GLOBAL STABILITY FOR A SYSTEM OF PARABOLIC CONSERVATION LAWS ARISING FROM A KELLER-SEGEL TYPE CHEMOTAXIS MODEL

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Abstract. In this paper, we investigate the time-asymptotically nonlinear stability to the initial-boundary value problem for a coupled system in (p, q) of parabolic conservation laws derived from a Keller-Segel type repulsive model for chemotaxis with singular sensitivity and nonlinear production rate of $g(p) = p^\gamma$, where $\gamma > 1$. The proofs are based on basic energy method without any smallness assumption. We also show the zero chemical diffusion limit ($\varepsilon \to 0$) of solutions in the case $\bar{p} = 0$.

1. Introduction

In this paper, we consider the global existence and long time behavior of initial-boundary value problems for a system of parabolic conservation laws

$$\begin{cases} p_t - (pq)_x = p_{xx}, \ x \in (0,1), t > 0, \\ q_t - (g(p) + \varepsilon q^2)_x = \varepsilon q_{xx}, \quad \varepsilon \geq 0. \end{cases} \tag{1.1}$$

By taking the transformation $p = n$, $q = [\ln(c)]_x$ and assuming $D = -\chi = 1$ without loss of generality since specific values of χ and D are not important in our analysis, we can derive this system from the following chemotactic model proposed in [1] with logarithmic sensitivity and nonlinear production rate

$$\begin{cases} n_t = Dn_{xx} - [\chi n(\ln(c))_x]_x, \ x \in (0,1), \ t > 0, \\ c_t = \varepsilon c_{xx} + g(n)c - \mu c, \ x \in (0,1), \ t > 0, \varepsilon \geq 0, \end{cases} \tag{1.2}$$

where n and c represent the cell density and the chemical signal concentration, respectively. The parameter D denotes cell diffusion rate ($D > 0$), ε describes chemical diffusion rate and χ stands for chemotactic sensitivity coefficient. If $\chi > 0$(the positive chemotaxis), the chemotaxis means to be attractive, while if $\chi < 0$(the negative chemotaxis), the chemotaxis is repulsive. The constant $\mu > 0$ stands for the natural degradation rate of the chemical signal. The function $\ln c$ denotes logarithmic chemotactic sensitivity function, which describes the signal detection mechanism of the cellular population. Such a kind of sensitivity function can be found in works [11–13]. The nonlinear function $g(n)$ denotes the chemical production rate, which satisfies $g'(n) \geq 0$ when $n \geq 0$.

When $g(p) = p$, the mathematical analysis about global well-posedness, long-time behavior, diffusion limit, boundary layer, stability of traveling wave, etc. of (1.1) with subject to various initial and/or boundary conditions in one and multiple space dimensions has been made in significant progresses in the past few years, please refer [2–7, 10, 14, 16–23, 29, 30, 32–35] and the references therein. On the other hand, when the chemical production rate is a nonlinear function, there are a few results. In [36], the global well-posedness for the Cauchy problem of (1.1) in one dimension space for general initial data under the assumption that $|g''(p)|$ is uniformly bounded was proved. Later, in [37], Zhu, Liu, Martinez and Zhao adopted a new Lyapunov functional and removed this assumption to get the global well-posedness, the long time behavior and the diffusion limit for the Cauchy problem of (1.1) with $g(p) = p^\gamma$ for all

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$\gamma > 1$. However, the well posedness, long-time behavior and diffusion limit to the system (1.1) in bounded domain remains open in the literature, although this type of boundary value problem is more meaningful than the Cauchy problem from the biological point view. The main purpose of this manuscript is to study the well posedness and long-time behavior for the system (1.1) with the following two kinds of initial and boundary conditions:

- Initial-Dirichlet boundary value
$$\begin{cases} (p,q)(x,0)=(p_0,q_0)(x),\ p_0(x)\geq 0,\ x\in I;\\ p|_{x=0,x=1}=\bar p\geq 0,\ q|_{x=0,x=1}=0,\ if\ \varepsilon>0;\\ p|_{x=0,x=1}=\bar p\geq 0,\ if\ \varepsilon=0, \end{cases} \tag{1.3}$$
- Neumann-Dirichlet boundary value
$$\begin{cases} (p,q)(x,0)=(p_0,q_0)(x), p_0(x)\geq 0,\ x\in I\\ p_x|_{x=0,x=1}=0,\ q|_{x=0,x=1}=0, \end{cases} \tag{1.4}$$
where $I=[0,1]$.

In order to understand the qualitative behavior of (1.1), based on similar ideas in [37] for the Cauchy problem, we use the following weak Lyapunov functional

$$\frac{d}{dt}\left(\frac{1}{\gamma-1}\int_I[p^\gamma-\bar p^\gamma-\gamma\bar p^{\gamma-1}(p-\bar p)]dx+\frac12\|q\|_{L^2}^2\right)+\frac{4}{\gamma}\|(p^{\frac{\gamma}{2}})_x\|_{L^2}^2+\varepsilon\|q_x\|_{L^2}^2=0. \tag{1.5}$$

Now it is the place to state our main results of this paper. The first result addresses the global well-posedness and long-time behavior of large-amplitude global solutions to (1.1) and (1.3).

Theorem 1.1. *Assume that the initial data satisfy $p_0\geq 0$ and $(p_0-\bar p,q_0)\in H^2(I)$ for some positive constant $\bar p$. Then there exists a global-in-time solution (p,q) to the initial-boundary value problem (1.1) and (1.3), such that*

- *For any $\varepsilon>0$ and $\gamma\geq 1$, it holds that*
$$(p-\bar p,q)\in C([0,\infty);H^2((0,1)))\cap L^2([0,\infty);H^3((0,1))),$$
and for all $t>0$,
$$\|(p-\bar p)(t)\|_{H^2}^2+\|q(t)\|_{H^2}^2+\int_0^t(\|p_x(\tau)\|_{H^2}^2+\|q_x(\tau)\|_{L^2}^2+\varepsilon\|q_x(\tau)\|_{H^2}^2)d\tau\leq C,$$
where the constant $C>0$ is independent of t. Moreover, it holds that
$$\|(p-\bar p)(t)\|_{H^2}^2+\|q(t)\|_{H^2}^2\leq\alpha e^{-\beta t}$$
for some positive constants α, β which are independent of t.
- *For $\varepsilon=0$ and $\gamma\geq 2$, it holds that*
$$(p-\bar p)\in C([0,\infty);H^2((0,1)))\cap L^2([0,\infty);H^2((0,1))),$$
$$q\in C([0,\infty);H^1((0,1)))\cap L^2([0,\infty);H^1((0,1))),$$
and, for all $t>0$,
$$\|(p-\bar p)(t)\|_{H^2}^2+\|q(t)\|_{H^1}^2+\int_0^t(\|(p-\bar p)(\tau)\|_{H^2}^2+\|q(\tau)\|_{H^1}^2)d\tau\leq C,$$
for some constant $C>0$ which is independent of t.
Moreover, when $\bar p>0$, it holds that
$$\|(p-\bar p)(t)\|_{H^2}^2+\|q(t)\|_{H^1}^2\leq\alpha_1e^{-\beta_1 t},$$
where α_1 and β_1 are positive constants which are independent of t.

Remark 1.1. *Compared with the Cauchy problem in [37], the distinction is that the energy estimate for the derivatives of the solution is inversely proportional to ε. We can't utilize the same method to estimate the first order spatial derivatives due to the influence of the boundary conditions.*

From the Theorem (1.1), we obtain the long-time dynamics for the original repulsive chemotaxis type model as follows.

Proposition 1.2. *(Long-time behavior of original model). Consider the following initial boundary value problem of the one-dimensional chemotaxis type model (1.2):*

$$\begin{cases} n_t = Dn_{xx} - \chi[n(\ln c)_x]_x, \\ c_t = \varepsilon c_{xx} + n^\gamma c - \mu c, \ \gamma > 1; \\ (n,c)(x,0) = (n_0, c_0)(x); \\ n|_{x=0,x=1} = \bar{n}, \quad \frac{c_x}{c}|_{x=0,x=1} = 0, \ if \ \varepsilon > 0; \\ n|_{x=0,x=1} = \bar{n}, \ if \ \varepsilon = 0, \end{cases} \tag{1.6}$$

$x \in (0,1)$, $t > 0$, where $D > 0$, $\chi < 0$, $\mu > 0$ and $\varepsilon \geq 0$ are constant parameters, and $\bar{n} \geq 0$ and $\hat{c}$ are constants. Suppose that the initial data are compatible with the boundary conditions and satisfy $u_0(x) \geq 0$, $0 < \underline{c} \leq c_0(x) \leq \bar{c} < \infty$ for some constants $\underline{c}$ and $\bar{c}$. Assume that $c_0 \in H^2((0,1))$ and $\ln c_0 \in H^3((0,1))$. Then there exists a unique global-in-time classical solution (n,c) to (1.6) such that

$$\|n(t) - \bar{n}\|_{L^\infty} \to 0, \ as \ t \to \infty,$$

and

$$\|c(t)\|_{L^\infty} \to \begin{cases} 0, \ if \ \bar{n}^\gamma < \mu, \\ +\infty, \ if \ \bar{n}^\gamma > \mu, as \ t \to \infty, \varepsilon \geq 0 \end{cases}$$

provided that either $\bar{n} \geq 0$, $\varepsilon \geq 0$, where the convergence rates are exponential in time.

The next result is concerned with the Global existence, zero chemical diffusion limit and convergence rate of the solution to (1.1)-(1.3).

Theorem 1.3. *Assume that the initial data satisfy $p_0 \geq 0$ and $(p_0, q_0) \in H^1(I)$ for $\bar{p} = 0$. Then for any $\varepsilon \geq 0$ and $\gamma \geq 2$, there exists a unique global-in-time strong solution (p,q) to (1.1)and (1.3)such that*

$$\|(p,q)(t)\|_{H^1}^2 + \int_0^t (\|p_x(\tau)\|_{H^1}^2 + \varepsilon \|q_x(\tau)\|_{H^1}^2) d\tau \leq C,$$

where the constant C is independent of ε. Moreover, let $(p^\varepsilon, q^\varepsilon)$ and (p^0, q^0) be the unique strong solution to (1.1) and (1.3) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively, then $(p^\varepsilon, q^\varepsilon)$ approaches (p^0, q^0) with the following convergence rate:

$$\|(p^\varepsilon - p^0)(t)\|_{L^2}^2 + \|(q^\varepsilon - q^0)(t)\|_{L^2}^2 \leq \alpha_2 e^{\beta_2 t} \varepsilon,$$

where α_2 and β_2 are positive constants which are independent of ε.

Remark 1.2. *When $\gamma = 1$, the result in [7] shows that the diffusive problem ($\varepsilon > 0$) does not converge to the non-diffusion problem as chemical diffusion coefficient ε tends to zero. However, in our result Theorem 1.3, which shows the diffusive problem converges to the non-diffusive problem, which implies that there is no boundary layer solution when $\bar{p} = 0$.*

Our next result shows that the global dynamics result can be obtained for the Neumann-Dirichlet boundary value problem.

Theorem 1.4. *(Neumann-Dirichlet problem). Consider the initial-boundary value problem*

$$\begin{cases} p_t = p_{xx} + (pq)_x, \\ q_t = \varepsilon q_{xx} + \varepsilon (q^2)_x + (p^\gamma)_x; \\ (p,q)(x,0) = (p_0,q_0)(x), \ (p_0 - \bar{p}, q_0) \in H^1((0,1)), \\ p_0(x) \geq 0, \ x \in [0,1]; \\ p_x|_{x=0,x=1} = 0, \ q|_{x=0,x=1} = 0, \end{cases} \tag{1.7}$$

where $\bar{p} = \int_0^1 p_0(x)dx > 0$. Suppose that the initial data are compatible with the boundary conditions. Then for any $\varepsilon > 0$ there exists a unique solution (p,q) to (1.7) such that

$$(p - \bar{p}, q) \in C([0,\infty); H^1((0,1))) \cap L^2([0,\infty); H^2((0,1)))$$

and

$$\|(p-\bar{p})(t)\|_{H^1}^2 + \|q(t)\|_{H^1}^2 + \int_0^t (\|(p-\bar{p})(\tau)\|_{H^2}^2 + \|q(\tau)\|_{H^1}^2 + \varepsilon\|q(\tau)\|_{H^2}^2) d\tau \leq C,$$

where C is independent of t and ε. Furthermore, it holds that

$$\|(p-\bar{p})(t)\|_{H^1}^2 + \|q(t)\|_{H^1}^2 \leq \alpha_3 e^{-\beta_3 t}$$

for some positive constants α_3, β_3 which are independent of t and ε.

Remark 1.3. *Here are some remarks concerning about Theorem 1.4:*

- *Note that the result in [18] focuses on the case when $\gamma = 1$. Compared with the result obtained in [18], the result in this paper is the first one concerning on the case when $\gamma > 1$ for the Neumannn-Dirichlet problem. The term $(p^\gamma)_x$ is strong nonlinear, luckily, we can overcome this difficulty by using similar method in [37].*
- *We emphasis that the bound C is independent on ε, so we can obtain the zero chemical diffusion limit and convergence rate of the solution obtained in Theorem 1.4 by virtue of the similar method in [27, 37], we omit the technical details in order to simplify the presentation.*

The rest of paper is organized as follows: we first give some preliminaries in Section 2. In section 3, we give the proof of Theorem1.1 Proposition 1.2 and Theorem1.3. Then the proof of Theorem1.4 will be deduced in Section 4.

Notations: Throughout this paper, we denote $\|\cdot\|_{L^2}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{H^s}$ by the usual norms of Lebesgue measurable spaces L^2, L^∞ and Hilbert's space H^s, respectively. The values of positive constants C may vary line by line according to the context. For two quantities A and B, we write $A \sim B$ if $C^{-1}A \leq B \leq CA$. The notation $A \lesssim B$ means that $A \leq CB$ for a universal constant $C > 0$ independent of time t.

2. PRELIMINARIES

In this section, we shall introduce some Algebraic inequalities which will be frequently used in the subsequent analysis (cf. [8, 9, 37]).

Lemma 2.1. *Let $a \geq -1$ and $\gamma \geq 2$. Then it holds that*

$$(a+1)^\gamma - 1 - \gamma a \geq \frac{\gamma}{2} a^2.$$

Lemma 2.2. *Let $a \geq -1$ and $\gamma \geq 2$. Then it holds that*

$$(a+1)^\gamma - 1 - \gamma a \geq |a|^\gamma.$$

Lemma 2.3. *Let $a \geq -1$ and $\gamma > 1$. Then it holds that*

$$(a+1)^\gamma \geq 1 + \gamma a.$$

Lemma 2.4. *Let $a \geq 0$ and $0 \leq \gamma \leq 1$. Then it holds that*

$$|a^\gamma - 1| \leq |a - 1|.$$

Lemma 2.5. *Let $a \geq -1$ and $1 < \gamma < 2$. Then it holds that*

$$(a+1)^\gamma - 1 - \gamma a \leq a^2.$$

3. Global dynamics when $\gamma > 1$

In this section, we are devoted to studying the dynamic of solutions to the problem (1.1) with Dirichlet Boundary condition. First, using the standard arguments (e.g. see [24–26, 31]), one can show the local existence of solutions to (1.1)-(1.3). We omit the technical details of the routine arguments in order to simplify the presentation. Next we derive some *a priori* uniform-in-t estimates of solutions, which not only extend the local solutions to global ones, but also play important role in investigating the long time behavior of solutions. For this goal, we shall first focus on study of the following reformulated problem:

$$\begin{cases} \tilde{p}_t - [(\tilde{p}+\bar{p})q]_x = \tilde{p}_{xx}, \\ q_t - [(\tilde{p}+\bar{p})^\gamma]_x = \varepsilon q_{xx} + \varepsilon (q^2)_x, \\ (\tilde{p}(x,0), q(x,0)) = (p_0 - \bar{p}, q_0), \\ \tilde{p}|_{x=0,x=1} = 0, \ \ q|_{x=0,x=1} = 0, \end{cases} \tag{3.1}$$

where $\tilde{p} = p - \bar{p}$.

