New Principles of Differential Equations II

Hong Lai Zhu *

School of Physics and Electronic Information,Huaibei Normal University, Anhui 235000, China

Abstract

This is the second part of the total paper. Three kinds of Z Transformations are used to get many laws for general solutions of mth-order linear partial differential equations with n variables in the present thesis. Some general solutions of first-order linear partial differential equations, which cannot be obtained by using the characteristic equation method, can be solved by the Z Transformations. By comparing, we find that general solutions of some first-order partial differential equations got by the characteristic equation method are not complete.

Keywords: Z Transformations; banal PDEs; non-banal PDEs; general solutions; the characteristic equation method.

Introduction

In recent years, many numerical methods [1-4] and analytical methods [5-7] have been developed to solve linear partial differential equations (PDEs), the existence [8], uniqueness [9, 10] and stability [11] of their solutions are also the focus of research.

In [12], we used new analytical methods to preliminarily study some laws of general solutions of linear PDEs. In present paper, we will use three kinds of Z Transforms to further study the mth-order linear PDEs with n variables.

1. New principles and methods

In order to obtain the general solutions of PDEs in various orthogonal coordinate systems, we proposed the concepts and laws of the independent variable transformational equations (IVTEs) and the dependent variable transformational equations (DVTEs) in [12]. About the IVTEs, there is an important new theorem:

Theorem 1. *In the domain D, $(D \subset \mathbb{R}^n)$, if $G\left(y_1, y_2, \ldots y_n, u, u_{y_1}, u_{y_2} \ldots u_{y_n}, u_{y_1 y_2}, u_{y_1 y_3} \ldots\right) = 0$ is an arbitrary independent variable transformational equation of a mth-order PDE $F(x_1, x_2 \ldots x_n, u, u_{x_1}, u_{x_2} \ldots u_{x_n}, u_{x_1 x_2}, u_{x_1 x_3} \ldots) = 0$, so*

1.If F=0 is a linear PDE, then G=0 is an mth-order linear PDE.

2.If F=0 is a nonlinear PDE, then G=0 is an mth-order nonlinear PDE.

*E-mail address: honglaizhu@qq.com; honglaizhu@163.com

Proof. Since

$$u_{x_i} = \sum_{p=1}^{n} u_{y_p} \frac{\partial y_p}{\partial x_i}, \tag{1}$$

$$u_{x_i x_j} = \sum_{p=1}^{n} u_{y_p} \frac{\partial^2 y_p}{\partial x_i \partial x_j} + \sum_{p=1}^{n}\sum_{q=1}^{n} u_{y_p y_q} \frac{\partial y_p}{\partial x_i}\frac{\partial y_q}{\partial x_j}, \tag{2}$$

$$\begin{aligned} u_{x_i x_j x_k} =& \sum_{p=1}^{n} u_{y_p} \frac{\partial^3 y_p}{\partial x_i \partial x_j \partial x_k} + \sum_{p=1}^{n}\sum_{q=1}^{n} u_{y_p y_q} \left(\frac{\partial^2 y_p}{\partial x_i \partial x_j}\frac{\partial y_q}{\partial x_k} + \frac{\partial^2 y_p}{\partial x_i \partial x_k}\frac{\partial y_q}{\partial x_j} + \frac{\partial y_p}{\partial x_i}\frac{\partial^2 y_q}{\partial x_j \partial x_k} \right) \\ &+ \sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n} u_{y_p y_q y_r} \frac{\partial y_p}{\partial x_i}\frac{\partial y_q}{\partial x_j}\frac{\partial y_r}{\partial x_k}, \end{aligned} \tag{3}$$

$$\cdots$$

where $i, j, k, p, q, r \in \{1, 2, \ldots n\}$, namely $u_{x_i}, u_{x_i x_j}, u_{x_i x_j x_k} \ldots$ are linear relationship with u_{y_p}, $u_{y_p y_q}$,$u_{y_p y_q y_r} \ldots$, and the highest order of the partial derivatives on both sides of the equations are equal. Therefore each linear term in $F = 0$ is transformed into a new linear term; every non-linear term in $F = 0$ is transformed into a new nonlinear term; and the highest order of the partial derivative of the dependent variable of each term is constant. So the independent variable transformation not only does not change the linearity or non-linearity of the original PDEs, but also does not change their order. Then Theorem 1 is proved. □

For the DVTEs, the specific lth-order transformation $v = h(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots)$ may be linear or non-linear. For the linear transformation, each linear term of an mth-order linear PDE is transformed into a new linear term, so a $(l+m)$th-order linear PDE will be transformed. For an mth-order nonlinear PDE, each nonlinear term will transform a new nonlinear term, normally it will be transformed a $(l+m)$th-order nonlinear PDE.

For the nonlinear transformation, an mth-order linear PDE is usually transformed into a $(l+m)$th-order nonlinear PDE, an mth-order nonlinear PDE may be transformed into a $(l+m)$th-order nonlinear PDE or a $(l+m)$th-order linear PDE.

In $\mathbb{R}^n$ whose independent variables are $x_1, x_2, \ldots x_n$, if the solution of a PDE contains an arbitrary function, the number of independent variables of the arbitrary function is generally not equal to n. Unless for any a smooth function, the essence of a PDE is $0 = 0$, such as

$$u_{x_i} - u_{x_i} = 0, \tag{4}$$

It is obvious that the general solution of Eq. (4) can be an arbitrary first differentiable function with n variables, but this PDE has no practical meaning. We call a PDE whose essence is $0 = 0$ a **banal PDE** and a PDE whose nature is not $0 = 0$ a **non-banal PDE** .

For non-banal PDEs, we propose a conjecture:

Conjecture 1. In $\mathbb{R}^n$, $(n \geq 2)$, if the solution of a non-banal PDE contains an arbitrary function, the number l of independent variables of the arbitrary function satisfies $1 \leq l \leq n-1$.

In [13], we presented Z_1-Z_3 transformations with the following contents.

Z_1 **Transformation.** *In the domain D, $(D \subset \mathbb{R}^n)$, any established mth-order PDE with n space variables $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = 0$, set $y_i = y_i(x_1, \ldots x_n)$ and $u = f(y_1, \ldots y_l)$ are both undetermined mth-differentiable functions $(u, y_i \in C^m(D), 1 \leq i \leq l \leq n)$, $y_1, y_2, \ldots y_l$ are independent of each other, then substitute $u = f(y_1, \ldots y_l)$ and its partial derivatives into*

$F = 0$
1. In case of working out $y_i = y_i(x_1, \ldots x_n)$ *and* $f(y_1, \ldots y_l)$ *then* $u = f(y_1, \ldots y_l)$ *is the solution of* $F = 0$,
2. In case of dividing out $u = f(y_1, \ldots y_l)$ *and its partial derivative, also working out* $y_i = y_i(x_1, \ldots x_n)$ *then* $u = f(y_1, \ldots y_l)$ *is the solution of* $F = 0$, *and* f *is an arbitrary mth-differentiable function,*
3. In case of dividing out $u = f(y_1, \ldots y_l)$ *and its partial derivative, also getting* $k = 0$, *but in fact* $k \neq 0$, *then* $u = f(y_1, \ldots y_l)$ *is not the solution of* $F = 0$, *and* f *is an arbitrary mth-differentiable function.*

Z_2 **Transformation.** *In the domain* D, $(D \subset \mathbb{R}^n)$, *any established mth-order PDE with* n *space variables* $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = 0$, *set* $y_i = y_i(x_1, \ldots x_n)$ *known and* $u = f(y_1, \ldots y_l)$ *undetermined* $(u, y_i \in C^m(D), 1 \leq i \leq l \leq n)$, $y_1, y_2, \ldots y_l$ *are independent of each other, then substitute* $u = f(y_1, \ldots y_l)$ *and its partial derivatives into* $F = 0$
1. In case of working out $f(y_1, \ldots y_l)$, *then* $u = f(y_1, \ldots y_l)$ *is the solution of* $F = 0$,
2. In case of dividing out $u = f(y_1, \ldots y_l)$ *and its partial derivative, also getting* $0 = 0$, *then* $u = f(y_1, \ldots y_l)$ *is the solution of* $F = 0$, *and* f *is an arbitrary mth-differentiable function,*
3. In case of dividing out $u = f(y_1, \ldots y_l)$ *and its partial derivative, also getting* $k = 0$, *but in fact* $k \neq 0$, *then* $u = f(y_1, \ldots y_l)$ *is not the solution of* $F = 0$, *and* f *is an arbitrary mth-differentiable function.*

Z_3 **Transformation.** *In the domain* D, $(D \subset \mathbb{R}^n)$, *any established mth-order PDE with* n *space variables* $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = 0$, *setting* $g(x_1, \ldots x_n)$, $h(y_1, \ldots y_l)$ *and* $y_i = y_i(x_1, \ldots x_n)$ *are all undetermined function,* $y_1, y_2, \ldots y_l$ *are independent of each other,* $(g, h, y_i \in C^m(D), 1 \leq i \leq l \leq n)$, *then substitute* $u = gh(y_1, \ldots y_l)$ *and its partial derivatives into* $F = 0$
1. In case of working out h, g *and* y_i, *then* $u = gh(y_1, \ldots y_l)$ *is the solution of* $F = 0$,
2. In case of dividing out h *and its partial derivative, also working out* g *and* y_i, *then* $u = gh(y_1, \ldots y_l)$ *is the solution of* $F = 0$, *and* h *is an arbitrary mth-differentiable function,*
3. In case of getting $k = 0$, *but in fact* $k \neq 0$, *then* $u = gh(y_1, \ldots y_l)$ *is not the solution of* $F = 0$.

In this paper, we upgrade the Z_3 transformation:

Z_3 **Transformation.** *In the domain* D, $(D \subset \mathbb{R}^n)$, *any established mth-order PDE with* n *space variables* $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = 0$, *setting* $y_i = y_i(x_1, \ldots x_n)$, $z_j = z_j(x_1, \ldots x_n)$, $g(y_1, \ldots y_k)$ *and* $h(z_1, \ldots z_l)$ *are all undetermined function,* $y_1, y_2, \ldots y_k$ *are independent of each other,* $z_1, z_2, \ldots z_l$ *are independent of each other too,* $(y_i, z_j, g, h \in C^m(D)$, $1 \leq i \leq k \leq n$, $1 \leq j \leq l \leq n)$, *then substitute* $u = g(y_1, \ldots y_k)h(z_1, \ldots z_l)$ *and its partial derivatives into* $F = 0$
1. In case of working out y_i, z_j, g *and* h, *then* $u = g(y_1, \ldots y_k)h(z_1, \ldots z_l)$ *is the solution of* $F = 0$,
2. In case of dividing out h *and its partial derivative, also working out* y_i, z_j *and* g, *then* $u = g(y_1, \ldots y_k)h(z_1, \ldots z_l)$ *is the solution of* $F = 0$, *and* h *is an arbitrary mth-differentiable function,*
3. In case of dividing out g,h and their partial derivative, also working out y_i *and* z_j, *then* $u = g(y_1, \ldots y_k)h(z_1, \ldots z_l)$ *is the solution of* $F = 0$, g *and* h *are arbitrary mth-differentiable functions,*
4. In case of getting $k = 0$, *but in fact* $k \neq 0$, *then* $u = g(y_1, \ldots y_k)h(z_1, \ldots z_l)$ *is not the solution of* $F = 0$.

2. General solutions' laws of linear partial differential equations with variable coefficients

In this section, if there is no special interpretation, the acquiescent independent variables of $\mathbb{R}^n$ are $x_1, x_2, \ldots x_n$. $a_i = a_i(x_1, \ldots x_k)$, $a_{j_i} = a_{j_i}(x_1, \ldots x_k)$, $a_{i_1 i_2 \ldots i_k} = a_{i_1 i_2 \ldots i_k}(x_1, \ldots x_k)$, $b_i = b_i(x_1, \ldots x_k)$, and $b_{j_i} = b_{j_i}(x_1, \ldots x_k)$ are arbitrary known functions. $y_i = y_i(x_1, \ldots x_k)$, $D_{x_i} \triangleq \frac{\partial}{\partial x_i}$, $u_{x_1 x_2 \ldots x_k}^{(i_1 i_2 \ldots i_k)} \triangleq \frac{\partial^{i_1}}{\partial x_1} \frac{\partial^{i_2}}{\partial x_2} \ldots \frac{\partial^{i_k}}{\partial x_k} u$. c_i and λ_i are arbitrary constants, f and f_i are arbitrary smooth functions, $(i, j = 1, 2, \ldots)$, $(1 \le k \le n)$.

Theorem 1. *In $\mathbb{R}^n$, if the exact solutions $u = y_i(x_1, \ldots x_k)$ of Eq. (5), which are independent of each other, are known, $(1 \le i \le l \le k-1)$,*

$$a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = 0, \tag{5}$$

then the general solution of Eq. (5) is

$$u = f(y_1, y_2, \ldots y_l, x_{k+1}, x_{k+2}, \ldots x_n). \tag{6}$$

Prove. By Z_1 Transformation, set $u(x_1, \ldots x_n) = f(y_1, y_2, \ldots y_l, x_{k+1}, x_{k+2}, \ldots x_n)$, then

$$\begin{aligned}
&a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} \\
&= a_1 \left(f_{y_1} y_{1x_1} + f_{y_2} y_{2x_1} + \ldots + f_{y_l} y_{lx_1}\right) + a_2 \left(f_{y_1} y_{1x_2} + f_{y_2} y_{2x_2} + \ldots + f_{y_l} y_{lx_2}\right) + \ldots \\
&+ a_k \left(f_{y_1} y_{1x_k} + f_{y_2} y_{2x_k} + \ldots + f_{y_l} y_{lx_k}\right) \\
&= f_{y_1} \left(a_1 y_{1x_1} + a_2 y_{1x_2} + \ldots + a_k y_{1x_k}\right) + f_{y_2} \left(a_1 y_{2x_1} + a_2 y_{2x_2} + \ldots + a_k y_{2x_k}\right) + \ldots \\
&+ f_{y_l} \left(a_1 y_{lx_1} + a_2 y_{lx_2} + \ldots + a_k y_{lx_k}\right) = 0.
\end{aligned}$$

Since $a_1 y_{ix_1} + a_2 y_{ix_2} + \ldots + a_k y_{ix_k} = 0$, $(1 \le i \le l \le k-1)$, the above equation is correct eternally. So Theorem 1 is proved. According to the characteristic equation method and Conjecture 1 we can know: Usually $l = k - 1$. □

According to Theorem 1, if the exact solutions $u = y_i(x_1, \ldots x_n)$ of $a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_n u_{x_n} = 0$, which are independent of each other, are known, $(1 \le i \le l \le n-1)$, then its general solution is $u = f(y_1, y_2, \ldots y_l)$.

Theorem 2. *In $\mathbb{R}^n$, if the general solution $u = f(y_1, y_2, \ldots y_l, x_{k+1}, x_{k+2}, \ldots x_n)$ of $(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k})\, u = 0$ is known, then*

$$(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^2 u = 0, \tag{7}$$

the general solution of Eq. (7) is

$$u = f_1(y_1, \ldots y_l, x_{k+1}, \ldots x_n) + (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)\, f_2(y_1, \ldots y_l, x_{k+1}, \ldots x_n). \tag{8}$$

Prove. Because the general solution $u = f(y_1, y_2, \ldots y_l, x_{k+1}, x_{k+2}, \ldots x_n)$ of $(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})\, u = 0$ is known, apparently $u = f(y_1, y_2, \ldots y_l, x_{k+1}, x_{k+2}, \ldots x_n)$ is also the solution of $(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^2 u = 0$, setting $u = g(x_1, \ldots x_k)\, f = \lambda_s x_s f$ is the solution of $(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^2 u = 0$ too, that is $= g(x_1, \ldots x_k) = \lambda_s x_s$, and

λ_s are arbitrary constants, $(s = 1, 2, \dots k)$, then

$$\begin{aligned}
&(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^2 u = (\sum_{i=1}^{k} b_i^2 D_{x_i}^2 + 2\sum_{i<j} b_ib_jD_{x_i}D_{x_j})(gf)\\
&= \sum_{i=1}^{k} b_i^2 \left(g_{x_ix_i}f + 2g_{x_i}f_{x_i} + gf_{x_ix_i}\right) + 2\sum_{i<j} b_ib_j \left(g_{x_ix_j}f + g_{x_i}f_{x_j} + g_{x_j}f_{x_i} + gf_{x_ix_j}\right)\\
&= g(\sum_{i=1}^{k} b_i^2 f_{x_ix_i} + 2\sum_{i<j} b_ib_jf_{x_ix_j}) + f(\sum_{i=1}^{k} b_i^2 g_{x_ix_i} + 2\sum_{i<j} b_ib_jg_{x_ix_j}) + 2\sum_{i=1}^{k} b_i^2 g_{x_if_{x_i}}\\
&+2\sum_{i<j} b_ib_j\left(g_{x_i}f_{x_j} + g_{x_j}f_{x_i}\right) = 2\sum_{i=1}^{k} b_i^2 g_{x_if_{x_i}} + 2\sum_{i<j} b_ib_jg_{x_if_{x_j}} + 2\sum_{i<j} b_ib_jg_{x_jf_{x_i}}\\
&= 2b_s^2\lambda_sf_{x_s} + 2\lambda_sb_s\sum_{s<j} b_jf_{x_j} + 2\lambda_sb_s\sum_{i<s} b_if_{x_i} = 2\lambda_sb_s\sum_{i=1}^{k} b_if_{x_i} = 0.
\end{aligned}$$

That $u = \lambda_s x_s f$ is the solution of $(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^2 u$ is proved. So its general solution is (8). $\square$

According to Theorem 2, we present Theorem 3.

