LIMITED POLYNOMIALS

T. AGAMA

Abstract. In this paper we study a particular class of polynomials. We study the distribution of their zeros, including the zeros of their derivatives as well as the interaction between this two. We prove a weak variant of the sendov conjecture in the case the zeros are real and are of the same sign.

1. Introduction and motivation

The sendov conjecture is the assertion that any complex coefficient polynomial $P_n(x)$ of degree $n \geq 2$ with sufficiently small zeros must lie in the same unit disk with some zero of $P_n'(x)$. More formally if $|a_i| < 1$ such that $P_n(a_i) = 0$, then there exist some b_k with $P_n'(b_k) = 0$ such that

$$|a_i - b_k| < 1.$$

There has been and still is a flurry of research devoted to this problem and manifestly the current literature contains dozens of papers just for the problem. There has really been substantive progress ever since it was posed. For instance, It has been shown in [5] that the conjecture holds for zeros near the unit circle. In [1], the conjecture has been verified for degree at most six. This was improved further to polynomials of degree at most seven in [2] and polynomials of degree at most eight in [4]. The best result thus far concerning sendov conjecture is found in [3], where it was verified to hold for sufficiently large degree polynomials.
In the sequel, we will study a particular class of complex polynomial. It turns out that such polynomial almost satisfies the sendov conjecture.

2. Limited complex-valued polynomials

In this section we introduce the concept of limited polynomials. We study various settings under which polynomials of this forms are preserved.

Definition 2.1. Let $P_n(z)$ be a complex-valued polynomial of degree n and let $\mathcal{Z}(P_n(z)) = \{a_1, a_2, \cdots, a_n\}$ be the set of zeros of $P_n(z)$. By the measure of $P_n(z)$ denoted $\mathcal{M}(P_n(z))$, we mean the value

$$\mathcal{M}(P_n(z)) = \prod_{i=1}^{n} |a_i|.$$

Date: July 13, 2020.
2000 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.
Key words and phrases. Sendov; critical; zeros; limited.

We say $P_n(z)$ is ϵ-limited if $\mathcal{M}(P_n(z)) < \epsilon$, for some $\epsilon > 0$, where $\epsilon = \sup\bigg(M(P_n(z)\bigg)$.

The concept of limited polynomial hinges essentially on the distribution of the zeros of a polynomial. This in some sense gives some information about the distribution of the zeros of their derivatives. In particular, the very notion that a polynomial is 1-limited, for instance, is quite suggestive in it's own right. In such a case, we could suspect that all the zeros of the polynomial in question are small, or almost all the zeros are somewhat large and the few exceptional ones are incredibly small with the tendency to absorb the large zeros upon taking their product. Next we examine some elementary properties of the concept of limited polynomials. We examine various settings under which this concept is preserved in the following sequel.

3. **Properties of limited polynomials**

In this section we examine some properties of limited polynomials. We examine the various settings under which this concept is preserved.

Proposition 1. *Let $P(z)$ and $Q(z)$ be any ϵ and δ -limited polynomials respectively. If $\mathcal{Z}(P(z)) \cap \mathcal{Z}(Q(z)) = \emptyset$, where $\mathcal{Z}(P(z))$ and $\mathcal{Z}(Q(z))$ are the set of zeros of $P(z)$ and $Q(z)$, respectively, then the product $P(z)Q(z)$ is $\epsilon\delta$-limited.*

Proof. Specify two limited polynomials $P(z)$ and $Q(z)$, where $P(z)$ is ϵ-limited and $Q(z)$ is δ-limited with zeros $\mathcal{Z}(P(z)) = \{a_1, a_2, \ldots a_n\}$ and $\mathcal{Z}(Q(z)) = \{b_1, b_2 \ldots, b_m\}$, then it follows that

$$\prod_{i=1}^{n} |a_i| < \epsilon \quad \text{and} \quad \prod_{j=1}^{m} |b_j| < \delta$$

for some $\epsilon, \delta > 0$, where $\epsilon = \sup\bigg(M(P(z)\bigg)$ and $\delta = \sup\bigg(M(Q(z)\bigg)$. Since the zeros of the product $P_n P_m$ is the union of the zeros of P_n and P_m. That is $\mathcal{Z}(P(z)Q(z)) = \mathcal{Z}(P(z)) \cup \mathcal{Z}(Q(z)) := \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots b_m\}$. It follows that