Lemma 3.1. *(Lyapunov functional) Under the conditions of Theorem 1.1, for any $\gamma > 1$, $\varepsilon \geq 0$, it holds that*

$$\begin{aligned} &\frac{1}{\gamma-1}\int_I [(\tilde{p}+\bar{p})^\gamma - \bar{p}^\gamma - \gamma \bar{p}^{\gamma-1}\tilde{p}](x,t)dx + \frac{1}{2}\|q(t)\|_{L^2}^2 \\ &\quad + \int_0^t \left(\gamma \int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2 dx + \varepsilon \|q_x\|_{L^2}^2\right)(\tau) d\tau \leq C, \end{aligned} \tag{3.2}$$

where the constant C depends only on $\bar{p}$, γ, and the initial data. Moreover, for any $\gamma \geq 2$, it holds that

$$\|\tilde{p}\|_{L^\gamma}^\gamma \leq (\gamma - 1)C. \tag{3.3}$$

Proof. Multiplying the first equation of (3.1) by $\frac{\gamma}{\gamma-1}((\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1})$, then integrating the result equation over I by parts, we obtain

$$\frac{1}{\gamma-1}\frac{d}{dt}\left(\int_I (\tilde{p}+\bar{p})^\gamma dx - \gamma \bar{p}^{\gamma-1}\int_I \tilde{p} dx\right) + \gamma \int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}+\bar{p})_x^2 dx = \int_I (\tilde{p}+\bar{p})^\gamma q_x dx. \tag{3.4}$$

Noting that (3.4) can be written as

$$\frac{1}{\gamma-1}\frac{d}{dt}\int_I (\tilde{p}+\bar{p})^\gamma - \bar{p}^\gamma - \gamma \bar{p}^{\gamma-1}\tilde{p} dx + \gamma \int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}+\bar{p})_x^2 dx = \int_I (\tilde{p}+\bar{p})^\gamma q_x dx. \tag{3.5}$$

Multiplying the second equation of (3.1) by q and integrating the resulting equation by parts over I, we have

$$\frac{1}{2}\frac{d}{dt}\|q\|_{L^2}^2 + \int_I (\tilde{p}+\bar{p})^\gamma q_x dx + \varepsilon \|q_x\|_{L^2}^2 = 0. \tag{3.6}$$

Adding (3.5) to (3.6), we obtain

$$\frac{d}{dt}\left(\frac{1}{\gamma-1}\int_I [(\tilde{p}+\bar{p})^\gamma - \bar{p}^\gamma - \gamma \bar{p}^{\gamma-1}\tilde{p}] dx + \frac{1}{2}\|q\|_{L^2}^2\right) + \gamma \int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2 dx + \varepsilon \|q_x\|_{L^2}^2 = 0. \tag{3.7}$$

Integrating (3.7) over $[0,t]$, we get

$$\frac{1}{\gamma-1}\int_I[(\tilde{p}+\bar{p})^\gamma-\bar{p}^\gamma-\gamma\bar{p}^{\gamma-1}\tilde{p}](x,t)dx+\frac{1}{2}\|q(t)\|_{L^2}^2$$
$$+\int_0^t\left(\gamma\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx+\varepsilon\|q_x\|_{L^2}^2\right)(\tau)d\tau$$
$$=\frac{1}{\gamma-1}\int_I[p_0^\gamma-\bar{p}^\gamma-\gamma\bar{p}^{\gamma-1}(p_0-\bar{p})](x)dx+\frac{1}{2}\|q_0\|_{L^2}^2.$$

By lemma 2.4, letting $a=\frac{p_0-\bar{p}}{\bar{p}}$, for $1<\gamma<2$, we have

$$\frac{1}{\gamma-1}\int_I[p_0^\gamma-\bar{p}^\gamma-\gamma\bar{p}^{\gamma-1}(p_0-\bar{p})]dx\le\frac{\bar{p}^{\gamma-2}}{\gamma-1}\|p_0-\bar{p}\|_{L^2}^2. \tag{3.8}$$

By Taylor's theorem, for $\gamma\ge2$, we have

$$\frac{1}{\gamma-1}\int_I[p_0^\gamma-\bar{p}^\gamma-\gamma\bar{p}^{\gamma-1}(p_0-\bar{p})]dx\le\frac{\gamma}{2}\int_I\hat{p}_0^{\gamma-2}(p_0-\bar{p})^2dx,$$

where $\hat{p}_0$ is between p_0 and $\bar{p}$. Hence, there is a constant C_0 depending only on γ and initial data, such that

$$\frac{1}{\gamma-1}\int_I[p_0^\gamma-\bar{p}^\gamma-\gamma\bar{p}^{\gamma-1}(p_0-\bar{p})]dx\le C_0\|p_0-\bar{p}\|_{L^2}^2. \tag{3.9}$$

By applying Lemma 2.1, we obtain

$$\frac{1}{\gamma-1}\int_I[(\tilde{p}+\bar{p})^\gamma-\bar{p}^\gamma-\gamma\bar{p}^{\gamma-1}\tilde{p}]\ge\frac{1}{\gamma-1}\|\tilde{p}\|_{L^\gamma}^\gamma. \tag{3.10}$$

Then the combination of (3.8), (3.8) and (3.9) completes the proof of Lemma 3.1. □

3.1. **Global classical solution when $\varepsilon>0$ and $\bar{p}>0$.** In this section, we are devoted to obtaining the global dynamical classical solution when $\varepsilon>0$ and $\bar{p}>0$. Firstly, we give the basic L^2-energy estimate, the idea are borrowed from [37].

Lemma 3.2. *Under the conditions of Theorem 1.1, for any $\gamma>2$, $\varepsilon\ge0$, it holds that*

$$\|\tilde{p}\|_{L^\gamma}^\gamma+\|\tilde{p}\|_{L^2}^2+\int_0^t\|\tilde{p}_x(\tau)\|_{L^2}^2d\tau\le C,$$

where C is a positive constant which is indepent of t and ε.

3.1.1. *L^2 Estimate when $2\le\gamma\le3$.*

Proof. Step 1. First of all, let $G(\tilde{p},\bar{p})=\frac{1}{\gamma-1}\int_I[(\tilde{p}+\bar{p})^\gamma-\bar{p}^\gamma-\gamma\bar{p}^{\gamma-1}\tilde{p}](x,t)dx$, then (3.7) can be rewritten as

$$\frac{d}{dt}\left(G(\tilde{p},\bar{p})+\frac{1}{2}\|q\|_{L^2}^2\right)+\gamma\int_I[(\tilde{p}+\bar{p})^{\gamma-2}-\bar{p}^{\gamma-2}](\tilde{p}_x)^2dx$$
$$+\gamma\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2+\varepsilon\|q_x\|_{L^2}^2=0. \tag{3.11}$$

Since $2\le\gamma\le3$, choosing $a=(\tilde{p}+\bar{p})/\bar{p}$, applying Lemma 2.3 and using Young's inequality, we deduce that

$$|(\tilde{p}+\bar{p})^{\gamma-2}-\bar{p}^{\gamma-2}|\le\bar{p}^{\gamma-3}|\tilde{p}|\le\frac{\bar{p}^{\gamma-4}\tilde{p}^2}{2}+\frac{\bar{p}^{\gamma-2}}{2},$$

which implies

$$(\tilde{p}+\bar{p})^{\gamma-2}-\bar{p}^{\gamma-2}\ge-\frac{\bar{p}^{\gamma-4}\tilde{p}^2}{2}-\frac{\bar{p}^{\gamma-2}}{2}. \tag{3.12}$$

Plugging (3.12) into (3.11), we obtain

$$\frac{d}{dt}\left(G(\tilde{p},\bar{p})+\frac{1}{2}\|q\|_{L^2}^2\right)+\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2-\frac{\gamma}{2}\bar{p}^{\gamma-4}\int_I(\tilde{p}\tilde{p}_x)^2dx+\varepsilon\|q_x\|_{L^2}^2\le0. \tag{3.13}$$

Step 2. To control the term $-\frac{\bar{p}^{\gamma-4}}{2}\int_I(\tilde{p}\tilde{p}_x)^2dx$, we multiplying the first equation of (3.1) by $4\tilde{p}^3$, then integrating by parts and using the Hölder's inequality to derive

$$\begin{aligned}
\frac{d}{dt}\|\tilde{p}\|_{L^4}^4+12\|\tilde{p}\tilde{p}_x\|_{L^2}^2 &= -12\int_I \tilde{p}^2\tilde{p}_x(\tilde{p}+\bar{p})qdx \\
&\le 12\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\left(\int_I(\tilde{p}+\bar{p})^{4-\gamma}\tilde{p}^4q^2dx\right)^{\frac12} \\
&= 12\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\left(\int_I(\tilde{p}^3+\bar{p}\tilde{p}^2)^{4-\gamma}|\tilde{p}|^{2\gamma-4}q^2dx\right)^{\frac12} \\
&\le 12\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\|\tilde{p}^3+\bar{p}\tilde{p}^2\|_{L^\infty}^{\frac{4-\gamma}{2}}\|\tilde{p}\|_{L^\infty}^{\gamma-2}\|q\|_{L^2} \\
&\le C\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\left(\|\tilde{p}^3\|_{L^\infty}^{\frac{4-\gamma}{2}}+\|\tilde{p}^2\|_{L^\infty}^{\frac{4-\gamma}{2}}\right)\|\tilde{p}\|_{L^\infty}^{\gamma-2}.
\end{aligned} \tag{3.14}$$

We have to estimate the L^∞ norms on the right-hand side of (3.14). Using Hölder's inequality and (3.3), we obtain

$$\begin{aligned}
|\tilde{p}^3(x,t)| = \left|3\int_0^x\tilde{p}^2\tilde{p}_xdx\right| &\le 3\int_I|\tilde{p}|^2|\tilde{p}_x|dx \\
&\le 3\left(\int_I|\tilde{p}|^\gamma dx\right)^{\frac12}\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac12} \\
&\le C\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac12},
\end{aligned}$$

which implies

$$\|\tilde{p}^3\|_{L^\infty}^{\frac{4-\gamma}{2}} \le C\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac{4-\gamma}{4}}. \tag{3.15}$$

Substituting (3.15) into (3.14), and by virtue of Sobolev's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{p}\|_{L^4}^4+12\|\tilde{p}\tilde{p}_x\|_{L^2}^2 \\
&\le C\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\left[\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac{4-\gamma}{4}}+\|\tilde{p}_x\|_{L^2}^{\frac{4-\gamma}{2}}\right]\|\tilde{p}_x\|_{L^2}^{\frac{\gamma-2}{2}} \\
&\le C(\delta)\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx\right)+\delta\left[\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac{4-\gamma}{2}}\|\tilde{p}_x\|_{L^2}^{\gamma-2}+\|\tilde{p}_x\|_{L^2}^2\right],
\end{aligned} \tag{3.16}$$

where $\delta>0$ is a constant to be determined. Noting that when $2\le\gamma\le3$, by employing Young's inequality, we derive that

$$\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac{4-\gamma}{2}}\|\tilde{p}_x\|_{L^2}^{\gamma-2} \le \int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx+\|\tilde{p}_x\|_{L^2}^2. \tag{3.17}$$

We employ Young's inequality and (3.17) to derive

$$\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac{4-\gamma}{2}}\|\tilde{p}_x\|_{L^2}^{\gamma-2} \le \|\tilde{p}\tilde{p}_x\|_{L^2}^2+2\|\tilde{p}_x\|_{L^2}^2. \tag{3.18}$$

Substituting (3.18) into (3.16), we can infer that

$$\frac{d}{dt}\|\tilde{p}\|_{L^4}^4 + 12\|\tilde{p}\tilde{p}_x\|_{L^2}^2 \le C\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2 dx + \delta(\|\tilde{p}\tilde{p}_x\|^2 + 3\|\tilde{p}_x\|_{L^2}^2). \tag{3.19}$$

Let

$$M_1 = \frac{4+\gamma\bar{p}^{\gamma-4}}{24}.$$

Multiplying (3.19) by M_1, then inserting the result to (3.13), we deduce that

$$\begin{aligned}&\frac{d}{dt}\left(G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2 + M_1\|\tilde{p}\|_{L^4}^4\right) + \frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + 2\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \varepsilon\|q_x\|_{L^2}^2 \\ &\le M_1 C(\delta)\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2 dx + \delta M_1(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + 3\|\tilde{p}_x\|_{L^2}^2).\end{aligned} \tag{3.20}$$

Letting

$$\delta = \frac{1}{M_1}\min\{1, \frac{\gamma}{12}\bar{p}^{\gamma-2}\},$$

then we have

$$\begin{aligned}&\frac{d}{dt}\left(G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2 + M_1\|\tilde{p}\|_{L^4}^4\right) + \frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \varepsilon\|q_x\|_{L^2}^2 \\ &\le C\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2 dx.\end{aligned} \tag{3.21}$$

We integrate (3.21) over $[0,t]$ and employ (3.2) to deduce

$$\begin{aligned}&G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2 + M_1\|\tilde{p}\|_{L^4}^4 + \int_0^t \left(\frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \varepsilon\|q_x\|_{L^2}^2\right) d\tau \\ &\le G(\tilde{p}_0,\bar{p}) + \frac{1}{2}\|q_0\|_{L^2}^2 + M_1\|\tilde{p}_0\|_{L^4}^4 + \frac{C}{\gamma}.\end{aligned} \tag{3.22}$$

□

3.1.2. L^2 *Estimate when* $3 \le \gamma \le 4$.

Proof. Step 1. Since $3 \le \gamma \le 4$, from Lemma 2.3, by choosing $a = \frac{\tilde{p}}{\bar{p}}$, we have

$$(\tilde{p}+\bar{p})^{\gamma-2} \ge \bar{p}^{\gamma-2} + (\gamma-2)\bar{p}^{\gamma-3}\tilde{p}. \tag{3.23}$$

Inserting (3.23) into (3.13), we obtain

$$\frac{d}{dt}\left(G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2\right) + \gamma\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \gamma(\gamma-2)\bar{p}^{\gamma-3}\int_I \tilde{p}(\tilde{p}_x)^2 dx + \varepsilon\|q_x\|_{L^2}^2 \le 0. \tag{3.24}$$

Noting that for any constant $\eta > 0$, it holds that

$$\tilde{p} \ge -|\tilde{p}| \ge -\frac{\tilde{p}^2}{2(\eta)^2} - \frac{(\eta)^2}{2}. \tag{3.25}$$

Plugging (3.25) into (3.24), one has

$$\begin{aligned}&\frac{d}{dt}\left(G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2\right) + \gamma\bar{p}^{\gamma-3}\left(\bar{p} - \frac{(\gamma-2)(\eta)^2}{2}\right)\|\tilde{p}_x\|_{L^2}^2 \\ &- \frac{\gamma(\gamma-2)}{2(\eta)^2}\bar{p}^{\gamma-3}\int_I (\tilde{p}\tilde{p}_x)^2 dx + \varepsilon\|q_x\|_{L^2}^2 \le 0.\end{aligned} \tag{3.26}$$

By choosing

$$\eta = \left(\frac{\bar{p}}{\gamma-2}\right)^{\frac{1}{2}},$$

we have

$$\frac{d}{dt}\left(G(\tilde{p},\bar{p})+\frac{1}{2}\|q\|_{L^2}^2\right)+\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 \\ -\frac{\gamma(\gamma-2)^2}{2}\bar{p}^{\gamma-4}\int_I(\tilde{p}\tilde{p}_x)^2dx+\varepsilon\|q_x\|_{L^2}^2\le 0. \tag{3.27}$$

