THE CONCEPT OF STRING

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Abstract. In this short note we introduce and develop the concept of a string. We examine the various elementary properties of a string. Further, we relate the concept of the string to the concept of continuity of a function. In fact we prove that the two are loosely connected.

1. Introduction and concept

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a real valued function. Then we say f is a string on $[a, b]$ with oscillation N if there exist some smallest $N > 0$ such that for any $x_i \in [a, b]$

$$f(x_i) = x_i \pm \gamma_i$$

for some $0 < \gamma_i \leq N$. A string with oscillation N on $[a, b]$ is said to taut if there exist some $x_0 \in [a, b]$ such that $f : [x_0 + a, x_0 + b] \longrightarrow \mathbb{R}$ is a string with oscillation $R > N$. It is said to swing to the right if $f : [a + x_1, b + x_1] \longrightarrow \mathbb{R}$ is still a string with oscillation N for any point $x_1 \in [a, b]$. Similarly it is said to swing to the left if $f : [a - x_1, b - x_1] \longrightarrow \mathbb{R}$ is still a string with oscillation N for any $x_1 \in [a, b]$. We say it is stationary if $f(x) = x + N$ for any $x \in [a, b]$.

In this short note we introduce and develop the concept of the string. We establish some elementary and analytic properties inherent in strings. We further relate the concept of the string to various notions of continuity.

2. Analytic properties of a string

Proposition 2.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a string on $[a, b]$, then f is uniformly bounded on $[a, b]$.*

Proof. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a string with oscillation N on $[a, b]$. Then it follows that for all $x \in [a, b]$

$$f(x) = x \pm \gamma$$

for $0 < \gamma \leq N$. It follows that $|x| - \gamma \leq f(x) \leq |x| + \gamma$ if and only if $a - N \leq f(x) \leq N + b$ and the result follows immediately. □

Proposition 2.2. *Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a string with oscillation N on $[a, b]$ that swings left to right. Then f is also a string with oscillation N on $[2a - b, 2b - a]$.*

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Proof. Let f be a string with oscillation N on $[a,b]$. Suppose f swings both left and right, then it follows that for any $\alpha \in [a,b]$, we have that $f:\mathbb{R} \longrightarrow \mathbb{R}$ is a string on $[a-\alpha, b-\alpha]$ and $[a+\alpha, b+\alpha]$ with the same oscillation N. Since $[a,b] \subset \mathbb{R}^{+}$, choose $\alpha = b-a$, then it follows that f is a string on $[2a-b, a]$, $[b, 2b-a]$ and $[a,b]$ with oscillation N. It follows that $f:\mathbb{R}^{+} \longrightarrow \mathbb{R}$ is a string on $[2a-b,a]\cup[a,b]\cup[b,2b-a]$ with oscillation N, and the result follows immediately. □

Consider any real valued function $f:[a,b] \longrightarrow \mathbb{R}$. Then we say f is approximately linear if for any $x_1, x_2, \ldots x_n \in [a,b]$

$$f(\sum_{i=1}^{n} x_i) \approx \sum_{i=1}^{n} f(x_i).$$

Similarly, we say $f:[a,b] \longrightarrow \mathbb{R}$ is an approximate homomorphism if

$$f(\prod_{1}^{n} x_i) \approx \prod_{i=1}^{n} f(x_i).$$

Next we prove that strings with sufficiently small oscillations are approximate linear and homomorphisms.

Proposition 2.3. *Let $\epsilon > 0$ and let $f:\mathbb{R} \longrightarrow \mathbb{R}$ be a string with oscillation N on $[a+\epsilon, b+\epsilon]$. If $N \approx 0$, then f is approximate linear and homomorphism.*

Proof. Let $f:\mathbb{R} \longrightarrow \mathbb{R}$ be a string with oscillation N on $[a+\epsilon, b+\epsilon]$ for any $\epsilon > 0$. Specify $x_1, x_2, \ldots, x_n \in [a+\epsilon, b+\epsilon]$, then it follows that

$$\begin{aligned} f(\sum_{i=1}^{n} x_i) &= f(x_1 + x_2 + \cdots + x_n) \\ &= \left(\sum_{i=1}^{n} x_i\right) \pm \gamma \end{aligned}$$

for some $0 < \gamma \leq N$. By taking $N \approx 0$, then $\gamma \approx 0$ and it follows that

$$f(\sum_{i=1}^{n} x_i) \approx \sum_{i=1}^{n} x_i \approx \sum_{i=1}^{n} f(x_i),$$

thereby showing that f is approximate linear. The proof for approximate homomorphism follows in a similar fashion. □

Remark 2.1. Next we relate the concept of the string to the concept of injectivity of functions on closed intervals of the form $[a,b] \subset \mathbb{R}$.

