Deriving the conventional quotient rule using alternative calculus-based techniques

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Abstract;

The quotient rule, used to determine the derivative of a function, is used abundantly across multiple branches of calculus. Naturally, there exist several simpler derivations, most of which use utilize the product and chain rules. In any event, this paper is an alternative approach to demonstrating the same result. In it, I use the truth of the product rule, as well as integration-based techniques, to systematically derive the quotient rule. It may be noted, that when completed initially, the proof was ordered in a reversed fashion. As a result, while all calculations are arranged coherently, their collective construction may not be intuitively apparent.

Phase 1:

We may begin, with a simple statement of the product rule; (when y and z are two arguments of x)

$$\frac{d}{dx}[yz] = z\frac{dy}{dx} + y\frac{dz}{dx}$$

If we integrate both sides of the equivalency;

$$\int \frac{d}{dx}[yz]\,dx = \int z\frac{dy}{dx} + y\frac{dz}{dx}dx$$

$$\int \frac{d}{dx}[yz]\,dx = yz + C; neglecting\ the\ arbitrary\ constant, we\ yield:$$

$$[yz] = \int z\frac{dy}{dx} + y\frac{dz}{dx}\ dx$$

$$+y\frac{dz}{dx} = -\frac{dz}{dx}(-y), so;$$

$$[yz] = \int z\frac{dy}{dx} - \frac{dz}{dx}(-y)\ dx$$

$$[yz] = \int z\frac{dy}{dx} - \frac{dz}{dx}(y - 2y)\ dx$$

$$[yz] = \int z\frac{dy}{dx} - y\frac{dz}{dx} + 2y\frac{dz}{dx}\ dx$$

$$[yz] = \int z\frac{dy}{dx} - y\frac{dz}{dx}dx + \int 2y\frac{dz}{dx}dx$$

Rearranging obtains:

$$[yz] - \int 2y\frac{dz}{dx}dx = \int z\frac{dy}{dx} - y\frac{dz}{dx}dx$$

Let this equation be titled *E1. We will return to this later.*

Phase 2: Proving that $[yz] - \int 2y\frac{dz}{dx}dx = \int z^2\frac{d}{dx}\frac{y}{z}dx$

In order to proceed on E1, we must simplify the left hand side of its expression ie. illustrate the equality stated above.

In doing this, we may first begin with

$$\int z^2\frac{d}{dx}\frac{y}{z}dx$$

and use integration by parts to simplify.

$$\int z^2 \frac{d}{dx}\frac{y}{z}dx$$

Using the conventional formula (for integration by parts) attributed to the product rule;

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

In our case, u = z^2 and v = $\frac{y}{z}$

$$\int z^2 \frac{d}{dx}\frac{y}{z}dx = z^2\frac{y}{z} - \int \frac{y}{z}\frac{dz^2}{dx}dx$$

$$\frac{dz^2}{dx} = \frac{dz^2}{dz}\frac{dz}{dx}[on\ account\ of\ the\ chain\ rule], therefore\ yielding:$$

$$\int z^2 \frac{d}{dx}\frac{y}{z}dx = z^2\frac{y}{z} - \int \frac{y}{z}\frac{dz^2}{dz}\frac{dz}{dx}dx$$

$$\int z^2 \frac{d}{dx}\frac{y}{z}dx = z^2\frac{y}{z} - \int \frac{y}{z}2z\frac{dz}{dx}dx$$

$$\int z^2 \frac{d}{dx}\frac{y}{z}dx = z^2\frac{y}{z} - \int 2y\frac{dz}{dx}dx$$

$$\int z^2 \frac{d}{dx}\frac{y}{z}dx = yz - \int 2y\frac{dz}{dx}dx$$

Thus, we have proven that the left hand side of the expression in *E1* is interchangeable with $\int z^2 \frac{d}{dx}\frac{y}{z}dx$;

Phase 3: Replacing E1's LHS with $\int \mathrm{z}^2 \frac{\mathrm{d}}{\mathrm{dx}}\frac{\mathrm{y}}{\mathrm{z}}\mathrm{dx}$

We may now revert back to E1, and substitute our obtained expression in place of its LHS.

$$[yz] - \int 2y\frac{dz}{dx}dx = \int z\frac{dy}{dx} - y\frac{dz}{dx}dx$$

$$\int z^2 \frac{d}{dx}\frac{y}{z}dx = \int z\frac{dy}{dx} - y\frac{dz}{dx}dx$$

Differentiating both sides (removing either integral) results in;

$$z^2\frac{d}{dx}\frac{y}{z} = z\frac{dy}{dx} - y\frac{dz}{dx}$$

$$\frac{d}{dx}\frac{y}{z} = \frac{z\frac{dy}{dx} - y\frac{dz}{dx}}{z^2}$$

Or;

$$\frac{d\,{}^{y}/_{z}}{dx} = \frac{z\frac{dy}{dx} - y\frac{dz}{dx}}{z^2}$$

Thus proving the quotient rule.

In reiteration, there exist far more straightforward techniques that or of utility in deriving the quotient rule. This technique, however, postulates the truth of the product rule, as well as certain integral conventions not oftentimes seen.

It may be a reasonable proposition that by orienting certain functions in this manner, repetitive iterations of a homogenous kind may allow for the derivation of a generalized product or quotient rule for integrals.