Step 2. Multiplying the first equation of (3.1) by $4\tilde{p}^3$, integrating by parts, and employing the Hölder's inequality, we deduce that

$$\begin{aligned}\frac{d}{dt}\|\tilde{p}\|_{L^4}^4+12\|\tilde{p}\tilde{p}_x\|_{L^2}^2&=-12\int_I\tilde{p}^2\tilde{p}_x(\tilde{p}+\bar{p})qdx\\&\le 12\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2\right)^{\frac{1}{2}}\left(\int_I(\tilde{p}+\bar{p})^{4-\gamma}\tilde{p}^4q^2dx\right)^{\frac{1}{2}}\\&=12\left((\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2\right)^{\frac{1}{2}}\left(\int_I(\tilde{p}^5+\bar{p}\tilde{p}^4)^{4-\gamma}|\tilde{p}|^{4\gamma-12}q^2dx\right)^{\frac{1}{2}}\\&\le 12\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac{1}{2}}\|\tilde{p}^5+\bar{p}\tilde{p}^4\|_{L^\infty}^{\frac{4-\gamma}{2}}\|\tilde{p}\|_{L^\infty}^{2\gamma-6}\|q\|_{L^2}.\end{aligned} \tag{3.28}$$

It suffices to bound the L^∞ norm of $\tilde{p}^5+\bar{p}\tilde{p}^4$ on the right-hand side of (3.28), for this purpose, we observe that

$$\begin{aligned}\tilde{p}^5(x,t)+\bar{p}\tilde{p}^4(x,t)&=5\int_0^x\tilde{p}^4\tilde{p}_xdx+4\bar{p}\int_0^x\tilde{p}^3\tilde{p}_xdx\\&\le 5\int_0^x|\tilde{p}|^3(\tilde{p}+\bar{p})|\tilde{p}_x|dx+\bar{p}\int_0^x|\tilde{p}|^3|\tilde{p}_x|dx\\&\le 5\left(\int_I\tilde{p}^6(\tilde{p}+\bar{p})dx\right)^{\frac{1}{2}}\left(\int_I(\tilde{p}+\bar{p})(\tilde{p}_x)^2dx\right)^{\frac{1}{2}}+\bar{p}\int_I|\tilde{p}|^3|\tilde{p}_x|dx.\end{aligned} \tag{3.29}$$

Noting that

$$5\left(\int_I\tilde{p}^6(\tilde{p}+\bar{p})dx\right)^{\frac{1}{2}}\le 5\|\tilde{p}^5+\bar{p}\tilde{p}^4\|_{L^\infty}^{\frac{1}{2}}\|\tilde{p}\|_{L^2},$$

we have

$$\|\tilde{p}^5+\bar{p}\tilde{p}^4\|_{L^\infty}\le 2C\int_I(\tilde{p}+\bar{p})(\tilde{p}_x)^2dx+2\bar{p}\int_I|\tilde{p}|^3|\tilde{p}_x|dx. \tag{3.30}$$

Substituting (3.30) into (3.28), we obtain

$$\begin{aligned}&\frac{d}{dt}\|\tilde{p}\|_{L^4}^4+12\|\tilde{p}\tilde{p}_x\|_{L^2}^2\\&\le 12\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac{1}{2}}\left(C\int_I(\tilde{p}+\bar{p})(\tilde{p}_x)^2dx+\|\tilde{p}_x\|_{L^2}^2\right)^{\frac{4-\gamma}{2}}\|\tilde{p}\|_{L^\infty}^{2\gamma-6}\|q\|_{L^2}\\&\le C\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac{1}{2}}\left(\int_I(\tilde{p}+\bar{p})(\tilde{p}_x)^2dx+\|\tilde{p}_x\|_{L^2}^2\right)^{\frac{4-\gamma}{2}}\|\tilde{p}_x\|_{L^2}^{\gamma-3}\\&\le C\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac{1}{2}}\left[\left(\int_I(\tilde{p}+\bar{p})(\tilde{p}_x)^2dx\right)^{\frac{4-\gamma}{2}}\|\tilde{p}_x\|_{L^2}^{\gamma-3}+\|\tilde{p}_x\|_{L^2}\right].\end{aligned} \tag{3.31}$$

We employ the Young's inequality to the right-hand side of (3.31) to obtain

$$\frac{d}{dt}\|\tilde{p}\|_{L^4}^4 + 12\|\tilde{p}\tilde{p}_x\|_{L^2}^2 \le C\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx + \delta\left[\left(\int_I (\tilde{p}+\bar{p})(\tilde{p}_x)^2\right)^{4-\gamma}\|\tilde{p}_x\|_{L^2}^{2(\gamma-3)} + \|\tilde{p}_x\|_{L^2}^2\right]. \tag{3.32}$$

When $3 \le \gamma \le 4$, by virtue of Young's inequality, we can get

$$\begin{aligned}\left(\int_I (\tilde{p}+\bar{p})(\tilde{p}_x)^2dx\right)^{4-\gamma}\|\tilde{p}_x\|_{L^2}^{2(\gamma-3)} &\le (4-\gamma)\int_I (\tilde{p}+\bar{p})(\tilde{p}_x)^2dx + (\gamma-3)\|\tilde{p}_x\|_{L^2}^2\\ &\le \int_I (\tilde{p}+\bar{p})(\tilde{p}_x)^2dx + \|\tilde{p}_x\|_{L^2}^2.\end{aligned} \tag{3.33}$$

We substitute (3.33) into (3.32), and use the elementary inequality $2|\tilde{p}| \le \tilde{p}^2 + 1$ to derive

$$\begin{aligned}&\frac{d}{dt}\|\tilde{p}\|_{L^4}^4 + 12\|\tilde{p}\tilde{p}_x\|_{L^2}^2\\ &\le C\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx + \delta\left(\int_I (\tilde{p}+\bar{p})(\tilde{p}_x)^2dx + 2\|\tilde{p}_x\|_{L^2}^2\right)\\ &\le C\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx + \delta(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + (\bar{p}+3)\|\tilde{p}_x\|_{L^2}^2).\end{aligned} \tag{3.34}$$

Step 3. Let

$$M_2 = \frac{4+\gamma(\gamma-2)^2\bar{p}^{\gamma-4}}{24}.$$

Multiplying (3.34) by M_2, then adding the result to (3.27), we have

$$\begin{aligned}&\frac{d}{dt}\left(G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2 + M_2\|\tilde{p}\|_{L^4}^4\right) + \frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + 2\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \varepsilon\|q_x\|_{L^2}^2\\ &\le M_2C(\delta)\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx + \delta M_2(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + (\bar{p}+3)\|\tilde{p}_x\|_{L^2}^2).\end{aligned} \tag{3.35}$$

Choosing

$$\delta = \frac{1}{M_2}\min\{1, \frac{\gamma\bar{p}^{\gamma-2}}{4(\bar{p}+3)}\},$$

we obtain from (3.35)

$$\begin{aligned}&\frac{d}{dt}\left(G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2 + M_2\|\tilde{p}\|_{L^4}^4\right) + \frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + 2\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \varepsilon\|q_x\|_{L^2}^2\\ &\le C\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx.\end{aligned} \tag{3.36}$$

Integrating (3.36) over $[0,t]$ and using (3.2), we have

$$\begin{aligned}&G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2 + M_2\|\tilde{p}\|_{L^4}^4 + \int_0^t (\frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \varepsilon\|q_x\|_{L^2}^2)d\tau\\ &\le G(\tilde{p}_0,\bar{p}) + \frac{1}{2}\|q_0\|_{L^2}^2 + M_2\|\tilde{p}_0\|_{L^4}^4 + \frac{C}{\gamma}.\end{aligned} \tag{3.37}$$

□

3.1.3. L^2 *estimate when* $\gamma > 4$.

Proof. Step 1. Since $\gamma > 4$, as an application of Lemma 2.3, by letting $a = \frac{\tilde{p}}{\bar{p}}$, we have

$$(\tilde{p}+\bar{p})^{\gamma-2} \geq \bar{p}^{\gamma-2} + (\gamma-2)\bar{p}^{\gamma-3}\tilde{p}. \tag{3.38}$$

Substituting (3.38) into (3.7), we have

$$\frac{d}{dt}\left(G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2\right) + \gamma\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \gamma(\gamma-2)\bar{p}^{\gamma-3}\int_I \tilde{p}(\tilde{p}_x)^2dx + \varepsilon\|q_x\|_{L^2}^2 \leq 0. \tag{3.39}$$

Since $\gamma > 4$, by applying the Young's inequality, we can show that

$$|\tilde{p}| = \eta_1 \cdot \frac{|\tilde{p}|}{\eta_1} \leq \left(\frac{\gamma-2}{\gamma-1}\right)\cdot \eta_1^{\frac{\gamma-1}{\gamma-2}} + \left(\frac{1}{\gamma-1}\right)\cdot\frac{|\tilde{p}|^{\gamma-1}}{\eta_1^{\gamma-1}} \leq \eta_1^{\frac{\gamma-1}{\gamma-2}} + \frac{|\tilde{p}|^{\gamma-1}}{\eta_1^{\gamma-1}},$$

which implies

$$\tilde{p} \geq -|\tilde{p}| \geq -\eta_1^{\frac{\gamma-1}{\gamma-2}} - \frac{|\tilde{p}|^{\gamma-1}}{\eta_1^{\gamma-1}}. \tag{3.40}$$

Substituting (3.40)into (3.39), we have

$$\begin{aligned}&\frac{d}{dt}\left(G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2\right) + \gamma\bar{p}^{\gamma-3}[\bar{p} - (\gamma-2)\eta_1^{\frac{\gamma-1}{\gamma-2}}]\|\tilde{p}_x\|_{L^2}^2\\ &- \frac{\gamma(\gamma-2)}{\eta_1^{\gamma-1}}\bar{p}^{\gamma-3}\int_I |\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx + \varepsilon\|q_x\|_{L^2}^2 \leq 0.\end{aligned} \tag{3.41}$$

Next, by choosing

$$\eta_1 = \left(\frac{\bar{p}}{2(\gamma-2)}\right)^{\frac{\gamma-2}{\gamma-1}},$$

we have

$$\begin{aligned}&\frac{d}{dt}\left(G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2\right) + \frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2\\ &- \frac{2^{\gamma-2}\gamma(\gamma-2)^{\gamma-1}}{\bar{p}}\int_I |\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx + \varepsilon\|q_x\|_{L^2}^2 \leq 0.\end{aligned} \tag{3.42}$$

Step 2. By multiplying the first equation of (3.1) by $(\gamma+1)|\tilde{p}|^{\gamma-1}\tilde{p}$ and integrating by parts with respect to x over I, we obtain

$$\frac{d}{dt}\left(\int_I |\tilde{p}|^{\gamma+1}dx\right) + \gamma(\gamma+1)\int_I |\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx = -\gamma(\gamma+1)\int_I (\tilde{p}+\bar{p})q|\tilde{p}|^{\gamma-1}\tilde{p}_xdx. \tag{3.43}$$

Since $\gamma > 4$ and $\tilde{p}+\bar{p} > 0$, by using the Holder's inequality, we estimate the right-hand side of (3.43) as

$$\begin{aligned}\left|\int_I (\tilde{p}+\bar{p})q|\tilde{p}|^{\gamma-1}\tilde{p}_xdx\right| &\leq \left(\int_I (\tilde{p}+\bar{p})^2|\tilde{p}_x|^{\frac{4}{\gamma-2}}|\tilde{p}_x|^{2\frac{\gamma-4}{\gamma-2}}dx\right)^{\frac{1}{2}}\|q\|_{L^2}\|\tilde{p}^{\gamma-1}\|_{L^\infty}\\ &\leq C\left(\int_I (\tilde{p}+\bar{p})^{\gamma-2}|\tilde{p}_x|^2dx\right)^{\frac{1}{\gamma-2}}\|\tilde{p}\|_{L^\infty}^{\gamma-1}\|\tilde{p}_x\|_{L^2}^{\frac{\gamma-4}{\gamma-2}}.\end{aligned} \tag{3.44}$$

To control the L^∞ norm of $\tilde{p}$ on the right-hand side of (3.44), we note that

$$\begin{aligned}
\tilde{p}(x,t)^{\gamma-1} &= \int_0^x \left((\tilde{p}^2)^{\frac{\gamma-1}{2}} \right)_x dx \\
&= (\gamma-1) \int_0^x |\tilde{p}|^{\gamma-3} \tilde{p} \tilde{p}_x dx \\
&\le (\gamma-1) \int_I |\tilde{p}|^{\gamma-2} |\tilde{p}_x| dx \\
&= (\gamma-1) \int_I |\tilde{p}|^{\frac{\gamma}{2}} |\tilde{p}|^{\frac{\gamma}{2}-2} |\tilde{p}_x| dx \\
&\le (\gamma-1) \left(\int_I |\tilde{p}|^\gamma dx \right)^{\frac{1}{2}} \left(\int_I |\tilde{p}|^{\gamma-4} |\tilde{p}_x|^2 dx \right)^{\frac{1}{2}},
\end{aligned}$$

which implies

$$\begin{aligned}
\|\tilde{p}^{\gamma-1}\|_{L^\infty} &\le (\gamma-1) \left(\int_I |\tilde{p}|^\gamma dx \right)^{\frac{1}{2}} \left(\int_I |\tilde{p}|^{\gamma-4} |\tilde{p}_x|^2 dx \right)^{\frac{1}{2}} \\
&\le C \left(\int_I |\tilde{p}|^{\gamma-4} |\tilde{p}_x|^2 dx \right)^{\frac{1}{2}} \\
&\le C \left(\int_I |\tilde{p}|^{\gamma-1} (\tilde{p}_x)^2 dx + \|\tilde{p}_x\|_{L^2}^2 \right)^{\frac{1}{2}}. \qquad (3.45)
\end{aligned}$$

Substituting (3.45) into (3.44), we have

$$\begin{aligned}
\left| \int_I (\tilde{p}+\bar{p}) q |\tilde{p}|^{\gamma-1} \tilde{p}_x dx \right| &\le C \left(\int_I (\tilde{p}+\bar{p})^{\gamma-2} |\tilde{p}_x|^2 dx \right)^{\frac{1}{\gamma-2}} \left(\int_I |\tilde{p}|^{\gamma-4} |\tilde{p}_x|^2 dx \right)^{\frac{1}{2}} \|\tilde{p}_x\|_{L^2}^{\frac{\gamma-4}{\gamma-2}} \\
&\le C(\delta) \left(\int_I (\tilde{p}+\bar{p})^{\gamma-2} |\tilde{p}_x|^2 dx \right) + \delta \left(\int_I |\tilde{p}|^{\gamma-4} |\tilde{p}_x|^2 dx \right)^{\frac{\gamma-2}{2(\gamma-3)}} \|\tilde{p}_x\|_{L^2}^{\frac{\gamma-4}{\gamma-3}} \\
&\le C(\delta) \int_I (\tilde{p}+\bar{p})^{\gamma-2} |\tilde{p}_x|^2 dx + \delta \left(\int_I |\tilde{p}|^{\gamma-4} |\tilde{p}_x|^2 dx + \|\tilde{p}_x\|_{L^2}^2 \right) \\
&\le C(\delta) \int_I (\tilde{p}+\bar{p})^{\gamma-2} |\tilde{p}_x|^2 dx + \delta \left(\int_I |\tilde{p}|^{\gamma-1} |\tilde{p}_x|^2 dx + 2\|\tilde{p}_x\|_{L^2}^2 \right), \quad (3.46)
\end{aligned}$$

where we have used the Young's inequality $|\tilde{p}|^{\gamma-4} \le |\tilde{p}|^{\gamma-1} + 1$ due to $\gamma > 4$. Substituting (3.46) into (3.43), we obtain

$$\begin{aligned}
&\frac{d}{dt} \left(\int_I |\tilde{p}|^{\gamma+1} dx \right) + \gamma(\gamma+1) \int_I |\tilde{p}|^{\gamma-1} (\tilde{p}_x)^2 dx \\
&\le C(\delta) \int_I (\tilde{p}+\bar{p})^{\gamma-2} |\tilde{p}_x|^2 dx + \delta \left(|\tilde{p}|^{\gamma-1} |\tilde{p}_x|^2 dx + 2\|\tilde{p}_x\|_{L^2}^2 \right). \qquad (3.47)
\end{aligned}$$

