}_{i_p^ri_p^1},\xi^{x_r}_{i_1^ri_1^1}>dx\end{aligned}$$

By changing the order of the summation, we can write this formula as:

$$\begin{aligned}Tr(T_p^r)\ &=\ \frac{1}{K^r}\int_{X^r}\sum_{i_1^1\ldots i_p^r}<\xi^{x_1}_{i_1^1i_1^2},\xi^{x_1}_{i_2^1i_2^2}>\ldots\ldots<\xi^{x_r}_{i_1^ri_1^1},\xi^{x_r}_{i_2^ri_2^1}>\\&\qquad\qquad\qquad\qquad\qquad\qquad\ldots\\&\qquad\qquad\qquad<\xi^{x_1}_{i_p^1i_p^2},\xi^{x_1}_{i_1^1i_1^2}>\ldots\ldots<\xi^{x_r}_{i_p^ri_p^1},\xi^{x_r}_{i_1^ri_1^1}>dx\end{aligned}$$

But this latter formula can be written as follows:

$$\begin{aligned}Tr(T_p^r)\ &=\ K^{p-r}\int_{X^r}\sum_{i_1^1\ldots i_p^r}\frac{1}{K}<\xi^{x_1}_{i_1^1i_1^2}\otimes\ldots\otimes\xi^{x_r}_{i_1^ri_1^1}\ ,\ \xi^{x_1}_{i_2^1i_2^2}\otimes\ldots\otimes\xi^{x_r}_{i_2^ri_2^1}>\\&\qquad\qquad\qquad\qquad\qquad\qquad\ldots\\&\qquad\qquad\frac{1}{K}<\xi^{x_1}_{i_p^1i_p^2}\otimes\ldots\otimes\xi^{x_r}_{i_p^ri_p^1}\ ,\ \xi^{x_1}_{i_1^1i_1^2}\otimes\ldots\otimes\xi^{x_r}_{i_1^ri_1^1}>dx\end{aligned}$$

In terms of the vectors in the statement, and of their Gram matrix G_r^x, we obtain:

$$\begin{aligned}
Tr(T_p^r) &= K^{p-r}\int_{X^r}\sum_{i_1^1\ldots i_p^r} < \xi^x_{i_1^1\ldots i_1^r},\xi^x_{i_2^1\ldots i_2^r} > \ldots\ldots < \xi^x_{i_p^1\ldots i_p^r},\xi^x_{i_1^1\ldots i_1^r} > dx\\
&= K^{p-r}\int_{X^r}\sum_{i_1^1\ldots i_p^r} (G_r^x)_{i_1^1\ldots i_1^r,i_2^1\ldots i_2^r}\ldots\ldots(G_r^x)_{i_p^1\ldots i_p^r,i_1^1\ldots i_1^r}\,dx\\
&= K^{p-r}\int_{X^r} Tr((G_r^x)^p)dx
\end{aligned}$$

Summarizing, the moments of the measure in the statement are given by:

$$\begin{aligned}
C_p &= \frac{1}{K^r}\int_{X^r} Tr((G_r^x)^p)dx\\
&= \left(tr\otimes\int_{X^r}\right)(G_r^p)
\end{aligned}$$

This gives the formula in the statement of the theorem. □

Following [30], let us discuss now some more subtle examples of stationary models, related to the Pauli matrices, and Weyl matrices, and physics. We first have:

Definition 12.11. *Given a finite abelian group H, the associated Weyl matrices are*

$$W_{ia}: e_b \to < i,b > e_{a+b}$$

where $i\in H$, $a,b\in\widehat{H}$, and where $(i,b)\to< i,b >$ is the Fourier coupling $H\times\widehat{H}\to\mathbb{T}$.

As a basic example, consider the cyclic group $H=\mathbb{Z}_2=\{0,1\}$. Here the Fourier coupling is given by $< i,b >= (-1)^{ib}$, and so the Weyl matrices act via $W_{00}: e_b\to e_b$, $W_{10}: e_b\to(-1)^b e_b$, $W_{11}: e_b\to(-1)^b e_{b+1}$, $W_{01}: e_b\to e_{b+1}$. Thus, we have:

$$W_{00}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\ ,\ W_{10}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\ ,\ W_{11}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\ ,\ W_{01}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

We recognize here, up to some multiplicative factors, the four Pauli matrices.