Proposition 2.4. *Let $f:\mathbb{R} \longrightarrow \mathbb{R}$ be a string on $[a,b]$ with oscillation N. If f is stationary, then f is injective.*

Proof. Let $f;\mathbb{R} \longrightarrow \mathbb{R}$ be a string on $[a,b]$ with oscillation N, and specify any $x_1, x_2 \in [a,b]$, then it follows that for $f(x_1) = f(x_2)$ and for f stationary, we have

$$x_1 + N = x_2 + N$$

and the result follows immediately. □

3. Elementary properties of a string

In the following section we study various elementary properties of a string. We study various context for which the notion of a string is preserved.

Theorem 3.1. *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be a strings with oscillations N_1 and N_2 respectively on $[a, b]$. Then $\frac{f+g}{2} : \mathbb{R} \longrightarrow \mathbb{R}$ is a string with oscillation $\frac{N_1+N_2}{2}$ on $[a, b]$.*

Proof. Suppose $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are strings with oscillations N_1 and N_2 respectively on $[a, b]$. Then it follows that for each $x_i \in [a, b]$

$$f(x_i) = x_i \pm \gamma_i$$

and

$$g(x_i) = x_i \pm \alpha_i$$

where $0 < \gamma_i \leq N_1$, $0 < \alpha_i \leq N_2$. It follows that

$$\begin{aligned}(f + g)(x_i) &= f(x_i) + g(x_i)\\ &= (x_i \pm \gamma_i) + (x_i \pm \alpha_i)\\ &= 2x_i + (\pm\gamma_i) + (\pm\alpha_i),\end{aligned}$$

and the result follows immediately. □

Theorem 3.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be string with oscillation N_1 on $[c+\epsilon, d+\epsilon]$ for any $\epsilon > 0$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be string with oscillation N_2 on $[c, d]$. Then the composites $f \circ g : \mathbb{R} \longrightarrow \mathbb{R}$ is also a string with oscillation $N_1 + N_2$ on $[c, d]$.*

Proof. Let $\epsilon > 0$ and suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a string with oscillation N_1 on $[c + \epsilon, d + \epsilon]$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a string with oscillations N_2 on $[c, d]$. It follows that

$$\begin{aligned}f \circ g(x_i) &= f(g(x_i))\\ &= f(x_i \pm \gamma_i)\\ &= x_i \pm \gamma_i \pm \alpha_i.\end{aligned}$$

where $0 < \gamma_i \leq N_1$ and $0 < \alpha_i \leq N_2$. It follows from this relation that $f \circ g$ is also a string with oscillation $N_1 + N_2$ on $[c, d]$. □

Remark 3.3. It is important to notice that the concatenation of two string is not a string. In other words the very notion of a string is not preserved under addition.

Proposition 3.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a string with oscillation N_1 on $[a, b]$ that tauts. If for any $\epsilon > 0$ $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a string with oscillation N_2 on $[\alpha + a + \epsilon + N_1, \alpha + b + \epsilon + N_1]$ for any $\alpha \in [a, b]$, then the composite $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$ tauts on $[\alpha + a, b + \alpha]$.*

Proof. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be strings with oscillations N_1 and N_2 respectively. If f tauts on $[a, b]$, then it follows that there exist some $x_0 \in [a, b]$ such that for some $y_0 \in [x_0 + a, x_0 + b]$, we have

$$f(y_0) = y_0 \pm \gamma_1,$$

for $\gamma_1 = N_1 + \beta$ for some $\beta > 0$. Since $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a string with oscillation N_2 on $[\alpha + a + \epsilon + N_1, \alpha + b + \epsilon + N_1]$ for any $\alpha \in [a, b]$, it follows that

$$\begin{aligned} g \circ f(y_0) &= g(y_0 \pm \gamma_1) \\ &= y_0 \pm \gamma_1 \pm \gamma_2 \end{aligned}$$

for $0 < \gamma_2 \leq N_2$. Since $\gamma_1 + \gamma_2 \leq N_2 + N_1 + \beta$, by Theorem 3.2, the result follows immediately. □

Remark 3.4. Proposition 3.1 reinforces the notion that decoupling part of a string formed by piecing together several strings decouples the longer string.

4. String and continuity

The notion of a string and the notion of continuity of any function $f : [a, b] \longrightarrow \mathbb{R}$ are very disparate. But we can make a loose connection if we impose some Lipchizt condition on the continuity of f. In quest of launching such a result, we review therefore the following definition [2].