Step 3. It suffices to bound the last term on the right-hand side. To this end, let

$$M_3 = \frac{2\bar{p} + 2^{\gamma-2}\gamma(\gamma-2)^{\gamma-1}}{\gamma(\gamma+1)\bar{p}}.$$

Multiplying (3.47) by M_3, then adding the result to (3.42), we get

$$\begin{aligned}&\frac{d}{dt}\left(G(\tilde{p},\bar{p})+\frac{1}{2}\|q\|_{L^2}^2+M_3\int_I|\tilde{p}|^{\gamma+1}dx\right)+\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2\\&\quad+2\int_I|\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx+\varepsilon\|q_x\|_{L^2}^2\\&\le C(\delta)\gamma(\gamma+1)M_3\int_I(\tilde{p}+\bar{p})^{\gamma-2}|\tilde{p}_x|^2dx+\delta\gamma(\gamma+1)M_3\left(\int_I|\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx+2\|\tilde{p}_x\|_{L^2}^2\right).\end{aligned}\tag{3.48}$$

Choosing

$$\delta=\frac{1}{\gamma(\gamma+1)M_3}\min\{\frac{\gamma}{8}\bar{p}^{\gamma-2},1\},$$

We obtain from (3.48)

$$\begin{aligned}&\frac{d}{dt}\left(G(\tilde{p},\bar{p})+\frac{1}{2}\|q\|_{L^2}^2+M_3\int_I|\bar{p}|^{\gamma+1}dx\right)+\frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2\\&\quad+\int_I|\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx+\varepsilon\|q_x\|_{L^2}^2\\&\le C\int_I(\tilde{p}+\bar{p})^{\gamma-2}|\tilde{p}_x|^2dx.\end{aligned}\tag{3.49}$$

Integrating (3.49) over $[0,t]$ and using (3.2), we end up with

$$\begin{aligned}&G(\tilde{p},\bar{p})+\frac{1}{2}\|q\|_{L^2}^2+M_3\int_I|\tilde{p}|^{\gamma+1}dx\\&\quad+\int_0^t\left(\frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2+\int_I|\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx+\varepsilon\|q_x\|_{L^2}^2\right)d\tau\\&\le\left(G(\tilde{p}_0,\bar{p})+\frac{1}{2}\|q_0\|_{L^2}^2+M_3\int_I|\tilde{p}_0|^{\gamma+1}dx\right)+\frac{C}{\gamma}.\end{aligned}\tag{3.50}$$

This completes the proof of Lemma 3.2. The following lemma is concerning on the estimate of the first order spatial derivatives of the solution. □

Lemma 3.3. *(H^1-estimate). Under the conditions of Theorem 1.1, for any $\gamma\ge 2$, $\varepsilon>0$, and $t>0$, it holds that*

$$\|\tilde{p}_x(t)\|_{L^2}^2+\|q_x(t)\|_{L^2}^2+\int_0^t(\|\tilde{p}_\tau(\tau)\|_{L^2}^2+\|q_\tau(\tau)\|_{L^2}^2)d\tau\le C,$$

where C is a positive constant which is independent of t.

Proof. Taking the L^2 inner product of the first equation of (3.1) with $\gamma\bar{p}^{\gamma-2}\tilde{p}_t$ and the second one with q_t, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2+\frac{1}{2}\varepsilon\|q_x\|_{L^2}^2\right)+\gamma\bar{p}^{\gamma-2}\|\tilde{p}_t\|_{L^2}^2+\|q_t\|_{L^2}^2\\
&=\gamma\bar{p}^{\gamma-2}\int_I(\tilde{p}q)_x\tilde{p}_tdx+\gamma\bar{p}^{\gamma-1}\int_I q_x\tilde{p}_tdx\\
&\quad+\gamma\int_I[(\tilde{p}+\bar{p})^{\gamma-1}-\bar{p}^{\gamma-1}]\tilde{p}_xq_tdx+\gamma\bar{p}^{\gamma-1}\int_I\tilde{p}_xq_tdx+2\varepsilon\int_I qq_xq_tdx\\
&\le\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_t\|_{L^2}^2+\frac{1}{2}\|q_t\|_{L^2}^2+4\gamma\bar{p}^{\gamma-2}\int_I|(\tilde{p}q)_x|^2dx+4\gamma\bar{p}^{\gamma}\|q_x\|_{L^2}^2\\
&\quad+\gamma\int_I([(\tilde{p}+\bar{p})^{\gamma-1}-\bar{p}^{\gamma-1}]\tilde{p}_x)^2dx+\gamma\bar{p}^{\gamma-1}\|\tilde{p}_x\|^2+\varepsilon\|(qq_x)\|_{L^2}^2\\
&=\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_t\|_{L^2}^2+\frac{1}{2}\|q_t\|_{L^2}^2+\sum_{i=1}^{5}J_i.
\end{aligned}\tag{3.51}
$$

Now we turn to estimate J_1, J_3 and J_5 term by term. First, we have from Poincaré inequality that

$$
\begin{aligned}
|J_1|&=\left|4\gamma\bar{p}^{\gamma-2}\int_I|(\tilde{p}q)_x|^2dx\right|\\
&\le 4\gamma\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2\|q_x\|_{L^2}^2.
\end{aligned}\tag{3.52}
$$

For J_3, by the mean value theorem, we have

$$
(\tilde{p}+\bar{p})^{\gamma-1}-\bar{p}^{\gamma-1}=(\gamma-1)(p^*)^{\gamma-2}\tilde{p},
$$

where p^* is between $\tilde{p}+\bar{p}$ and $\bar{p}$, satisfying $|p^*|\le|\tilde{p}|+|\bar{p}|$. Then we have

$$
\begin{aligned}
|J_3|&=\gamma\int_I([(\tilde{p}+\bar{p})^{\gamma-1}-\bar{p}^{\gamma-1}]\tilde{p}_x)^2dx\\
&\le 2\gamma(\gamma-1)\int_I(((|\tilde{p}|+\bar{p})^{\gamma-2}\tilde{p}\tilde{p}_x)^2dx\\
&\le 2\gamma(\gamma-1)\|\tilde{p}\|_{L^\infty}^{2\gamma-2}\|\tilde{p}_x\|_{L^2}^2+2\gamma(\gamma-1)\bar{p}^{2(\gamma-2)}\|\tilde{p}\|_{L^\infty}^2\|\tilde{p}_x\|_{L^2}^2\\
&\le 2\gamma(\gamma-1)\|\tilde{p}\|_{L^2}^{\gamma-1}\|\tilde{p}_x\|_{L^2}^{\gamma-1}\|\tilde{p}_x\|_{L^2}^2+2\gamma(\gamma-1)\bar{p}^{2(\gamma-2)}\|\tilde{p}\|_{L^2}\|\tilde{p}_x\|_{L^2}\|\tilde{p}_x\|_{L^2}^2\\
&\le C\|\tilde{p}_x\|_{L^2}^{\gamma-1}\|\tilde{p}_x\|_{L^2}^2+C\|\tilde{p}_x\|_{L^2}^2\|\tilde{p}_x\|_{L^2}^2.
\end{aligned}\tag{3.53}
$$

For the case $2\le\gamma<3$, by Young's inequality, we have

$$
\|\tilde{p}_x\|_{L^2}^{(\gamma-1)}\le\frac{\xi_1}{2}+C(\xi_1)\|\tilde{p}_x\|_{L^2}^2.\tag{3.54}
$$

Substituting (3.54) into (3.53), we obtain

$$
|J_3|\le\xi_1\|\tilde{p}_x\|_{L^2}^2+C(\xi_1)\|\tilde{p}_x\|_{L^2}^2\int_I|\tilde{p}_x|^2dx.\tag{3.55}
$$

For the case $3\le\gamma\le4$, by using the fact $\|\tilde{p}\|_{L^\infty}^{2\gamma-2}\le C\int_I|\tilde{p}|^{2(\gamma-3)}|\tilde{p}_x|^2dx$ and $|\tilde{p}|^{2(\gamma-3)}\le|\tilde{p}|^2+1$, we have

$$
\|\tilde{p}\|_{L^\infty}^{2\gamma-2}\le C(\|\tilde{p}\tilde{p}_x\|_{L^2}^2+\|\tilde{p}_x\|_{L^2}^2).\tag{3.56}
$$

Substituting (3.56) into (3.53), we obtain

$$
|J_3|\le C\|\tilde{p}\tilde{p}_x\|_{L^2}^2\int_I\tilde{p}_x^2dx+C\|\tilde{p}_x\|_{L^2}^2\int_I\tilde{p}_x^2dx.\tag{3.57}
$$

For the case $\gamma > 4$, Noting that $|\tilde{p}|^{\gamma-4} \leq |\tilde{p}|^{\gamma-1} + 1$, combing (3.45), we have

$$\begin{aligned}|J_3| &\leq \|\tilde{p}\|_{L^\infty}^{2\gamma-2} \int_I \tilde{p}_x^2 dx + C\|\tilde{p}_x\|_{L^2}^2 \int_I \tilde{p}_x^2 dx \\ &\leq (\gamma-1)^2 \int_I |\tilde{p}|^\gamma dx \int_I |\tilde{p}|^{\gamma-4}|\tilde{p}_x|^2 dx \|\tilde{p}_x\|_{L^2}^2 + C\int_I |\tilde{p}_x|^2 dx \|\tilde{p}_x\|_{L^2}^2 \\ &\leq C\left(\int_I |\tilde{p}|^{\gamma-1}|\tilde{p}_x^2| dx + \int_I |\tilde{p}_x|^2 dx\right)\|\tilde{p}_x\|_{L^2}^2. \end{aligned} \tag{3.58}$$

For J_5, using Sobolev's embedding inequality, we have

$$\begin{aligned}|J_5| &= \varepsilon\|(qq_x)\|_{L^2}^2 \\ &\leq \varepsilon\|q_x\|_{L^2}^2\|q_x\|_{L^2}^2. \end{aligned} \tag{3.59}$$

From (3.51)-(3.59), we have the following inequalities. For $2 \leq \gamma \leq 3$, we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \frac{\varepsilon}{2}\|q_x\|_{L^2}^2\right) + \frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_t\|_{L^2}^2 + \frac{1}{2}\|q_t\|_{L^2}^2 \\ &\leq C(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2)(\|\tilde{p}_x\|_{L^2}^2 + \varepsilon\|q_x\|_{L^2}^2) + (\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2). \end{aligned} \tag{3.60}$$

For the case $3 \leq \gamma \leq 4$, we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \frac{\varepsilon}{2}\|q_x\|_{L^2}^2\right) + \frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_t\|_{L^2}^2 + \frac{1}{2}\|q_t\|_{L^2}^2 \\ &\leq C(\|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2)(\|\tilde{p}_x\|_{L^2}^2 + \varepsilon\|q_x\|_{L^2}^2) + C(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2). \end{aligned} \tag{3.61}$$

For the case $\gamma \geq 4$, we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \frac{\varepsilon}{2}\|q_x\|_{L^2}^2\right) + \frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_t\|_{L^2}^2 + \frac{1}{2}\|q_t\|_{L^2}^2 \\ &\leq C(\|\tilde{p}_x\|_{L^2}^2 + \int_I \tilde{p}^{\gamma-1}(\tilde{p}_x)^2 dx + \|q_x\|_{L^2}^2)(\|\tilde{p}_x\|_{L^2}^2 + \varepsilon\|q_x\|_{L^2}^2) + C(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2). \end{aligned} \tag{3.62}$$

We have from (3.60), (3.61) and (3.62) together with Gronwall's inequality and (3.22), (3.37) and (3.50) that for all $\gamma \geq 2$ it holds that

$$\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \int_0^t (\|\tilde{p}_t\|_{L^2}^2 + \|q_t\|_{L^2}^2) d\tau \leq C. \tag{3.63}$$

Combing (3.63) with the equations (1.1), we obtain

$$\int_0^t \|\tilde{p}_{xx}\|_{L^2}^2 d\tau + \int_0^t \|q_{xx}\|_{L^2}^2 d\tau \leq C. \tag{3.64}$$

This completes the proof of Lemma 3.3. □

3.1.4. *Global dynamical when* $1 < \gamma < 2$. We first derive uniform energy estimates for the zeroth and first order spatial derivative of the perturbation equation (3.1) by using the same method as in [37, Lemmas 5.1, 5.2]. For convenience and simplicity, we only state the results here but omit the specific proofs.

Lemma 3.4. *Under the conditions of Theorem 1.3, for any* $\varepsilon > 0$ *and* $t > 0$*, it holds that*

$$\|\tilde{p}\|_{L^2}^2 + \|\tilde{p}\|_{L^{\gamma+1}}^{\gamma+1} + \|\tilde{p}\|_{L^4}^4 + \|q\|_{L^2}^2 + \int_0^t (\|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \varepsilon\|q_x\|_{L^2}^2) d\tau \leq C. \tag{3.65}$$

Lemma 3.5. *Under the conditions of Theorem 1.1, for any* $1 < \gamma < 2$, $\varepsilon > 0$ *and* $t > 0$*, it holds that*

$$\|\tilde{p}_x(t)\|_{L^2}^2 + \|q_x(t)\|_{L^2}^2 + \int_0^t (\|\tilde{p}_{xx}(\tau)\|_{L^2}^2 + \varepsilon\|q_{xx}(\tau)\|_{L^2}^2) d\tau \leq C, \tag{3.66}$$

where C *is a positive constant which is independent of* t*.*

Based on the Lemmas 3.2, 3.3, 3.4 and 3.5, we can obtain the second-order spatial derivative of $(\tilde{p}, q)$.

Lemma 3.6. *(H^2 estimate) Let (p, q) be a solution to (1.1). Then for any $\varepsilon > 0$ and $\gamma > 1$, it holds that*

$$\|\tilde{p}_{xx}(t)\|_{L^2}^2 + \|q_{xx}(t)\|_{L^2}^2 + \int_0^t (\|\tilde{p}_{xt}\|_{L^2}^2 + \|q_{xt}\|_{L^2}^2) d\tau \le C. \tag{3.67}$$

Proof. Taking ∂_t to the two equations in (3.1), then multiplying the resulting equations with $\gamma \bar{p}^{\gamma-2}\partial_t \tilde{p}$ and $\partial_t q$, respectively and integrating by parts over I, we deduce

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\gamma}{2}\bar{p}^{\gamma-2}\int_I |\tilde{p}_t|^2 + \frac{1}{2}|q_t|^2 dx\right) + \gamma\bar{p}^{\gamma-2}\int_I |\tilde{p}_{xt}|^2 + \varepsilon\int_I |q_{xt}|^2 \\
&= -\gamma\bar{p}^{\gamma-2}\int_I (\tilde{p}q)_t \tilde{p}_{xt} - \varepsilon\int_I (q^2)_t q_{xt} dx + \gamma\int_I ([(\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}]\tilde{p}_x)_t q_t dx \\
&= J_6 + J_7 + J_8.
\end{aligned} \tag{3.68}$$

By using Cauchy-Schwarz inequality and Sobolev's inequality, we have

$$\begin{aligned}
|J_6| &\le \frac{1}{4}\|\tilde{p}_{xt}\|^2 + C(\|\tilde{p}\|_{L^\infty}^2\|q_t\|_{L^2}^2 + \|q\|_{L^\infty}^2\|\tilde{p}_t\|_{L^2}^2) \\
&\le \frac{1}{4}\|\tilde{p}_{xt}\|^2 + C(\|\tilde{p}_x\|^2\|q_t\|^2 + \|q_x\|^2\|\tilde{p}_t\|^2).
\end{aligned} \tag{3.69}$$

$$\begin{aligned}
|J_7| &\le \frac{\varepsilon}{2}\|q_{xt}\|^2 + C\|q\|_{L^\infty}^2\|q_t\|^2 \\
&\le \frac{\varepsilon}{2}\|q_{xt}\|^2 + C\|q_x\|^2\|q_t\|^2.
\end{aligned} \tag{3.70}$$

$$\begin{aligned}
|J_8| &\le \gamma\int_I (\tilde{p}+\bar{p})^{\gamma-2}\tilde{p}_t\tilde{p}_x q_t + \gamma\int_I [(\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}]\tilde{p}_{xt} q_t dx \\
&\le C\|p_t\|_{L^\infty}\|\tilde{p}_x\|_{L^2}\|q_t\|_{L^2} + \int_I (|\tilde{p}| + \bar{p})^{\gamma-2}|\tilde{p}||\tilde{p}_{xt}||q_t| dx \\
&\le \delta\|\tilde{p}_{xt}\|_{L^2}^2 + C\|\tilde{p}_x\|_{L^2}^2\|q_t\|_{L^2}^2 + C\|\tilde{p}_x\|_{L^2}^2\|q_t\|_{L^2}^2.
\end{aligned} \tag{3.71}$$

From (3.68)-(3.71), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\gamma}{2}\bar{p}^{\gamma-2}\int_I |\tilde{p}_t|^2 + \frac{1}{2}|q_t|^2 dx\right) + \gamma\bar{p}^{\gamma-2}\int_I |\tilde{p}_{xt}|^2 + \varepsilon\int_I |q_{xt}|^2 \\
&\le C(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2)(\|\tilde{p}_t\|_{L^2}^2 + \|q_t\|_{L^2}^2),
\end{aligned} \tag{3.72}$$

which together with Gronwall's inequality and (3.22), implies

$$\|\tilde{p}_t\|_{L^2}^2 + \|q_t\|_{L^2}^2 + \int_0^t (\|\tilde{p}_{xt}\|_{L^2}^2 + \|q_{xt}\|_{L^2}^2) d\tau \le C. \tag{3.73}$$

which together with the equations (3.1), we have

$$\|\tilde{p}_{xx}\|_{L^2}^2 + \|q_{xx}\|_{L^2}^2 + \int_0^t (\|\tilde{p}_{xxx}(\tau)\|^2 + \|q_{xxx}(\tau)\|^2) d\tau \le C. \tag{3.74}$$

Next, we prove the decay property recorded in Theorem1.1.