$$\begin{aligned}\mathcal{M}(P(z)Q(z)) &= \prod_{i=1}^{n} |a_i| \prod_{j=1}^{m} |b_j| \\ &< \epsilon\delta\end{aligned}$$

and the result follows immediately. □

Proposition 2. *Let $\lambda \in \mathbb{C}$ and $P(x)$ be an ϵ-limited polynomial. Then $\lambda P(x)$ is also ϵ-limited.*

Proposition 3. *Let $P(x) = \prod_{i=1}^{n}(x - a_i)$ be an ϵ-limited polynomial. Then the polynomial $Q(x) = \prod_{i=1}^{n}(x - \overline{a}_i)$ is also ϵ-limited. In particular, $P(x)Q(x)$ is also ϵ^2-limited polynomial.*

Proof. The result follows follows by applying Proposition 1. □

Proposition 4. *Let* $P(x) = \prod_{i=1}^{n}(x - a_i)$ *be an* ϵ*-limited polynomial of degree* $n \geq 2$*. If* $a_i = \lambda_j b_j$*, then* $Q(x) = \prod_{j=1}^{n}(x - b_j)$ *is* $\left(\frac{\epsilon}{\prod_{j=1}^{n}|\lambda_j|}\right)$*-limited.*

Proof. Suppose $P(x) = \prod_{i=1}^{n}(x - a_i)$ be an ϵ-limited polynomial, and let $a_i = \lambda_j b_j$, then it follows that

$$\begin{aligned}\prod_{j=1}^{n}|b_j| &= \frac{\prod_{i=1}^{n}|a_i|}{\prod_{j=1}^{n}|\lambda_j|}\\ &< \frac{\epsilon}{\prod_{j=1}^{n}|\lambda_j|},\end{aligned}$$

since $P_n(x)$ is ϵ-limited. □

4. **Critical points of limited polynomials**

In this section we prove some preliminary results concerning the existence of zeros of limited polynomials with real zeros and the zeros of their derivatives, as well as the interplay between the two concerning their local and global distributions. We begin with the following elementary inequalities and identities. These two forms a rich tool-box for further studies in the sequel.

Lemma 4.1. *Let* $A, B \in \mathbb{R}^+$*. If* $AB < 1$*, then* $\min\{A, B\} < 1$*.*

Proof. Suppose $A, B \in \mathbb{R}^+$. Then using the elementary relation, we have

$$\begin{aligned}(A + B)^2 &= A^2 + B^2 + 2AB\\ &< A^2 + B^2 + 2.\end{aligned}$$

This inequality can only be valid if $\min\{A, B\} < 1$, thereby ending the proof. □

Lemma 4.2. *Let* $P(x) = \prod_{i=1}^{n}(x - a_i)$ *be a monic polynomial of degree* n*, with* $a_i \in \mathbb{R}^+$*, then there exist some* s_j *for* $j = 1, 2 \ldots n - 1$ *such*

$$\begin{aligned}\prod_{i=1}^{n}(x - a_i) &= (x - a_j)^n + s_{n-1}(x - a_j)^{n-1} + s_{n-2}(x - a_j)^{n-2} + \cdots + s_1(x - a_j)\\ &= \sum_{k=1}^{n} s_k(x - a_j)^k,\end{aligned}$$

where $a_j = \min\{a_i\}_{i=1}^{n}$ *and* $|s_j| < \prod_{\substack{i=1\\ i\neq j}}^{n}|a_i|$*.*

Proof. Specify the monic polynomial

$$P(x)=\prod_{i=1}^{n}(x-a_i).$$

Now choose $a_j=\min\{a_i\}_{i=1}^{n}$. Then we can write

$$\begin{aligned}\prod_{i=1}^{n}(x-a_i)&=(x-a_1)(x-a_2)\cdots(x-a_j)\cdots(x-a_n)\\&=(x-a_j)(x-a_j-r_1)(x-a_j-r_2)\cdots(x-a_j-r_{n-2})(x-a_j-r_{n-1})\\&=(x-a_j)((x-a_j)-r_1)((x-a_j)-r_2)\cdots((x-a_j)-r_{n-2})((x-a_j)-r_{n-1})\\&=\sum_{k=1}^{n}s_k(x-a_j)^k\end{aligned}$$

thereby establishing the relation. □

Remark 4.3. Next we prove an analogue of Lemma 4.2.