Now back to the general case, we have the following well-known result:

Proposition 12.12. *The Weyl matrices are unitaries, and satisfy:*

(1) $W_{ia}^* =< i,a > W_{-i,-a}$.
(2) $W_{ia}W_{jb} =< i,b > W_{i+j,a+b}$.
(3) $W_{ia}W_{jb}^* =< j-i,b > W_{i-j,a-b}$.
(4) $W_{ia}^*W_{jb} =< i,a-b > W_{j-i,b-a}$.

Proof. The unitary follows from (3,4), and the rest of the proof goes as follows:

(1) We have indeed the following computation:

$$\begin{aligned} W_{ia}^* &= \left(\sum_b < i,b > E_{a+b,b}\right)^* = \sum_b < -i,b > E_{b,a+b} \\ &= \sum_b < -i,b-a > E_{b-a,b} = < i,a > W_{-i,-a} \end{aligned}$$

(2) Here the verification goes as follows:

$$\begin{aligned} W_{ia}W_{jb} &= \left(\sum_d < i,b+d > E_{a+b+d,b+d}\right)\left(\sum_d < j,d > E_{b+d,d}\right) \\ &= \sum_d < i,b >< i+j,d > E_{a+b+d,d} = < i,b > W_{i+j,a+b} \end{aligned}$$

(3,4) By combining the above two formulae, we obtain:

$$\begin{aligned} W_{ia}W_{jb}^* &= < j,b > W_{ia}W_{-j,-b} = < j,b >< i,-b > W_{i-j,a-b} \\ W_{ia}^*W_{jb} &= < i,a > W_{-i,-a}W_{jb} = < i,a >< -i,b > W_{j-i,b-a} \end{aligned}$$

But this gives the formulae in the statement, and we are done. □

Observe that, with $n = |H|$, we can use an isomorphism $l^2(\widehat{H}) \simeq \mathbb{C}^n$ as to view each W_{ia} as a usual matrix, $W_{ia} \in M_n(\mathbb{C})$, and hence as a usual unitary, $W_{ia} \in U_n$.

Given a vector ξ, we denote by $Proj(\xi)$ the orthogonal projection onto $\mathbb{C}\xi$.

Now let $N = n^2$, and consider Wang's quantum permutation algebra $C(S_N^+)$, with standard generators denoted $w_{ia,jb}$, using double indices. We have:

Proposition 12.13. *Given a closed subgroup $E \subset U_n$, we have a representation*

$$\pi_H : C(S_N^+) \to M_N(C(E)) \quad : \quad w_{ia,jb} \to [U \to Proj(W_{ia}UW_{jb}^*)]$$

where $n = |H|, N = n^2$, and where W_{ia} are the Weyl matrices associated to H.

Proof. The Weyl matrices being given by $W_{ia} : e_b \to < i,b > e_{a+b}$, we have:

$$tr(W_{ia}) = \begin{cases} 1 & \text{if } (i,a) = (0,0) \\ 0 & \text{if } (i,a) \neq (0,0) \end{cases}$$

Together with the formulae in Proposition 12.12, this shows that the Weyl matrices are pairwise orthogonal with respect to the scalar product $< x,y >= tr(x^*y)$ on $M_n(\mathbb{C})$. Thus, these matrices form an orthogonal basis of $M_n(\mathbb{C})$, consisting of unitaries:

$$W = \left\{ W_{ia} \middle| i \in H, a \in \widehat{H} \right\}$$

Thus, each row and each column of the matrix $\xi_{ia,jb} = W_{ia}UW_{jb}^*$ is an orthogonal basis of $M_n(\mathbb{C})$, and so the corresponding projections form a magic unitary, as claimed. □

We will need the following well-known result:

Proposition 12.14. *With $T = Proj(x_1)\ldots Proj(x_p)$ and $||x_i|| = 1$ we have*

$$<\xi, T\eta>=<\xi, x_1><x_1, x_2>\ldots<x_{p-1}, x_p><x_p, \eta>$$

for any ξ, η. In particular, $Tr(T) =<x_1, x_2><x_2, x_3>\ldots<x_p, x_1>$.