Lemma 3.7. *(Exponential decay). Let (p, q) be a solution to (3.2). Then for any $\varepsilon > 0$,*

$$\|(\tilde{p}, q)\|_{H^2}^2 \le \alpha e^{-\beta t},$$

where the constants α, β are independent of t.

Proof. First, for the case $2 \leq \gamma \leq 3$, from (3.60), (3.63) and (3.72)-(3.22), we deduce that

$$\frac{1}{2}\frac{d}{dt}\left(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2\right) + \|\tilde{p}_t\|_{L^2}^2 + \|q_t\|_{L^2}^2 \leq C_1(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2) \tag{3.75}$$

and

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\gamma}{2}\bar{p}^{\gamma-2}\int_I |\tilde{p}_t|^2 + \frac{1}{2}|q_t|^2 dx\right) + \frac{\gamma}{2}\bar{p}^{\gamma-2}\int_I |\tilde{p}_{xt}|^2 dx + \frac{\varepsilon}{2}\int_I |q_{xt}|^2 dx \\ &\leq C_2(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2) + \xi\|q_t\|_{L^2}^2.\end{aligned} \tag{3.76}$$

Next, we fix a constant $K_1 > 0$ such that

$$K_1 \times \min\{\frac{\gamma}{4}\bar{p}^{\gamma-2}, \varepsilon\} \geq C_1 + C_2 + 1.$$

Multiplying (3.21) by K_1, then adding the resulting estimate to (3.75) + (3.76), we have

$$\frac{d}{dt}M(t) + N(t) \leq K_1 C_3 \int_I (\tilde{p} + \bar{p})^{\gamma-2}(\tilde{p}_x)^2 dx \tag{3.77}$$

where

$$M(t) = K_1 G(t) + \frac{1}{2}(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \gamma\bar{p}^{\gamma-2}\|\tilde{p}_t\|_{L^2}^2 + \|q_t\|_{L^2}^2),$$

$$\begin{aligned}N(t) = {}& K_1(\frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \varepsilon\|q_x\|_{L^2}^2) + \frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_t\|_{L^2}^2 + \frac{1}{2}\|q_t\|_{L^2}^2 \\ &+ \frac{\gamma}{2}\bar{p}^{\gamma-2}\int_I |\tilde{p}_{xt}|^2 dx + \frac{\varepsilon}{2}\int_I |q_{xt}|^2 dx - (C_1 + C_2)(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2) \\ &- \xi\|q_t\|_{L^2}^2.\end{aligned} \tag{3.78}$$

Now, we fix another constant $K_2 > 0$ such that

$$K_2 \geq K_1 C_3 + 1.$$

Multiply (3.7) by K_2 and adding the resulting estimate to (3.77), we get

$$\begin{aligned}&\frac{d}{dt}\underbrace{\left(K_2 G(\tilde{p}, \bar{p}) + \frac{K_2}{2}\|q\|_{L^2}^2 + K_1 M(t)\right)}_{X(t)} \\ &\quad + \underbrace{(K_2 - K_1 C_{15})\int_I (\tilde{p} + \bar{p})^{\gamma-2}(\tilde{p}_x)^2 dx + K_2\varepsilon\|q_x\|_{L^2}^2 + N(t)}_{Y(t)} \leq 0.\end{aligned} \tag{3.79}$$

By employing Poincar$\acute{e}$s inequality, we can easily deduce that there exists a t-independent constant C such that

$$X(t) \leq CY(t),$$

which implies the exponential decaying of $X(t)$. Finally, the exponential decaying of $\|(\tilde{p}, q)(t)\|_{H^2}^2$ follows from the fact that $\|(\tilde{p}, q)(t)\|_{H^2}^2 \leq CX(t)$. For the cases $1 < \gamma < 2$, $3 \leq \gamma \leq 4$ and $\gamma > 4$, the exponential decay rate of the perturbations can be proved by using exactly the same idea as in the proof for the case $2 \leq \gamma \leq 3$, we omit further details here for brevity. This completes the proof. $\square$

3.2. **Dynamics of transformed system with $\varepsilon = 0$ and $\bar{p} > 0$.** In this section, we prove the global dynamics of large-amplitude solutions to the non-diffusive problem. We shall see from below that the proof of the non-diffusive problem relies on a non-homogeneous damping equation equation of q_x. Let us consider the following initial-boundary value problem

$$\begin{cases} \tilde{p}_t - [(\tilde{p}+\bar{p})q]_x = \tilde{p}_{xx}, \\ q_t - \gamma[(\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}]\tilde{p}_x - \gamma\bar{p}^{\gamma-1}\tilde{p}_x = 0, \\ (\tilde{p}, q)(x,0) = (p_0 - \bar{p}, q_0)(x); \\ \tilde{p}|_{\{x=0,x=1\}} = 0. \end{cases} \tag{3.80}$$

First of all, let's recall that the bound about L^2-estimate is independent on ε in the subsection 3.1.1, so let $\varepsilon = 0$, we have the following estimates.

$$\begin{aligned} &\frac{1}{\gamma-1}\int_I [(\tilde{p}+\bar{p})^\gamma - \bar{p}^\gamma - \gamma\bar{p}^{\gamma-1}\tilde{p}](x,t)dx + \frac{1}{2}\|q(t)\|_{L^2}^2 \\ &\quad + \int_0^t \left(\gamma\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2 dx\right)(\tau)d\tau \le C, \end{aligned} \tag{3.81}$$

$$\begin{aligned} &\frac{d}{dt}\left(G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2 + M_1\|\tilde{p}\|_{L^4}^4\right) + \frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2 \\ &\le C\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2 dx. \end{aligned} \tag{3.82}$$

If $2 \le \gamma \le 3$, we have

$$\begin{aligned} &G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2 + M_1\|\tilde{p}\|_{L^4}^4 + \int_0^t (\frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2)d\tau \\ &\le G(\tilde{p}_0,\bar{p}) + \frac{1}{2}\|q_0\|_{L^2}^2 + M_1\|\tilde{p}_0\|_{L^4}^4 + \frac{C}{\gamma}. \end{aligned} \tag{3.83}$$

If $3 \le \gamma \le 4$, we have

$$\begin{aligned} &G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2 + M_2\|\tilde{p}\|_{L^4}^4 + \int_0^t (\frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2)d\tau \\ &\le G(\tilde{p}_0,\bar{p}) + \frac{1}{2}\|q_0\|_{L^2}^2 + M_2\|\tilde{p}_0\|_{L^4}^4 + \frac{C}{\gamma}. \end{aligned} \tag{3.84}$$

If $\gamma > 4$, we have

$$\begin{aligned} &G(\tilde{p},\bar{p}) + \frac{1}{2}\|q\|_{L^2}^2 + M_3\|\tilde{p}\|_{L^{\gamma+1}}^{\gamma+1} + \int_0^t (\frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \int_I |\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2 dx)d\tau \\ &\le G(\tilde{p}_0,\bar{p}) + \frac{1}{2}\|q_0\|_{L^2}^2 + M_3\|\tilde{p}_0\|_{L^{\gamma+1}}^{\gamma+1} + \frac{C}{\gamma}. \end{aligned} \tag{3.85}$$

Lemma 3.8. *(H^1-estimate). Under the conditions of Theorem 1.1, for any $\gamma \ge 2$, and $t > 0$, it holds that*

$$\|\tilde{p}_x(t)\|_{L^2}^2 + \|q_x(t)\|_{L^2}^2 + \int_0^t \|\tilde{p}_{xx}\|_{L^2}^2 + \|q_x\|_{L^2}^2 \le C, \tag{3.86}$$

where C is a positive constant which is independent of t.

Taking ∂_x to the second equation of (3.80), then subtracting the resulting equation from the first equation of (3.80), we obtain

$$q_{xt} = \gamma\bar{p}^{\gamma-1}[\tilde{p}_t - (\tilde{p}q)_x - \bar{p}q_x] + \gamma(\gamma-1)(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2 + \gamma((\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1})\tilde{p}_{xx}. \tag{3.87}$$

Multiplying (3.87) by q_x and integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|q_x\|_{L^2}^2+\gamma\bar{p}^{\gamma}\|q_x\|_{L^2}^2\\
&=\gamma\bar{p}^{\gamma-1}\frac{d}{dt}\int_I\tilde{p}q_xdx-\gamma\bar{p}^{\gamma-1}\int_I\tilde{p}q_{xt}dx\\
&\quad-\gamma\bar{p}^{\gamma-1}\int_I(\tilde{p}q)_xq_xdx+\gamma(\gamma-1)\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2q_xdx\\
&\quad+\gamma\int_I[(\tilde{p}+\bar{p})^{\gamma-1}-\bar{p}^{\gamma-1}]\tilde{p}_{xx}q_xdx\\
&=\gamma\bar{p}^{\gamma-1}\frac{d}{dt}\int_I\tilde{p}q_xdx+\sum_{i=9}^{12}J_i.
\end{aligned}\tag{3.88}$$

Next we carry out energy estimates for J_9, J_{10}, J_{11}, J_{12}. For technical reasons, we divide the proof into three subsections for the cases:$2\le\gamma<3$, $3\le\gamma\le4$, and $\gamma>4$.

3.2.1. H^1 *Estimate when* $2\le\gamma<3$.

Proof. Since $2\le\gamma<3$, by Young's inequality, one has

$$(\tilde{p}+\bar{p})^{\gamma-1}\le2^{\gamma-2}(|\tilde{p}|^{\gamma-1}+\bar{p}^{\gamma-1})\le2^{\gamma-2}(|\tilde{p}|^2+1+\bar{p}^{\gamma-1}),$$

which combing the second equation of (1.2), we have

$$\begin{aligned}
|J_9|&=\gamma\bar{p}^{\gamma-1}|\int_I\tilde{p}q_{xt}dx|\\
&=\gamma\bar{p}^{\gamma-1}|\int_I\tilde{p}_xq_tdx|\\
&\le\gamma^2\bar{p}^{\gamma-1}|\int_I(\tilde{p}+\bar{p})^{\gamma-1}(\tilde{p}_x)^2dx|\\
&\le2^{\gamma-2}\gamma^2\bar{p}^{\gamma-1}\|\tilde{p}\tilde{p}_x\|_{L^2}^2+2^{\gamma-2}\gamma^2\bar{p}^{\gamma-1}(1+\bar{p}^{\gamma-1})\|\tilde{p}_x\|_{L^2}^2.
\end{aligned}\tag{3.89}$$

For J_{10}, using Cauchy-Schwarz and Gagliardo-Nirenberg inequalities, we have

$$\begin{aligned}
|J_{10}|&=\gamma\bar{p}^{\gamma-1}|\int_I(\tilde{p}q)_xq_xdx|\\
&\le\xi\|q_x\|_{L^2}^2+C(\xi)\|\tilde{p}_x\|_{L^2}^2\|q_x\|_{L^2}^2.
\end{aligned}\tag{3.90}$$

J_{11} can be estimated by Cauchy-Schwarz inequality and Gagliardo-Nirenberg inequality as

$$\begin{aligned}
|J_{11}|&=2\gamma(\gamma-1)|\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2q_xdx|\\
&\le C(\int_I|\tilde{p}|^{\gamma-2}|\tilde{p}_x|^2|q_x|dx+\int_I|\tilde{p}_x|^2|q_x|dx)\\
&\le C\left(\int_I|\tilde{p}|^{2(\gamma-2)}(\tilde{p}_x)^2dx\right)^{\frac{1}{2}}\|\tilde{p}_x\|_{L^\infty}\|q_x\|_{L^2}+C\|\tilde{p}_x\|_{L^\infty}\|\tilde{p}_x\|_{L^2}\|q_x\|_{L^2}\\
&\le C(\|\tilde{p}\tilde{p}_x\|_{L^2}^2+\|\tilde{p}_x\|_{L^2}^2)^{\frac{1}{2}}(\|\tilde{p}_x\|_{L^2}^{\frac{1}{2}}\|\tilde{p}_{xx}\|_{L^2}^{\frac{1}{2}}+\|\tilde{p}_x\|_{L^2})\|q_x\|_{L^2}\\
&\quad+C(\|\tilde{p}_x\|_{L^2}^{\frac{1}{2}}\|\tilde{p}_{xx}\|_{L^2}^{\frac{1}{2}}+\|\tilde{p}_x\|_{L^2})\|\tilde{p}_x\|_{L^2}\|q_x\|_{L^2}\\
&\le\delta\|\tilde{p}_{xx}\|_{L^2}^2+C(\delta)[(\|\tilde{p}\tilde{p}_x\|_{L^2}^2+\|\tilde{p}_x\|_{L^2}^2)\|q_x\|_{L^2}^2+\|\tilde{p}\tilde{p}_x\|_{L^2}^2+\|\tilde{p}_x\|_{L^2}^2].
\end{aligned}\tag{3.91}$$

Recall that $(\tilde{p}+\bar{p})^{\gamma-1} \leq (\gamma-1)(|\tilde{p}|+\bar{p})^{\gamma-2}|\tilde{p}|$, we can estimate J_{12} as

$$\begin{aligned}|J_{12}| &= |2\gamma\int_I [(\tilde{p}+\bar{p})^{\gamma-1}-\bar{p}^{\gamma-1}]\tilde{p}_{xx}q_x dx| \\ &\leq 2\gamma(\gamma-1)\int_I (|\tilde{p}|+\bar{p})^{\gamma-2}|\tilde{p}||\tilde{p}_{xx}||q_x|dx \\ &\leq 2\gamma(\gamma-1)\|\tilde{p}\|_{L^\infty}^{\gamma-1}\|\tilde{p}_{xx}\|_{L^2}\|q_x\|_{L^2} + 2\gamma(\gamma-1)\bar{p}^{\gamma-2}\|\tilde{p}\|_{L^\infty}\|\tilde{p}_{xx}\|_{L^2}\|q_x\|_{L^2} \\ &\leq C(\|\tilde{p}_x\|_{L^2}^{\frac{\gamma-1}{2}} + \|\tilde{p}_x\|_{L^2}^{\frac{1}{2}})\|\tilde{p}_{xx}\|_{L^2}\|q_x\|_{L^2} \\ &\leq \delta\|\tilde{p}_{xx}\|_{L^2}^2 + \xi\|q_x\|_{L^2}^2 + C(\xi,\delta)\|\tilde{p}_x\|_{L^2}^2\|q_x\|_{L^2}^2, \end{aligned} \tag{3.92}$$

where we have use the Young's inequality $C(\delta)\|\tilde{p}_x\|_{L^2}^{\gamma-1} \leq \frac{\xi}{2} + C(\delta,\xi)\|\tilde{p}_x\|_{L^2}^2$ due to $2 \leq \gamma < 3$ in the last inequality. From (3.88)-(3.92), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|q_x\|_{L^2}^2 - \gamma\bar{p}^{\gamma-1}\frac{d}{dt}\int_I \tilde{p}q_x dx + \gamma\bar{p}^\gamma\|q_x\|_{L^2}^2 \\ &\leq \delta\|\tilde{p}_{xx}\|_{L^2}^2 + \xi\|q_x\|_{L^2}^2 + C(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2)\|q_x\|_{L^2}^2 + C\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + C\|\tilde{p}_x\|_{L^2}^2. \end{aligned} \tag{3.93}$$