Theorem 3. *In $\mathbb{R}^n$, if the general solution $u = f\left(y_1, y_2, \dots y_l, x_{k+1}, x_{k+2}, \dots x_n\right)$ of $(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})\, u = 0$ is known, then the general solution of*

$$(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^m u = 0, \tag{9}$$

is

$$u = \sum_{j=1}^{m} (\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_kx_k)^{j-1} f_j\left(y_1, \dots y_l, x_{k+1}, \dots x_n\right). \tag{10}$$

Prove. Two mathematical formulas are required to prove the theorem:

$$(a_1 + a_2 + \cdots + a_k)^m = \sum_{l_1+l_2+\ldots+l_k=m} \frac{m!}{l_1!l_2!\cdots l_k!} a_1^{l_1}a_2^{l_2}\dots a_k^{l_k}, \tag{11}$$

$$D_{x_i}^n(uv) = \sum_{k=0}^{n} C_n^k u_{x_i}^{(n-k)} v_{x_i}^{(k)}. \tag{12}$$

Then

$$(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^m u = \sum_{l_1+l_2+\ldots+l_k=m} \frac{m!}{l_1!l_2!\dots l_k!} b_1^{l_1}b_2^{l_2}\dots b_k^{l_k} D_{x_1}^{l_1}D_{x_2}^{l_2}\dots D_{x_k}^{l_k} u. \tag{13}$$

According to Theorem 2, we know that $u = f_1\left(y_1, \dots y_l, x_{k+1}, \dots x_n\right) + (\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_kx_k)$ $f_2\left(y_1, y_2, \dots y_l, x_{k+1}, \dots x_n\right)$ is the general solution of $(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^2 u = 0$. Using mathematical induction, we set

$$u = \Gamma = \sum_{j=1}^{m} (\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_kx_k)^{j-1} f_j\left(y_1, \dots y_l, x_{k+1}, \dots x_n\right), \tag{14}$$

is the general solution of $(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^m u = D^{(m)}u = 0$, namely $D^{(m)}\Gamma = 0$.

$$(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^{m+1} u = D^{(m+1)}u = 0. \tag{15}$$

Note $u = \Gamma$ is also the solution of Eq. (15). Setting $u = g(x_1, \dots x_k)\Gamma = \lambda_s x_s \Gamma$ is the solution of Eq. (15) too, that is $g(x_1, \dots x_k) = \lambda_s x_s$, and λ_s are arbitrary constants, $(s = 1, 2, \dots k)$, then

$$D_{x_s}^{n}(g\Gamma) = \sum_{k=0}^{n} C_n^k g_{x_s}^{(n-k)} \Gamma_{x_s}^{(k)} = \lambda_s x_s \Gamma_{x_s}^{(n)} + n\lambda_s \Gamma_{x_s}^{(n-1)}$$

$$D_{x_i}^{l}(g\Gamma) = \sum_{k=0}^{l} C_l^k g_{x_i}^{(l-k)} \Gamma_{x_i}^{(k)} = \lambda_s x_s \Gamma_{x_i}^{(l)}$$

$$D_{x_i}^{l} D_{x_s}^{n}(g\Gamma) = D_{x_s}^{n} D_{x_i}^{l}(g\Gamma) = \lambda_s x_s \Gamma_{x_i x_s}^{(l,n)} + n\lambda_s \Gamma_{x_i x_s}^{(l,n-1)}, (s \neq i).$$

The same can be obtained:

$$D_{x_i}^{l_i} D_{x_s}^{l_s}(g\Gamma) = D_{x_s}^{l_s} D_{x_i}^{l_i}(g\Gamma) = \lambda_s x_s \Gamma_{x_i x_s}^{(l_i,l_s)} + l_s\lambda_s \Gamma_{x_i x_s}^{(l_i,l_s-1)}, (s \neq i) \tag{16}$$

$$D_{x_j}^{l_j} D_{x_i}^{l_i} D_{x_s}^{l_s}(g\Gamma) = D_{x_s}^{l_s} D_{x_i}^{l_i} D_{x_j}^{l_j}(g\Gamma) = \lambda_s x_s \Gamma_{x_i x_j x_s}^{(l_i,l_j,l_s)} + l_s\lambda_s \Gamma_{x_i x_j x_s}^{(l_i,l_j,l_s-1)}, (s \neq i, j)$$

$$D_{x_1}^{l_1} D_{x_2}^{l_2} \dots D_{x_s}^{l_s} \dots D_{x_k}^{l_k}(g\Gamma) = \lambda_s x_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1,l_2,\dots l_s,\dots l_k)} + l_s\lambda_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1,l_2,\dots l_s-1,\dots l_k)}. \tag{17}$$

Then

$$\begin{aligned}
&(b_1 D_{x_1} + b_2 D_{x_2} + \dots + b_k D_{x_k})^{m+1} \\
&= (b_1 D_{x_1} + b_2 D_{x_2} + \dots + b_k D_{x_k}) \left(\sum_{l_1+l_2+\dots+l_k=m} \tfrac{m!}{l_1! l_2! \dots l_k!} b_1^{l_1} b_2^{l_2} \dots b_k^{l_k} D_{x_1}^{l_1} D_{x_2}^{l_2} \dots D_{x_k}^{l_k} \right) \\
&= \sum_{l_1+l_2+\dots+l_k=m} \tfrac{m!}{l_1! l_2! \dots l_k!} b_1^{l_1} b_2^{l_2} \dots b_k^{l_k} (b_1 D_{x_1} + b_2 D_{x_2} + \dots + b_k D_{x_k}) D_{x_1}^{l_1} D_{x_2}^{l_2} \dots D_{x_k}^{l_k} \\
&= \sum_{l_1+l_2+\dots+l_k=m} \tfrac{m!}{l_1! l_2! \dots l_k!} (b_1^{l_1+1} b_2^{l_2} \dots b_k^{l_k} D_{x_1}^{l_1+1} D_{x_2}^{l_2} \dots D_{x_k}^{l_k} + b_1^{l_1} b_2^{l_2+1} \dots b_k^{l_k} D_{x_1}^{l_1} D_{x_2}^{l_2+1} \dots D_{x_k}^{l_k} \\
&+ \dots + b_1^{l_1} b_2^{l_2} \dots b_s^{l_s+1} \dots b_k^{l_k} D_{x_1}^{l_1} D_{x_2}^{l_2} \dots D_{x_s}^{l_s+1} \dots D_{x_k}^{l_k} + \dots + b_1^{l_1} b_2^{l_2} \dots b_k^{l_k+1} D_{x_1}^{l_1} D_{x_2}^{l_2} \dots D_{x_k}^{l_k+1}).
\end{aligned}$$

Because of

$$(b_1 D_{x_1} + b_2 D_{x_2} + \dots + b_k D_{x_k})^{m+1} \Gamma = 0.$$

So

$$\begin{aligned}
&(b_1 D_{x_1} + b_2 D_{x_2} + \dots + b_k D_{x_k})^{m+1} (g\Gamma) \\
&= \sum_{l_1+l_2+\dots+l_k=m} \tfrac{m!}{l_1! l_2! \dots l_k!} (b_1^{l_1+1} b_2^{l_2} \dots b_k^{l_k} D_{x_1}^{l_1+1} D_{x_2}^{l_2} \dots D_{x_k}^{l_k} + b_1^{l_1} b_2^{l_2+1} \dots b_k^{l_k} D_{x_1}^{l_1} D_{x_2}^{l_2+1} \dots D_{x_k}^{l_k} \\
&+ \dots + b_1^{l_1} b_2^{l_2} \dots b_s^{l_s+1} \dots b_k^{l_k} D_{x_1}^{l_1} D_{x_2}^{l_2} \dots D_{x_s}^{l_s+1} \dots D_{x_k}^{l_k} + \dots + b_1^{l_1} b_2^{l_2} \dots b_k^{l_k+1} D_{x_1}^{l_1} D_{x_2}^{l_2} \dots D_{x_k}^{l_k+1})(g\Gamma) \\
&= \sum_{l_1+l_2+\dots+l_k=m} \tfrac{m!}{l_1! l_2! \dots l_k!} (b_1^{l_1+1} b_2^{l_2} \dots b_k^{l_k} (\lambda_s x_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1+1,l_2,\dots l_s,\dots l_k)} + l_s\lambda_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1+1,l_2,\dots l_s-1,\dots l_k)}) \\
&+ b_1^{l_1} b_2^{l_2+1} \dots b_k^{l_k} (\lambda_s x_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1,l_2+1,\dots l_s,\dots l_k)} + l_s\lambda_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1,l_2+1,\dots l_s-1,\dots l_k)}) + \dots \\
&+ b_1^{l_1} b_2^{l_2} \dots b_s^{l_s+1} \dots b_k^{l_k} (\lambda_s x_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1,l_2,\dots l_s+1,\dots l_k)} + l_s\lambda_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1,l_2,\dots l_s,\dots l_k)}) + \dots \\
&+ b_1^{l_1} b_2^{l_2} \dots b_k^{l_k+1} (\lambda_s x_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1,l_2,\dots l_s,\dots l_k+1)} + l_s\lambda_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1,l_2,\dots l_s-1,\dots l_k+1)})).
\end{aligned}$$

Notice:

$$\begin{aligned}
&\sum_{l_1+l_2+\dots+l_k=m} \tfrac{m!}{l_1! l_2! \dots l_k!} (b_1^{l_1+1} b_2^{l_2} \dots b_k^{l_k} (\lambda_s x_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1+1,l_2,\dots l_s,\dots l_k)}) + b_1^{l_1} b_2^{l_2+1} \dots b_k^{l_k} (\lambda_s x_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1,l_2+1,\dots l_s,\dots l_k)}) \\
&+ \dots + b_1^{l_1} b_2^{l_2} \dots b_s^{l_s+1} \dots b_k^{l_k} (\lambda_s x_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1,l_2,\dots l_s+1,\dots l_k)}) + \dots + b_1^{l_1} b_2^{l_2} \dots b_k^{l_k+1} (\lambda_s x_s \Gamma_{x_1 x_2 \dots x_s \dots x_k}^{(l_1,l_2,\dots l_s,\dots l_k+1)}) \\
&= \lambda_s x_s \sum_{l_1+l_2+\dots+l_k=m} \tfrac{m!}{l_1! l_2! \dots l_k!} (b_1^{l_1+1} b_2^{l_2} \dots b_k^{l_k} D_{x_1}^{l_1+1} D_{x_2}^{l_2} \dots D_{x_k}^{l_k} + b_1^{l_1} b_2^{l_2+1} \dots b_k^{l_k} D_{x_1}^{l_1} D_{x_2}^{l_2+1} \dots D_{x_k}^{l_k} \\
&+ \dots + b_1^{l_1} b_2^{l_2} \dots b_k^{l_k+1} D_{x_1}^{l_1} D_{x_2}^{l_2} \dots D_{x_k}^{l_k+1})\Gamma \\
&= \lambda_s x_s (b_1 D_{x_1} + b_2 D_{x_2} + \dots + b_k D_{x_k})^{m+1} \Gamma = 0.
\end{aligned}$$

Then

$$
\begin{aligned}
&(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k})^{m+1}(g\Gamma)\\
&=\lambda_s\sum_{l_1+l_2+\ldots+l_k=m}\tfrac{m!l_s}{l_1!l_2!\ldots l_k!}(b_1^{l_1+1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1+1,l_2,\ldots l_s-1,\ldots l_k)}\\
&+b_1^{l_1}b_2^{l_2+1}\ldots b_s^{l_s}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2+1,\ldots l_s-1,\ldots l_k)}+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s,\ldots l_k)}\\
&+\cdots+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k+1}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s-1,\ldots l_k+1)}).
\end{aligned}
$$

Since

$$
\begin{aligned}
&(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k})^{m}\Gamma\\
&=(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k})\,(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k})^{m-1}\Gamma\\
&=(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k})\sum_{l_1+l_2+\cdots+(l_s-1)+\cdots+l_k=m-1}\tfrac{(m-1)!}{l_1!l_2!\ldots(l_s-1)!\ldots l_k!}b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s-1}\\
&\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s-1,\ldots l_k)}\\
&=\sum_{l_1+l_2+\cdots+(l_s-1)+\cdots+l_k=m-1}\tfrac{(m-1)!}{l_1!l_2!\ldots(l_s-1)!\ldots l_k!b_s}b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k}\,(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k})\\
&\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s-1,\ldots l_k)}\\
&=\sum_{l_1+l_2+\cdots+(l_s-1)+\cdots+l_k=m-1}\tfrac{(m-1)!}{l_1!l_2!\ldots(l_s-1)!\ldots l_k!b_s}(b_1^{l_1+1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1+1,l_2,\ldots l_s-1,\ldots l_k)}\\
&+b_1^{l_1}b_2^{l_2+1}\ldots b_s^{l_s}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2+1,\ldots l_s-1,\ldots l_k)}+\cdots+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s,\ldots l_k)}\cdots\\
&+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k+1}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s-1,\ldots l_k+1)})=0.
\end{aligned}
$$

That is

$$
\begin{aligned}
&\sum_{l_1+l_2+\cdots+(l_s-1)+\ldots+l_k=m-1}\tfrac{(m-1)!}{l_1!l_2!\ldots(l_s-1)!\ldots l_k!}(b_1^{l_1+1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1+1,l_2,\ldots l_s-1,\ldots l_k)}\\
&+b_1^{l_1}b_2^{l_2+1}\ldots b_s^{l_s}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2+1,\ldots l_s-1,\ldots l_k)}+\cdots+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s,\ldots l_k)}+\cdots\\
&+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k+1}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s-1,\ldots l_k+1)})=0.
\end{aligned}
$$

So

$$
\begin{aligned}
&\lambda_s\sum_{l_1+l_2+\cdots+l_k=m}\tfrac{m!l_s}{l_1!l_2!\ldots l_k!}(b_1^{l_1+1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1+1,l_2,\ldots l_s-1,\ldots l_k)}\\
&+b_1^{l_1}b_2^{l_2+1}\ldots b_s^{l_s}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2+1,\ldots l_s-1,\ldots l_k)}+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s,\ldots l_k)}\\
&+\cdots+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k+1}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s-1,\ldots l_k+1)})\\
&=\lambda_s m\sum_{l_1+l_2+\cdots+(l_s-1)+\cdots+l_k=m-1}\tfrac{(m-1)!}{l_1!l_2!\ldots(l_s-1)!\ldots l_k!}(b_1^{l_1+1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1+1,l_2,\ldots l_s-1,\ldots l_k)}\\
&+b_1^{l_1}b_2^{l_2+1}\ldots b_s^{l_s}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2+1,\ldots l_s-1,\ldots l_k)}+\cdots+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_k^{l_k}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s,\ldots l_k)}+\cdots\\
&+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k+1}\Gamma_{x_1x_2\ldots x_s\ldots x_k}^{(l_1,l_2,\ldots l_s-1,\ldots l_k+1)})=0.
\end{aligned}
$$

Namely

$$(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k})^{m+1}(\lambda_s x_s\Gamma)=0. \tag{18}$$

Since $s=1,2,\ldots k$, according to the principle of linear superposition, the general solution of Eq. (9) is (10), so the theorem is proved. □

In $\mathbb{R}^n$, for the mth-order linear PDE with variable coefficients

$$\sum_{i_1+i_2+\ldots+i_k=m} a_{i_1i_2\ldots i_k}u_{x_1x_2\ldots x_k}^{(i_1i_2\ldots i_k)}=0, \tag{19}$$

where i_j are natural number, $1\le j\le k\le n$, If Eq. (19) can be translated into

$$
\begin{aligned}
&(b_{1_1}D_{x_1}+b_{1_2}D_{x_2}+\cdots+b_{1_k}D_{x_k})\,(b_{2_1}D_{x_1}+b_{2_2}D_{x_2}+\cdots+b_{2_k}D_{x_k})\ldots\\
&(b_{m_1}D_{x_1}+b_{m_2}D_{x_2}+\cdots+b_{m_k}D_{x_k})\,u=0.
\end{aligned} \tag{20}
$$

For

$$(b_{j_1}D_{x_1}+b_{j_2}D_{x_2}+\cdots+b_{j_k}D_{x_k})\,u=b_{j_1}u_{x_1}+b_{j_2}u_{x_2}+\cdots+b_{j_k}u_{x_k}=0,(j=1,2,\ldots m). \quad (21)$$

If the particular solutions $u=y_{j_s}(x_1,\ldots x_k)$ of Eq. (21), which are independent of each other, are all known, $(1\le s\le l_j)$, by Theorem 1 the general solution of Eq. (19) is

$$u=\sum_{j=1}^{m} f_j\left(y_{j_1},y_{j_2},\ldots y_{j_{l_j}},x_{k+1},x_{k+2},\ldots x_n\right). \quad (22)$$

If Eq. (19) can be translated into:

$$\prod_{j=1}^{q}(b_{j_1}D_{x_1}+b_{j_2}D_{x_2}+\cdots+b_{j_k}D_{x_k})^{p_j}u=0, \quad (23)$$

where $\sum\limits_{j=1}^{q}p_j=m$, its general solution can be written by Theorem 3.

In $\mathbb{R}^n$, for the mth-order linear PDE with variable coefficients

$$\sum_{0\le i_1+i_2+\cdots+i_k\le m} a_{i_1i_2\ldots i_k}u^{(i_1i_2\ldots i_k)}_{x_1x_2\ldots x_k}=A\left(x_1,x_2,\ldots x_n\right), \quad (24)$$

where $A(x_1,x_2,\ldots x_n)$ is an arbitrary known function. In general, we need to solve the particular solution of (24) first, and then use the general solution of its homogeneous equation to obtain its general solution. Such as

Theorem 4. *In $\mathbb{R}^n$, if particular solutions $u=y_i\left(x_1,\ldots x_k\right)$ of $a_1u_{x_1}+a_2u_{x_2}+\cdots+a_ku_{x_k}=0$ are all known, $(1\le i\le k-1)$, then*

$$a_1u_{x_1}+a_2u_{x_2}+\cdots+a_ku_{x_k}=A\left(x_1,x_2,\ldots x_n\right), \quad (25)$$

the general solution of Eq. (25) is

$$u=f\left(y_1,\ldots y_{k-1},x_{k+1},\ldots x_n\right)+\int\frac{A\left(y_1,\ldots y_k,x_{k+1},\ldots x_n\right)dy_k}{a_1y_{k_{x_1}}+a_2y_{k_{x_2}}+\cdots+a_ky_{k_{x_k}}}, \quad (26)$$

where $y_1,y_2,\ldots y_k$ are independent of each other, and $x_j=x_j(y_1,y_2,\ldots y_k)$ should be got, $(1\le j\le k)$.