Proof. For $||x|| = 1$ we have $Proj(x)\eta = x<x, \eta>$, and this gives:

$$\begin{aligned}
T\eta &= Proj(x_1)\ldots Proj(x_p)\eta\\
&= Proj(x_1)\ldots Proj(x_{p-1})x_p<x_p, \eta>\\
&= Proj(x_1)\ldots Proj(x_{p-2})x_{p-2}<x_{p-1}, x_p><x_p, \eta>\\
&= \ldots\\
&= x_1<x_1, x_2>\ldots<x_{p-1}, x_p><x_p, \eta>
\end{aligned}$$

Now by taking the scalar product with ξ, this gives the first assertion. As for the second assertion, this follows from the first assertion, by summing over $\xi = \eta = e_i$. ☐

Now back to the Weyl matrix models, let us first compute T_p. We have:

Proposition 12.15. *We have the formula*

$$\begin{aligned}
(T_p)_{ia,jb} &= \frac{1}{N}<i_1, a_1 - a_2>\ldots<i_p, a_p - a_1><j_2, b_2 - b_1>\ldots<j_1, b_1 - b_p>\\
&\int_E tr(W_{i_2-i_1,a_2-a_1}UW_{j_1-j_2,b_1-b_2}U^*)\ldots tr(W_{i_1-i_p,a_1-a_p}UW_{j_p-j_1,b_p-b_1}U^*)dU
\end{aligned}$$

with all the indices varying in a cyclic way.

Proof. By using the trace formula in Proposition 12.14 above, we obtain:

$$\begin{aligned}
(T_p)_{ia,jb} &= \left(tr\otimes\int_E\right)\left(Proj(W_{i_1a_1}UW_{j_1b_1}^*)\ldots Proj(W_{i_pa_p}UW_{j_pb_p}^*)\right)\\
&= \frac{1}{N}\int_E<W_{i_1a_1}UW_{j_1b_1}^*, W_{i_2a_2}UW_{j_2b_2}^*>\ldots<W_{i_pa_p}UW_{j_pb_p}^*, W_{i_1a_1}UW_{j_1b_1}^*>dU
\end{aligned}$$

In order to compute now the scalar products, observe that we have:

$$\begin{aligned}
<W_{ia}UW_{jb}^*, W_{kc}UW_{ld}^*> &= tr(W_{jb}U^*W_{ia}^*W_{kc}UW_{ld}^*)\\
&= tr(W_{ia}^*W_{kc}UW_{ld}^*W_{jb}U^*)\\
&= <i, a - c><l, d - b>tr(W_{k-i,c-a}UW_{j-l,b-d}U^*)
\end{aligned}$$

By plugging these quantities into the formula of T_p, we obtain the result. ☐

Consider now the Weyl group $W = \{W_{ia}\} \subset U_n$, that we already met in the proof of Proposition 12.13 above. We have the following result, from [30]:

Theorem 12.16. *For any compact group $W\subset E\subset U_n$, the model*

$$\pi_H:C(S_N^+)\to M_N(C(E))\quad:\quad w_{ia,jb}\to[U\to Proj(W_{ia}UW_{jb}^*)]$$

constructed above is stationary on its image.

Proof. We must prove that we have $T_p^2=T_p$. We have:

$$\begin{aligned}
(T_p^2)_{ia,jb} &= \sum_{kc}(T_p)_{ia,kc}(T_p)_{kc,jb}\\
&= \frac{1}{N^2}\sum_{kc}<i_1,a_1-a_2>\ldots<i_p,a_p-a_1><k_2,c_2-c_1>\ldots<k_1,c_1-c_p>\\
&<k_1,c_1-c_2>\ldots<k_p,c_p-c_1><j_2,b_2-b_1>\ldots<j_1,b_1-b_p>\\
&\int_E tr(W_{i_2-i_1,a_2-a_1}UW_{k_1-k_2,c_1-c_2}U^*)\ldots tr(W_{i_1-i_p,a_1-a_p}UW_{k_p-k_1,c_p-c_1}U^*)dU\\
&\int_E tr(W_{k_2-k_1,c_2-c_1}VW_{j_1-j_2,b_1-b_2}V^*)\ldots tr(W_{k_1-k_p,c_1-c_p}VW_{j_p-j_1,b_p-b_1}V^*)dV
\end{aligned}$$