Lemma 4.4. *Let* $P(x)=\prod\limits_{i=1}^{n}(x+a_i)$ *be a monic polynomial of degree* n*, with* $a_i\in\mathbb{R}^+$*, then there exist some* s_j *for* $j=1,2\ldots n-1$ *such*

$$\begin{aligned}\prod_{i=1}^{n}(x+a_i)&=(x+a_j)^n+s_{n-1}(x+a_j)^{n-1}+s_{n-2}(x+a_j)^{n-2}+\cdots+s_1(x+a_j)\\&=\sum_{k=1}^{n}s_k(x+a_j)^k,\end{aligned}$$

where $a_j=\max\{a_i\}_{i=1}^{n}$ *and* $|s_j|<|a_j|^n$*.*

Proof. Specify the monic polynomial

$$P(x)=\prod_{i=1}^{n}(x+a_i).$$

Now choose $a_j=\max\{a_i\}_{i=1}^{n}$. Then we can write

$$\begin{aligned}\prod_{i=1}^{n}(x+a_i)&=(x+a_1)(x+a_2)\cdots(x+a_j)\cdots(x+a_n)\\&=(x+a_j)(x+a_j-r_1)(x+a_j-r_2)\cdots(x+a_j-r_{n-2})(x+a_j-r_{n-1})\\&=(x+a_j)((x+a_j)-r_1)((x+a_j)-r_2)\cdots((x+a_j)-r_{n-2})((x+a_j)-r_{n-1})\\&=\sum_{k=1}^{n}s_k(x+a_j)^k\end{aligned}$$

thereby establishing the relation. □

Remark 4.5. By leveraging Lemma 4.1 and Lemma 4.2, we prove a weak variant of the sendov conjecture. Before then we launch the following definition.

Definition 4.6. Let $P(x) := \prod_{i=1}^{n}(x-a_i)$ be a polynomial of degree n with $a_i > 0$. Then by the local expansion of $P(x)$ we mean the finite sum of the form

$$P(x) = \sum_{k=1}^{n} s_k (x-a_j)^k,$$

where $a_j = \min\{a_i\}_{i=1}^{n}$ and $|s_j| < \prod_{\substack{i=1\\i\neq j}}^{n} |a_i|$. We call s_j for $j = 1, \ldots n-1$ the index of expansion.

Theorem 4.7. *Let $P(x)$ be a monic polynomial with real coefficients with index $|s_t| = \frac{1}{t}$ for $t = 1 \ldots, n-1$. Let $\mathcal{Z}(P(x)) = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{R}^+$ and $\mathcal{C}(P(x)) = \{b_1, b_2, \ldots, b_{n-1}\}$ be the set of zeros and the set of critical points respectively. If $\frac{P(x)}{x-a_j}$ is 1-limited, then for each b_i*

$$|a_j - b_i| < 1$$

where $a_j = \min\{a_i\}_{i=1}^{n}$.

Proof. Specify the monic polynomial of degree $n \geq 2$ with real coefficients, given by

$$P(x) = \prod_{i=1}^{n}(x-a_i).$$

Then by Lemma 4.2, we have the decomposition

$$\prod_{i=1}^{n}(x-a_i) = \sum_{k=1}^{n} s_k (x-a_j)^k,$$

where $a_j = \min\{a_i\}_{i=1}^{n}$ and $|s_j| < \prod_{\substack{i=1\\i\neq j}}^{n} |a_i|$. It follows that

$$\begin{aligned}
\frac{dP(x)}{dx} &= \sum_{k=1}^{n} k s_k (x-a_j)^{k-1} \\
&= \sum_{k=2}^{n} k s_k (x-a_j)^{k-1} + s_1 \\
&= (x-a_j)\left(\sum_{k=2}^{n} k s_k (x-a_j)^{k-2}\right) + s_1.
\end{aligned}$$