Prove. By Z_1 Transformation, set $u(x_1,x_2,\ldots x_n)=u(y_1,y_2,\ldots y_k,x_{k+1},x_{k+2},\ldots x_n)$ and $a_i(x_1,x_2,\ldots x_n)=a_i(y_1,y_2,\ldots y_k,x_{k+1},x_{k+2},\ldots x_n)$, where $y_s=y_s(x_1,x_2,\ldots x_k)$,$(s=1,2,\ldots k)$, and $y_1,y_2,\ldots y_k$ are independent of each other, then

$$\begin{aligned}&a_1u_{x_1}+a_2u_{x_2}+\cdots+a_ku_{x_k}\\&=a_1\sum_{s=1}^{k}y_{s_{x_1}}u_{y_s}+a_2\sum_{s=1}^{k}y_{s_{x_2}}u_{y_s}+\cdots+a_k\sum_{s=1}^{k}y_{s_{x_k}}u_{y_s}\\&=\left(a_1y_{1_{x_1}}+a_2y_{1_{x_2}}+\cdots+a_ky_{1_{x_k}}\right)u_{y_1}+\left(a_1y_{2_{x_1}}+a_2y_{2_{x_2}}+\cdots+a_ky_{2_{x_k}}\right)u_{y_2}+\cdots\\&+\left(a_1y_{k_{x_1}}+a_2y_{k_{x_2}}+\cdots+a_ky_{k_{x_k}}\right)u_{y_k}=A\left(x_1,x_2,\ldots x_n\right).\end{aligned} \quad (27)$$

If the particular solutions $u = y_i(x_1, \ldots x_k)$ of $a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = 0$ are all known $(1 \le i \le k-1)$, then Eq. (27) can be translated into

$$\left(a_1 y_{k_{x_1}} + a_2 y_{k_{x_2}} + \cdots + a_k y_{k_{x_k}}\right) u_{y_k} = A\left(x_1, x_2, \ldots x_n\right). \tag{28}$$

A particular solution of Eq. (28) is

$$u = \int \frac{A\left(y_1, \ldots y_k, x_{k+1}, \ldots x_n\right) dy_k}{a_1 y_{k_{x_1}} + a_2 y_{k_{x_2}} + \cdots + a_k y_{k_{x_k}}}. \tag{29}$$

To compute (29) we need to find the first differentiable $y_k = y_k(x_1, x_2, \ldots x_k)$ and $y_1, y_2, \ldots y_k$ independent of each other, and can get $x_j = x_j(y_1, y_2, \ldots y_k)$, then the general solution of Eq. (25) is (26). So Theorem 4 is proved. □

Theorem 5. *In $\mathbb{R}^n$, if particular solutions $u = y_i(x_1, \ldots x_k)$ of $b_1 u_{x_1} + b_2 u_{x_2} + \cdots + b_k u_{x_k} = 0$ are all known, $(1 \le i \le k-1)$, then*

$$\left(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}\right)^m u = A\left(x_1, \ldots x_n\right), \tag{30}$$

the general solution of Eq. (30) is

$$\begin{aligned} u = &\sum_{j=1}^{m} \left(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k\right)^{j-1} f_j\left(y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n\right) \\ &+ \frac{\int \ldots \int A\left(y_1, \ldots y_k, x_{k+1}, \ldots x_n\right) d^m y_k}{\left(b_1 c_1 + b_2 c_2 + \cdots + b_k c_k\right)^m}, \end{aligned} \tag{31}$$

where $y_1, y_2, \ldots y_k$ are independent of each other, $y_k = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k$ and $x_j = x_j$ $(y_1, y_2, \ldots y_k)$ can be got, $(1 \le j \le k)$.

Prove. By Z_1 Transformation, set $u(x_1, x_2, \ldots x_n) = u(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n)$ and $a_i(x_1, x_2, \ldots x_n) = a_i(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n)$, where $y_s = y_s(x_1, x_2, \ldots x_k)$,$(s = 1, 2, \ldots k)$, $y_1, y_2, \ldots y_k$ are independent of each other, and

$$y_k = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k.$$

Then

$$\begin{aligned} &\left(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}\right)^m u \\ &= \left(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}\right)^{m-1} \left(b_1 \sum_{s=1}^{k} y_{s_{x_1}} u_{y_s} + b_2 \sum_{s=1}^{k} y_{s_{x_2}} u_{y_s} + \cdots + b_k \sum_{s=1}^{k} y_{s_{x_k}} u_{y_s}\right) \\ &= \left(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}\right)^{m-1} (\left(b_1 y_{1_{x_1}} + b_2 y_{1_{x_2}} + \cdots + b_k y_{1_{x_k}}\right) u_{y_1} \\ &+ \left(b_1 y_{2_{x_1}} + b_2 y_{2_{x_2}} + \cdots + b_k y_{2_{x_k}}\right) u_{y_2} + \cdots + \left(b_1 y_{k_{x_1}} + b_2 y_{k_{x_2}} + \cdots + b_k y_{k_{x_k}}\right) u_{y_k}) \\ &= \left(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}\right)^{m-1} \left(\left(b_1 y_{k_{x_1}} + b_2 y_{k_{x_2}} + \cdots + b_k y_{k_{x_k}}\right) u_{y_k}\right) \\ &= \left(b_1 c_1 + b_2 c_2 + \cdots + b_k c_k\right) \left(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}\right)^{m-1} u_{y_k} \\ &= \left(b_1 c_1 + b_2 c_2 + \cdots + b_k c_k\right) \left(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}\right)^{m-2} (b_1 \sum_{s=1}^{k} y_{s_{x_1}} u_{y_k y_s} + \\ &b_2 \sum_{s=1}^{k} y_{s_{x_2}} u_{y_k y_s} + \cdots + b_k \sum_{s=1}^{k} y_{s_{x_k}} u_{y_k y_s}) \\ &= \left(b_1 c_1 + b_2 c_2 + \cdots + b_k c_k\right) \left(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}\right)^{m-2} \\ &(\left(b_1 y_{1_{x_1}} + b_2 y_{1_{x_2}} + \cdots + b_k y_{1_{x_k}}\right) u_{y_k y_1} + \left(b_1 y_{2_{x_1}} + b_2 y_{2_{x_2}} + \cdots + b_k y_{2_{x_k}}\right) u_{y_k y_2} + \cdots \\ &+ \left(b_1 y_{k_{x_1}} + b_2 y_{k_{x_2}} + \cdots + b_k y_{k_{x_k}}\right) u_{y_k y_k}) \\ &= \left(b_1 c_1 + b_2 c_2 + \cdots + b_k c_k\right)^2 \left(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}\right)^{m-2} u_{y_k y_k} \\ &= \ldots = \left(b_1 c_1 + b_2 c_2 + \cdots + b_k c_k\right)^m u_{y_k}^{(m)}. \end{aligned}$$

Namely

$$(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^m u = (b_1 c_1 + b_2 c_2 + \cdots + b_k c_k)^m u_{y_k}^{(m)} = A\left(y_1, \ldots y_k, x_{k+1}, \ldots x_n\right). \tag{32}$$

The particular solution of Eq. (32) is

$$u = \frac{\int \ldots \int A\left(y_1, \ldots y_k, x_{k+1}, \ldots x_n\right) d^m y_k}{\left(b_1 c_1 + b_2 c_2 + \cdots + b_k c_k\right)^m}.$$

According to Theorem 3, the general solution of Eq. (30) is (31), so the theorem is proved. □

Theorem 6. *In $\mathbb{R}^n$, if the particular solution $g = g(x_1, \ldots x_k)$ of $a_1 g_{x_1} + a_2 g_{x_2} + \cdots + a_k g_{x_k} + a_{k+1} g = 0$ and exact solutions $u = y_i(x_1, \ldots x_k)$ of $a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = 0$, which are independent of each other, are all known, $(1 \le i \le l \le k-1)$, then*

$$a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u = 0, \tag{33}$$

the general solution of Eq. (33) is

$$u = g\left(x_1, \ldots x_k\right) f\left(y_1, \ldots y_l, x_{k+1}, \ldots x_n\right). \tag{34}$$

Prove. By Z_3 Transformation, set $u = g(x_1, \ldots x_k) f(y_1, \ldots y_l, x_{k+1}, \ldots x_n)$, then

$$\begin{aligned}
& a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u \\
& = a_1 \left(f g_{x_1} + g \sum_{i=1}^{l} f_{y_i} y_{i x_1} \right) + a_2 \left(f g_{x_2} + g \sum_{i=1}^{l} f_{y_i} y_{i x_2} \right) + \cdots + a_k \left(f g_{x_k} + g \sum_{i=1}^{l} f_{y_i} y_{i x_k} \right) \\
& + a_{k+1} g f \\
& = f\left(a_1 g_{x_1} + a_2 g_{x_2} + \cdots + a_k g_{x_k} + a_{k+1} g\right) + g \sum_{i=1}^{l} f_{y_i} \left(a_1 y_{i x_1} + a_2 y_{i x_2} + \cdots + a_k y_{i x_k}\right) \\
& = 0.
\end{aligned}$$

Since $a_1 g_{x_1} + a_2 g_{x_2} + \cdots + a_k g_{x_k} + a_{k+1} g = 0$ and $a_1 y_{i x_1} + a_2 y_{i x_2} + \cdots + a_k y_{i x_k} = 0$. The above equation is correct eternally, so Theorem 6 is proved. □

Theorem 7. *In $\mathbb{R}^n$, if the general solution $u = g(x_1, \ldots x_k) f(y_1, \ldots y_l, x_{k+1}, \ldots x_n)$ of $(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1}) u = 0$ is known, then*

$$\left(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1}\right)^2 u = 0, \tag{35}$$

the general solution of Eq. (35) is

$$\begin{aligned}
u = & g\left(x_1, \ldots x_k\right) \\
& \left(f_1\left(y_1, \ldots y_l, x_{k+1}, \ldots x_n\right) + \left(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k\right) f_2\left(y_1, \ldots y_l, x_{k+1}, \ldots x_n\right)\right).
\end{aligned} \tag{36}$$

Prove. Because the general solution $u = g(x_1, \ldots x_k) f(y_1, \ldots y_l, x_{k+1}, \ldots x_n)$ of $(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1}) u = 0$ is known, apparently $u = g(x_1, \ldots x_k) f(y_1, \ldots y_l, x_{k+1}, \ldots x_n)$ is also the solution of $(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^2 u = 0$, setting

$$u = h\Lambda = \lambda_s x_s g\left(x_1, \ldots x_k\right) f\left(y_1, \ldots y_l, x_{k+1}, \ldots x_n\right), (1 \le s \le k), \tag{37}$$

where $h=\lambda_s x_s, \Lambda=g(x_1,\ldots x_k)f(y_1,\ldots y_l,x_{k+1},\ldots x_n)$, and λ_s are arbitrary constants. Assuming (37) is a solution of (35), then

$$
\begin{aligned}
&\left(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1}\right)^2u\\
&=\left(\sum_{i=1}^{k}b_i^2D_{x_i}^2+b_{k+1}^2+2\sum_{1\le i<j\le k}b_ib_jD_{x_i}D_{x_j}+2b_{k+1}\sum_{i=1}^{k}b_iD_{x_i}\right)(h\Lambda)\\
&=\sum_{i=1}^{k}b_i^2\left(h_{x_ix_i}\Lambda+2h_{x_i}\Lambda_{x_i}+h\Lambda_{x_ix_i}\right)+b_{k+1}^2h\Lambda\\
&+2\sum_{1\le i<j\le k}b_ib_j\left(h_{x_ix_j}\Lambda+h_{x_i}\Lambda_{x_j}+h_{x_j}\Lambda_{x_i}+h\Lambda_{x_ix_j}\right)+2b_{k+1}\sum_{i=1}^{k}b_i\left(h_{x_i}\Lambda+h\Lambda_{x_i}\right)\\
&=h\left(\sum_{i=1}^{k}b_i^2\Lambda_{x_ix_i}+b_{k+1}^2\Lambda+2\sum_{1\le i<j\le k}b_ib_j\Lambda_{x_ix_j}+2b_{k+1}\sum_{i=1}^{k}b_i\Lambda_{x_i}\right)+\Lambda\sum_{i=1}^{k}b_i^2h_{x_ix_i}\\
&+2\sum_{i=1}^{k}b_i^2h_{x_i}\Lambda_{x_i}+2\Lambda\sum_{1\le i<j\le k}b_ib_jh_{x_ix_j}+2\sum_{1\le i<j\le k}b_ib_jh_{x_i}\Lambda_{x_j}+2\sum_{1\le i<j\le k}b_ib_jh_{x_j}\Lambda_{x_i}\\
&+2b_{k+1}\Lambda\sum_{i=1}^{k}b_ih_{x_i}\\
&=2\sum_{i=1}^{k}b_i^2h_{x_i}\Lambda_{x_i}+2\sum_{1\le i<j\le k}b_ib_jh_{x_i}\Lambda_{x_j}+2\sum_{1\le i<j\le k}b_ib_jh_{x_j}\Lambda_{x_i}+2b_{k+1}\Lambda\sum_{i=1}^{k}b_ih_{x_i}\\
&=2b_s^2h_{x_s}\Lambda_{x_s}+2b_sh_{x_s}\sum_{1\le s<j\le k}b_j\Lambda_{x_j}+2b_sh_{x_s}\sum_{1\le i<s\le k}b_i\Lambda_{x_i}+2b_{k+1}b_sh_{x_s}\Lambda\\
&=2b_sh_{x_s}\sum_{i=1}^{k}b_i\Lambda_{x_i}+2b_{k+1}b_sh_{x_s}\Lambda=2b_sh_{x_s}\left(\sum_{i=1}^{k}b_i\Lambda_{x_i}+b_{k+1}\Lambda\right)=0.
\end{aligned}
$$

That (37) is a solution of (35) is proved. So its general solution is (36). □

According to Theorem 7, we present Theorem 8.

Theorem 8. *In $\mathbb{R}^n$, if the general solution $u=g(x_1,\ldots x_k)f(y_1,\ldots y_l,x_{k+1},\ldots x_n)$ of $(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})u=0$ is known, then the general solution of*

$$\left(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1}\right)^mu=0,(m\ge 2), \tag{38}$$

is

$$u=g(x_1,\ldots x_k)\sum_{j=1}^{m}\left(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_kx_k\right)^{j-1}f_j\left(y_1,\ldots y_l,x_{k+1},\ldots x_n\right). \tag{39}$$

Prove. Using Eq. (11), we get

$$
\begin{aligned}
&\left(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1}\right)^mu\\
&=\sum_{l_1+l_2+\cdots+l_{k+1}=m}\frac{m!}{l_1!l_2!\ldots l_k!l_{k+1}!}b_1^{l_1}b_2^{l_2}\ldots b_k^{l_k}b_{k+1}^{l_{k+1}}D_{x_1}^{l_1}D_{x_2}^{l_2}\ldots D_{x_k}^{l_k}u.
\end{aligned}
\tag{40}
$$

According to Theorem 7, we know that $u=g(x_1,\ldots x_k)(f_1(y_1,\ldots y_l,x_{k+1},\ldots x_n)+(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_kx_k)f_2(y_1,\ldots y_l,x_{k+1},\ldots x_n))$ is the general solution of $(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})^2u=0$. Using mathematical induction, we set

$$u=\Gamma=g(x_1,\ldots x_k)\sum_{j=1}^{m}\left(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_kx_k\right)^{j-1}f_j\left(y_1,\ldots y_l,x_{k+1},\ldots x_n\right). \tag{41}$$

is the general solution of $(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})^m u = D^{(m)}u = 0$, namely $D^{(m)}\Gamma = 0$, so $u = \Gamma$ is also the solution of $(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})^{m+1} u = D^{(m+1)}u = 0$.

Setting $u = h(x_1, \ldots x_k)\Gamma = \lambda_s x_s \Gamma$ is the solution of $D^{(m+1)}u = 0$ yet, that is $h(x_1, \ldots x_k) = \lambda_s x_s$, and λ_s are arbitrary constants, $(s = 1, 2, \ldots k)$, by Eq. (17) we obtain

$$D^{l_1}_{x_1} D^{l_2}_{x_2} \ldots D^{l_s}_{x_s} \ldots D^{l_k}_{x_k} (h\Gamma) = \lambda_s x_s \Gamma^{(l_1,l_2,\ldots l_s,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k} + l_s\lambda_s \Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}. \tag{42}$$

Then

$$\begin{aligned}
&(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})^{m+1}\\
&= (b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}\\
&+b_{k+1})\left(\sum_{l_1+l_2+\cdots+l_{k+1}=m} \tfrac{m!}{l_1!l_2!\ldots l_k!l_{k+1}!} b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} D^{l_1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_k}_{x_k}\right)\\
&= \sum_{l_1+l_2+\cdots+l_{k+1}=m} \tfrac{m!}{l_1!l_2!\ldots l_k!l_{k+1}!} b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} (b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}\\
&+b_{k+1}) D^{l_1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_k}_{x_k}\\
&= \sum_{l_1+l_2+\cdots+l_{k+1}=m} \tfrac{m!}{l_1!l_2!\ldots l_k!l_{k+1}!} (b_1^{l_1+1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} D^{l_1+1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_k}_{x_k}+\cdots\\
&+b_1^{l_1} b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_k^{l_k} D^{l_1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_s+1}_{x_s}\ldots D^{l_k}_{x_k}+\cdots+ b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k+1} b_{k+1}^{l_{k+1}} D^{l_1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_k+1}_{x_k}\\
&+b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}+1} D^{l_1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_k}_{x_k}).
\end{aligned}$$

Because of

$$(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})^{m+1}\Gamma = 0. \tag{43}$$

So

$$\begin{aligned}
&(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})^{m+1}(h\Gamma)\\
&= \sum_{l_1+l_2+\cdots+l_{k+1}=m} \tfrac{m!}{l_1!l_2!\ldots l_k!l_{k+1}!} (b_1^{l_1+1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} D^{l_1+1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_k}_{x_k}+\cdots\\
&+b_1^{l_1} b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_k^{l_k} D^{l_1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_s+1}_{x_s}\ldots D^{l_k}_{x_k}+\cdots+ b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k+1} b_{k+1}^{l_{k+1}} D^{l_1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_k+1}_{x_k}\\
&+b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}+1} D^{l_1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_k}_{x_k})(h\Gamma)\\
&= \sum_{l_1+l_2+\ldots+l_{k+1}=m} \tfrac{m!}{l_1!l_2!\ldots l_k!l_{k+1}!} (b_1^{l_1+1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} (\lambda_s x_s \Gamma^{(l_1+1,l_2,\ldots l_s,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k} + l_s\lambda_s \Gamma^{(l_1+1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k})\\
&+\cdots+ b_1^{l_1} b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} (\lambda_s x_s \Gamma^{(l_1,l_2,\ldots l_s+1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k} + l_s\lambda_s \Gamma^{(l_1,l_2,\ldots l_s,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k})+\cdots\\
&+b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k+1} b_{k+1}^{l_{k+1}} (\lambda_s x_s \Gamma^{(l_1,l_2,\ldots l_s,\ldots l_k+1)}_{x_1x_2\ldots x_s\ldots x_k} + l_s\lambda_s \Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k+1)}_{x_1x_2\ldots x_s\ldots x_k})\\
&+b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}+1} (\lambda_s x_s \Gamma^{(l_1,l_2,\ldots l_s,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k} + l_s\lambda_s \Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k})).
\end{aligned}$$