By rearranging the terms, this formula becomes:

$$\begin{aligned}
(T_p^2)_{ia,jb} &= \frac{1}{N^2}<i_1,a_1-a_2>\ldots<i_p,a_p-a_1><j_2,b_2-b_1>\ldots<j_1,b_1-b_p>\\
&\int_E\int_E\sum_{kc}<k_1-k_2,c_1-c_2>\ldots<k_p-k_1,c_p-c_1>\\
&tr(W_{i_2-i_1,a_2-a_1}UW_{k_1-k_2,c_1-c_2}U^*)tr(W_{k_2-k_1,c_2-c_1}VW_{j_1-j_2,b_1-b_2}V^*)\\
&\qquad\qquad\ldots\ldots\\
&tr(W_{i_1-i_p,a_1-a_p}UW_{k_p-k_1,c_p-c_1}U^*)tr(W_{k_1-k_p,c_1-c_p}VW_{j_p-j_1,b_p-b_1}V^*)dUdV
\end{aligned}$$

Let us denote by I the above double integral. By using $W_{kc}^*=<k,c>W_{-k,-c}$ for each of the couplings, and by moving as well all the U^* variables to the left, we obtain:

$$\begin{aligned}
I &= \int_E\int_E\sum_{kc}tr(U^*W_{i_2-i_1,a_2-a_1}UW_{k_1-k_2,c_1-c_2})tr(W^*_{k_1-k_2,c_1-c_2}VW_{j_1-j_2,b_1-b_2}V^*)\\
&\qquad\qquad\ldots\ldots\\
&tr(U^*W_{i_1-i_p,a_1-a_p}UW_{k_p-k_1,c_p-c_1})tr(W^*_{k_p-k_1,c_p-c_1}VW_{j_p-j_1,b_p-b_1}V^*)dUdV
\end{aligned}$$

In order to perform now the sums, we use the following formula:

$$\begin{aligned} tr(AW_{kc})tr(W_{kc}^*B) &= \frac{1}{N}\sum_{qrst} A_{qr}(W_{kc})_{rq}(W_{kc}^*)_{st}B_{ts} \\ &= \frac{1}{N}\sum_{qrst} A_{qr} < k,q > \delta_{r-q,c} < k,-s > \delta_{t-s,c}B_{ts} \\ &= \frac{1}{N}\sum_{qs} < k, q-s > A_{q,q+c}B_{s+c,s} \end{aligned}$$

If we denote by A_x, B_x the variables which appear in the formula of I, we have:

$$\begin{aligned} I &= \frac{1}{N^p}\int_E\int_E\sum_{kcqs} < k_1-k_2, q_1-s_1 > \ldots < k_p-k_1, q_p-s_p > \\ & \quad (A_1)_{q_1,q_1+c_1-c_2}(B_1)_{s_1+c_1-c_2,s_1}\ldots(A_p)_{q_p,q_p+c_p-c_1}(B_p)_{s_p+c_p-c_1,s_p} \\ &= \frac{1}{N^p}\int_E\int_E\sum_{kcqs} < k_1, q_1-s_1-q_p+s_p > \ldots < k_p, q_p-s_p-q_{p-1}+s_{p-1} > \\ & \quad (A_1)_{q_1,q_1+c_1-c_2}(B_1)_{s_1+c_1-c_2,s_1}\ldots(A_p)_{q_p,q_p+c_p-c_1}(B_p)_{s_p+c_p-c_1,s_p} \end{aligned}$$

Now observe that we can perform the sums over $k_1,\ldots,k_p$. We obtain in this way a multiplicative factor n^p, along with the condition $q_1-s_1=\ldots=q_p-s_p$. Thus we must have $q_x=s_x+a$ for a certain a, and the above formula becomes:

$$I=\frac{1}{n^p}\int_E\int_E\sum_{csa}(A_1)_{s_1+a,s_1+c_1-c_2+a}(B_1)_{s_1+c_1-c_2,s_1}\ldots(A_p)_{s_p+a,s_p+c_p-c_1+a}(B_p)_{s_p+c_p-c_1,s_p}$$

Consider now the variables $r_x=c_x-c_{x+1}$, which altogether range over the set Z of multi-indices having sum 0. By replacing the sum over c_x with the sum over r_x, which creates a multiplicative n factor, we obtain the following formula:

$$I=\frac{1}{n^{p-1}}\int_E\int_E\sum_{r\in Z}\sum_{sa}(A_1)_{s_1+a,s_1+r_1+a}(B_1)_{s_1+r_1,s_1}\ldots(A_p)_{s_p+a,s_p+r_p+a}(B_p)_{s_p+r_p,s_p}$$

