A Premise for the Natural Density of Prime Number Power Multisets: Continuum Hypothesis and Canonical Countability Assertions Refuted

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Abstract: The current canonical treatment of multisets to our knowledge does not discuss their powersets, but such powersets have a natural connection linking prime and natural numbers. Using the natural definition of continuity, and the natural extension of the formula for counting elements in a powerset to count power multisets, we find prime numbers are countably infinite, with an infinitude of infinities with natural density specified by the Riemann zeta function, between the primes and the continuous natural numbers.

Introduction. The powerset operation over the set S, denoted $\mathcal{P}_1(S)$, we know, results in a set with $2^{|S|}$ elements, where absolute value bars denote the cardinality of the enclosed set. It is less known that this formula is a material equation, and generalizes to enumerate the cardinality of powersets taken to allow up to multiplicity, m, other than one. We use the subscript of $\mathcal{P}_m$ to denote the maximum multiplicity in a power multiset taken over the argument. Recall that the equation $|\mathcal{P}_1(S)| = 2^{|S|}$ comes from the fact that each element in S, with respect to each element in its' powerset, has a status in $\{1\ 0\}$. That is, it is either present or absent in a given subset. Since there are two possible states for each element, we have $2^{|S|}$ total elements including the empty set in the set of all subsets[i] of S, i.e. its' powerset. From this, it follows that if we have a power multiset with maximum multiplicity 2, i.e., $\mathcal{P}_2(S)$, to find its' cardinality we use $|\mathcal{P}_2(S)| = 3^{|S|}$. Now, because there are three possible states of being, $\{0\ 1\ 2\}$, the elements from S produce a power multiset with cardinality $3^{|S|}$. For example, with $S = \{7\ \&\}$, we have, $|\mathcal{P}_2(S)| = 3^{|S|}$. And because there are two elements, $3^{|S|} = 3^2$. Which gives us 9.

$\aleph_1$	$\aleph_{00}$	$\aleph_{0.2}$	$\aleph_{0.3}$	$\aleph_{0.4}$	$\aleph_{0.5}$
1		1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4			4	4	4
5	5	5	5	5	5
6		6	6	6	6
7	7	7	7	7	7
8				8	8
9			9	9	9
10		10	10	10	10
11	11	11	11	11	11
12			12	12	12
13	13	13	13	13	13
14		14	14	14	14
15		15	15	15	15
16					16
17	17	17	17	17	17
18			18	18	18
19	19	19	19	19	19
20			20	20	20
21		21	21	21	21
22		22	22	22	22
23	23	23	23	23	23
24				24	24
25			25	25	25
26		26	26	26	26
27				27	27
28			28	28	28
29	29	29	29	29	29
30		30	30	30	30

Table 1. The infinite sets spanning the primes and the natural numbers from 1 to 30.

Set: $\{\ 7\ \&\ \}$.

Powerset: $\mathcal{P}_1\{7\ \&\} = \{\{7\}\{\&\}\{7\ \&\}\{\emptyset\}\}$

Power Multiset $m = 2$: $\mathcal{P}_2\{7\ \&\} = \{\{7\}\{\&\}\{7\ \&\}\{7\ 7\}\{\&\ \&\}\{7\ 7\ \&\}\{\&\ \&\ 7\}\{\&\ 7\ \&\ 7\}\{\emptyset\}\}$.

Thus, to find the cardinality of a power multiset, we raise the multiplicity plus one, to the number of elements in S. Notice that, from a set of k elements, we find the sequence of sizes of power multisets with increasing multiplicity is the sequence of k^{th} powers. The density of a power multiset among n continuous elements can be expressed leveraging the Riemann zeta function (see below) in such a way that we write the generalized powerset sizing equation,

$$1)\qquad |\mathcal{P}_m(S)| = (m+1)^{|S|} \equiv \frac{n}{\zeta(m+1)}.$$

Computation: The prime numbers make up a discontinuous subset of the natural numbers, using the naïvely intuitive interpretation of continuous, where continuity implies 1,2,3 … without missing a natural number. When the set S in equation 1 is the prime numbers, the powerset transparently corresponds to the square free numbers, which remain a subset of the natural numbers. Increasing the multiplicity m of the powerset formed from the primes, establishes continuity up to 2^m, after which point, discontinuity ensues. We see this in table 1. By increasing the multiplicity of the powersets to five, we are able to reach continuity among the first thirty natural numbers. But this will only last until 2^m, i.e., 32.