Notice:

$$\begin{aligned}
&\sum_{l_1+l_2+\ldots+l_{k+1}=m} \tfrac{m!}{l_1!l_2!\ldots l_k!l_{k+1}!} (b_1^{l_1+1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} (\lambda_s x_s \Gamma^{(l_1+1,l_2,\ldots l_s,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k})+\cdots\\
&+b_1^{l_1} b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} (\lambda_s x_s \Gamma^{(l_1,l_2,\ldots l_s+1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k})+\cdots\\
&+b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k+1} b_{k+1}^{l_{k+1}} (\lambda_s x_s \Gamma^{(l_1,l_2,\ldots l_s,\ldots l_k+1)}_{x_1x_2\ldots x_s\ldots x_k}) + b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}+1} (\lambda_s x_s \Gamma^{(l_1,l_2,\ldots l_s,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k})\\
&= \lambda_s x_s \sum_{l_1+l_2+\ldots+l_{k+1}=m} \tfrac{m!}{l_1!l_2!\ldots l_k!l_{k+1}!} (b_1^{l_1+1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} D^{l_1+1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_k}_{x_k}\\
&+b_1^{l_1} b_2^{l_2+1}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} D^{l_1}_{x_1} D^{l_2+1}_{x_2}\ldots D^{l_k}_{x_k}+\cdots+ b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k+1} b_{k+1}^{l_{k+1}} D^{l_1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_k+1}_{x_k}\\
&+b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}+1} D^{l_1}_{x_1} D^{l_2}_{x_2}\ldots D^{l_k}_{x_k})\Gamma = \lambda_s x_s (b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})^{m+1}\Gamma\\
&= 0.
\end{aligned}$$

So

$$\begin{aligned}
&(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})^{m+1}(h\Gamma)\\
&= \sum_{l_1+l_2+\cdots+l_{k+1}=m} \tfrac{m!l_s\lambda_s}{l_1!l_2!\ldots l_k!l_{k+1}!} (b_1^{l_1+1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} \Gamma^{(l_1+1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}\\
&+b_1^{l_1} b_2^{l_2+1}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} \Gamma^{(l_1,l_2+1,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}+\cdots+ b_1^{l_1} b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}} \Gamma^{(l_1,l_2,\ldots l_s,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}+\cdots\\
&+b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k+1} b_{k+1}^{l_{k+1}} \Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k+1)}_{x_1x_2\ldots x_s\ldots x_k} + b_1^{l_1} b_2^{l_2}\ldots b_k^{l_k} b_{k+1}^{l_{k+1}+1} \Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}).
\end{aligned}$$

Since

$$\begin{aligned}
&(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})^m\Gamma\\
&=(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})^{m-1}\Gamma.\\
&=(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}\\
&+b_{k+1})\sum_{l_1+l_2+\cdots+(l_s-1)+\cdots+l_{k+1}=m-1}\tfrac{(m-1)!}{l_1!l_2!\ldots(l_s-1)!\ldots l_{k+1}!}b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s-1}\ldots b_{k+1}^{l_{k+1}}\Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}\\
&=0.
\end{aligned}$$

We get

$$\begin{aligned}
&\sum_{l_1+l_2+\ldots+(l_s-1)+\ldots+l_{k+1}=m-1}\tfrac{(m-1)!}{l_1!l_2!\ldots(l_s-1)!\ldots l_{k+1}!}(b_1^{l_1+1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_{k+1}^{l_{k+1}}\Gamma^{(l_1+1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}\\
&+b_1^{l_1}b_2^{l_2+1}\ldots b_s^{l_s}\ldots b_{k+1}^{l_{k+1}}\Gamma^{(l_1,l_2+1,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}+\cdots+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_{k+1}^{l_{k+1}}\Gamma^{(l_1,l_2,\ldots l_s,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}+\cdots\\
&+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k+1}b_{k+1}^{l_{k+1}}\Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k+1)}_{x_1x_2\ldots x_s\ldots x_k}+b_1^{l_1}b_2^{l_2}\ldots b_k^{l_k}b_{k+1}^{l_{k+1}+1}\Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k})=0.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{l_1+l_2+\ldots+l_{k+1}=m}\tfrac{m!l_s\lambda_s}{l_1!l_2!\ldots l_k!l_{k+1}!}(b_1^{l_1+1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_{k+1}^{l_{k+1}}\Gamma^{(l_1+1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}\\
&+b_1^{l_1}b_2^{l_2+1}\ldots b_s^{l_s}\ldots b_{k+1}^{l_{k+1}}\Gamma^{(l_1,l_2+1,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}+\cdots+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_{k+1}^{l_{k+1}}\Gamma^{(l_1,l_2,\ldots l_s,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}+\cdots\\
&+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k+1}b_{k+1}^{l_{k+1}}\Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k+1)}_{x_1x_2\ldots x_s\ldots x_k}+b_1^{l_1}b_2^{l_2}\ldots b_k^{l_k}b_{k+1}^{l_{k+1}+1}\Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k})\\
&=\lambda_s m\sum_{l_1+l_2+\ldots+(l_s-1)+\ldots+l_{k+1}=m-1}\tfrac{(m-1)!}{l_1!l_2!\ldots(l_s-1)!\ldots l_{k+1}!}(b_1^{l_1+1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_{k+1}^{l_{k+1}}\Gamma^{(l_1+1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}\\
&+b_1^{l_1}b_2^{l_2+1}\ldots b_s^{l_s}\ldots b_{k+1}^{l_{k+1}}\Gamma^{(l_1,l_2+1,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}+\cdots+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s+1}\ldots b_{k+1}^{l_{k+1}}\Gamma^{(l_1,l_2,\ldots l_s,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k}+\cdots\\
&+b_1^{l_1}b_2^{l_2}\ldots b_s^{l_s}\ldots b_k^{l_k+1}b_{k+1}^{l_{k+1}}\Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k+1)}_{x_1x_2\ldots x_s\ldots x_k}+b_1^{l_1}b_2^{l_2}\ldots b_k^{l_k}b_{k+1}^{l_{k+1}+1}\Gamma^{(l_1,l_2,\ldots l_s-1,\ldots l_k)}_{x_1x_2\ldots x_s\ldots x_k})=0.
\end{aligned}$$

Namely

$$(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k}+b_{k+1})^{m+1}(\lambda_sx_s\Gamma)=0. \tag{44}$$

Since $s=1,2,\ldots k$, according to the principle of linear superposition, the general solution of Eq. (38) is (39), so the theorem is proved. □

In $\mathbb{R}^n$, for the mth-order linear PDE with variable coefficients

$$\sum_{0\le i_1+i_2+\cdots+i_k\le m}a_{i_1i_2\ldots i_k}u^{(i_1i_2\ldots i_k)}_{x_1x_2\ldots x_k}=0. \tag{45}$$

where i_r are natural number, $1\le r\le k\le n$. If Eq. (45) can be translated into

$$\begin{aligned}
&\left(b_{1_1}D_{x_1}+b_{1_2}D_{x_2}+\cdots+b_{1_k}D_{x_k}+b_{1_{k+1}}\right)\left(b_{2_1}D_{x_1}+b_{2_2}D_{x_2}+\cdots+b_{2_k}D_{x_k}+b_{2_{k+1}}\right)\ldots\\
&\left(b_{m_1}D_{x_1}+b_{m_2}D_{x_2}+\cdots+b_{m_k}D_{x_k}+b_{m_{k+1}}\right)u=0.
\end{aligned} \tag{46}$$

If the particular solution $u=g_j(x_1,\ldots x_k)$ of $(b_{j_1}D_{x_1}+b_{j_2}D_{x_2}+\cdots+b_{j_k}D_{x_k}+b_{j_{k+1}})u=0$ and exact solutions $u=y_{j_s}(x_1,\ldots x_k)$ of $(b_{j_1}D_{x_1}+b_{j_2}D_{x_2}+\cdots+b_{j_k}D_{x_k})u=0$, which are independent of each other, are all known, $(1\le j\le m, 1\le s\le l_j\le k-1)$, by Theorem 6 the general solution of Eq. (45) is

$$u\left(x_1,\ldots x_n\right)=\sum_{j=1}^{m}\left(g_j\left(x_1,\ldots x_k\right)f_j\left(y_{j_1},y_{j_2},\ldots y_{j_{l_j}},x_{k+1},x_{k+2},\ldots x_n\right)\right). \tag{47}$$

If Eq. (45) can be translated into:

$$\prod_{j=1}^{q}\left(b_{j_1}D_{x_1}+b_{j_2}D_{x_2}+\cdots+b_{j_n}D_{x_n}+b_{j_{n+1}}\right)^{p_j}u=0, \tag{48}$$

where $\sum_{j=1}^{q} p_j = m$, its general solution can be written by Theorem 8.

Next we propose Theorem 9.

Theorem 9. *In $\mathbb{R}^n$, if the particular solution $g = g(x_1, \dots x_k)$ of $a_1 g_{x_1} + a_2 g_{x_2} + \dots + a_k g_{x_k} + a_{k+1} g = 0$ and the exact solutions $u = y_i(x_1, \dots x_k)$ of $a_1 u_{x_1} + a_2 u_{x_2} + \dots + a_k u_{x_k} = 0$, which are independent of each other, are all known, $(1 \le i \le k-1)$, then*

$$a_1 u_{x_1} + a_2 u_{x_2} + \dots + a_k u_{x_k} + a_{k+1} u = A\left(x_1, \dots x_n\right), \tag{49}$$

the general solution of Eq. (49) is

$$u = g\left(x_1, \dots x_k\right) \left(f\left(y_1, \dots y_l, x_{k+1}, \dots x_n\right) + \int \frac{A\left(y_1, \dots y_k, x_{k+1}, \dots x_n\right) dy_k}{\left(a_1 y_{k x_1} + a_2 y_{k x_2} + \ldots + a_k y_{k x_k}\right) g} \right), \tag{50}$$

where $y_1, y_2, \dots y_k$ are independent of each other.

Prove. By Z_3 Transformation, set $u = g(x_1, \dots x_k) f(y_1, \dots y_k, x_{k+1}, \dots x_n)$, and $a_s(x_1, \dots x_n) = a_s(y_1, \dots y_k, x_{k+1}, \dots x_n), (1 \le s \le k+1)$, where $y_t = y_t(x_1, \dots x_k), (t = 1, 2, \dots k), y_1, y_2, \dots y_k$ are independent of each other, and $x_j = x_j(y_1, \dots y_k), (1 \le j \le k)$, note that $u = y_k(x_1, \dots x_k)$ is not the exact solutions of $a_1 u_{x_1} + a_2 u_{x_2} + \dots + a_k u_{x_k} = 0$, then

$$\begin{aligned}
&a_1 u_{x_1} + a_2 u_{x_2} + \dots + a_k u_{x_k} + a_{k+1} u \\
&= a_1 \left(f g_{x_1} + g \sum_{t=1}^{k} f_{y_t} y_{t x_1} \right) + a_2 \left(f g_{x_2} + g \sum_{t=1}^{k} f_{y_t} y_{t x_2} \right) + \cdots \\
&+ a_k \left(f g_{x_k} + g \sum_{t=1}^{k} f_{y_t} y_{t x_k} \right) + a_{k+1} g f \\
&= f\left(a_1 g_{x_1} + \dots + a_k g_{x_k} + a_{k+1} g\right) + g \sum_{t=1}^{k} f_{y_t} \left(a_1 y_{t x_1} + a_2 y_{t x_2} + \dots + a_k y_{t x_k}\right).
\end{aligned}$$

Since $a_1 g_{x_1} + \dots + a_k g_{x_k} + a_{k+1} g = 0$ and $a_1 y_{i x_1} + a_2 y_{i x_2} + \dots + a_k y_{i x_k} = 0$,$(1 \le i \le k-1)$, so

$$\begin{aligned}
a_1 u_{x_1} + a_2 u_{x_2} + \dots + a_k u_{x_k} + a_{k+1} u &= g f_{y_k} \left(a_1 y_{k x_1} + a_2 y_{k x_2} + \dots + a_k y_{k x_k}\right) \\
&= A\left(y_1, \dots y_k, x_{k+1}, \dots x_n\right).
\end{aligned}$$

We get

$$f = \int \frac{A\left(y_1, \dots y_k, x_{k+1}, \dots x_n\right) dy_k}{\left(a_1 y_{k x_1} + a_2 y_{k x_2} + \dots + a_k y_{k x_k}\right) g}.$$

A particular solution of Eq. (34) is

$$u = g \int \frac{A\left(y_1, \dots y_k, x_{k+1}, \dots x_n\right) dy_k}{\left(a_1 y_{k x_1} + a_2 y_{k x_2} + \dots + a_k y_{k x_k}\right) g}. \tag{51}$$

To compute (51) we need to find the first differentiable $y_k = y_k(x_1, x_2, \dots x_k)$ and $y_1, y_2, \dots y_k$ independent of each other, and can get $x_j = x_j(y_1, y_2, \dots y_k)$. According to Theorem 6, the general solution of Eq. (49) is (50), so Theorem 9 is proved. □

Note that Theorem 9 obtains the general solution of the variable coefficient transport equation.

We propose Theorem 10 as follows.

Theorem 10. *In $\mathbb{R}^n$, if the particular solution $g = g(x_1, \dots x_k)$ of $b_1 g_{x_1} + b_2 g_{x_2} + \dots + b_k g_{x_k} +$*

$b_{k+1}g = 0$ and exact solutions $u = y_i(x_1, \dots x_k)$ of $b_1 u_{x_1} + b_2 u_{x_2} + \cdots + b_k u_{x_k} = 0$, which are independent of each other, are all known, $(1 \le i \le k-1)$, then

$$(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^m u = A\left(x_1, \dots x_n\right), \tag{52}$$

the general solution of Eq. (52) is

$$\begin{aligned} u = g(x_1, \dots x_k) \sum_{j=1}^{m} \left(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k\right)^{j-1} f_j\left(y_1, \dots y_{k-1}, x_{k+1}, \dots x_n\right) \\ + \frac{g \int \dots \int g^{-1} A\left(y_1, \dots y_k, x_{k+1}, \dots x_n\right) d^m y_k}{(b_1 c_1 + b_2 c_2 + \cdots + b_k c_k)^m}, \end{aligned} \tag{53}$$

where $y_1, y_2, \dots y_k$ are independent of each other, $y_k = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k$, and $x_j = x_j(y_1, y_2, \dots y_k)$ can be got, $(1 \le j \le k)$.

Prove. By Z_3 Transformation, set $u = g(x_1, \dots x_k) f(y_1, \dots y_k, x_{k+1}, \dots x_n)$ and $a_i(x_1, x_2, \dots x_n) = a_i(y_1, y_2, \dots y_k, x_{k+1}, x_{k+2}, \dots x_n)$. $y_1, y_2, \dots y_k$ are independent of each other, $y_s = y_s(x_1, x_2, \dots x_k)$, $(s = 1, 2, \dots k)$, and

$$y_k = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k. \tag{54}$$

Note that $u = y_k(x_1, \dots x_k)$ is not the exact solutions of $b_1 u_{x_1} + b_2 u_{x_2} + \cdots + b_k u_{x_k} = 0$, then

$$\begin{aligned}
&(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^m u \\
&= (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m-1} (b_1 \left(f g_{x_1} + g \textstyle\sum_{i=1}^{k} f_{y_i} y_{i x_1}\right) \\
&+ b_2 \left(f g_{x_2} + g \textstyle\sum_{i=1}^{k} f_{y_i} y_{i x_2}\right) + \cdots + b_k \left(f g_{x_k} + g \textstyle\sum_{i=1}^{k} f_{y_i} y_{i x_k}\right) + b_{k+1} g f) \\
&= (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m-1} (f\left(b_1 g_{x_1} + b_2 g_{x_2} + \cdots + b_k g_{x_k} + b_{k+1} g\right) \\
&+ g \textstyle\sum_{i=1}^{k} f_{y_i}\left(b_1 y_{i x_1} + b_2 y_{i x_2} + \cdots + b_k y_{i x_k}\right)) \\
&= (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m-1} (g f_{y_k}\left(b_1 y_{k x_1} + b_2 y_{k x_2} + \cdots + b_k y_{k x_k}\right)) \\
&= (b_1 c_1 + b_2 c_2 + \cdots + b_k c_k)(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m-1} (g f_{y_k}) \\
&= (b_1 c_1 + b_2 c_2 + \cdots \\
&+ b_k c_k)(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m-2} (b_1 \left(f_{y_k} g_{x_1} + g \textstyle\sum_{i=1}^{k} f_{y_k y_i} y_{i x_1}\right) \\
&+ b_2 \left(f_{y_k} g_{x_2} + g \textstyle\sum_{i=1}^{k} f_{y_k y_i} y_{i x_2}\right) + \cdots + b_k \left(f_{y_k} g_{x_k} + g \textstyle\sum_{i=1}^{k} f_{y_k y_i} y_{i x_k}\right) + b_{k+1} g f_{y_k}) \\
&= (b_1 c_1 + b_2 c_2 + \cdots \\
&+ b_k c_k)(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m-2} (f_{y_k}\left(b_1 g_{x_1} + b_2 g_{x_2} + \cdots + b_k g_{x_k} + b_{k+1} g\right) \\
&+ g \textstyle\sum_{i=1}^{k} f_{y_k y_i}\left(b_1 y_{i x_1} + b_2 y_{i x_2} + \cdots + b_k y_{i x_k}\right)) \\
&= (b_1 c_1 + b_2 c_2 + \cdots \\
&+ b_k c_k)(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m-2} (g f_{y_k y_k}\left(b_1 y_{k x_1} + b_2 y_{k x_2} + \cdots + b_k y_{k x_k}\right)) \\
&= (b_1 c_1 + b_2 c_2 + \cdots + b_k c_k)^2 (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m-2} (g f_{y_k y_k}) = \cdots \\
&= (b_1 c_1 + b_2 c_2 + \cdots + b_k c_k)^m g f_{y_k}^{(m)}.
\end{aligned}$$

Then

$$\begin{aligned} (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^m u &= (b_1 c_1 + b_2 c_2 + \cdots + b_k c_k)^m g f_{y_k}^{(m)} \\ &= A\left(y_1, \dots y_k, x_{k+1}, \dots x_n\right). \end{aligned} \tag{55}$$

We get

$$f = \frac{\int \cdots \int g^{-1} A\left(y_1, \ldots y_k, x_{k+1}, \ldots x_n\right) d^m y_k}{\left(b_1 c_1 + b_2 c_2 + \cdots + b_k c_k\right)^m}.$$

So the particular solution of Eq. (52) is

$$u = \frac{g \int \cdots \int g^{-1} A\left(y_1, \ldots y_k, x_{k+1}, \ldots x_n\right) d^m y_k}{\left(b_1 c_1 + b_2 c_2 + \cdots + b_k c_k\right)^m}. \tag{56}$$

By Theorem 8, the general solution of Eq. (52) is (53). So Theorem 10 is proved. □

2. General solutions' laws of linear partial differential equations with constant coefficients

Here we will research the general solutions' laws of linear PDEs with constant coefficients, which are the special cases of linear PDEs with variable coefficients. In this section, if there is no special interpretation, the acquiescent independent variables of $\mathbb{R}^n$ are $x_1, x_2, \ldots x_n$; $D_{x_i} \triangleq \frac{\partial}{\partial x_i}$, $2 \leq k \leq n$; a_i, b_i, b_{i_j} and $a_{i_1 i_2 \ldots i_n}$ are arbitrary known constants, $c_i, c_{ij}, c_{i_s j}, l_i, l_{j_i}, \lambda_i, C_i$ and C are arbitrary constants, f and f_i are arbitrary smooth functions $(i, j, s = 1, 2, \ldots)$.