By setting $\frac{dP(x)}{dx} = 0$, it follows that

$$\left|(x-a_j)\left(\sum_{k=2}^{n} k s_k (x-a_j)^{k-2}\right)\right| = |s_1|.$$

We remark that the values of x that satisfies this relation are the zeros of the polynomial $P_n'(x)$. That is, for each $b_i \in \mathcal{C}(P(x)) = \{b_1, b_2, \ldots, b_{n-1}\}$ the relation is valid

$$\Bigg|(b_i - a_j)\bigg(\sum_{k=2}^{n} ks_k(b_i - a_j)^{k-2}\bigg)\Bigg| = |s_1|.$$

Since $\frac{P(x)}{x-a_j}$ is 1-limited, it follows that $|s_j| < \prod_{\substack{i=1\\i\neq j}}^{n} |a_i| < 1$ for each $j = 1, \ldots n-1$ so that

$$|b_i - a_j|\Bigg|\bigg(\sum_{k=2}^{n} ks_k(b_i - a_j)^{k-2}\bigg)\Bigg| < 1.$$

Since $|s_k| = \frac{1}{k}$, it follows by applying Lemma 4.1 that

$$|b_i - a_j| < 1.$$

For suppose $|b_i - a_j| \geq 1$, then it follows that

$$\begin{aligned} 1 &< |b_i - a_j|\Bigg|\bigg(\sum_{k=2}^{n} ks_k(b_i - a_j)^{k-2}\bigg)\Bigg| \\ &< 1, \end{aligned}$$

which is absurd. That is, all the critical values are sufficiently close to the least zero of the polynomial, thereby ending the proof. □

Remark 4.8. Theorem 4.7 can be considered as a weak variant of the result of the real case of the sendov conjecture. The only compromise in this case is the assumption that all the zeros are positive, in which case we have found that indeed the smallest zero of $P(x)$ must be close to all the zeros of the polynomial $P'(x)$. A similar approach could be adapted for the case all the zeros are negative. Next we prove a result that in some sense exposes the distribution of the zeros of limited polynomials.

Theorem 4.9. *Let $P(x) = \prod_{i=1}^{n}(x - a_i)$ be a polynomial of degree n for $n \geq 2$, with $a_i \in \mathbb{R}^+$ and $a_i > 0$. If $\frac{P(x)}{x-a_j}$ is ϵ-limited, where $a_j = \min\{a_i\}_{i=1}^{n}$, then*

$$\sum_{s=1}^{n}\left|\frac{d^s P(a_j)}{dx^s}\right| < \epsilon\sqrt{2\pi}\sum_{k=1}^{n} e^{-k}k^{k+\frac{1}{2}}.$$

Proof. Let $P(x) = \prod_{i=1}^{n}(x - a_i)$ be a polynomial of degree n, with $a_i > 0$, then by definition 4.6 we can write

$$P(x) = \sum_{k=1}^{n} s_k(x - a_j)^k,$$

where $a_j = \min\{a_i\}_{i=1}^n$ and $|s_j| < \prod_{\substack{i=1 \\ i \neq j}}^n |a_i|$. It follows that

$$\begin{aligned}\frac{dP(x)}{dx} &= \sum_{k=1}^n k s_k (x-a_j)^{k-1} \\ &= \sum_{k=2}^n k s_k (x-a_j)^{k-1} + s_1.\end{aligned}$$

It follows that $\frac{dP(a_j)}{dx} = s_1$. Again, by taking the second derivative $P''(x)$, we have

$$\begin{aligned}\frac{d^2P(x)}{dx^2} &= \sum_{k=2}^n k(k-1) s_k (x-a_j)^{k-2} \\ &= \sum_{k=3}^n k(k-1) s_k (x-a_j)^{k-2} + 2s_2.\end{aligned}$$