Since for an arbitrary multi-index r we have $\delta_{\sum_i r_i,0}=\frac{1}{n}\sum_i < i,r_1 > \ldots < i,r_p >$, we can replace the sum over $r\in Z$ by a full sum, as follows:

$$\begin{aligned} I &= \frac{1}{n^p}\int_E\int_E\sum_{rsia} < i,r_1 > (A_1)_{s_1+a,s_1+r_1+a}(B_1)_{s_1+r_1,s_1} \\ & \qquad \ldots\ldots \\ & \qquad < i,r_p > (A_p)_{s_p+a,s_p+r_p+a}(B_p)_{s_p+r_p,s_p} \end{aligned}$$

In order to "absorb" now the indices i,a, we can use the following formula:

$$\begin{aligned} W_{ia}^*AW_{ia} &= \left(\sum_b <i,-b> E_{b,a+b}\right)\left(\sum_{bc} E_{a+b,a+c}A_{a+b,a+c}\right)\left(\sum_c <i,c> E_{a+c,c}\right) \\ &= \sum_{bc} <i,c-b> E_{bc}A_{a+b,a+c} \end{aligned}$$

Thus we have $(W_{ia}^*AW_{ia})_{bc} =<i,c-b> A_{a+b,a+c}$, and our formula becomes:

$$\begin{aligned} I &= \frac{1}{n^p}\int_E\int_E\sum_{rsia}(W_{ia}^*A_1W_{ia})_{s_1,s_1+r_1}(B_1)_{s_1+r_1,s_1}\ldots(W_{ia}^*A_pW_{ia})_{s_p,s_p+r_p}(B_p)_{s_p+r_p,s_p} \\ &= \int_E\int_E\sum_{ia} tr(W_{ia}^*A_1W_{ia}B_1)\ldots\ldots tr(W_{ia}^*A_pW_{ia}B_p) \end{aligned}$$

Now by replacing A_x,B_x with their respective values, we obtain:

$$\begin{aligned} I &= \int_E\int_E\sum_{ia} tr(W_{ia}^*U^*W_{i_2-i_1,a_2-a_1}UW_{ia}VW_{j_1-j_2,b_1-b_2}V^*) \\ &\qquad\qquad\ldots\ldots \\ &\qquad tr(W_{ia}^*U^*W_{i_1-i_p,a_1-a_p}UW_{ia}VW_{j_p-j_1,b_p-b_1}V^*)dUdV \end{aligned}$$

By moving the $W_{ia}^*U^*$ variables at right, we obtain, with $S_{ia}=UW_{ia}V$:

$$\begin{aligned} I &= \sum_{ia}\int_E\int_E tr(W_{i_2-i_1,a_2-a_1}S_{ia}W_{j_1-j_2,b_1-b_2}S_{ia}^*) \\ &\qquad\qquad\ldots\ldots \\ &\qquad tr(W_{i_1-i_p,a_1-a_p}S_{ia}W_{j_p-j_1,b_p-b_1}S_{ia}^*)dUdV \end{aligned}$$

Now since S_{ia} is Haar distributed when U,V are Haar distributed, we obtain:

$$I=N\int_E\int_E tr(W_{i_2-i_1,a_2-a_1}UW_{j_1-j_2,b_1-b_2}U^*)\ldots tr(W_{i_1-i_p,a_1-a_p}UW_{j_p-j_1,b_p-b_1}U^*)dU$$

But this is exactly N times the integral in the formula of $(T_p)_{ia,jb}$, from Proposition 12.15 above. Since the N factor cancels with one of the two N factors that we found in the beginning of the proof, when first computing $(T_p^2)_{ia,jb}$, we are done. □

As an illustration for the above result, going back to [19], we have:

Theorem 12.17. *We have a stationary matrix model*

$$\pi:C(S_4^+)\subset M_4(C(SU_2))$$

given on the standard coordinates by the formula

$$\pi(u_{ij})=[x\to Proj(c_ixc_j)]$$

where $x\in SU_2$, and c_1,c_2,c_3,c_4 are the Pauli matrices.