Theorem 11. *In* $\mathbb{R}^n$,

$$a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = A\left(x_1, x_2, \ldots x_n\right), (2 \leq k \leq n), \tag{57}$$

the general solution of Eq. (57) is

$$u = f\left(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n\right) + \frac{\int A\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right) d y_k}{a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}}, \tag{58}$$

where

$$y_i = c_{i1} x_1 + c_{i2} x_2 + \cdots + c_{ik} x_k, (1 \leq i \leq k), \tag{59}$$

$$\frac{\partial\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right)}{\partial\left(x_1, x_2, \ldots x_n\right)} = \frac{\partial\left(y_1, y_2, \ldots y_k\right)}{\partial\left(x_1, x_2, \ldots x_k\right)} \neq 0, \tag{60}$$

$$c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \cdots - a_k c_{ik}}{a_1}, (1 \leq i \leq k-1). \tag{61}$$

Prove. According to Z_1 Transformation, set $u(x_1, x_2, \ldots x_n) = u(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n)$, $A(x_1, x_2, \ldots x_n) = A(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n)$ and

$$\begin{cases} y_1 = c_{11} x_1 + c_{12} x_2 + \cdots + c_{1k} x_k \\ y_2 = c_{21} x_1 + c_{22} x_2 + \cdots + c_{2k} x_k \\ \cdots \\ y_k = c_{k1} x_1 + c_{k2} x_2 + \cdots + c_{kk} x_k, \end{cases} \tag{62}$$

and

$$\frac{\partial\left(y_1, y_2, \ldots y_k\right)}{\partial\left(x_1, x_2, \ldots x_k\right)} \neq 0, \tag{60}$$

where c_{ij} are undetermined constants. According to (60, 62), $x_i = x_i(y_1, y_2, \ldots y_k)$ always has a unique solution $(1 \le i, j \le k)$. So

$$\begin{aligned}
& a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} \\
& = a_1 \sum_{i=1}^{k} c_{i1} u_{y_i} + a_2 \sum_{i=1}^{k} c_{i2} u_{y_i} + \cdots + a_k \sum_{i=1}^{k} c_{ik} u_{y_i} \\
& = \left(a_1 c_{11} + a_2 c_{12} + \cdots + a_k c_{1k}\right) u_{y_1} + \left(a_1 c_{21} + a_2 c_{22} + \cdots + a_k c_{2k}\right) u_{y_2} + \cdots \\
& + \left(a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}\right) u_{y_k}.
\end{aligned}$$

Set

$$\begin{aligned}
& a_1 c_{11} + a_2 c_{12} + \cdots + a_k c_{1k} = a_1 c_{21} + a_2 c_{22} + \cdots + a_k c_{2k} \\
& = \cdots = a_1 c_{(k-1)1} + a_2 c_{(k-1)2} + \cdots + a_k c_{(k-1)k} = 0.
\end{aligned}$$

We obtain

$$c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \cdots - a_k c_{ik}}{a_1}, (1 \le i \le k-1). \tag{61}$$

Then

$$\begin{aligned}
a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} & = \left(a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}\right) u_{y_k} \\
& = A\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right).
\end{aligned} \tag{63}$$

The general solution of (63) is

$$u = f\left(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n\right) + \frac{\int A\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right) dy_k}{a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}}.$$

So Theorem 11 is proved. □

According to Theorem 11, in $\mathbb{R}^n$,

$$a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_n u_{x_n} = A\left(x_1, x_2, \ldots x_n\right), \tag{64}$$

the general solution of Eq. (64) is

$$u = f\left(y_1, y_2, \ldots y_{n-1}\right) + \frac{\int A\left(y_1, y_2, \ldots y_n\right) dy_n}{a_1 c_{n1} + a_2 c_{n2} + \cdots + a_n c_{nn}}, \tag{65}$$

where

$$y_i = c_{i1} x_1 + c_{i2} x_2 + \cdots + c_{in} x_n, \tag{66}$$

$$c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \cdots - a_n c_{in}}{a_1}, (1 \le i \le n-1). \tag{67}$$

According to Theorem 11 we can get Theorem 12:

Theorem 12. *In* $\mathbb{R}^n$,

$$a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = 0, \tag{68}$$

the general solution of Eq. (68) *is*

$$u = f\left(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n\right), \tag{69}$$

where y_i *satisfy* $(59-61)$.

The proof of Theorem 12 is not complicated, the readers may try it.

According to Theorem 2 and 12, we can get directly Theorem 13:

Theorem 13. *In $\mathbb{R}^n$,*

$$(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^2 u = 0, \tag{70}$$

the general solution of Eq. (70) *is*

$$u = f_1 (y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n) + (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) f_2 (y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n). \tag{71}$$

where y_i satisfy $(59, 60)$ *and*

$$c_{i1} = \frac{-b_2 c_{i2} - b_3 c_{i3} - \cdots - b_k c_{ik}}{a_1}, (1 \leq i \leq k-1). \tag{72}$$

According to Theorem 3, we can present Theorem 14:

Theorem 14. *In $\mathbb{R}^n$, the general solution of*

$$(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^m u = 0, \tag{73}$$

is

$$u = \sum_{i=1}^{m} (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)^{i-1} f_i (y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n). \tag{74}$$

where y_i satisfy $(59, 60)$ *and* (72).

In $\mathbb{R}^n$, for the mth-order linear PDE with constant coefficients

$$\sum_{i_1 + i_2 + \cdots + i_k = m} a_{i_1 i_2 \ldots i_k} u_{x_1 x_2 \ldots x_k}^{(i_1 i_2 \ldots i_k)} = 0, \tag{75}$$

where i_j are natural number, $1 \leq j \leq k \leq n$, If Eq. (75) can be translated into

$$\begin{aligned}&(b_{1_1} D_{x_1} + b_{1_2} D_{x_2} + \cdots + b_{1_k} D_{x_k}) (b_{2_1} D_{x_1} + b_{2_2} D_{x_2} + \cdots + b_{2_k} D_{x_k}) \ldots (b_{m_1} D_{x_1} + b_{m_2} D_{x_2} \\ &+ \cdots + b_{m_k} D_{x_k}) u = 0.\end{aligned} \tag{76}$$

According to Theorem 12, we can get the general solution of Eq. (75) is

$$u = \sum_{r=1}^{m} f_r \left(y_{r_1}, y_{r_2}, \ldots y_{r_{k-1}}, x_{k+1}, \ldots x_n\right), (1 \leq r \leq m), \tag{77}$$

where

$$y_{r_s} = c_{r_{s1}} x_1 + c_{r_{s2}} x_2 + \ldots + c_{r_{sk}} x_k, \tag{78}$$

$$c_{r_{s1}} = \frac{-b_{s_2} c_{r_{s1}} - b_{s_3} c_{r_{s2}} - \cdots - b_{s_k} c_{r_{sk}}}{b_{s1}}, (1 \leq s \leq k-1). \tag{79}$$

If Eq. (75) can be translated into:

$$\prod_{j=1}^{q} (b_{j_1} D_{x_1} + b_{j_2} D_{x_2} + \cdots + b_{j_k} D_{x_k})^{p_j} u = 0, \tag{80}$$

where $\sum\limits_{j=1}^{q} p_j = m$, its general solution can be written by Theorem 14.

According to Theorem 5 and 12, we can present Theorem 15.

Theorem 15. *In* $\mathbb{R}^n$,

$$(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k})^m u = A(x_1,\ldots x_n), \tag{81}$$

the general solution of Eq. (81) *is*

$$\begin{aligned} u = &\frac{\int\ldots\int A(y_1,\ldots y_k,x_{k+1},\ldots x_n)\,d^m y_k}{(b_1c_{k1}+b_2c_{k2}+\cdots+b_kc_{kk})^m} \\ &+\sum_{j=1}^{m}(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_kx_k)^{j-1}f_j(y_1,\ldots y_{k-1},x_{k+1},\ldots x_n), \end{aligned} \tag{82}$$

where y_i *satisfy* $(59,60)$ *and* (72).

Next we propose Theorem 16.

Theorem 16. *In* $\mathbb{R}^n$,

$$a_1u_{x_1}+a_2u_{x_2}+\cdots+a_ku_{x_k}+a_{k+1}u=0, \tag{83}$$

the general solution of Eq. (83) *is*

$$u=f(y_1,y_2,\ldots y_{k-1},x_{k+1},x_{k+2},\ldots x_n)\sum_{i=1}^{k}l_ie^{\frac{-a_{k+1}x_i}{a_i}}, \tag{84}$$

where l_i *are arbitrary constants and* y_i *satisfy* $(59-61)$.

Prove. According to Z_3 Transformation, set $u(x_1,\ldots x_n)=g(x_1,\ldots x_k)h(y_1,\ldots y_k,x_{k+1},\ldots x_n)$, $y_i=c_{i1}x_1+c_{i2}x_2+\cdots+c_{ik}x_k$, and

$$\frac{\partial(y_1,y_2,\ldots y_k)}{\partial(x_1,x_2,\ldots x_k)}\neq 0. \tag{60}$$

So

$$\begin{aligned} &a_1u_{x_1}+a_2u_{x_2}+\cdots+a_ku_{x_n}+a_{k+1}u \\ &=a_1g_{x_1}h+a_1g\sum_{i=1}^{k}c_{i1}h_{y_i}+a_2g_{x_2}h+a_2g\sum_{i=1}^{k}c_{i2}h_{y_i}+\cdots+a_kg_{x_k}h+a_kg\sum_{i=1}^{k}c_{ik}h_{y_i} \\ &+a_{k+1}gh \\ &=(a_1c_{11}+a_2c_{12}+\ldots+a_kc_{1k})\,gh_{y_1}+(a_1c_{21}+a_2c_{22}+\cdots+a_kc_{2k})\,gh_{y_2}+\cdots \\ &+(a_1c_{k1}+a_2c_{k2}+\cdots+a_kc_{kk})\,gh_{y_k}+(a_1g_{x_1}+a_2g_{x_2}+\cdots+a_kg_{x_k}+a_{k+1}g)\,h=0. \end{aligned}$$

Set

$$c_{i1}=\frac{-a_2c_{i2}-a_3c_{i3}-\cdots-a_kc_{ik}}{a_1},(1\le i\le k-1), \tag{61}$$

$$a_1g_{x_1}+a_2g_{x_2}+\cdots+a_kg_{x_k}+a_{k+1}g=0. \tag{85}$$

And set $g(x_1,\ldots x_k)=g(x_i),(i=1,2,\ldots k)$, Then

$$\begin{aligned} &a_1g_{x_1}+a_2g_{x_2}+\cdots+a_kg_{x_k}+a_{k+1}g=a_ig_{x_i}+a_{k+1}g=0 \\ &\Rightarrow g(x_i)=l_ie^{\frac{-a_{k+1}x_i}{a_i}}, \end{aligned} \tag{86}$$

where l_i are arbitrary constants. Namely $g\left(x_1,\dots x_k\right)=\sum\limits_{i=1}^{k} l_i e^{\frac{-a_{k+1}x_i}{a_i}}$ is a particular solution of $a_1 g_{x_1}+a_2 g_{x_2}+\cdots+a_k g_{x_k}+a_{k+1}g=0$, thus

$$a_1 u_{x_1}+a_2 u_{x_2}+\cdots+a_k u_{x_n}+a_{k+1}u=\left(a_1 c_{k1}+a_2 c_{k2}+\cdots+a_k c_{kk}\right) g h_{y_k}=0 \Rightarrow h_{y_k}=0.$$

We get

$$h=f\left(y_1,y_2,\dots y_{k-1},x_{k+1},x_{k+2},\dots x_n\right),$$

where f is an arbitrary first differentiable function, so the general solution of Eq. (83) is

$$u=gh=f\left(y_1,y_2,\dots y_{k-1},x_{k+1},x_{k+2},\dots x_n\right)\sum_{i=1}^{k} l_i e^{\frac{-a_{k+1}x_i}{a_i}}.$$

□

According to Theorem 8 and 16, we can present Theorem 17:

Theorem 17. *In* $\mathbb{R}^n$,

$$\left(b_1 D_{x_1}+b_2 D_{x_2}+\cdots+b_k D_{x_k}+b_{k+1}\right)^m u=0, \tag{87}$$

the general solution of Eq. (87) *is*

$$u=\left(\sum_{j=1}^{m}\left(\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_k x_k\right)^{j-1} f_j\left(y_1,\dots y_{k-1},x_{k+1},\dots x_n\right)\right)\sum_{i=1}^{k} l_i e^{\frac{-b_{k+1}x_i}{b_i}}, \tag{88}$$

where l_i,λ_j *are arbitrary constants,* y_j *satisfy* (59, 60) *and* (72).

In $\mathbb{R}^n$, for the mth-order linear PDE with constant coefficients

$$\sum_{0\le i_1+i_2+\ldots+i_k\le m} a_{i_1 i_2\ldots i_k} u^{(i_1 i_2\ldots i_k)}_{x_1 x_2\ldots x_k}=0, \tag{89}$$

where i_j are natural number, $1\le j\le k\le n$, If Eq. (89) can be translated into

$$\begin{aligned}&\left(b_{1_1}D_{x_1}+b_{1_2}D_{x_2}+\cdots+b_{1_k}D_{x_k}+b_{1_{k+1}}\right)\left(b_{2_1}D_{x_1}+b_{2_2}D_{x_2}+\cdots+b_{2_k}D_{x_k}+b_{2_{k+1}}\right)\\&\cdots\left(b_{m_1}D_{x_1}+b_{m_2}D_{x_2}+\cdots+b_{m_k}D_{x_k}+b_{m_{k+1}}\right)u=0.\end{aligned} \tag{90}$$

By Theorem 16, the general solution of Eq. (89) is

$$u=\sum_{r=1}^{m}\left(f_r\left(y_{r_1},y_{r_2},\dots y_{r_{k-1}},x_{k+1},\dots x_n\right)\sum_{i=1}^{k} l_{r_i} e^{\frac{-b_{r_{k+1}}x_i}{b_{r_i}}}\right), \tag{91}$$

where y_{r_s} satisfies (78, 79), $(1\le s\le k-1)$.

If Eq. (89) can be translated into:

$$\prod_{r=1}^{q}\left(b_{r_1}D_{x_1}+b_{r_2}D_{x_2}+\cdots+b_{r_k}D_{x_k}+b_{r_{k+1}}\right)^{p_r}u=0, \tag{92}$$

where $\sum\limits_{r=1}^{q} p_r=m$, its general solution can be written by Theorem 17.

Using Theorem 9 and 16, we can get Theorem 18.

Theorem 18. *In* $\mathbb{R}^n$,

$$a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u = A\left(x_1, x_2, \ldots x_n\right), (2 \leq k \leq n), \tag{93}$$

the general solution of Eq. (93) *is*

$$u = e^{\frac{-a_{k+1}x_1}{a_1}} \left(f\left(y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n\right) + \frac{\int e^{\frac{a_{k+1}x_1}{a_1}} A\left(y_1, \ldots y_k, x_{k+1}, \ldots x_n\right) dy_k}{a_1 c_{k1} + \cdots + a_k c_{kk}} \right), \tag{94}$$

where y_i *satisfy* $(59 - 61)$.

According to Theorem 9 and 16,

$$u = \sum_{i=1}^{k} l_i e^{\frac{-a_{k+1}x_i}{a_i}} \left(f\left(y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n\right) + \int \frac{A\left(y_1, \ldots y_k, x_{k+1}, \ldots x_n\right)}{\left(a_1 c_{k1} + \ldots + a_k c_{kk}\right) \sum_{i=1}^{k} l_i e^{\frac{-a_{k+1}x_i}{a_i}}} dy_k \right), \tag{95}$$

is the general solution of Eq. (93) too, however, in the specific calculation, (95) may be more complicated. In order to simplify the calculation, in Theorem 18 we choose

$$g\left(x_1, \ldots x_k\right) = e^{\frac{-a_{k+1}x_1}{a_1}}.$$

So (95) is simplified to (94).

Below we use a new method to calculate the general solution of Eq. (93) and propose Theorem 19.

Theorem 19. *In* $\mathbb{R}^n$,

$$a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u = A\left(x_1, x_2, \ldots x_n\right), (2 \leq k \leq n), \tag{93}$$

the general solution of Eq. (93) *is*

$$\begin{aligned} u = {} & f\left(y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n\right) e^{\frac{-a_{k+1}x_1}{a_1}} \\ & + e^{-a_{k+1}By_k} \left(C - B \int e^{a_{k+1}By_k} A\left(y_1, \ldots y_k, x_{k+1}, \ldots x_n\right) dy_k \right), \end{aligned} \tag{96}$$

where y_i *satisfy* $(59 - 61)$, *and*

$$B = \left(a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}\right)^{-1}. \tag{97}$$

Prove. According to Z_1 Transformation, set $u(x_1, x_2, \ldots x_n) = u(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n)$, $A(x_1, x_2, \ldots x_n) = A(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n)$ and

$$\begin{cases} y_1 = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1k}x_k \\ y_2 = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2k}x_k \\ \cdots \\ y_k = c_{k1}x_1 + c_{k2}x_2 + \cdots + c_{kk}x_k, \end{cases} \tag{62}$$

and

$$\frac{\partial\left(y_1, y_2, \ldots y_k\right)}{\partial\left(x_1, x_2, \ldots x_k\right)} \neq 0, \tag{60}$$

where c_{ij} are undetermined constants. According to (60, 62), $x_i = x_i(y_1, y_2, \ldots y_k)$ always has a unique solution $(1 \leq i, j \leq k)$. So

$$\begin{aligned}
&a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u \\
&= a_1 \sum_{i=1}^{k} c_{i1} u_{y_i} + a_2 \sum_{i=1}^{k} c_{i2} u_{y_i} + \cdots + a_k \sum_{i=1}^{k} c_{ik} u_{y_i} + a_{k+1} u \\
&= \left(a_1 c_{11} + a_2 c_{12} + \cdots + a_k c_{1k}\right) u_{y_1} + \left(a_1 c_{21} + a_2 c_{22} + \cdots + a_k c_{2k}\right) u_{y_2} + \cdots \\
&+ \left(a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}\right) u_{y_k} + a_{k+1} u.
\end{aligned}$$

Set

$$\begin{aligned}
a_1 c_{11} + a_2 c_{12} + \cdots + a_k c_{1k} &= a_1 c_{21} + a_2 c_{22} + \cdots + a_k c_{2k} \\
&= a_1 c_{(k-1)1} + a_2 c_{(k-1)2} + \cdots + a_k c_{(k-1)k} = 0.
\end{aligned}$$

We obtain

$$c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \cdots - a_k c_{ik}}{a_1}, (1 \leq i \leq k-1). \tag{61}$$

Then

$$\begin{aligned}
a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u &= \left(a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}\right) u_{y_k} + a_{k+1} u \\
&= A\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right).
\end{aligned}$$

Namely

$$\left(a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}\right) u_{y_k} + a_{k+1} u = A\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right). \tag{98}$$

According to the general solution of the first-order linear ordinary differential equation, it is not difficult to find the solution of Eq. (98) is

$$u = e^{\frac{-a_{k+1} y_k}{a_1 c_{k1} + a_2 c_{k2} + \ldots + a_k c_{kk}}} \left(C + \int e^{\frac{a_{k+1} y_k}{a_1 c_{k1} + a_2 c_{k2} + \ldots + a_k c_{kk}}} \frac{A\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right)}{a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}} dy_k\right).$$

Combined with Theorem 16, the general solution of Eq. (93) is (96), so the theorem is proved. □

Theorems 9, 18, and 19 have special significance because they reveal the laws of the general solution of the transport equation. Since the transport equation has important applications in many fields, it is one of the most active linear PDEs being studied at present. Current research methods mainly use numerical methods to solve definite solution problems, such as Petrov-Galerkin methods [14], Discontinuous Galerkin Method [15, 16], the finite difference method [17,18], etc., the existence [19, 20] and uniqueness [21, 22] of solutions are also important areas of research.