Thus it follows that $\frac{d^2P(a_j)}{dx^2} = 2s_2$. By induction, it follows that

$$\begin{aligned}\sum_{s=1}^n \left|\frac{d^sP(a_j)}{dx^s}\right| &= \sum_{k=1}^n |k! s_k| \\ &\leq \sum_{k=1}^n k! |s_k| \\ &< \epsilon \sum_{k=1}^n k!\end{aligned}$$

since $\frac{P(x)}{x-a_j}$ is ϵ-limited. Applying stirlings formula we can write

$$\sum_{s=1}^n \left|\frac{d^sP(a_j)}{dx^s}\right| < \epsilon\sqrt{2\pi}\sum_{k=1}^n e^{-k} k^{k+\frac{1}{2}}.$$

The result follows immediately from the relation. □

The above inequality tells a lot about the relationship between the zeros of a given polynomial $P(x)$ and the higher order derivatives $P^k(x)$ for $k = 1, \ldots n$ in terms of the distribution. Indeed for any fixed n, by shrinking $\epsilon > 0$ to be very close zero, then it follows that the least zeros of $P(x)$ are sufficiently close to the zeros of each $P^k(x)$. Conversely, If we take $\epsilon > 0$ some what large for a fixed n then certainly, the least zero of $P(x)$ is far from at least one of the zeros of $P'(x)$. This property is archetypal of very rare class of polynomials of which limited polynomials is a sub-class. The ensuing result is a strengthening of Theorem 4.7.

Corollary 1. Let $P(x) = \prod_{i=1}^n (x - a_i)$ be a polynomial of degree n for $n \geq 2$, with $a_i > 0$ and let $\epsilon > 0$ be arbitrary. If $\frac{P_n(x)}{x-a_j}$ is ϵ-limited, where $a_j = \min\{a_i\}_{i=1}^n$, then $|a_j - b_{ik}| < \delta$, for any $\delta > 0$ and $P^k(b_{ik}) = 0$ $(i = 1, \ldots, n)$.

Proof. The result follows by varying the magnitude of ϵ in Theorem 4.9, since ϵ is arbitrary. □

Corollary 1 reveals quite well that the zeros of any real coefficient ϵ-limited polynomial $P_n(x)$ can be made close to the zeros of their derivative $P_n'(x)$ by shrinking the size of ϵ. In such a case we could take

$$\epsilon = \frac{1}{\sqrt{2\pi}\sum\limits_{k=1}^{n} e^{-k}(k+1)^{k+1}}.$$

Similarly the zeros of $P_n(x)$ can be made to be far from the zeros of $P_n'(x)$ by taking $\epsilon > 0$ to be smaller than

$$\sqrt{2\pi}\sum_{k=1}^{n} e^{-k}k^{k+\frac{1}{2}}.$$

Carefully tweaking ϵ this way gives some information about the local distribution of the zeros of limited polynomials and the derivatives.

Corollary 2. Let $P(x) = \prod\limits_{i=1}^{n}(x-a_i)$ be a polynomial of degree n for $n \geq 2$, with $a_i > 0$. If $\frac{P(x)}{x-a_j}$ is ϵ-limited, where $a_j = \min\{a_i\}_{i=1}^{n}$, then

$$\left|\sum_{\sigma:[1,n]\longrightarrow[1,n]}\prod_{\sigma(i)\neq\sigma(k)}(a_j - a_{\sigma(i)})_{n-1}\right| < \epsilon\frac{\sqrt{2\pi}}{e},$$

where $\prod\limits_{i\in[1,n]}(a_i)_k$ denotes the product of k terms with index belonging to the set $[1,n]$.

Proof. Let $P(x) = \prod\limits_{i=1}^{n}(x-a_i)$ be a polynomial of degree n for $n \geq 2$, with $a_i > 0$. Then we observe that we can write

$$P'(x) = \sum_{\sigma:[1,n]\longrightarrow[1,n]}\prod_{\sigma(i)\neq\sigma(k)}(x - a_{\sigma(i)})_{n-1},$$

where the sum runs over all permutations. By taking $n = 1$ in Theorem 4.9, we observe on the other hand that

$$\frac{dP(x)}{dx} < \epsilon\frac{\sqrt{2\pi}}{e}.$$

It follows that

$$\sum_{\sigma:[1,n]\longrightarrow[1,n]}\prod_{\sigma(i)\neq\sigma(k)}(a_j - a_{\sigma(i)})_{n-1} < \epsilon\frac{\sqrt{2\pi}}{e}$$

and the result follows immediately. □

Remark 4.10. Corollary 2 is quite suggestive. Roughly speaking, it tells us that once a polynomial is ϵ-limited for any $\epsilon > 0$, then by neccessity the zeros should be at most within ϵ distance away from the least zero.

