On the coloring of efl graph using colors equal to size of maximal clique

Prajnanaswaroopa S

Abstract

In this short note, we give a proof for the fact that the chromatic number of the EFL graph formed by the adjoining of k cliques such that any two cliques share at most one vertex is k.

1 Introduction

In graph theory, one of the famous conjectures is Erdos-Faber-Lovasz (EFL) conjecture. It states that, if $\boldsymbol{H}$ is a linear hypergraph consisting of n edges of cardinality n, then it is possible to color the edges of $\boldsymbol{H}$ with n colors so that no two edges with same color are incident with the same vertex. Reducing to the case of simple graphs, it is equivalent to the fact that the union of n pairwise edge-disjoint complete graphs with n vertices is n-colorable[8].

Infact in 1975, Erdos offered 50 USD and in 1981, offered 500 USD for the proof or disproof of the conjecture. Many people worked on this and have some results. Chang et al.[4] showed that for any simple hypergraph $\mathbf{H}$ on n vertices, the chromatic index of H is at most [1.5n-2]. Kahn [3] proved that the chromatic number of $\boldsymbol{H}$ is at most $n + o(n)$. Jackson et al. [1] showed that the conjecture is true when the partial hypergraph S of $\boldsymbol{H}$ determined by the edges of size at least three can be Δ_S-edge-colored and satisfies $\Delta_S \leq 3$. In particular, the conjecture holds when S is unimodular and $\Delta_S \leq 3$. Viji Paul and Germina [6] established the truth of the conjecture for all linear hypergraphs on n vertices and $\Delta(\mathbf{H}) \leq \sqrt{n + \sqrt{n} + 1}$. Sanchez-Arroyo [7] proved the conjecture is true for dense hypergraphs. Faber [5] proved that for fixed degree, there can be only finitely many counterexamples to EFL on both regular and uniform hypergraphs. Hegde et al. [2] gave a method for assigning colors to the graphs which satisfies the hypothesis of the EFL and every complete graphs has at most $\frac{n}{2}$ vertices of clique degree greater than one using symmetric latin squares and clique degrees of the vertices of the graph.

2 Use of Chromatic Polynomials

Let G be a graph consisting of k cliques C_i of size k and any two cliques share at most one vertex. This graph is called Erdos-Faber-Lovasz graph.
The set of shared vertices of the graph G is denoted by $C^{(2)} = \cup_{i \neq j\ , i,j \leq k} C_i \cap C_j$. We denote Λ as a graph formed by taking the cliques as vertices and two vertices adjacent in Λ iff the corresponding cliques meet at a single vertex in G. $L(\Lambda)'$ denotes the induced graph formed by the contact vertices. Also, we denote the falling factorial by $x_{(k)} = \prod_{i=1}^{k}(x - i - 1)$.

Proposition 1. *The induced subgraph formed by the contact vertices, $L(\Lambda)'$ has chromatic number at most k.*

Proof. Let us denote the clique vertex graph M, that is, the graph formed in which each clique is considered a vertex and two vertices are adjacent if they share a contact vertex. Now, we assume that the graph $L(\Lambda)'$ is connected, for the other case follows from the connected case. Let us consider the connected components of those contact vertices that have degree 2 (degree corresponds to the number of vertices incident on that contact vertex). Now, consider the subgraph of M that contains these contact vertices and call them M_i. The induced graph formed by the contact vertices in these components is in fact the line graphs of the corresponding M_is. As the maximum degree of these M_is is at most $m_i - 1$, where m_i be the number of cliques in the i th component, hence, Vizing's theorem, the required number of colors is at most m_i. Now, let us the consider the connected components formed by the induced graphs of the contact vertices with degree greater than 2 and call them L_i. Definitely, in thse components, the number of contact vertices will be less than the number of cliques, as for each contact vertices, we require at least 3 cliques, with each new contact vertex requiring at the least two extra cliques. Hence we can say that the chromatic number of these components cannot exceed l_i where $l_i$is the number of cliques in each component L_i. Thus, reasoning, the total number of colors required in coloring the full graph $L(\Lambda)'$ is at most the sum of colors of all the connected components above which equals the number of cliques k. $\square$

Theorem 2.1. *The graph G consisting of k cliques of order k such that any two cliques share at most one vertex is k colorable.*

Proof. We begin with x colors to start with. Let C_i denote the distinct cliques with i ranging from 1 to k. We observe that the chromatic polynomial of the graph so formed is actually

$$P_G(x) = P_{L(\Lambda)'}(x) \prod_{i=1}^{k} (x - |C_i \cap C^{(2)}|)_{(k - |C_i \cap C^{(2)}|)}$$

, where $P_{L(\Lambda)'}$ is the chromatic polynomial of the induced subgraph formed by the contact vertices of the cliques in the graph and $C^{(2)} = \cup_{i \neq j} C_i \cap C_j$, the set of contact vertices.

This is because, in forming the chromatic polynomial of the graph, we first color the vertices the induced subgraph of the graph formed by the contact vertices of the cliques, which is given by the chromatic polynomial of the induced subgraph, namely, $P_{L(\Lambda)'}$. Now, the remaining graph consists of independent subgraphs of cliques. The vertices which were a part of the contact of two cliques were already colored in the above process of coloring the induced subgraph formed by the contact vertices. Therefore, the remaining portion of the each individual clique C_i can be colored in $(x - |C_i \cap C^{(2)}|)_{(k-|C_i \cap C^{(2)}|)}$. Now, multiplying the above chromatic polynomials of independent parts gives us that the chromatic polynomial of the graph to be

$$P_{L(\Lambda)'} \prod_{i=1}^{k} (x - |C_i \cap C^{(2)}|)_{(k-|C_i \cap C^{(2)}|)}$$

.

Now, by the previous proposition, the chromatic number of $l(\Lambda)'$ is at most k. This, along with the fact that the clique number of the graph G is k implies that the value of the chromatic polynomial $P_G(x)$ is nonzero at $x = k$, or in other words, the graph G is k colorable.

□

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