Proof. As already explained in the comments following Definition 12.11, the Pauli matrices appear as particular cases of the Weyl matrices. By working out the details, we conclude that Theorem 12.16 produces in this case the model in the statement. □

Observe that, since the matrix $Proj(c_i x c_j)$ depends only on the image of x in the quotient $SU_2 \to SO_3$, we can replace the model space SU_2 by the smaller space SO_3, if we want to. This is something that can be used in conjunction with the isomorphism $S_4^+ \simeq SO_3^{-1}$ from section 4 above, and as explained in [8], our model becomes in this way something quite conceptual, algebrically speaking, as follows:

$$\pi : C(SO_3^{-1}) \subset M_4(C(SO_3))$$

Finally, let us discuss the Hadamard models, which are of particular importance. Let us start with the following well-known definition:

Definition 12.18. *A complex Hadamard matrix is a square matrix*

$$H \in M_N(\mathbb{C})$$

whose entries are on the unit circle, and whose rows are pairwise orthogonal.

Observe that the orthogonality condition tells us that the rescaled matrix $U = H/\sqrt{N}$ must be unitary. Thus, these matrices form a real algebraic manifold, given by:

$$X_N = M_N(\mathbb{T}) \cap \sqrt{N} U_N$$

The basic example is the Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$. More generally, we have as example the Fourier coupling of any finite abelian group G, regarded via the isomorphism $G \simeq \widehat{G}$ as a square matrix, $F_G \in M_G(\mathbb{C})$:

$$F_G =< i, j >_{i \in G, j \in \widehat{G}}$$

Observe that for the cyclic group $G = \mathbb{Z}_N$ we obtain in this way the above standard Fourier matrix F_N. In general, we obtain a tensor product of Fourier matrices F_N.

There are many other examples of Hadamard matrices, some being elementary, some other fairly exotic, appearing in various branches of mathematics and physics. The idea is that the complex Hadamard matrices can be though of as being "generalized Fourier matrices", and this is where the interest in these matrices comes from. See [83].

In relation with the quantum groups, the starting observation is as follows:

Proposition 12.19. *If $H \in M_N(\mathbb{C})$ is Hadamard, the rank one projections*

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H, form a magic unitary.

Proof. This is clear, the verification for the rows being as follows:

$$\left\langle\frac{H_i}{H_j},\frac{H_i}{H_k}\right\rangle=\sum_l\frac{H_{il}}{H_{jl}}\cdot\frac{H_{kl}}{H_{il}}=\sum_l\frac{H_{kl}}{H_{jl}}=N\delta_{jk}$$

The verification for the columns is similar. □

We can proceed now in the same way as we did with the Weyl matrices, namely by constructing a model of $C(S_N^+)$, and performing the Hopf image construction:

Definition 12.20. *To any Hadamard matrix $H\in M_N(\mathbb{C})$ we associate the quantum permutation group $G\subset S_N^+$ given by the fact that $C(G)$ is the Hopf image of*

$$\pi:C(S_N^+)\to M_N(\mathbb{C})\quad,\quad u_{ij}\to Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1,\ldots,H_N\in\mathbb{T}^N$ are the rows of H.

Summarizing, we have a construction $H\to G$, and our claim is that this construction is something really useful, with G encoding the combinatorics of H. To be more precise, our claim is that "H can be thought of as being a kind of Fourier matrix for G".

This is of course quite interesting, philosophically speaking. There are several results supporting this, with the main evidence coming from the following result, which collects the basic known results regarding the construction:

Theorem 12.21. *The construction $H\to G$ has the following properties:*

1. *For a Fourier matrix $H=F_G$ we obtain the group G itself, acting on itself.*
2. *For $H\notin\{F_G\}$, the quantum group G is not classical, nor a group dual.*
3. *For a tensor product $H=H'\otimes H''$ we obtain a product, $G=G'\times G''$.*

Proof. All this material is standard, and elementary, as follows:

(1) In the cyclic group case, $H=F_N$, all the objects involved in the construction $H\to G$ have an obvious cyclic structure, and we obtain from this $G=\mathbb{Z}_N$. We can pass then to the general case by using (3), whose proof is independent of this.

(2) This is something more tricky, needing some general study of the representations whose Hopf images are commutative, or cocommutative. For details here, along with a number of supplementary facts on the construction $H\to G$, we refer to [11].

(3) This is elementary, the idea being that the tensor products of matrix models are matrix models, and that at the level of corresponding Hopf images, under suitable assumptions, the compatibility holds as well. Once again, we refer here to [11]. □

Going beyond the above result is an interesting question, and we refer here to [10], and to follow-up papers. There are several computations available here, for the most regarding the deformations of the Fourier models. The unification of all this with the Weyl matrix models is a very good question, related to many interesting things.

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