By Cantor's theorem, these sets have greater cardinality than the primes, but are still discontinuous among the naturals. We know [1] by the the Riemann zeta function,

$$\zeta(s) = \lim_{n \to \infty} \prod_{n \in \mathbb{N}}^{\infty} \frac{1}{1 - {p_n}^{-s}}, \qquad s > 1, p \in \mathbb{P}$$

that the corresponding natural density of the m^{th} power free numbers is given by $n/\zeta(m+1)$.

Results:

We verify its practical relevance empirically in table 2 for n of 30. We notice that to form the continuous natural numbers requires the power multiset of infinite multiplicity taken of the infinite set of primes. This is two levels of infinitude, suggesting that the natural numbers and prime numbers have distinct cardinality. Following convention, the smallest countably infinite set is given cardinality $\aleph_0$. This is consistent with current theory. However by Cantor's theorem, the powerset of a set has cardinality greater than that of the set. Since present theory equates $\aleph_1$ with the continuum, and we encounter continuity only with infinite multiplicity, i.e., $|\mathcal{P}_\infty(\mathbb{P})|$; it stands to reason that the natural numbers are uncountably infinite, while the prime numbers are countably infinite, and we have the natural numbers define the continuum. That is, $|\mathcal{P}_\infty(\mathbb{P})| = |\mathbb{N}| = \aleph_1$, a result that is kinder to intuition than presently canonical beliefs. Therefore, we have identified an infinitude of distinct cardinality infinite sets in between $\aleph_0$ as the set of primes. and the natural number continuum, henceforth $\aleph_1$, disproving the continuum hypothesis independently from questions about the axiom of choice.

	$\aleph_{0.2}$	$\aleph_{0.3}$	$\aleph_{0.4}$	$\aleph_{0.5}$
Empirical Measure From Table 1	19/30	26/15	29/30	1
$30/\zeta(m+1)$	18/30	25/30	29/30	1

Table 2. Empirical and theoretical densities of infinite sets with asymptotic cardinalities bewteen 1 and 30.

Discussion

We have shown that the continuous uncountably infinite set, contrary to canonical opinion, in the context provided by canonical principles, is the natural numbers. This makes sense because a set with the capacity to count any given set, including infinite sets like the prime numbers, must itself have a cardinality greater than any particular set it is counting. We can see that sets of numbers with cardinality less than $\mathbb{N}$ are not continuous because they are too few. With cardinality greater than the natural numbers, we have supersaturation, if we have unlimited precision of measurement, we can always find intervening quantities. In contrast to existing proofs [1] asserting a bijection between the powerset of the natural numbers and the real numbers, and the powerset of the natural numbers, and the continuum, which are too opaque for us to consciously opine, the validity of the bijection between the power multiset of the primes and the natural numbers is established by the fundamental theorem of arithmetic. To see this, from the power multiset with infinite multiplicity of the primes, map the empty set to one, and each element to the natural number equal to the product of its constituent subsets.

References

[1] Wikipedia, "Natural Density," Wikipedia, 2020. [Online]. Available: https://en.wikipedia.org/wiki/Natural_density. [Accessed 6 March 2020].

[2] "Continuum equals Cardinality of Power Set of Naturals," 12 August 2020. [Online]. Available: https://proofwiki.org/wiki/Continuum_equals_Cardinality_of_Power_Set_of_Naturals.

[i] We do not include the empty set when counting $|S|$