According to Theorem 10 and 18, we can obtain Theorem 20.

Theorem 20. *In* $\mathbb{R}^n$,

$$\left(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1}\right)^m u = A\left(x_1, \ldots x_n\right), \tag{99}$$

the general solution of Eq. (99) *is*

$$\begin{aligned}
u = {} & e^{\frac{-a_{k+1} x_1}{a_1}} \sum_{j=1}^{m} \left(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k\right)^{j-1} f_j\left(y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n\right) \\
& + \frac{e^{\frac{-a_{k+1} x_1}{a_1}} \int \ldots \int e^{\frac{a_{k+1} x_1}{a_1}} A\left(y_1, \ldots y_k, x_{k+1}, \ldots x_n\right) d^m y_k}{\left(b_1 c_{k1} + b_2 c_{k2} + \cdots + b_k c_{kk}\right)^m},
\end{aligned} \tag{100}$$

where y_i satisfy $(59-61)$.

If we choose $\sum\limits_{i=1}^{k} l_i e^{\frac{-a_{k+1}x_i}{a_i}}$ as $g(x_1,\dots x_k)$, the general solution form of Eq. (99) will be more complicated.

Next we propose Theorem 21.

Theorem 21. *In $\mathbb{R}^2$,*

$$a_1u_{xx}+a_2u_{yy}+a_3u_{xy}+a_4u_x+a_5u_y+a_6u=0, \tag{101}$$

the general solution of Eq. (101) *is,*

$$u=\left(C_1e^{\lambda_1 v_1}+C_2e^{\lambda_2 v_1}\right)h_1\left(l_1x+l_2y+l_3\right)+\left(C_3e^{\lambda_3 v_2}+C_4e^{\lambda_4 v_2}\right)h_2\left(l_4x+l_5y+l_6\right), \tag{102}$$

where a_i are arbitrary known constants, h_1 and h_2 are arbitrary second differentiable functions, and

$$v_1=k_1x+k_2y+k_3, v_2=k_4x+k_5y+k_6, \tag{103}$$

$$\lambda_1=\frac{-a_4k_1-a_5k_2+\sqrt{\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}, \tag{104}$$

$$\lambda_2=\frac{-a_4k_1-a_5k_2-\sqrt{\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}, \tag{105}$$

$$\lambda_3=\frac{-a_4k_4-a_5k_5+\sqrt{\left(a_4k_4+a_5k_5\right)^2-4a_6\left(a_1k_4^2+a_2k_5^2+a_3k_4k_5\right)}}{2\left(a_1k_4^2+a_2k_5^2+a_3k_4k_5\right)}, \tag{106}$$

$$\lambda_4=\frac{-a_4k_4-a_5k_5-\sqrt{\left(a_4k_4+a_5k_5\right)^2-4a_6\left(a_1k_4^2+a_2k_5^2+a_3k_4k_5\right)}}{2\left(a_1k_4^2+a_2k_5^2+a_3k_4k_5\right)}, \tag{107}$$

where k_3,k_6,l_3 and l_6 are arbitrary constants, $k_1,k_2,k_4,k_5,l_1,l_2,l_4$ and l_5 are constants which satisfy

$$l_1=\frac{-a_3l_2+\sqrt{a_3^2l_2^2-4a_1a_2l_2^2}}{2a_1}, l_4=\frac{-a_3l_5-\sqrt{a_3^2l_5^2-4a_1a_2l_5^2}}{2a_1}, \tag{108}$$

$$k_1\lambda_1\left(2a_1l_1+a_3l_2\right)+k_2\lambda_1\left(2a_2l_2+a_3l_1\right)+a_4l_1+a_5l_2=0, \tag{109}$$

$$k_1\lambda_2\left(2a_1l_1+a_3l_2\right)+k_2\lambda_2\left(2a_2l_2+a_3l_1\right)+a_4l_1+a_5l_2=0, \tag{110}$$

$$k_4\lambda_3\left(2a_1l_4+a_3l_5\right)+k_5\lambda_3\left(2a_2l_5+a_3l_4\right)+a_4l_4+a_5l_5=0, \tag{111}$$

$$k_4\lambda_4\left(2a_1l_4+a_3l_5\right)+k_5\lambda_4\left(2a_2l_5+a_3l_4\right)+a_4l_4+a_5l_5=0, \tag{112}$$

$$\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)>0, \tag{113}$$

$$\left(a_4k_4+a_5k_5\right)^2-4a_6\left(a_1k_4^2+a_2k_5^2+a_3k_4k_5\right)>0. \tag{114}$$

Prove. By Z_1 Transformation, set

$$u\left(x,y\right)=f\left(v\right)=f\left(k_1x+k_2y+k_3\right), \tag{115}$$

where $v(x,y)=k_1x+k_2y+k_3, k_1-k_3$ are undetermined constants, f is an undetermined second differentiable function, so

$$\begin{aligned}&a_1u_{xx}+a_2u_{yy}+a_3u_{xy}+a_4u_x+a_5u_y+a_6u\\&=\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)f_v^{''}+\left(a_4k_1+a_5k_2\right)f_v^{'}+a_6f=0.\end{aligned}\tag{116}$$

The characteristic equation of Eq. (116) is

$$\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)\lambda^2+\left(a_4k_1+a_5k_2\right)\lambda+a_6=0.\tag{117}$$

If

$$\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)>0,\tag{118}$$

the particular solution of Eq. (101) is

$$u=f=C_1e^{\lambda_1v}+C_2e^{\lambda_2v},\tag{119}$$

where C_1 and C_2 are arbitrary constants, and

$$\lambda_1=\frac{-a_4k_1-a_5k_2+\sqrt{\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)},$$

$$\lambda_2=\frac{-a_4k_1-a_5k_2-\sqrt{\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}.$$

For getting the general solution of Eq. (101), using Z_3 Transformation, we set

$$u\left(x,y\right)=gh\left(w\right)=g\left(x,y\right)h\left(l_1x+l_2y+l_3\right),\tag{120}$$

where $w(x,y)=l_1x+l_2y+l_3$, l_1-l_3 are undetermined constants, h and g are undetermined second differentiable functions, so

$$\begin{aligned}&a_1u_{xx}+a_2u_{yy}+a_3u_{xy}+a_4u_x+a_5u_y+a_6u\\&=a_1\left(g_{xx}h+2l_1g_xh_w^{'}+l_1^2gh_w^{''}\right)+a_2\left(g_{yy}h+2l_2g_yh_w^{'}+l_2^2gh_w^{''}\right)\\&+a_3\left(g_{xy}h+l_2g_xh_w^{'}+l_1g_yh_w^{'}+l_1l_2gh_w^{''}\right)+a_4\left(g_xh+l_1gh_w^{'}\right)+a_5\left(g_yh+l_2gh_w^{'}\right)+a_6gh.\end{aligned}$$

Namely

$$\begin{aligned}&\left(a_1l_1^2+a_2l_2^2+a_3l_1l_2\right)gh_w^{''}+\left(\left(2a_1l_1+a_3l_2\right)g_x+\left(2a_2l_2+a_3l_1\right)g_y+\left(a_4l_1+a_5l_2\right)g\right)h_w^{'}\\&+\left(a_1g_{xx}+a_2g_{yy}+a_3g_{xy}+a_4g_x+a_5g_y+a_6g\right)h=0.\end{aligned}\tag{121}$$

Set $h(w)$ is an arbitrary second differentiable function, and

$$a_1l_1^2+a_3l_2l_1+a_2l_2^2=0\Longrightarrow l_1=\frac{-a_3l_2\pm\sqrt{a_3^2l_2^2-4a_1a_2l_2^2}}{2a_1},\tag{122}$$

$$\left(2a_1l_1+a_3l_2\right)g_x+\left(2a_2l_2+a_3l_1\right)g_y+\left(a_4l_1+a_5l_2\right)g=0,\tag{123}$$

$$a_1g_{xx}+a_2g_{yy}+a_3g_{xy}+a_4g_x+a_5g_y+a_6g=0.\tag{124}$$

By Eq. (119), the particular solution of Eq. (124) is

$$g=C_1e^{\lambda_1v}+C_2e^{\lambda_2v}.\tag{125}$$

Substituting (125) into (123) we get

$$\begin{aligned}&(2a_1l_1+a_3l_2)\,g_x+(2a_2l_2+a_3l_1)\,g_y+(a_4l_1+a_5l_2)\,g\\&=(C_1k_1\lambda_1\,(2a_1l_1+a_3l_2)+C_1k_2\lambda_1\,(2a_2l_2+a_3l_1)+C_1\,(a_4l_1+a_5l_2))\,e^{\lambda_1 v}\\&+(C_2k_1\lambda_2\,(2a_1l_1+a_3l_2)+C_2k_2\lambda_2\,(2a_2l_2+a_3l_1)+C_2\,(a_4l_1+a_5l_2))\,e^{\lambda_2 v}=0.\end{aligned}$$

Then

$$k_1\lambda_1\,(2a_1l_1+a_3l_2)+k_2\lambda_1\,(2a_2l_2+a_3l_1)+a_4l_1+a_5l_2=0,$$

$$k_1\lambda_2\,(2a_1l_1+a_3l_2)+k_2\lambda_2\,(2a_2l_2+a_3l_1)+a_4l_1+a_5l_2=0.$$

So the general solution of Eq. (101) is

$$\begin{aligned}u&=g_1h_1\,(w_1)+g_2h_2\,(w_2)\\&=\left(C_1e^{\lambda_1 v_1}+C_2e^{\lambda_2 v_1}\right)h_1\,(l_1x+l_2y+l_3)+\left(C_3e^{\lambda_3 v_2}+C_4e^{\lambda_4 v_2}\right)h_2\,(l_4x+l_5y+l_6)\,,\end{aligned}$$

where $v_1, v_2, \lambda_1-\lambda_4, k_1, k_2, k_4, k_5, l_1, l_2, l_4$ and l_5 satisfy Eqs. (103-114), so the theorem is proved. □

In Theorem 21, if

$$\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)<0,$$

$$\left(a_4k_4+a_5k_5\right)^2-4a_6\left(a_1k_4^2+a_2k_5^2+a_3k_4k_5\right)<0,$$

or

$$\begin{aligned}&\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)\\&=\left(a_4k_4+a_5k_5\right)^2-4a_6\left(a_1k_4^2+a_2k_5^2+a_3k_4k_5\right)=0,\end{aligned}$$

By analogous calculation, we may have Theorem 21 and 23.

Theorem 22. *In* $\mathbb{R}^2$,

$$a_1u_{xx}+a_2u_{yy}+a_3u_{xy}+a_4u_x+a_5u_y+a_6u=0, \tag{101}$$

the general solution of Eq. (101) *is*

$$\begin{aligned}u=&\left(C_1\mathrm{sin}q_1v_1+C_2\mathrm{cos}q_1v_1\right)e^{p_1v_1}h_1\,(l_1x+l_2y+l_3)\\&+\left(C_3\mathrm{sin}q_2v_2+C_4\mathrm{cos}q_2v_2\right)e^{p_2v_2}h_2\,(l_4x+l_5y+l_6)\,,\end{aligned} \tag{126}$$

where a_i are arbitrary known constants, h_1 and h_2 are arbitrary second differentiable functions, and

$$v_1=k_1x+k_2y+k_3, v_2=k_4x+k_5y+k_6, \tag{103}$$

$$p_1=\frac{-a_4k_1-a_5k_2}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}, q_1=\frac{\sqrt{\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}, \tag{127}$$

$$p_2=\frac{-a_4k_4-a_5k_5}{2\left(a_1k_4^2+a_2k_5^2+a_3k_4k_5\right)}, q_2=\frac{\sqrt{\left(a_4k_4+a_5k_5\right)^2-4a_6\left(a_1k_4^2+a_2k_5^2+a_3k_4k_5\right)}}{2\left(a_1k_4^2+a_2k_5^2+a_3k_4k_5\right)}, \tag{128}$$

where k_3, k_6, l_3 and l_6 are arbitrary constants, $k_1, k_2, k_4, k_5, l_1, l_2, l_4$ and l_5 are constants which satisfy

$$l_1=\frac{-a_3l_2+\sqrt{a_3^2l_2^2-4a_1a_2l_2^2}}{2a_1}, l_4=\frac{-a_3l_5-\sqrt{a_3^2l_5^2-4a_1a_2l_5^2}}{2a_1}, \tag{108}$$

$$(C_1p_1 - C_2q_1)(2a_1k_1l_1 + a_3k_1l_2 + 2a_2k_2l_2 + a_3k_2l_1) + C_1(a_4l_1 + a_5l_2) = 0, \tag{129}$$

$$k_1(C_1q_1 + C_2p_1)(2a_1l_1 + a_3l_2) + k_2(C_1q_1 - C_2p_1)(2a_2l_2 + a_3l_1) + C_2(a_4l_1 + a_5l_2) = 0, \tag{130}$$

$$(C_3p_2 - C_4q_2)(2a_1k_4l_1 + a_3k_4l_2 + 2a_2k_5l_2 + a_3k_5l_1) + C_3(a_4l_1 + a_5l_2) = 0, \tag{131}$$

$$k_4(C_3q_2 + C_4p_2)(2a_1l_1 + a_3l_2) + k_5(C_3q_2 - C_4p_2)(2a_2l_2 + a_3l_1) + C_4(a_4l_1 + a_5l_2) = 0, \tag{132}$$

$$(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right) < 0, \tag{133}$$

$$(a_4k_4 + a_5k_5)^2 - 4a_6\left(a_1k_4^2 + a_2k_5^2 + a_3k_4k_5\right) < 0. \tag{134}$$

Theorem 23. *In* $\mathbb{R}^2$*,*

$$a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u = 0, \tag{101}$$

the general solution of Eq. (101) *is*

$$u = C_1v_1e^{\lambda_1v_1}h_1(l_1x + l_2y + l_3) + C_2v_2e^{\lambda_2v_2}h_2(l_4x + l_5y + l_6), \tag{135}$$

where a_i are arbitrary known constants, h_1 and h_2 are arbitrary second differentiable functions, and

$$v_1 = k_1x + k_2y + k_3, v_2 = k_4x + k_5y + k_6, \tag{103}$$

$$\lambda_1 = \frac{-a_4k_1 - a_5k_2}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}, \lambda_2 = \frac{-a_4k_4 - a_5k_5}{2\left(a_1k_4^2 + a_2k_5^2 + a_3k_4k_5\right)}, \tag{136}$$

where k_3, k_6, l_3 and l_6 are arbitrary constants, $k_1, k_2, k_4, k_5, l_1, l_2, l_4$ and l_5 are constants which satisfy

$$l_1 = \frac{-a_3l_2 + \sqrt{a_3^2l_2^2 - 4a_1a_2l_2^2}}{2a_1}, l_4 = \frac{-a_3l_5 - \sqrt{a_3^2l_5^2 - 4a_1a_2l_5^2}}{2a_1}, \tag{108}$$

$$(2a_1l_1 + a_3l_2)(k_1 + v_1\lambda_1k_1) + (2a_2l_2 + a_3l_1)(k_2 + v_1\lambda_1k_2) + (a_4l_1 + a_5l_2)v_1 = 0, \tag{137}$$

$$(2a_1l_4 + a_3l_5)(k_4 + v_2\lambda_2k_4) + (2a_2l_5 + a_3l_4)(k_5 + v_2\lambda_2k_5) + (a_4l_4 + a_5l_5)v_2 = 0, \tag{138}$$

$$(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right) = 0, \tag{139}$$

$$(a_4k_4 + a_5k_5)^2 - 4a_6\left(a_1k_4^2 + a_2k_5^2 + a_3k_4k_5\right) = 0. \tag{140}$$

We propose Theorem 24 as follows.