Corollary 3. Let $P(x) = \prod_{i=1}^{n}(x - a_i)$ be a polynomial of degree n for $n \geq 2$, with $a_i > 0$. If $\frac{P(x)}{x-a_j}$ is ϵ-limited, where $a_j = \min\{a_i\}_{i=1}^{n}$, then

$$\sum_{s=0}^{n-1}\left|\frac{d^s}{dx^s}\Bigg(\sum_{\sigma:[1,n]\longrightarrow[1,n]}\prod_{\sigma(i)\neq\sigma(k)}(x-a_{\sigma(i)})_{n-1}\Bigg)(a_j)\right| < \epsilon\sqrt{2\pi}\sum_{k=1}^{n}e^{-k}k^{k+\frac{1}{2}},$$

where $\prod_{i\in[1,n]}(a_i)_k$ denotes the product of k terms with index belonging to the set $[1,n]$.

Proof. Let $P(x) = \prod_{i=1}^{n}(x - a_i)$ for $n \geq 2$ with $a_i > 0$, then we can write

$$P'(x) = \sum_{\sigma:[1,n]\longrightarrow[1,n]}\prod_{\sigma(i)\neq\sigma(k)}(x-a_{\sigma(i)})_{n-1}.$$

The result follows by applying Theorem 4.9. □

5. Distribution of zeros and critical points for complex limited-polynomials

In this section, we discuss how this method could in principle be used to give an analogue proof of the result in Theorem 4.7 for the case the polynomial in question is allowed to take at least some zeros from the complex plane $\mathbb{C}$.
Given any complex coefficient polynomial of the form

$$P(z) := \prod_{i=1}^{n}(z-a_i),$$

the first step is to basically convert any such complex coefficient polynomial to a polynomial with real coefficient of the form

$$R(x) = \prod_{i=1}^{n}(x-|a_i|)$$

Applying Theorem 4.7, we can then get information about the distribution of the zeros and the critical points of the polynomial. Given that the function

$$|\cdot| : \mathbb{R}^+\times\mathbb{R}^+ \subset \mathbb{C} \longrightarrow \mathbb{R}^+$$

is bijective, we can then pull back this information to the complex coefficient polynomial by observing that the zero $|a_j|$ corresponds to the zero a_j of the polynomial $P_n(x)$ and the number $c_i \in \mathbb{R}^+$ which is the zero of $R'(x)$ corresponds to some number $b \in \mathbb{R}^+\times\mathbb{R}^+ \subset \mathbb{C}$ such that $|b - a_j| < 1 + \epsilon$ for some small $\epsilon > 0$. A really good inverse theorem will suffice to improve the upper bound to

$$|b - a_j| < 1$$

and to show that b satisfies $P'(b) = 0$. [1]

1

.

References

1. Brown, Johnny E *On the Sendov conjecture for sixth degree polynomials.*, Proceedings of the American mathematical Society, vol. 175:4, 1991, pp. 939–946.
2. Borcea, Lulius *The Sendov conjecture for polynomials with at most seven distinct zeros*, Analysis, vol.16:2, Oldenbourge wissenschaftsverlag, 1996, pp. 137–160.
3. Dégot, Jérôme, *Sendov conjecture for high degree polynomials*, Proceedings of the American Mathematical Society, vol.142:4 , 2014, pp.1337–1349.
4. Brown, Johnny E and Xiang, Guangping *Proof of the Sendov conjecture for polynomials of degree at most eight*, Journal of mathematical analysis and applications, vol.232:2, Elsevier, 1991, 272–292.
5. Miller, Michael J *On Sendov's conjecture for roots near the unit circle*, Journal of mathematical analysis and applications, vol.175, academic press Inc ,1993, pp. 632–632.

Department of Mathematics, African Institute for Mathematical science, Ghana

E-mail address: `theophilus@aims.edu.gh/emperordagama@yahoo.com`