A Short Proof Pertaining to the Euler/De-Moivre Complex Identity

Aryamoy Mitra

Abstract

This paper is a succinct demonstration of an equality derived from the differentiated expressions corresponding to the relation between Euler's and De-Moivre's formulations of complex numbers.

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This proof will demonstrate, that for any value of θ; | $e^{i\theta} = \frac{1}{2cos\theta - e^{i\theta}}$ |

Phase 1: Consider the classic equivalency between De Moivre's and Euler's formulae

$$e^{i\theta} = cis\theta$$

$$e^{i\theta} = cos\theta + isin\theta$$

$$isin\theta = e^{i\theta} - cos\theta$$

$$\boldsymbol{i = \frac{e^{i\theta} - cos\theta}{sin\theta}}$$

Let this equation be titled *E1.*

Phase 2: Consider the same theorem; but differentiate the expression with respect to θ on both sides.

$$e^{i\theta} = cos\theta + isin\theta$$

$$\frac{d[e^{i\theta}]}{d\theta} = \frac{d[cos\theta + isin\theta]}{d\theta}$$

Since $\frac{d[e^{i\theta}]}{d\theta}$ = $ie^{i\theta}(derivatives\ of\ exponential\ functions)$;

$$ie^{i\theta} = \frac{d[cos\theta + isin\theta]}{d\theta}$$

$$ie^{i\theta} = \frac{d[cos\theta]}{d\theta} + \frac{d[isin\theta]}{d\theta}$$

$$ie^{i\theta} = \frac{d[cos\theta]}{d\theta} + i\frac{d[sin\theta]}{d\theta}$$

$$ie^{i\theta} = -sin\theta + icos\theta$$

$$ie^{i\theta} - icos\theta = -sin\theta$$

If one were to multiply both sides by a factor equivalent to -1;

$$-1(ie^{i\theta} - icos\theta) = -1(-sin\theta)$$

$$icos\theta - ie^{i\theta} = sin\theta$$

$$i(cos\theta - e^{i\theta}) = sin\theta$$

$$\boldsymbol{i = \frac{sin\theta}{cos\theta - e^{i\theta}}}$$

Let this equation be titled *E2.*

Phase 3: Equivalency;

Since *E1* and *E2* both describe i in terms of trigonometric functions, they can be equated with one another;

$$i = \frac{e^{i\theta} - cos\theta}{sin\theta} = \frac{sin\theta}{cos\theta - e^{i\theta}}$$

Cross-multiplying yields;

$$(sin\theta)(sin\theta) = (e^{i\theta} - cos\theta)(cos\theta - e^{i\theta})$$

$$\sin^2\theta = (e^{i\theta} - cos\theta)(cos\theta - e^{i\theta})$$

$$\sin^2\theta = (e^{i\theta} - cos\theta)(cos\theta - e^{i\theta})$$

$$\sin^2\theta = e^{i\theta}cos\theta - (e^{i\theta})^2 - \cos^2\theta + e^{i\theta}cos\theta$$

$$\sin^2\theta = 2e^{i\theta}cos\theta - (e^{i\theta})^2 - \cos^2\theta$$

$$\sin^2\theta + \cos^2\theta = 2e^{i\theta}cos\theta - (e^{i\theta})^2$$

Since $sin^2\theta + cos^2\theta = 1$;

$$2e^{i\theta}cos\theta - (e^{i\theta})^2 = 1$$

$$2e^{i\theta}cos\theta - (e^{i\theta})^2 = 1$$

$$e^{i\theta}(2cos\theta - e^{i\theta}) = 1$$

$$\boldsymbol{e^{i\theta} = \frac{1}{2cos\theta - e^{i\theta}}}$$

On account of the original theorem, θ must be expressed in degrees in the term '$2cos\theta'$', and radians in the term $e^{i\theta}$;

Confirming this equality: For $\theta = \pi\ radians$:

$$e^{i\theta} = e^{i\pi} = -1$$

$$\frac{1}{2cos\theta - e^{i\theta}} = \frac{1}{2cos180 - e^{i\pi}}$$

$$= \frac{1}{-2-(-1)}$$

$$\frac{1}{-1} = -1 = e^{i\pi}$$