Theorem 24. *In* $\mathbb{R}^2$*,*

$$a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u = A(x, y), \tag{141}$$

the general solution of Eq. (141) *is*

$$\begin{aligned} u = {} & C_1v_1e^{\lambda_1v_1}h_1(l_1x + l_2y + l_3) + C_2v_2e^{\lambda_2v_2}h_2(l_4x + l_5y + l_6) \\ & + v_3e^{\lambda_3v_3}\int\int \frac{A(p,q)\,dpdq}{v_3e^{\lambda_3v_3}\left(2a_1c_1c_3 + 2a_2c_2c_4 + a_3(c_1c_4 + c_2c_3)\right)}, \end{aligned} \tag{142}$$

where a_i are arbitrary known constants, h_1 and h_2 are arbitrary second differentiable functions, and

$$v_1 = k_1x + k_2y + k_3, v_2 = k_4x + k_5y + k_6, v_3 = k_7x + k_8y + k_9, \tag{143}$$

$$\lambda_1 = \frac{-a_4k_1 - a_5k_2}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}, \lambda_2 = \frac{-a_4k_4 - a_5k_5}{2\left(a_1k_4^2 + a_2k_5^2 + a_3k_4k_5\right)}, \lambda_3 = \frac{-a_4k_7 - a_5k_8}{2\left(a_1k_7^2 + a_2k_8^2 + a_3k_7k_8\right)}, \tag{144}$$

$$p = c_1x + c_2y, q = c_3x + c_4y, \tag{145}$$

$$x = \frac{pc_4 - qc_2}{c_1c_4 - c_2c_3}, y = \frac{qc_1 - pc_3}{c_1c_4 - c_2c_3}, \tag{146}$$

where $C_1, C_2, k_3, k_6, k_9, l_3$ *and* l_6 *are arbitrary constants,* $k_1, k_2, k_4, k_5, k_7, k_8, l_1, l_2, l_4, l_5$ *and* $c_1 - c_4$ *are constants which satisfy*

$$c_1c_4 - c_2c_3 \neq 0, \tag{147}$$

$$l_1 = \frac{-a_3l_2 + \sqrt{a_3^2l_2^2 - 4a_1a_2l_2^2}}{2a_1}, l_4 = \frac{-a_3l_5 - \sqrt{a_3^2l_5^2 - 4a_1a_2l_5^2}}{2a_1}, \tag{108}$$

$$c_1 = \frac{-a_3c_2 \pm \sqrt{a_3^2c_2^2 - 4a_1a_2c_2^2}}{2a_1}, c_3 = \frac{-a_3c_4 \pm \sqrt{a_3^2c_4^2 - 4a_1a_2c_4^2}}{2a_1}, \tag{148}$$

$$(2a_1l_1 + a_3l_2)(k_1 + v_1\lambda_1k_1) + (2a_2l_2 + a_3l_1)(k_2 + v_1\lambda_1k_2) + (a_4l_1 + a_5l_2)v_1 = 0, \tag{137}$$

$$(2a_1l_4 + a_3l_5)(k_4 + v_2\lambda_2k_4) + (2a_2l_5 + a_3l_4)(k_5 + v_2\lambda_2k_5) + (a_4l_4 + a_5l_5)v_2 = 0, \tag{138}$$

$$(2a_1c_1 + a_3c_2)(1 + \lambda_3v_3)k_7 + (2a_2c_2 + a_3c_1)(1 + \lambda_3v_3)k_8 + (a_4c_1 + a_5c_2)v_3 = 0, \tag{149}$$

$$(2a_1c_3 + a_3c_4)(1 + \lambda_3v_3)k_7 + (2a_2c_4 + a_3c_3)(1 + \lambda_3v_3)k_8 + (a_4c_3 + a_5c_4)v_3 = 0, \tag{150}$$

$$(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right) = 0, \tag{139}$$

$$(a_4k_4 + a_5k_5)^2 - 4a_6\left(a_1k_4^2 + a_2k_5^2 + a_3k_4k_5\right) = 0. \tag{140}$$

$$(a_4k_7 + a_5k_8)^2 - 4a_6\left(a_1k_7^2 + a_2k_8^2 + a_3k_7k_8\right) = 0. \tag{151}$$

Prove. By Z_3 Transformation, we set

$$u(x, y) = g(x, y)\,h(p, q), \tag{152}$$

$$p = c_1x + c_2y, q = c_3x + c_4y,$$

where $c_1 - c_4$ are undetermined constants, g and h are undetermined second differentiable function, and

$$x = \frac{pc_4 - qc_2}{c_1c_4 - c_2c_3}, y = \frac{qc_1 - pc_3}{c_1c_4 - c_2c_3},$$

$$c_1c_4 - c_2c_3 \neq 0.$$

By Eq. (152), we get

$$\begin{aligned}
&a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u \\
&= a_1\left(hg_{xx} + 2g_x(c_1h_p + c_3h_q) + g\left(c_1^2h_{pp} + c_3^2h_{qq} + 2c_1c_3h_{pq}\right)\right) \\
&+a_2\left(hg_{yy} + 2g_y(c_2h_p + c_4h_q) + g\left(c_2^2h_{pp} + c_4^2h_{qq} + 2c_2c_4h_{pq}\right)\right) \\
&+a_3\left(hg_{xy} + g_x(c_2h_p + c_4h_q) + g_y(c_1h_p + c_3h_q) + g\left(c_1c_2h_{pp} + c_3c_4h_{qq} + (c_1c_4 + c_2c_3)h_{pq}\right)\right) \\
&+a_4\left(g_xh + g(c_1h_p + c_3h_q)\right) + a_5\left(g_yh + g(c_2h_p + c_4h_q)\right) + a_6gh \\
&= \left(a_1c_1^2 + a_2c_2^2 + a_3c_1c_2\right)gh_{pp} + \left(a_1c_3^2 + a_2c_4^2 + a_3c_3c_4\right)gh_{qq} \\
&+ \left(2a_1c_1c_3 + 2a_2c_2c_4 + a_3(c_1c_4 + c_2c_3)\right)gh_{pq} \\
&+ \left((2a_1c_1 + a_3c_2)g_x + (2a_2c_2 + a_3c_1)g_y + (a_4c_1 + a_5c_2)g\right)h_p \\
&+ \left((2a_1c_3 + a_3c_4)g_x + (2a_2c_4 + a_3c_3)g_y + (a_4c_3 + a_5c_4)g\right)h_q \\
&+ \left(a_1g_{xx} + a_2g_{yy} + a_3g_{xy} + a_4g_x + a_5g_y + a_6g\right)h = A(x, y).
\end{aligned}$$

Set

$$a_1 g_{xx} + a_2 g_{yy} + a_3 g_{xy} + a_4 g_x + a_5 g_y + a_6 g = 0, \tag{153}$$

$$(2a_1c_1 + a_3c_2)\, g_x + (2a_2c_2 + a_3c_1)\, g_y + (a_4c_1 + a_5c_2)\, g = 0, \tag{154}$$

$$(2a_1c_3 + a_3c_4)\, g_x + (2a_2c_4 + a_3c_3)\, g_y + (a_4c_3 + a_5c_4)\, g = 0. \tag{155}$$

We set $g = g(v_3)$ and $v_3(x,y) = k_7 x + k_8 y + k_9$, then

$$\begin{aligned} &a_1 g_{xx} + a_2 g_{yy} + a_3 g_{xy} + a_4 g_x + a_5 g_y + a_6 g \\ &= \left(a_1 k_7^2 + a_2 k_8^2 + a_3 k_7 k_8\right) g_v'' + (a_4 k_7 + a_5 k_8)\, g_v' + a_6 g = 0. \end{aligned}$$

If

$$(a_4 k_7 + a_5 k_8)^2 - 4a_6 \left(a_1 k_7^2 + a_2 k_8^2 + a_3 k_7 k_8\right) = 0, \tag{156}$$

the particular solution of Eq. (153) is

$$g = C_3 v_3 e^{\lambda_3 v_3}, \tag{157}$$

$$\lambda_3 = \frac{-a_4 k_7 - a_5 k_8}{2\left(a_1 k_7^2 + a_2 k_8^2 + a_3 k_7 k_8\right)}. \tag{158}$$

Substituting (157) into (154) we get

$$\begin{aligned} &(2a_1c_1 + a_3c_2)\, g_x + (2a_2c_2 + a_3c_1)\, g_y + (a_4c_1 + a_5c_2)\, g \\ &= (2a_1c_1 + a_3c_2)(1 + \lambda_3 v_3)\, C_3 k_7 e^{\lambda_3 v_3} + (2a_2c_2 + a_3c_1)(1 + \lambda_3 v_3)\, C_3 k_8 e^{\lambda_3 v_3} \\ &+ (a_4c_1 + a_5c_2)\, C_3 v_3 e^{\lambda_3 v_3} = 0. \end{aligned}$$

Namely

$$(2a_1c_1 + a_3c_2)(1 + \lambda_3 v_3)\, k_7 + (2a_2c_2 + a_3c_1)(1 + \lambda_3 v_3)\, k_8 + (a_4c_1 + a_5c_2)\, v_3 = 0$$

Substituting (157) into (155) we obtain

$$\begin{aligned} &(2a_1c_3 + a_3c_4)\, g_x + (2a_2c_4 + a_3c_3)\, g_y + (a_4c_3 + a_5c_4)\, g \\ &= (2a_1c_3 + a_3c_4)(1 + \lambda_3 v_3)\, C_3 k_7 e^{\lambda_3 v_3} + (2a_2c_4 + a_3c_3)(1 + \lambda_3 v_3)\, C_3 k_8 e^{\lambda_3 v_3} \\ &+ (a_4c_3 + a_5c_4)\, C_3 v_3 e^{\lambda_3 v_3} = 0. \end{aligned}$$

That is

$$(2a_1c_3 + a_3c_4)(1 + \lambda_3 v_3)\, k_7 + (2a_2c_4 + a_3c_3)(1 + \lambda_3 v_3)\, k_8 + (a_4c_3 + a_5c_4)\, v_3 = 0.$$

So

$$\begin{aligned} &a_1 u_{xx} + a_2 u_{yy} + a_3 u_{xy} + a_4 u_x + a_5 u_y + a_6 u \\ &= \left(a_1 c_1^2 + a_2 c_2^2 + a_3 c_1 c_2\right) g h_{pp} + \left(a_1 c_3^2 + a_2 c_4^2 + a_3 c_3 c_4\right) g h_{qq} \\ &+ (2a_1c_1c_3 + 2a_2c_2c_4 + a_3(c_1c_4 + c_2c_3))\, g h_{pq} = A(x,y). \end{aligned} \tag{159}$$

Set

$$a_1 c_1^2 + a_2 c_2^2 + a_3 c_1 c_2 = a_1 c_3^2 + a_2 c_4^2 + a_3 c_3 c_4 = 0. \tag{160}$$

Namely

$$c_1 = \frac{-a_3 c_2 \pm \sqrt{a_3^2 c_2^2 - 4a_1 a_2 c_2^2}}{2a_1},$$

$$c_3 = \frac{-a_3 c_4 \pm \sqrt{a_3^2 c_4^2 - 4a_1 a_2 c_4^2}}{2a_1}.$$

Then

$$\begin{aligned}&a_1u_{xx}+a_2u_{yy}+a_3u_{xy}+a_4u_x+a_5u_y+a_6u\\&=(2a_1c_1c_3+2a_2c_2c_4+a_3(c_1c_4+c_2c_3))\,gh_{pq}=A(x,y)\,,\end{aligned}$$

the particular solution of h is

$$h=\int\int\frac{A(p,q)\,dpdq}{(2a_1c_1c_3+2a_2c_2c_4+a_3(c_1c_4+c_2c_3))\,g}. \tag{161}$$

Combining Theorem 23, we can have the general solution of Eq. (141) is

$$\begin{aligned}&u=C_1v_1e^{\lambda_1v_1}h_1(l_1x+l_2y+l_3)+C_2v_2e^{\lambda_2v_2}h_2(l_4x+l_5y+l_6)\\&+v_3e^{\lambda_3v_3}\int\int\frac{A(p,q)dpdq}{v_3e^{\lambda_3v_3}(2a_1c_1c_3+2a_2c_2c_4+a_3(c_1c_4+c_2c_3))}.\end{aligned}$$

So the theorem is proved. □

Eqs. (101, 141) are very important linear partial differential equations. One-dimensional homogeneous and non-homogeneous wave equations, heat equations, two-dimensional Helmholtz equations and reaction-diffusion-convection equation [23-25], etc. are all special cases of them. The general solution or exact solution of these equations can be obtained by using Theorem 21-24.

Next we propose Theorem 25.

Theorem 25. *In* $\mathbb{R}^3$*,*

$$a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{tx}+a_5u_{ty}+a_6u_{xy}+a_7u_t+a_8u_x+a_9u_y+a_{10}u=A(t,x,y)\,, \tag{162}$$

the general solution of Eq. (162) *is*

$$u=g(h_1(p,r)+h_2(q,r))+\frac{g}{B}\int\frac{A(p,q,r)\,dpdq}{g}, \tag{163}$$

where a_i are arbitrary known constants, h_1 and h_2 are arbitrary second differentiable functions, and

$$g=C_1e^{\lambda_1v}+C_2e^{\lambda_2v},v=k_1t+k_2x+k_3y+k_4, \tag{164}$$

$$\lambda_1=\frac{-b+\sqrt{b^2-4aa_{10}}}{2a},\lambda_2=\frac{-b-\sqrt{b^2-4aa_{10}}}{2a}, \tag{165}$$

$$a=a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_1k_3+a_6k_2k_3, \tag{166}$$

$$b=a_7k_1+a_8k_2+a_9k_3, \tag{167}$$

$$p=l_1t+l_2x+l_3y,q=l_4t+l_5x+l_6y,r=l_7t+l_8x+l_9y, \tag{168}$$

$$B=2a_1l_1l_4+2a_2l_2l_5+2a_3l_3l_6+a_4(l_1l_5+l_2l_4)+a_5(l_1l_6+l_3l_4)+a_6(l_2l_6+l_3l_5)\,, \tag{169}$$

where C_1,C_2 and k_4 are arbitrary constants, k_1-k_3 and l_1-l_9 are constants which satisfy

$$b^2-4aa_{10}>0, \tag{170}$$

$$-l_3l_5l_7+l_2l_6l_7+l_3l_4l_8-l_1l_6l_8-l_2l_4l_9+l_1l_5l_9\neq0, \tag{171}$$

$$\begin{aligned}&k_1\lambda_1(2a_1l_1+a_4l_2+a_5l_3)+k_2\lambda_1(2a_2l_2+a_4l_1+a_6l_3)+k_3\lambda_1(2a_3l_3+a_5l_1+a_6l_2)\\&+a_7l_1+a_8l_2+a_9l_3=0,\end{aligned} \tag{172}$$

$$\begin{aligned}&k_1\lambda_2\left(2a_1l_1+a_4l_2+a_5l_3\right)+k_2\lambda_2\left(2a_2l_2+a_4l_1+a_6l_3\right)+k_3\lambda_2\left(2a_3l_3+a_5l_1+a_6l_2\right)\\&+a_7l_1+a_8l_2+a_9l_3=0,\end{aligned}\tag{173}$$

$$\begin{aligned}&k_1\lambda_1\left(2a_1l_4+a_4l_5+a_5l_6\right)+k_2\lambda_1\left(2a_2l_5+a_4l_4+a_6l_6\right)+k_3\lambda_1\left(2a_3l_6+a_5l_4+a_6l_5\right)\\&+a_7l_4+a_8l_5+a_9l_6=0,\end{aligned}\tag{174}$$

$$\begin{aligned}&k_1\lambda_2\left(2a_1l_4+a_4l_5+a_5l_6\right)+k_2\lambda_2\left(2a_2l_5+a_4l_4+a_6l_6\right)+k_3\lambda_2\left(2a_3l_6+a_5l_4+a_6l_5\right)\\&+a_7l_4+a_8l_5+a_9l_6=0,\end{aligned}\tag{175}$$

$$\begin{aligned}&k_1\lambda_1\left(2a_1l_7+a_4l_8+a_5l_9\right)+k_2\lambda_1\left(2a_2l_8+a_4l_7+a_6l_9\right)+k_3\lambda_1\left(2a_3l_9+a_5l_7+a_6l_8\right)\\&+a_7l_7+a_8l_8+a_9l_9=0,\end{aligned}\tag{176}$$

$$\begin{aligned}&k_1\lambda_2\left(2a_1l_7+a_4l_8+a_5l_9\right)+k_2\lambda_2\left(2a_2l_8+a_4l_7+a_6l_9\right)+k_3\lambda_2\left(2a_3l_9+a_5l_7+a_6l_8\right)\\&+a_7l_7+a_8l_8+a_9l_9=0,\end{aligned}\tag{177}$$

$$a_1l_1^2+a_2l_2^2+a_3l_3^2+a_4l_1l_2+a_5l_1l_3+a_6l_2l_3=0,\tag{178}$$

$$a_1l_4^2+a_2l_5^2+a_3l_6^2+a_4l_4l_5+a_5l_4l_6+a_6l_5l_6=0,\tag{179}$$

$$a_1l_7^2+a_2l_8^2+a_3l_9^2+a_4l_7l_8+a_5l_7l_9+a_6l_8l_9=0,\tag{180}$$

$$2a_1l_1l_7+2a_2l_2l_8+2a_3l_3l_9+a_4\left(l_1l_8+l_2l_7\right)+a_5\left(l_1l_9+l_3l_7\right)+a_6\left(l_2l_9+l_3l_8\right)=0,\tag{181}$$

$$2a_1l_4l_7+2a_2l_5l_8+2a_3l_6l_9+a_4\left(l_4l_8+l_5l_7\right)+a_5\left(l_4l_9+l_6l_7\right)+a_6\left(l_5l_9+l_6l_8\right)=0.\tag{182}$$

Prove. By Z_1 Transformation, we set

$$u\left(t,x,y\right)=f\left(v\right)=f\left(k_1t+k_2x+k_3y+k_4\right),\tag{183}$$

where $v(t,x,y)=k_1t+k_2x+k_3y+k_4, k_1-k_4$ are undetermined constants, f is an undetermined second differentiable function, so

$$\begin{aligned}&a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{tx}+a_5u_{ty}+a_6u_{xy}+a_7u_t+a_8u_x+a_9u_y+a_{10}u\\&=\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_1k_3+a_6k_2k_3\right)f_v^{''}+\left(a_7k_1+a_8k_2+a_9k_3\right)f_v^{'}+a_{10}f\\&=0.\end{aligned}\tag{184}$$

The characteristic equation of Eq. (184) is

$$\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_1k_3+a_6k_2k_3\right)\lambda^2+\left(a_7k_1+a_8k_2+a_9k_3\right)\lambda+a_{10}=0.\tag{185}$$

So

$$\lambda=\frac{-b\pm\sqrt{b^2-4aa_{10}}}{2a},$$
$$a=a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_1k_3+a_6k_2k_3,$$
$$b=a_7k_1+a_8k_2+a_9k_3.$$

If

$$b^2-4aa_{10}>0,$$

the particular solution of Eq. (162) is

$$u=f=C_1e^{\lambda_1v}+C_2e^{\lambda_2v},\tag{186}$$

where

$$\lambda_1 = \frac{-b+\sqrt{b^2-4aa_{10}}}{2a}, \lambda_2 = \frac{-b-\sqrt{b^2-4aa_{10}}}{2a}. \tag{165}$$

For getting the general solution of Eq. (162), using Z_3 Transformation, we set

$$u = g(t,x,y)\,h(p,q,r)\,, \tag{187}$$

and

$$p = l_1t + l_2x + l_3y, q = l_4t + l_5x + l_6y, r = l_7t + l_8x + l_9y, \tag{168}$$

where $l_1 - l_9$ are undetermined constants, h and g are undetermined second differentiable functions, and

$$\frac{\partial(p,q,r)}{\partial(t,x,y)} = \begin{vmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{vmatrix} = -l_3l_5l_7 + l_2l_6l_7 + l_3l_4l_8 - l_1l_6l_8 - l_2l_4l_9 + l_1l_5l_9 \neq 0,$$