Multiplying the first equation of (1.2) by $-\gamma\bar{p}^{\gamma-2}\tilde{p}_{xx}$, we have

$$\begin{aligned}\frac{d}{dt}\frac{\gamma}{2}|\bar{p}|^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \gamma\bar{p}^{\gamma-2}\|\tilde{p}_{xx}\|_{L^2}^2 &= -\gamma\bar{p}^{\gamma-2}\int_I (\tilde{p}q)_x\tilde{p}_{xx}dx - \gamma\bar{p}^{\gamma-1}\int_I q_x\tilde{p}_{xx}dx \\ &\leq \frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_{xx}\|_{L^2}^2 + C(\|\tilde{p}_x\|_{L^2}^2\|q_x\|_{L^2}^2 + \|q_x\|_{L^2}^2). \end{aligned} \tag{3.94}$$

Adding (3.94) into (3.93), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 - \gamma\bar{p}^{\gamma-1}\int_I \tilde{p}q_x dx\right) \\ &\quad + \frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_{xx}\|_{L^2}^2 + \frac{\gamma}{2}\bar{p}^\gamma\|q_x\|_{L^2}^2 \\ &\leq C[(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2)\|q_x\|_{L^2}^2 + \|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2]. \end{aligned} \tag{3.95}$$

Multiplying (3.82) by M_4 and adding the resulting equation into (3.95), we obtain

$$\begin{aligned}\frac{d}{dt}G_1(t) + H_1(t) &\leq C[(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2)\|q_x\|_{L^2}^2 \\ &\quad + \|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2] + M_4C\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx, \end{aligned} \tag{3.96}$$

where

$$\begin{aligned}G_1(t) &= \frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 - \gamma\bar{p}^{\gamma-2}\int_I \tilde{p}q_x dx \\ &\quad + \frac{M_4}{\gamma-1}\int_I [(\tilde{p}+\bar{p})^\gamma - \bar{p}^\gamma - \gamma\bar{p}^{\gamma-1}\tilde{p}]dx + \frac{M_4}{2}\|q\|_{L^2}^2 + M_4M_1\|\tilde{p}\|_{L^4}^4, \end{aligned}$$

$$H_1(t) = \frac{1}{2}\gamma\bar{p}^{\gamma-2}\|\tilde{p}_{xx}\|_{L^2}^2 + \frac{1}{2}\gamma\bar{p}^\gamma\|q_x\|_{L^2}^2 + \frac{M_4}{4}\gamma\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + M_4\|\tilde{p}\tilde{p}_x\|_{L^2}^2.$$

By Cauchy inequality, we have

$$-\gamma\bar{p}^{\gamma-1}\int_I \tilde{p}q_x dx \geq -\frac{1}{2}\|q_x\|_{L^2}^2 - \frac{(\gamma\bar{p}^{\gamma-1})^2}{2}\|\tilde{p}\|_{L^2}^2. \tag{3.97}$$

By virtue of Lemma 2.1, we have

$$\frac{M_4}{\gamma-1}\int_I [(\tilde{p}+\bar{p})^\gamma - \bar{p}^\gamma - \gamma\bar{p}^{\gamma-1}\tilde{p}]dx \geq 2(\gamma-1)\bar{p}^{\gamma-2}\|\tilde{p}\|_{L^2}^2. \tag{3.98}$$

$$G_1(t) \geq \frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \frac{1}{2}\|q_x\|_{L^2}^2 + \|\tilde{p}\|_{L^2}^2 + \frac{M_4}{2}\|q\|_{L^2}^2 + M_1M_4\|\tilde{p}\|_{L^4}^4,$$

which implies

$$\|q_x\|_{L^2}^2 \leq 2G_1(t). \tag{3.99}$$

Substituting (3.99) into (3.96), we have

$$\begin{aligned} \frac{d}{dt}G_1(t) + H_1(t) \leq{}& C(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2)G_1(t) \\ &+ C(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2) + K_2C\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx. \end{aligned} \tag{3.100}$$

Applying Gronwall's inequality to (3.100) and using (3.81) and (3.83), we obtain

$$G_1(t) + \int_0^t H_1(\tau)d\tau \leq C, \tag{3.101}$$

which implies that

$$\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \int_0^t(\|\tilde{p}_{xx}\|_{L^2}^2 + \|q_x\|_{L^2}^2)d\tau \leq C$$

for some constant which is independent of t. □

3.2.2. H^1 *Estimate when* $3 \leq \gamma \leq 4$.

Proof. By using (3.56), we can estimate J_{16} as

$$\begin{aligned} |J_9| &= \gamma^2\bar{p}^{\gamma-1}|\int_I(\tilde{p}+\bar{p})^{\gamma-1}(\tilde{p}_x)^2dx| \\ &\leq C(\|\tilde{p}\|_{L^\infty}^{\gamma-1} + 1)\|\tilde{p}_x\|_{L^2}^2 \\ &\leq C[(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2)^{\frac{1}{2}} + 1]\|\tilde{p}_x\|_{L^2}^2 \\ &\leq C(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2)\|\tilde{p}_x\|_{L^2}^2 + C\|\tilde{p}_x\|_{L^2}^2. \end{aligned} \tag{3.102}$$

For J_{10}, by using Cauchy-Schwarz inequality and Sobolev inequality, we can show that

$$\begin{aligned} |J_{10}| &= \gamma\bar{p}^{\gamma-1}|\int_I(\tilde{p}q)_xq_xdx| \\ &\leq \xi\|q_x\|_{L^2}^2 + C(\xi)\|\tilde{p}_x\|_{L^2}^2\|q_x\|_{L^2}^2. \end{aligned} \tag{3.103}$$

For J_{11}, it is straightforward to check that

$$
\begin{aligned}
|J_{11}| &= 2\gamma(\gamma-1)|\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2 q_x dx| \\
&\le C(\int_I |\tilde{p}|^{\gamma-2}|\tilde{p}_x|^2|q_x|dx + \int_I |\tilde{p}_x|^2|q_x|dx) \\
&\le C\left(\|\tilde{p}\|_{L^\infty}\|\tilde{p}_x\|_{L^\infty}\int_I |\tilde{p}|^{\gamma-3}|\tilde{p}_x||q_x|dx + \|\tilde{p}_x\|_{L^\infty}\int_I |\tilde{p}_x||q_x|dx\right) \\
&\le C[\|\tilde{p}\|_{L^\infty}\|\tilde{p}_x\|_{L^\infty}\left(\int_I |\tilde{p}|^{2(\gamma-3)}|\tilde{p}_x|^2dx\right)^{\frac{1}{2}}\|q_x\|_{L^2} + \|\tilde{p}_x\|_{L^\infty}\|\tilde{p}_x\|_{L^2}\|q_x\|_{L^2}] \\
&\le C\Big[(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2)^{\frac{1}{2}}\|\tilde{p}_x\|_{L^2}^{\frac{1}{2}}(\|\tilde{p}_x\|^{\frac{1}{2}}\|\tilde{p}_{xx}\|^{\frac{1}{2}} + \|\tilde{p}_x\|_{L^2})\|q_x\|_{L^2} \\
&\quad +(\|\tilde{p}_x\|_{L^2}^{\frac{3}{2}}\|\tilde{p}_{xx}\|_{L^2}^{\frac{1}{2}} + \|\tilde{p}_x\|_{L^2}^2)\|q_x\|_{L^2}\Big] \\
&\le \delta\|\tilde{p}_{xx}\|_{L^2}^2 + C[(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2)^{\frac{2}{3}}\|\tilde{p}_x\|^{\frac{4}{3}}\|q_x\|^{\frac{4}{3}} + \|\tilde{p}_x\|_{L^2}^2\|q_x\|_{L^2}^{\frac{4}{3}}] \\
&\le \delta\|\tilde{p}_{xx}\|_{L^2}^2 + C(\delta)[(\|\tilde{p}\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2)(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2) + \|\tilde{p}_x\|_{L^2}^2]. \qquad (3.104)
\end{aligned}
$$

For J_{12}, similar to (3.53), by virtue of (3.56), we obtain

$$
\begin{aligned}
|J_{12}| &= |2\gamma\int_I [(\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}]\tilde{p}_{xx}q_x dx| \\
&\le 2\gamma(\gamma-1)\int_I (|\tilde{p}|+\bar{p})^{\gamma-2}|\tilde{p}||\tilde{p}_{xx}||q_x|dx \\
&\le 2\gamma(\gamma-1)\|\tilde{p}\|_{L^\infty}^{\gamma-1}\|\tilde{p}_{xx}\|_{L^2}\|q_x\|_{L^2} + 2\gamma(\gamma-1)\bar{p}^{\gamma-2}\|\tilde{p}\|_{L^\infty}\|\tilde{p}_{xx}\|_{L^2}\|q_x\|_{L^2} \\
&\le \delta\|\tilde{p}_{xx}\|_{L^2}^2 + C(\delta)(\|\tilde{p}_x\|_{L^2}^{\gamma-1} + \|\tilde{p}_x\|_{L^2})\|q_x\|_{L^2}^2 \\
&\le \delta\|\tilde{p}_{xx}\|_{L^2}^2 + \xi\|q_x\|_{L^2}^2 + C(\xi_1,\delta)\|\tilde{p}_x\|_{L^2}^2)\|q_x\|_{L^2}^2. \qquad (3.105)
\end{aligned}
$$

Step 2. The subsequent coupling are identical to those presented in section 3.2.1 for the case when $3<\gamma<4$. We omit the technical details for brevity. □

3.2.3. H^1 *Estimate when* $\gamma > 4$.

Proof. Step 1. By using integration by parts and (3.45), we obtain

$$
\begin{aligned}
|J_9| &= \gamma\bar{p}^{\gamma-1}|\int_I \tilde{p}q_{xt}dx| \\
&= \gamma\bar{p}^{\gamma-1}|\int_I \tilde{p}_x q_t dx| \\
&\le \gamma^2\bar{p}^{\gamma-1}|\int_I (\tilde{p}+\bar{p})^{\gamma-1}(\tilde{p}_x)^2dx| \\
&\le C(\int_I |\tilde{p}|^{\gamma-1}|\tilde{p}_x|^2dx + \|\tilde{p}_x\|_{L^2}^2). \qquad (3.106)
\end{aligned}
$$

J_{10} can be estimated as

$$
\begin{aligned}
|J_{10}| &= \gamma\bar{p}^{\gamma-1}|\int_I (\tilde{p}q)_x q_x dx| \\
&\le \xi\|q_x\|_{L^2}^2 + C(\xi)\|\tilde{p}_x\|_{L^2}^2\|q_x\|_{L^2}^2. \qquad (3.107)
\end{aligned}
$$

Recalling (3.104) and using (3.45), we have

$$\begin{aligned}|J_{11}| &= 2\gamma(\gamma-1)|\int_I (\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2 q_x dx| \\ &\le C(\int_I |\tilde{p}|^{\gamma-2}|\tilde{p}_x|^2|q_x|dx + \int_I |\tilde{p}_x|^2|q_x|dx) \\ &\le \delta\|\tilde{p}_{xx}\|_{L^2}^2 + C(\delta)\left[(\int_I |\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx + \|\tilde{p}_x\|_{L^2}^2)(\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2) + \|\tilde{p}_x\|_{L^2}^2\right]. \end{aligned} \tag{3.108}$$

For J_{12}, similar to (3.105), by using (3.45), we obtain

$$\begin{aligned}|J_{12}| &= |2\gamma\int_I [(\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}]\tilde{p}_{xx}q_x dx| \\ &\le C(\|\tilde{p}\|_{L^\infty}^{\gamma-1} + \|\tilde{p}\|_{L^\infty})\|\tilde{p}_{xx}\|_{L^2}\|q_x\|_{L^2} \\ &\le \delta\|\tilde{p}_{xx}\|_{L^2}^2 + \xi\|q_x\|_{L^2}^2 + C(\xi,\delta)(\int_I |\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx + \|\tilde{p}_x\|_{L^2}^2)\|q_x\|_{L^2}^2. \end{aligned} \tag{3.109}$$

Step 2. Again, the subsequent coupling are identical to those presented in section 3.2.1 for the case when $2 < \gamma < 3$. The details are omitted. Hence we conclude that for any $\gamma \ge 2$, the following estimate holds

$$\|q_x\|_{L^2}^2 + \|\tilde{p}_x\|_{L^2}^2 + \int_0^t (\|q_x\|_{L^2}^2 + \|\tilde{p}_{xx}\|_{L^2}^2) \le C. \tag{3.110}$$

Combing (3.110) with equation (3.80), we obtain immediately

$$\int_0^t \|\tilde{p}_t(\tau)\|_{L^2}^2 + \|q_t(\tau)\|_{L^2}^2 d\tau \le C, \tag{3.111}$$

which implies

$$\int_0^t \|\tilde{p}_{xx}\|_{L2}^2 \le C.$$

□

Taking ∂_t to the two equations in (3.80), then taking the L^2 inner products of the resulting equations with the first order temporal derivatives of the solution, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\gamma}{2}\bar{p}^{\gamma-2}\int_I |\tilde{p}_t|^2 + \frac{1}{2}|q_t|^2\right) + \gamma\bar{p}^{\gamma-2}\int_I |\tilde{p}_{xt}|^2 \\ &= -\gamma\bar{p}^{\gamma-2}\int_I (\tilde{p}\tilde{q})_t\tilde{p}_{xt} + \gamma(\gamma-1)\int_I (\tilde{p}+\bar{p})^{\gamma-2}\tilde{p}_t\tilde{p}_x qtdx + \gamma\int_I ([(\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}]\tilde{p}_{xt})q_t dx \\ &= J_{13} + J_{14} + J_{15}. \end{aligned} \tag{3.112}$$

Let us estimate J_{13}-J_{15} term by term.

$$J_{13} \le \|\tilde{p}_{tx}\|_{L^2}^2 + (\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2)(\|\tilde{p}_t\|_{L^2}^2 + \|q_t\|_{L^2}^2). \tag{3.113}$$

$$\begin{aligned}J_{14} &= \gamma(\gamma-1)\int_I (\tilde{p}+\bar{p})^{\gamma-2}\tilde{p}_t\tilde{p}_x q_t dx \\ &\le C\|\tilde{p}_t\|_{L^2}^2 + (\|\tilde{p}_x\|_{L^2}^2 + \|\tilde{p}_{xx}\|_{L^2}^2)\|q_t\|_{L^2}^2. \end{aligned} \tag{3.114}$$

$$J_{15} \le \frac{\gamma\bar{p}^{\gamma-2}}{4}\|\tilde{p}_{tx}\|_{L^2}^2 + C\|q_t\|_{L^2}^2. \tag{3.115}$$