Definition 4.1. Let $f : [a, b] \longrightarrow \mathbb{R}$, then we say f is lipchizt continous on the support $[a, b]$ with lipchizt constant C_0 if f is continous, and for any $x_1, x_2 \in [a, b]$, then

$$|f(x_1) - f(x_2)| \leq C_0|x_1 - x_2|.$$

Remark 4.2. Definition 4.1 tells us that once a function is lipchitz continous, then the deviation of the conductors of any two points in the support of f can be controlled by the deviation of the points themselves in the support.

Theorem 4.3. *Let $f : [a, b] \subset \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be lipchitz continous with lipchitz constant C_0. Specify $\sup\{f(x) : x \in [a, b]\} = M$ and $\inf\{f(x) : x \in [a, b]\} = N$. Then $\frac{1}{C_0} f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a string with oscillation $\frac{M}{C_0} + b$ on $[a, b]$.*

Proof. Let $f : [a, b] \longrightarrow \mathbb{R}^+$ and specify $x_1 \in [a, b]$ for any $x \in [a, b]$ then

$$|f(x) - f(x_1)| \leq C_0|x - x_1|,$$

since f is lipchizt continous. It follows that

$$\frac{f(x_1)}{C_0} - |x_1| \leq \frac{f(x)}{C_0} - x \leq \frac{f(x_1)}{C_0} + |x_1|.$$

It follows from this relation

$$\frac{\mathrm{Inf}(f(x))}{C_0} - b < \frac{f(x)}{C_0} - x < \frac{\sup(f(x))}{C_0} + b$$

and it follows that $\frac{f}{C_0}(x) = x \pm \gamma$ for $0 < \gamma \leq \frac{\sup(f(x))}{C_0} + b$, thereby ending the proof. □

Theorem 4.3 tells us that the class of lipchitz continous function constitute a good class of strings. Given that the oscillation of any lipchitz continous function depends greatly on the lipchitz constant, it follows that if the lipchitz constant of a liptschitz continous function is small enough then it must have somewhat large oscillation. On the other hand, if the lipchitz constant is somewhat large, then the oscillation of the string it represents must be small enough.

Recall that a function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be strongly continous if for any $\epsilon > 0$ there exist an $\delta > 0$ so that for

$$\sum_{i=1}^{n} |y_i - x_i| < \delta$$

then

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \epsilon$$

for any $y_i, x_i \in [a, b]$ (See [1]). Next we prove that any string can be made to be strongly continous by making their oscillation negligible.

Theorem 4.4. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a string with oscillation N on $[a, b]$. If $\sum\limits_{i=1}^{n} N \approx 0$, then f is strongly continous.*

Proof. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a string with oscillation N. Let $\epsilon > 0$ and choose $\delta = \epsilon - \alpha$ for any $\alpha \approx 0$, then for any $x_i, y_i \in [a, b]$ such that

$$\sum_{i=1}^{n} |x_i - y_i| < \delta$$

then

$$\begin{aligned}\sum_{i=1}^{n} |f(x_i) - f(y_i)| &= \sum_{i=1}^{n} |x_i \pm \gamma_i + y_i \pm \beta_i| \\ &\leq \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} \gamma_i + \sum_{i=1}^{n} \beta_i \\ &< \delta + \sum_{i=1}^{n} \gamma_i + \sum_{i=1}^{n} \beta_i.\end{aligned}$$

It follows that

$$\sum_{i=1}^{n} \gamma_i + \sum_{i=1}^{n} \beta_i \approx 0,$$

for $0 < \alpha_i, \beta_i \leq N$ and hence

$$\sum_{i=1}^{n} |f(x_i) - f(y_i)| < \epsilon,$$

thereby ending the proof. □

Corollary 4.1. Let $f : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a string with sufficiently small oscillation, then f is continous.

Remark 4.5. Next we prove a converse of Theorem 4.3, but with an extra condition that the string in question has a sufficiently small oscillation.

Proposition 4.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a string with sufficiently small oscillation. Then f is lipchitz continous with lipchitz constant $C_0 = 2$.*

Proof. Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is a string with oscillation $N \approx 0$. Then it follows by Corollary 4.1 that f is continous. Specify $x_i, x_j \in [a, b]$, then it follows that

$$\begin{aligned} |f(x_i) - f(x_j)| &= |x_i \pm \gamma_i - (x_j \pm \gamma_j)| \\ &= |x_i - x_j \pm \gamma_i \pm \gamma_j| \\ &\leq |x_i - x_j| + \gamma_i + \gamma_j \\ &\approx |x_i - x_j|, \end{aligned}$$

and the result follows immediately. □

[1].

References

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