By Eq. (168), we get

$$t = -\frac{-rl_3l_5 + rl_2l_6 + ql_3l_8 - pl_6l_8 - ql_2l_9 + pl_5l_9}{l_3l_5l_7 - l_2l_6l_7 - l_3l_4l_8 + l_1l_6l_8 + l_2l_4l_9 - l_1l_5l_9}, \tag{188}$$

$$x = -\frac{rl_3l_4 - rl_1l_6 - ql_3l_7 + pl_6l_7 + ql_1l_9 - pl_4l_9}{l_3l_5l_7 - l_2l_6l_7 - l_3l_4l_8 + l_1l_6l_8 + l_2l_4l_9 - l_1l_5l_9}, \tag{189}$$

$$y = \frac{rl_2l_4 - rl_1l_5 - ql_2l_7 + pl_5l_7 + ql_1l_8 - pl_4l_8}{l_3l_5l_7 - l_2l_6l_7 - l_3l_4l_8 + l_1l_6l_8 + l_2l_4l_9 - l_1l_5l_9}. \tag{190}$$

Then

$$\begin{aligned}
&a_1u_{tt} + a_2u_{xx} + a_3u_{yy} + a_4u_{tx} + a_5u_{ty} + a_6u_{xy} + a_7u_t + a_8u_x + a_9u_y + a_{10}u \\
&= a_1(g_{tt}h + 2g_t(l_1h_p + l_4h_q + l_7h_r) \\
&+g\left(l_1^2h_{pp} + l_4^2h_{qq} + l_7^2h_{rr} + 2l_1l_4h_{pq} + 2l_1l_7h_{pr} + 2l_4l_7h_{qr}\right)) \\
&+a_2(g_{xx}h + 2g_x(l_2h_p + l_5h_q + l_8h_r) \\
&+g\left(l_2^2h_{pp} + l_5^2h_{qq} + l_8^2h_{rr} + 2l_2l_5h_{pq} + 2l_2l_8h_{pr} + 2l_5l_8h_{qr}\right)) \\
&+a_3(g_{yy}h + 2g_y(l_3h_p + l_6h_q + l_9h_r) \\
&+g\left(l_3^2h_{pp} + l_6^2h_{qq} + l_9^2h_{rr} + 2l_3l_6h_{pq} + 2l_3l_9h_{pr} + 2l_6l_9h_{qr}\right)) \\
&+a_4(g_{tx}h + g_t(l_2h_p + l_5h_q + l_8h_r) + g_x(l_1h_p + l_4h_q + l_7h_r) \\
&+g\left(l_1l_2h_{pp} + l_4l_5h_{qq} + l_7l_8h_{rr} + (l_1l_5 + l_2l_4)h_{pq} + (l_1l_8 + l_2l_7)h_{pr} + (l_4l_8 + l_5l_7)h_{qr}\right)) \\
&+a_5(g_{ty}h + g_t(l_3h_p + l_6h_q + l_9h_r) + g_y(l_1h_p + l_4h_q + l_7h_r) \\
&+g\left(l_1l_3h_{pp} + l_4l_6h_{qq} + l_7l_9h_{rr} + (l_1l_6 + l_3l_4)h_{pq} + (l_1l_9 + l_3l_7)h_{pr} + (l_4l_9 + l_6l_7)h_{qr}\right)) \\
&+a_6(g_{xy}h + g_x(l_3h_p + l_6h_q + l_9h_r) + g_y(l_2h_p + l_5h_q + l_8h_r) \\
&+g\left(l_2l_3h_{pp} + l_5l_6h_{qq} + l_8l_9h_{rr} + (l_2l_6 + l_3l_5)h_{pq} + (l_2l_9 + l_3l_8)h_{pr} + (l_5l_9 + l_6l_8)h_{qr}\right)) \\
&+a_7(g_th + g(l_1h_p + l_4h_q + l_7h_r)) + a_8(g_xh + g(l_2h_p + l_5h_q + l_8h_r)) \\
&+a_9(g_yh + g(l_3h_p + l_6h_q + l_9h_r)) + a_{10}gh \\
&= \left(a_1l_1^2 + a_2l_2^2 + a_3l_3^2 + a_4l_1l_2 + a_5l_1l_3 + a_6l_2l_3\right)gh_{pp} \\
&+ \left(a_1l_4^2 + a_2l_5^2 + a_3l_6^2 + a_4l_4l_5 + a_5l_4l_6 + a_6l_5l_6\right)gh_{qq} \\
&+ \left(a_1l_7^2 + a_2l_8^2 + a_3l_9^2 + a_4l_7l_8 + a_5l_7l_9 + a_6l_8l_9\right)gh_{rr} \\
&+ \left(2a_1l_1l_4 + 2a_2l_2l_5 + 2a_3l_3l_6 + a_4(l_1l_5 + l_2l_4) + a_5(l_1l_6 + l_3l_4) + a_6(l_2l_6 + l_3l_5)\right)gh_{pq} \\
&+ \left(2a_1l_1l_7 + 2a_2l_2l_8 + 2a_3l_3l_9 + a_4(l_1l_8 + l_2l_7) + a_5(l_1l_9 + l_3l_7) + a_6(l_2l_9 + l_3l_8)\right)gh_{pr} \\
&+ \left(2a_1l_4l_7 + 2a_2l_5l_8 + 2a_3l_6l_9 + a_4(l_4l_8 + l_5l_7) + a_5(l_4l_9 + l_6l_7) + a_6(l_5l_9 + l_6l_8)\right)gh_{qr} \\
&+((2a_1l_1 + a_4l_2 + a_5l_3)g_t + (2a_2l_2 + a_4l_1 + a_6l_3)g_x + (2a_3l_3 + a_5l_1 + a_6l_2)g_y \\
&+ (a_7l_1 + a_8l_2 + a_9l_3)g)h_p \\
&+((2a_1l_4 + a_4l_5 + a_5l_6)g_t + (2a_2l_5 + a_4l_4 + a_6l_6)g_x + (2a_3l_6 + a_5l_4 + a_6l_5)g_y \\
&+ (a_7l_4 + a_8l_5 + a_9l_6)g)h_q \\
&+((2a_1l_7 + a_4l_8 + a_5l_9)g_t + (2a_2l_8 + a_4l_7 + a_6l_9)g_x + (2a_3l_9 + a_5l_7 + a_6l_8)g_y \\
&+ (a_7l_7 + a_8l_8 + a_9l_9)g)h_r \\
&+ \left(a_1g_{tt} + a_2g_{xx} + a_3g_{yy} + a_4g_{tx} + a_5g_{ty} + a_6g_{xy} + a_7g_t + a_8g_x + a_9g_y + a_{10}g\right)h.
\end{aligned}$$

Set

$$a_1 g_{tt} + a_2 g_{xx} + a_3 g_{yy} + a_4 g_{tx} + a_5 g_{ty} + a_6 g_{xy} + a_7 g_t + a_8 g_x + a_9 g_y + a_{10} g = 0, \tag{191}$$

$$\begin{aligned}&(2a_1l_1 + a_4l_2 + a_5l_3)\, g_t + (2a_2l_2 + a_4l_1 + a_6l_3)\, g_x + (2a_3l_3 + a_5l_1 + a_6l_2)\, g_y \\ &+ (a_7l_1 + a_8l_2 + a_9l_3)\, g = 0,\end{aligned} \tag{192}$$

$$\begin{aligned}&(2a_1l_4 + a_4l_5 + a_5l_6)\, g_t + (2a_2l_5 + a_4l_4 + a_6l_6)\, g_x + (2a_3l_6 + a_5l_4 + a_6l_5)\, g_y \\ &+ (a_7l_4 + a_8l_5 + a_9l_6)\, g = 0,\end{aligned} \tag{193}$$

$$\begin{aligned}&(2a_1l_7 + a_4l_8 + a_5l_9)\, g_t + (2a_2l_8 + a_4l_7 + a_6l_9)\, g_x + (2a_3l_9 + a_5l_7 + a_6l_8)\, g_y \\ &+ (a_7l_7 + a_8l_8 + a_9l_9)\, g = 0,\end{aligned} \tag{194}$$

By Eq. (186), the particular solution of Eq. (191) is

$$g = C_1 e^{\lambda_1 v} + C_2 e^{\lambda_2 v}, \tag{195}$$

where λ_1 and λ_2 satisfy Eq. (165). Substituting (195) into (192) we get

$$\begin{aligned}&(2a_1l_1 + a_4l_2 + a_5l_3)\, g_t + (2a_2l_2 + a_4l_1 + a_6l_3)\, g_x + (2a_3l_3 + a_5l_1 + a_6l_2)\, g_y \\ &+ (a_7l_1 + a_8l_2 + a_9l_3)\, g \\ &= (2a_1l_1 + a_4l_2 + a_5l_3)\left(C_1k_1\lambda_1e^{\lambda_1 v} + C_2k_1\lambda_2e^{\lambda_2 v}\right) \\ &+ (2a_2l_2 + a_4l_1 + a_6l_3)\left(C_1k_2\lambda_1e^{\lambda_1 v} + C_2k_2\lambda_2e^{\lambda_2 v}\right) \\ &+ (2a_3l_3 + a_5l_1 + a_6l_2)\left(C_1k_3\lambda_1e^{\lambda_1 v} + C_2k_3\lambda_2e^{\lambda_2 v}\right) \\ &+ (a_7l_1 + a_8l_2 + a_9l_3)\left(C_1e^{\lambda_1 v} + C_2e^{\lambda_2 v}\right) \\ &= C_1e^{\lambda_1 v}(k_1\lambda_1\,(2a_1l_1 + a_4l_2 + a_5l_3) + k_2\lambda_1\,(2a_2l_2 + a_4l_1 + a_6l_3) \\ &+ k_3\lambda_1\,(2a_3l_3 + a_5l_1 + a_6l_2) + a_7l_1 + a_8l_2 + a_9l_3) \\ &+ C_2e^{\lambda_2 v}(k_1\lambda_2\,(2a_1l_1 + a_4l_2 + a_5l_3) + k_2\lambda_2\,(2a_2l_2 + a_4l_1 + a_6l_3) \\ &+ k_3\lambda_2\,(2a_3l_3 + a_5l_1 + a_6l_2) + a_7l_1 + a_8l_2 + a_9l_3) = 0.\end{aligned}$$

Namely

$$\begin{aligned}&k_1\lambda_1\,(2a_1l_1 + a_4l_2 + a_5l_3) + k_2\lambda_1\,(2a_2l_2 + a_4l_1 + a_6l_3) + k_3\lambda_1\,(2a_3l_3 + a_5l_1 + a_6l_2) \\ &+ a_7l_1 + a_8l_2 + a_9l_3 = 0,\end{aligned}$$

$$\begin{aligned}&k_1\lambda_2\,(2a_1l_1 + a_4l_2 + a_5l_3) + k_2\lambda_2\,(2a_2l_2 + a_4l_1 + a_6l_3) + k_3\lambda_2\,(2a_3l_3 + a_5l_1 + a_6l_2) \\ &+ a_7l_1 + a_8l_2 + a_9l_3 = 0.\end{aligned}$$

Substituting (195) into (192,193) respectively, we get

$$\begin{aligned}&k_1\lambda_1\,(2a_1l_4 + a_4l_5 + a_5l_6) + k_2\lambda_1\,(2a_2l_5 + a_4l_4 + a_6l_6) + k_3\lambda_1\,(2a_3l_6 + a_5l_4 + a_6l_5) \\ &+ a_7l_4 + a_8l_5 + a_9l_6 = 0,\end{aligned}$$

$$\begin{aligned}&k_1\lambda_2\,(2a_1l_4 + a_4l_5 + a_5l_6) + k_2\lambda_2\,(2a_2l_5 + a_4l_4 + a_6l_6) + k_3\lambda_2\,(2a_3l_6 + a_5l_4 + a_6l_5) \\ &+ a_7l_4 + a_8l_5 + a_9l_6 = 0,\end{aligned}$$

$$\begin{aligned}&k_1\lambda_1\,(2a_1l_7 + a_4l_8 + a_5l_9) + k_2\lambda_1\,(2a_2l_8 + a_4l_7 + a_6l_9) + k_3\lambda_1\,(2a_3l_9 + a_5l_7 + a_6l_8) \\ &+ a_7l_7 + a_8l_8 + a_9l_9 = 0,\end{aligned}$$

$$\begin{aligned}&k_1\lambda_2\,(2a_1l_7 + a_4l_8 + a_5l_9) + k_2\lambda_2\,(2a_2l_8 + a_4l_7 + a_6l_9) + k_3\lambda_2\,(2a_3l_9 + a_5l_7 + a_6l_8) \\ &+ a_7l_7 + a_8l_8 + a_9l_9 = 0.\end{aligned}$$

Case 1: Set

$$a_1l_1^2+a_2l_2^2+a_3l_3^2+a_4l_1l_2+a_5l_1l_3+a_6l_2l_3=0, \tag{178}$$

$$a_1l_4^2+a_2l_5^2+a_3l_6^2+a_4l_4l_5+a_5l_4l_6+a_6l_5l_6=0, \tag{179}$$

$$a_1l_7^2+a_2l_8^2+a_3l_9^2+a_4l_7l_8+a_5l_7l_9+a_6l_8l_9=0, \tag{180}$$

$$2a_1l_1l_7+2a_2l_2l_8+2a_3l_3l_9+a_4\left(l_1l_8+l_2l_7\right)+a_5\left(l_1l_9+l_3l_7\right)+a_6\left(l_2l_9+l_3l_8\right)=0, \tag{181}$$

$$2a_1l_4l_7+2a_2l_5l_8+2a_3l_6l_9+a_4\left(l_4l_8+l_5l_7\right)+a_5\left(l_4l_9+l_6l_7\right)+a_6\left(l_5l_9+l_6l_8\right)=0. \tag{182}$$

Then

$$\begin{aligned}&a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{tx}+a_5u_{ty}+a_6u_{xy}+a_7u_t+a_8u_x+a_9u_y+a_{10}u\\&=\left(2a_1l_1l_4+2a_2l_2l_5+2a_3l_3l_6+a_4\left(l_1l_5+l_2l_4\right)+a_5\left(l_1l_6+l_3l_4\right)+a_6\left(l_2l_6+l_3l_5\right)\right)gh_{pq}\\&=A\left(p,q,r\right).\end{aligned}$$

That is

$$h_{pq}=\frac{A\left(p,q,r\right)}{\left(2a_1l_1l_4+2a_2l_2l_5+2a_3l_3l_6+a_4\left(l_1l_5+l_2l_4\right)+a_5\left(l_1l_6+l_3l_4\right)+a_6\left(l_2l_6+l_3l_5\right)\right)g}, \tag{196}$$

the general solution of Eq. (196) is

$$h=h_1\left(p,r\right)+h_2\left(q,r\right)+\int\frac{A\left(p,q,r\right)dpdq}{Bg}, \tag{197}$$

where

$$B=2a_1l_1l_4+2a_2l_2l_5+2a_3l_3l_6+a_4\left(l_1l_5+l_2l_4\right)+a_5\left(l_1l_6+l_3l_4\right)+a_6\left(l_2l_6+l_3l_5\right), \tag{169}$$

So the general solution of Eq. (162) is

$$u=gh=g\left(h_1\left(p,r\right)+h_2\left(q,r\right)\right)+\frac{g}{B}\int\frac{A\left(p,q,r\right)dpdq}{g}.$$

Whereupon the theorem is proved. □

Case 2: Set

$$a_1l_1^2+a_2l_2^2+a_3l_3^2+a_4l_1l_2+a_5l_1l_3+a_6l_2l_3=0,$$

$$a_1l_4^2+a_2l_5^2+a_3l_6^2+a_4l_4l_5+a_5l_4l_6+a_6l_5l_6=0,$$

$$a_1l_7^2+a_2l_8^2+a_3l_9^2+a_4l_7l_8+a_5l_7l_9+a_6l_8l_9=0,$$

$$2a_1l_1l_4+2a_2l_2l_5+2a_3l_3l_6+a_4\left(l_1l_5+l_2l_4\right)+a_5\left(l_1l_6+l_3l_4\right)+a_6\left(l_2l_6+l_3l_5\right)=0,$$

$$2a_1l_4l_7+2a_2l_5l_8+2a_3l_6l_9+a_4\left(l_4l_8+l_5l_7\right)+a_5\left(l_4l_9+l_6l_7\right)+a_6\left(l_5l_9+l_6l_8\right)=0.$$

Then

$$\begin{aligned}&a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{tx}+a_5u_{ty}+a_6u_{xy}+a_7u_t+a_8u_x+a_9u_y+a_{10}u\\&=\left(2a_1l_1l_7+2a_2l_2l_8+2a_3l_3l_9+a_4\left(l_1l_8+l_2l_7\right)+a_5\left(l_1l_9+l_3l_7\right)+a_6\left(l_2l_9+l_3l_8\right)\right)gh_{pr}\\&=A\left(p,q,r\right).\end{aligned}$$

Namely

$$h_{pr}=\frac{A\left(p,q,r\right)}{Eg}, \tag{198}$$

$$E=2a_1l_1l_7+2a_2l_2l_8+2a_3l_3l_9+a_4\left(l_1l_8+l_2l_7\right)+a_5\left(l_1l_9+l_3l_7\right)+a_6\left(l_2l_9+l_3l_8\right). \tag{199}$$

The general solution of Eq. (198) is

$$h = h_1(p,q) + h_2(q,r) + \int \frac{A(p,q,r)\,dpdr}{Eg}.$$

So the general solution of Eq. (162) is

$$u = gh = g(h_1(p,q) + h_2(q,r)) + \frac{g}{E}\int \frac{A(p,q,r)\,dpdr}{g}. \tag{200}$$

For other cases and $b^2 - 4aa_{10} = 0, b^2 - 4aa_{10} < 0$, similar calculations can be done.

According to Theorem 25, we can get Theorem 26.

Theorem 26. *In* $\mathbb{R}^3$,

$$a_1u_{tt} + a_2u_{xx} + a_3u_{yy} + a_4u_{tx} + a_5u_{ty} + a_6u_{xy} + a_7u_t + a_8u_x + a_9u_y + a_{10}u = 0, \tag{201}$$

the general solution of Eq. (201) *is*

$$u = g(h_1(p,r) + h_2(q,r)), \tag{202}$$

where a_i *are arbitrary known constants,* h_1 *and* h_2 *are arbitrary second differentiable functions,* g, p, q *and* r *satisfy* $(164-168)$ *and* $(170-182)$.

Eqs. (162, 201) are very significant linear partial differential equations. Two-dimensional wave equation, heat equation, Fokker-Planck Equation [26-28], Telegraph Equation [29-31], etc. are all special cases of them.

4. Conclusions

In this paper, we first prove a new theorem for the independent variable transformational equations, that is, the independent variable transformation not only does not change the linearity or non-linearity of the original PDEs, but also does not change their order.

We propose the concept of the banal PDE and the non-banal PDE, then use the proposed three kinds of Z Transformations to obtain the plentiful laws of general solutions of linear PDEs with variable coefficients and constant coefficients.

The characteristic equation method is a basic method to solve first order linear and quasilinear PDEs [32]. By comparing with Z Transformations, we can find that it has some limitations, such as using it cannot get the general solutions of the first order linear PDEs (25, 57, 93), cannot obtain the complete general solution of (68) and so on.

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