Plugging (3.114)-(3.115) into (3.112), we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\gamma}{2}\bar{p}^{\gamma-2}\int_I|\tilde{p}_t|^2+\frac{1}{2}|q_t|^2\right)+\frac{\gamma\bar{p}^{\gamma-2}}{2}\int_I|\tilde{p}_{xt}|^2\\&\le(\|\tilde{p}_x\|_{L^2}^2+\|q_x\|_{L^2}^2)(\|\tilde{p}_t\|_{L^2}^2+\|q_t\|_{L^2}^2)+C\|\tilde{p}_t\|_{L^2}^2+C\|q_t\|_{L^2}^2.\end{aligned}\tag{3.116}$$

By virtue of Gronwall' inequality and (3.81), (3.110) and (3.111), we obtain

$$\|\tilde{p}_t\|_{L^2}^2+\|q_t\|_{L^2}^2\le C,\tag{3.117}$$

which implies

$$\|\tilde{p}_{xx}\|_{L^2}^2\le C.\tag{3.118}$$

Remark 3.1. *When $\varepsilon=0$, the proof of the exponential decay rate of $\|\tilde{p}(t)\|_{H^2}^2+\|q(t)\|_{H^1}^2\le\alpha_1e^{-\beta_1t}$ is essentially identical to that of the diffusive problem by using the same method in Lemma 3.7. We omit further details here for brevity. This completes the proof of Theorem 1.1.*

□

3.3. **Dynamics of original functions.** In this section we investigate the long-time behavior of the original chemotaxis model. Noticing that the transformed and pre-transformed systems have the same quantity $n=p$ which is clear from the above sections, here we are left to consider the dynamics of original function c. At frst, we consider the following equation

$$[\ln(c)]_t=(\tilde{n}+\bar{n})^\gamma-\mu+\varepsilon q_x+\varepsilon q^2,$$

where $\tilde{n}=n-\bar{n}$, $q=[\ln(c)]_x$. Solving the above equation, we have

$$c(x,t)=c(x,0)\exp\left\{\int_0^t((\tilde{n}+\bar{n})^\gamma-\bar{n}^\gamma+\varepsilon q_x+\varepsilon q^2)d\tau\right\}\exp\{(\bar{n}^\gamma-\mu)t\},\forall t>0.$$

Because of the exponential decaying of $\|(\tilde{n},q)(t)\|_{H^2}^2$, we easily see that there exist $t-$ independent constant c_1, c_2 such that

$$c_1\le\exp\left\{\int_0^t((\tilde{n}+\bar{n})^\gamma-\bar{n}^\gamma+\varepsilon q_x+\varepsilon q^2)d\tau\right\}\le c_2.$$

Moreover, since $0<\underline{c}\le c(x,0)\le\bar{c}<\infty$, it holds that

$$\underline{c}c_1\exp{(\bar{n}^\gamma-\mu)t}\le c(x,t)\le\bar{c}c_2\exp\{(\bar{n}^\gamma-\mu)t\},$$

which implies that $c(x,t)>0$ for all $t>0$, and

$$c(x,t)\to0\ as\ t\to\infty, when\ \bar{n}^\gamma<\mu,$$
$$c(x,t)\to+\infty\ as\ t\ \to\infty, when\ \bar{n}^\gamma>\mu,$$

for $\varepsilon\ge0$. This completes the proof of Proposition 1.2.

3.4. **The case $\bar{p}=0$, $\varepsilon\ge0$.** In this section, we give a ketch of proof for the case $\bar{p}=0$ and $\varepsilon\ge0$. For convenience, we restate the following IBVP:

$$\begin{cases}p_t-(pq)_x=p_{xx},\\q_t-(p^\gamma)_x=\varepsilon q_{xx}+\varepsilon(q^2)_x,\\p|_{x=0,x=1}=0,\ q|_{x=0,x=1}=0,\ \varepsilon>0\\p|_{x=0,x=1}=0,\ \varepsilon=0\\(p,q)(x,0)=(p_0,q_0)(x).\end{cases}\tag{3.119}$$

We divide the subsequent proof into several steps.

Step 1. Testing $(3.119)_1$ with $\frac{\gamma}{\gamma-1}p^{\gamma-1}$ and $(3.119)_2$ with q, and adding the results, we have

$$\int_Ip^\gamma dx+\|q\|_{L^2}^2+\gamma\int_0^t\int_Ip^{\gamma-2}p_x^2dx+\varepsilon\|q_x\|_{L^2}^2d\tau\le C,\tag{3.120}$$

where C is independent on t and ε. Multiplying the first equation of (3.119) by p, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|p\|^2+\|p_x\|^2&=-\int_I(pqp_x)dx\\&\le\frac{1}{4}\|p_x\|^2+\|p\|_{L^\infty}^2\|q\|^2\\&\le\frac{1}{4}\|p_x\|^2+2\|p\|\|p_x\|\|q\|^2\\&\le\frac{1}{2}\|p_x\|^2+4\|p\|^2\|q\|^4,\end{aligned}$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\|p\|^2+\frac{1}{2}\|p_x\|^2\le C\|p\|^2. \tag{3.121}$$

Applying Gronwall's inequality to (3.121), we infer that

$$\|p(t)\|^2+\int_0^t\|p_x(\tau)\|^2d\tau\le C, \tag{3.122}$$

where C is an increasing function of t, but independent on ε.
Similar to the idea (3.19) in section 3.1.1 and (3.34) in section 3.1.2, for any $2\le\gamma\le4$, we also have

$$\frac{d}{dt}\|\tilde p\|_{L^4}^4+12\|\tilde p\tilde p_x\|_{L^2}^2\le C\int_I(\tilde p+\bar p)^{\gamma-2}(\tilde p_x)^2dx+\delta(\|\tilde p\tilde p_x\|^2+3\|\tilde p_x\|_{L^2}^2). \tag{3.123}$$

Let δ small enough, then integrating the resulting equation over $(0,t)$, together with (3.122), we obtain

$$\|\tilde p\|_{L^4}^4+\int_0^t\|\tilde p\tilde p_x\|_{L^2}^2d\tau\le C, \tag{3.124}$$

where C is dependent on t, but independent on ε.

Lemma 3.9. *Under the conditions of Theorem 1.3, for any $\gamma\ge2$, $\varepsilon\ge0$, and $t>0$, it holds that*

$$\|\tilde p_x(t)\|_{L^2}^2+\|q_x(t)\|_{L^2}^2+\int_0^t(\|\tilde p_{xx}(\cdot,\tau)\|_{L^2}^2+\varepsilon\|q_{xx}(\cdot,\tau)\|_{L^2}^2)d\tau\le C,$$

where C is a positive constant which is independent of ε.

Proof. Multiplying the equation $(3.119)_1$ and $(3.119)_2$ by $-p_{xx}$ and $-q_{xx}$, respectively. Then integrating by parts and summing up the resulting equations we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|p_x\|^2+\|q_x\|^2)+\|p_{xx}\|^2+\varepsilon\|q_{xx}\|^2\\&=-\int_I(pq)_xp_{xx}dx-2\varepsilon\int_Iqq_xq_{xx}dx-\gamma\int_Ip^{\gamma-1}p_xq_{xx}dx\\&=-\int_Ip_xqp_{xx}dx-\int_Ipq_xp_{xx}dx-2\varepsilon\int_Iqq_xq_{xx}dx+\gamma(\gamma-1)\int_Ip^{\gamma-2}(p_x)^2q_xdx\\&\quad+\gamma\int_Ip^{\gamma-1}p_{xx}q_xdx\\&=\sum_{i=16}^{20}J_i.\end{aligned} \tag{3.125}$$

For J_{16}, by using Sobolev inequality and Young's inequality, we have

$$\begin{aligned} J_{16} &\le \|p_x\|_{L^\infty}\|q\|\|p_{xx}\| \\ &\le C\|p_x\|^{\frac{1}{2}}\|q\|\|p_{xx}\|^{\frac{3}{2}} \\ &\le \delta\|p_{xx}\|^2 + C(\delta)\|p_x\|^2. \end{aligned} \tag{3.126}$$

Similarly, J_{17} can be estimated

$$\begin{aligned} J_{17} &\le \|p\|_{L^\infty}\|q_x\|\|p_{xx}\| \\ &\le \delta\|p_{xx}\|^2 + C(\delta)\|p_x\|^2\|q_x\|^2. \end{aligned} \tag{3.127}$$

$$|J_{18}| \le \frac{\varepsilon}{2}\|q_{xx}\|^2 + 2\varepsilon\|q\|^2\|q_x\|^2. \tag{3.128}$$

For J_{19}, when $2 \le \gamma \le 3$, we have

$$\begin{aligned} J_{19} &\le \gamma(\gamma-1)\left(\int_I p^{2(\gamma-2)}(p_x)^2 dx\right)^{\frac{1}{2}}\|p_x\|_{L^\infty}\|q_x\| \\ &\le \gamma(\gamma-1)(\|pp_x\|^2 + \|p_x\|^2)^{\frac{1}{2}}\|p_x\|^{\frac{1}{2}}\|p_{xx}\|^{\frac{1}{2}}\|q_x\| \\ &\le \delta\|p_{xx}\|^2 + C(\delta)[(\|pp_x\|^2 + \|p_x\|^2)\|q_x\|^2 + \|p_x\|^2] \end{aligned} \tag{3.129}$$

$$\begin{aligned} J_{20} &\le \gamma\|p\|_{L^\infty}^{\gamma-1}\|p_{xx}\|\|q_x\| \\ &\le C\|p_x\|^{\frac{\gamma-1}{2}}\|p_{xx}\|\|q_x\| \\ &\le \delta\|p_{xx}\|^2 + C(\delta)\|p_x\|^{\gamma-1}\|q_x\|^2 \\ &\le \delta\|p_{xx}\|^2 + \xi\|q_x\|^2 + C(\delta,\xi)\|p_x\|^2\|q_x\|^2. \end{aligned} \tag{3.130}$$

when $3 \le \gamma \le 4$,

$$\begin{aligned} J_{19} &\le C\|p\|_{L^\infty}\|p_x\|_{L^\infty}\int_I |p|^{\gamma-3}|p_x||q_x| dx \\ &\le C\|p\|_{L^\infty}\|p_x\|_{L^\infty}\left(\int_I |p|^{2(\gamma-3)}|p_x|^2 dx\right)^{\frac{1}{2}}\|q_x\| \\ &\le \delta\|p_{xx}\|^2 + C(\|pp_x\|^2 + \|p_x\|^2)(\|p_x\|^2 + \|q_x\|^2) \end{aligned} \tag{3.131}$$

$$\begin{aligned} J_{20} &\le C\|p\|_{L^\infty}^{\gamma-1}\|p_{xx}\|\|q_x\| \\ &\le C(\|pp_x\|^2 + \|p_x\|^2)^{\frac{1}{2}}\|p_{xx}\|\|q_x\| \\ &\le \delta\|p_{xx}\|^2 + C(\delta)(\|pp_x\|^2 + \|p_x\|^2)\|q_x\|^2. \end{aligned} \tag{3.132}$$

When $\gamma > 4$, we have

$$\begin{aligned}
J_{19} &\le C\int_I |p|^{\frac{\gamma}{2}}|p|^{\frac{\gamma}{2}-2}|p_x|^2|q_x|dx \\
&\le C\left(\int_I |p|^{\gamma-4}(p_x)^2dx\right)^{\frac{1}{2}}\|p\|_{L^\infty}^{\frac{\gamma}{2}}\|p_x\|_{L^\infty}\|q_x\|_{L^2} \\
&\le C\left(\int_I p^{\gamma-2}(p_x)^2dx+\|p_x\|^2\right)^{\frac{1}{2}}\|p\|_{L^\infty}^{\frac{\gamma}{2}}\|p_x\|_{L^\infty}\|q_x\| \\
&\le C\left(\int_I |p|^{\gamma-2}(p_x)^2dx+\|p_x\|^2\right)^{\frac{3}{4}}\|p_x\|^{\frac{1}{2}}\|p_{xx}\|^{\frac{1}{2}}\|q_x\| \\
&\le \delta\|p_{xx}\|^2 + C(\delta)\left(\int_I |p|^{\gamma-2}(p_x)^2dx+\|p_x\|^2\right)(\|p_x\|^2+\|q_x\|^2).
\end{aligned} \tag{3.133}$$

$$\begin{aligned}
J_{20} &\le \|p\|_{L^\infty}^{\gamma-1}\|p_{xx}\|\|q_x\| \\
&\le \left(\int_I p^{\gamma-2}(p_x)^2dx+\|p_x\|^2\right)^{\frac{1}{2}}\|p_{xx}\|\|q_x\| \\
&\le \delta\|p_{xx}\|^2 + C\left(\int_I |p|^{\gamma-2}(p_x)^2dx+\|p_x\|^2\right)\|q_x\|^2.
\end{aligned} \tag{3.134}$$

When $2 \le \gamma \le 3$, inserting (3.126)-(3.130) into (3.125), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|p_x\|^2+\|q_x\|^2)+\frac{1}{2}\|p_{xx}\|^2+\frac{\varepsilon}{2}\|q_{xx}\|^2 \\
&\le C(\|pp_x\|^2+\|p_x\|^2)\|q_x\|^2 + C\|p_x\|^2 + C\varepsilon\|q_x\|^2,
\end{aligned} \tag{3.135}$$

which together with (3.120), (3.122) and Gronwall's inequality, we have

$$\|p_x\|^2+\|q_x\|^2+\int_0^t \|p_{xx}\|^2+\varepsilon\|q_{xx}\|^2d\tau \le C, \tag{3.136}$$

where the constant C is independent of ε. When $3 \le \gamma \le 4$, combining (3.125)-(3.128), (3.131) with (3.132), when $\gamma \ge 4$, combining (3.125)-(3.128), (3.133)and (3.134), then using the same method as the case $2 \le \gamma \le 3$, we can also obtain the result (3.136). This completes the proof of Lemma 3.9. Then using the same idea as in [37, Theorem 1.2], we can show that

$$\|(p^\varepsilon - p^0)(t)\|^2 + \|(q^\varepsilon - p^0)(t)\|^2 \le \alpha_2 e^{\beta_2 t}\varepsilon. \tag{3.137}$$

We omit the details of proof for brevity. This completes the proof of Theorem 1.3. □

4. Proof of Theorem 1.4

In this section, we are devoted to the study of subject to the Neumann boundary condition for p. For the reader's convenience, we rewrite the following initial-boundary value problem:

$$\begin{cases}
p_t - (pq)_x = p_{xx} \\
q_t - (p^\gamma)_x = \varepsilon q_{xx} + \varepsilon(q^2)_x \\
(p,q)(x,0) = (p_0,q_0)(x), \ (p_0-\bar{p},q_0) \in H^1(0,1), \\
p_0(x) \ge 0, \ x \in [0,1], \\
p_x|_{x=0,x=1} = 0, \ q|_{x=0,x=1} = 0.
\end{cases} \tag{4.1}$$

where $\bar{p} = \int_0^1 p_0(x)dx > 0$ denotes the spatial average of the cell density, which is a conserved quantity.

First of all, under the Neumann-Dirichlet boundary condition, we note that Lemma 3.1 still holds true for (4.1). That is to say, we have the following energy estimate:

$$\begin{aligned}
&\frac{1}{\gamma-1}\int_I[(\tilde{p}+\bar{p})^\gamma-\bar{p}^\gamma-\gamma\bar{p}^{\gamma-1}\tilde{p}](x,t)dx+\frac{1}{2}\|q(t)\|_{L^2}^2\\
&\quad+\int_0^t\left(\gamma\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2dx+\varepsilon\|q_x\|_{L^2}^2\right)(\tau)d\tau\le C,
\end{aligned}\tag{4.2}$$

Let $\tilde{p}=p-\bar{p}$, we have

$$\begin{cases}
\tilde{p}_t-(\tilde{p}q)_x-\bar{p}q_x=\tilde{p}_{xx},\\
q_t-\gamma[(\tilde{p}+\bar{p})^{\gamma-1}-\bar{p}^{\gamma-1}]\tilde{p}_x-\gamma\bar{p}^{\gamma-1}\tilde{p}_x=\varepsilon q_{xx}+\varepsilon(q^2)_x,\\
(\tilde{p}_0,q_0)(x)=(p_0-\bar{p},q_0)(x),\ (\tilde{p}_0,q_0)\in H^1(0,1),\\
\tilde{p}_x|_{x=0,x=1}=0,\ q|_{x=0,x=1}=0.
\end{cases}\tag{4.3}$$

In particular, making use of the fact $\int_I p(x,t)dx=\int_I p_0(x)dx=\bar{p}$, we conclude that there exists a $x^*\in I$ such that $p(x^*,t)=\bar{p}$ which implies $\tilde{p}(x^*,t)=0$, so we only need to make some small modification in the proof of Lemma 3.2. Then we have the following lemma.

Lemma 4.1. *Under the conditions of Theorem 1.3, for any $\gamma\ge 2$, $\varepsilon\ge 0$, and $t>0$, it holds that If $2\le\gamma\le 3$, we have*

$$\begin{aligned}
&G(\tilde{p},\bar{p})+\frac{1}{2}\|q\|_{L^2}^2+M_1\|\tilde{p}\|_{L^4}^4+\int_0^t(\frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2+\|\tilde{p}\tilde{p}_x\|_{L^2}^2+\varepsilon\|q_x\|_{L^2}^2)d\tau\\
&\le G(\tilde{p}_0,\bar{p})+\frac{1}{2}\|q_0\|_{L^2}^2+M_1\|\tilde{p}_0\|_{L^4}^4+\frac{C}{\gamma}.
\end{aligned}\tag{4.4}$$

If $3\le\gamma\le 4$, we have

$$\begin{aligned}
&G(\tilde{p},\bar{p})+\frac{1}{2}\|q\|_{L^2}^2+M_2\|\tilde{p}\|_{L^4}^4+\int_0^t(\frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2+\|\tilde{p}\tilde{p}_x\|_{L^2}^2+\varepsilon\|q_x\|_{L^2}^2)d\tau\\
&\le G(\tilde{p}_0,\bar{p})+\frac{1}{2}\|q_0\|_{L^2}^2+M_2\|\tilde{p}_0\|_{L^4}^4+\frac{C}{\gamma}.
\end{aligned}\tag{4.5}$$

If $\gamma>4$, we have

$$\begin{aligned}
&G(\tilde{p},\bar{p})+\frac{1}{2}\|q\|_{L^2}^2+M\|\tilde{p}\|_{L^{\gamma+1}}^{\gamma+1}+\int_0^t(\frac{\gamma}{4}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2+\int_I|\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx+\varepsilon\|q_x\|_{L^2}^2)d\tau\\
&\le G(\tilde{p}_0,\bar{p})+\frac{1}{2}\|q_0\|_{L^2}^2+M\|\tilde{p}_0\|_{L^{\gamma+1}}^{\gamma+1}+\frac{C}{\gamma}.
\end{aligned}\tag{4.6}$$

To simplify the presentation, we omit details of the proof of Lemma 4.1 here. The following lemma gives the H^1 estimate.

Lemma 4.2. *Let $(\tilde{p},q)$ be a solution to (4.3). Then for any $\varepsilon>0$, it holds that*

$$\|(\tilde{p}_x,q_x)(t)\|^2+\int_0^t\|\tilde{p}_{xx}(\tau)\|^2+\|q_x(\tau)\|^2+\varepsilon(\|q_x(\tau)\|^2+\|q_{xx}(\tau)\|^2)d\tau\le C,\tag{4.7}$$

where the constant C is independent of t and ε.

Proof. By using the boundary conditions we can deduce that $\varepsilon q_{xx}|_{x=0,x=1}=(q_t-\gamma\bar{p}^{\gamma-1}\tilde{p}_x-2\varepsilon qq_x)|_{x=0,x=1}=0$, then taking ∂_x to the second equation of (1.2), we have

$$\begin{aligned}
q_{xt}=&\gamma\bar{p}^{\gamma-1}[\tilde{p}_t-(\tilde{p}q)_x-\bar{p}q_x]+\varepsilon q_{xxx}+\varepsilon(q^2)_{xx}\\
&+\gamma(\gamma-1)(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2+\gamma((\tilde{p}+\bar{p})^{\gamma-1}-\bar{p}^{\gamma-1})\tilde{p}_{xx}.
\end{aligned}\tag{4.8}$$

Multiplying (4.8) by q_x and integrating by parts, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|q_x\|_{L^2}^2\right) + \gamma\bar{p}^\gamma\|q_x\|_{L^2}^2 + \varepsilon\|q_{xx}\|_{L^2}^2 \\
&= \gamma\bar{p}^{\gamma-1}\frac{d}{dt}\int_I \tilde{p}q_x dx - \gamma\bar{p}^{\gamma-1}\int_I \tilde{p}q_{xt}dx - \gamma\bar{p}^{\gamma-1}\int_I(\tilde{p}q)_x q_x dx - 2\varepsilon\int_I qq_xq_{xx}dx \\
&+ \gamma(\gamma-1)\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2 q_x dx + \gamma\int_I[(\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}]\tilde{p}_{xx}q_x dx \\
&= \gamma\bar{p}^{\gamma-1}\frac{d}{dt}\int_I \tilde{p}q_x dx + \gamma^2\bar{p}^{\gamma-1}\int_I(\tilde{p}+\bar{p})^{\gamma-1}(\tilde{p}_x)^2dx + \varepsilon\gamma\bar{p}^{\gamma-1}\int_I\tilde{p}_xq_{xx}dx \\
&\quad + 2\varepsilon\gamma\bar{p}^{\gamma-1}\int_I qq_x\tilde{p}_x dx - \gamma\bar{p}^{\gamma-1}\int_I(\tilde{p}q)_xq_xdx - 2\varepsilon\int_I qq_xq_{xx}dx \\
&\quad + \gamma(\gamma-1)\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2q_xdx + \gamma\int_I[(\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}]\tilde{p}_{xx}q_xdx. \qquad (4.9)
\end{aligned}
$$

Multiplying the first equation of (4.1) by $-\gamma\bar{p}^{\gamma-2}\tilde{p}_{xx}$, we obtain

$$
\frac{d}{dt}\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \gamma\bar{p}^{\gamma-2}\|\tilde{p}_{xx}\|_{L^2}^2 = -\gamma\bar{p}^{\gamma-2}\int_I(\tilde{p}q)_x\tilde{p}_{xx} - \gamma\bar{p}^{\gamma-1}\int_I q_x\tilde{p}_{xx}. \qquad (4.10)
$$

Taking the L^2 inner product of $(4.1)_2$ with $-q_{xx}$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|q_x\|_{L^2}^2 + \varepsilon\|q_{xx}\|_{L^2}^2 \\
&= \gamma\int_I[(\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}]\tilde{p}_{xx}q_xdx + \gamma(\gamma-1)\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2q_xdx \\
&\quad - \gamma\bar{p}^{\gamma-1}\int_I\tilde{p}_xq_{xx} - 2\varepsilon\int_I qq_xq_{xx}dx. \qquad (4.11)
\end{aligned}
$$

Adding (4.9),(4.10) into (4.11), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\gamma}{2}\bar{p}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \|q_x\|_{L^2}^2 - \gamma\bar{p}^{\gamma-1}\int_I\tilde{p}q_xdx\right) \\
&+ \gamma\bar{p}^{\gamma-2}\|\tilde{p}_{xx}\|_{L^2}^2 + \gamma\bar{p}^\gamma\|q_x\|_{L^2}^2 + 2\varepsilon\|q_{xx}\|_{L^2}^2 \\
&= \gamma^2\bar{p}^{\gamma-1}\int_I(\tilde{p}+\bar{p})^{\gamma-1}(\tilde{p}_x)^2dx + 2\gamma\int_I[(\tilde{p}+\bar{p})^{\gamma-1} - \bar{p}^{\gamma-1}]\tilde{p}_{xx}q_xdx \\
&+ 2\gamma(\gamma-1)\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2q_xdx - \gamma\bar{p}^{\gamma-2}\int_I(\tilde{p}q)_x\tilde{p}_{xx}dx - \gamma\bar{p}^{\gamma-1}\int_I(\tilde{p}q)_xq_xdx \\
&+ \varepsilon\gamma\bar{p}^{\gamma-1}\int_I\tilde{p}_xq_{xx}dx + 2\varepsilon\gamma\bar{p}^{\gamma-1}\int_I qq_x\tilde{p}_xdx - 4\varepsilon\int_I qq_xq_{xx}dx \\
&= \sum_{i=21}^{28} J_i. \qquad (4.12)
\end{aligned}
$$

Then using the similar method as in [37, lemma 3.3], we get the result, we omit the detail for brevity. The exponential decay rate of the perturbations can be proved by using exactly the same idea as in the proof of Lemma 3.7 through coupling together the energy inequalities (4.4)-(4.7). To simplify the presentation, we omit further details here, the finial estimate reads

$$
\|\tilde{p}(t)\|_{H^1}^2 + \|q(t)\|_{H^1}^2 \le \alpha_3 e^{-\beta_3 t}.
$$

for some positive constant α_3, β_3 which are independent of time. This completes the proof of Theorem 1.4. $\square$

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References

[1] J. Adler, *Chemotaxis in bacteria*, Science, 153 (1966), pp. 708–716.

[2] C. Deng and T. Li, *Well-posedness of a 3D parabolic-hyperbolic Keller-Segel system in the Sobolev space framework*, J Differential Equations, 257 (2014), pp. 1311–1332.

[3] J. Fan and K. Zhao, *Blow up criteria for a hyperbolic-parabolic system arising from chemotaxis*, J. Math. Anal. Appl., 394 (2012), pp. 687–695.

[4] M.A. Fontelos, A. Friedman, and B. Hu, *Mathematical analysis of a model for the initiation of angiogenesis*, SIAM J. Math. Anal., 33 (2002), pp. 1330–1355.

[5] J. Guo, J. Xiao, H. Zhao, and C. Zhu, *Global solutions to a hyperbolic-parabolic coupled system with large initial data*, Acta Math. Sci. Ser. B Engl. Ed., 29 (2009), pp. 629–641.

[6] C. Hao, *Global well-posedness for a multidimensional chemotaxis model in critical Besov spaces*, Z. Angew. Math. Phys., 63 (2012), pp. 825–834.

[7] Q. Hou, Z. Wang, and K. Zhao, *Boundary layer problem on a hyperbolic system arising from chemotaxis*, J. Differential Equations, 261 (2016), pp. 5035–5070.

[8] F. Huang, P. Marcati, and R. Pan, *Convergence to the Barenblatt solutions for the compressible Euler equations with damping and vacuum*, Arch. Ration. Mech. Anal., 176 (2005), pp. 1–24.

[9] F. Huang, R. Pan, and Z. Wang, *L^1 Convergence to the Barenblatt solution for compressible Euler equations with damping*, Arch. Ration. Mech. Anal., 200 (2011), pp. 665–689.

[10] H. Jin, J. Li, and Z. Wang, *Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity*, J. Differential Equations, 255 (2013), pp. 193–219.

[11] E.F. Keller and L.A. Segel, *Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol.*, 26 (1970), pp. 399–415.

[12] E.F. Keller and L.A. Segel, *Model for chemotaxis*, J. Theoret. Biol., 30 (1971), pp. 225–234.

[13] E.F. Keller and L.A. Segel, *Traveling bands of chemotactic bacteria: A theoretical analysis*, J. Theoret. Biol., 26 (1971), pp. 235–248.

[14] H.A. Levine and B.D. Sleeman, *A system of reaction diffusion equations arising in the theory of reinforced random walks*, SIAM J. Appl. Math., 57 (1997), pp. 683–730.

[15] H.A. Levine, B.D. Sleeman, and M. Nilsen-Hamilton, *A mathematical model for the roles of pericytes and macrophages in the initiation of angiogenesis. I. The role of protease inhibitors in preventing angiogenesis*, Math. Biosci., 168 (2000), pp. 71–115.

[16] D. Li, T. Li, and K. Zhao, *On a hyperbolic-parabolic system modeling chemotaxis*, Math. Models Methods Appl. Sci., 21 (2011), pp. 1631–1650.

[17] D. Li, R. Pan, and K. Zhao, *Quantitative decay of a hybrid type chemotaxis model with large data*, Nonlinearity, 28 (2015), pp. 2181–2210.

[18] H. Li and K. Zhao, *Initial boundary value problems for a system of hyperbolic balance laws arising from chemotaxis*, J. Differential Equations, 258 (2015), pp. 302–338.

[19] T. Li, R. Pan, and K. Zhao, *Global dynamics of a hyperbolic-parabolic model arising from chemotaxis*, SIAM J. Appl. Math., 72 (2012), pp. 417–443.

[20] T. Li and Z. Wang, *Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis*, SIAM J. Appl. Math., 70 (2009), pp. 1522–1541.

[21] T. Li and Z. Wang, *Nonlinear stability of large amplitude viscous shock waves of a hyperbolicparabolic system arising in chemotaxis*, Math. Models Methods Appl. Sci., 20 (2010), pp. 1967–1998.

[22] T. Li and Z. Wang, *Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis*, J. Differential Equations, 250 (2011), pp. 1310–1333.

[23] T. Li and Z. Wang, *Steadily propagating waves of a chemotaxis model*, Math. Biosci., 240 (2012), pp. 161–168.

[24] P.L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. I: Incompressible Models*, Oxford Lecture Ser. Math. Appl., Oxford University Press, Oxford, 1996.

[25] P.L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. II: Compressible Models*, Oxford Lecture Ser. Math. Appl., Oxford University Press, Oxford, 1998.

[26] Z. Luo, *Local existence of classical solutions to the two-dimensional viscous compressible flows with vacuum*, Commun. Math. Sci., 10 (2012), pp. 527–554.

[27] V. Martinez, Z. Wang, and K. Zhao, *Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology*, Indiana Univ. Math. J., 27 (2018), pp. 1383–1424.

[28] H. Othmer and A. Stevens, *Aggregation, blowup and collapse: The ABCs of taxis in reinforced random walks*, SIAM J. Appl. Math., 57 (1997), pp. 1044–1081.

[29] H. Peng, H. Wen, and C. Zhu, *Global well-posedness and zero diffusion limit of classical solutions to 3D conservation laws arising in chemotaxis*, Z. Angew. Math. Phys., 65 (2014), pp. 1167–1188.

[30] Y. Tao, L. Wang, and Z. Wang, *Large-time behavior of a parabolic-parabolic chemotaxis model with logarithmic sensitivity in one dimension*, Discrete Contin. Dyn. Syst. Ser. B., 18 (2013), pp. 821–845.

[31] D.-H. Wang and C. Yu, *Global weak solution and large-time behavior for the compressible flow of liquid crystals*, Arch. Ration. Mech. Anal., 204 (2012), pp. 881–915.

[32] Z. Wang and T. Hillen, *Shock formation in a chemotaxis model*, Math. Methods Appl. Sci., 31 (2008), pp. 45–70.

[33] Z. Wang, Z. Xiang, and P. Yu, *Asymptotic dynamics on a singular chemotaxis system modeling onset of tumor angiogenesis*, J. Differential Equations, 260 (2016), pp. 2225–2258.

[34] Z. Wang and K. Zhao, *Global dynamics and diffusion limit of a parabolic system arising from repulsive chemotaxis*, Commun. Pure Appl. Anal., 12 (2013), pp. 3027–3046.

[35] M. Zhang and C. Zhu, *Global existence of solutions to a hyperbolic-parabolic system*, Proc. Amer. Math. Soc., 135 (2006), pp. 1017–1027.

[36] Y. Zhang, Z. Tan, and M. Sun, *Global existence and asymptotic behavior of smooth solutions to a coupled hyperbolic-parabolic system*, Nonlinear Anal., 14 (2013), pp. 465–482.

[37] N. Zhu, Z R. Liu, v. Martinez and K. Zhao, *Global Cauchy problem of a system of parabolic conservation laws arising from a Keller-Segel type chemotaxis model*, SIAM J. Math. Anal. 50 (2018), pp. 5380–5425.

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