THE WARING RANK OF THE 3×3 PERMANENT

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Abstract. Let f be a homogeneous polynomial of degree d with coefficients in a field $\mathbb{F}$ satisfying $\operatorname{char}\mathbb{F} = 0$ or $\operatorname{char}\mathbb{F} > d$. The *Waring rank* of f is the smallest integer r such that f is a linear combination of r powers of $\mathbb{F}$-linear forms. We show that the Waring rank of the polynomial

$$x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 + x_2y_3z_1 + x_3y_1z_2 + x_3y_2z_1$$

is at least 16, which matches the known upper bound.

1. Introduction

Let $\mathbb{F}$ be a field. A *linear form* is a homogeneous polynomial of degree 1, where the word *homogeneous* means that the non-zero terms of a polynomial should have equal degrees. A homogeneous polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d has *Waring rank* r if there are linear forms $l_1, \ldots, l_r$ and $c_1, \ldots, c_r \in \mathbb{F}$ such that

$$f = c_1 l_1^d + \ldots + c_r l_r^d, \tag{1.1}$$

and r is the smallest number of terms in such a decomposition. If the characteristic of $\mathbb{F}$ was finite and did not exceed d, the monomial $m_d = x_1x_2 \ldots x_d$ would appear with zero coefficient in the dth power of any linear form, so the Waring rank of m_d would not be defined in this case. So it is natural to assume that either $\operatorname{char}\mathbb{F} = 0$ or $\operatorname{char}\mathbb{F} > d$, which guarantees the existence of a decomposition of the form (1.1).

Waring rank is NP-hard to compute [17], and its exact value is unknown for many families of polynomials as well as several relevant sporadic examples. This paper is devoted to the permanent, that is, the polynomial $\operatorname{per}_d \in \mathbb{F}[x_{11}, \ldots, x_{dd}]$ obtained from the determinant of a generic matrix (x_{ij}) by replacing every -1 coefficient with a $+1$. The Waring ranks of both the determinant and permanent grow exponentially with d, but the ratios between the known lower and upper bounds are still exponential [9, 11]. In fact, no exact value was known for the Waring rank of these polynomials except the trivial cases of $d = 1$ (the rank equals 1) and $d = 2$ (the rank equals 4). The main result of this paper is as follows.

Theorem 1.1. *The Waring rank of* per_3 *equals* 16.

In what follows, we denote the Waring rank of a polynomial f by $\mathrm{WR}(f)$, so the statement of Theorem 1.1 can be written as $\mathrm{WR}(\operatorname{per}_3) = 16$. We finalize the introductory section with a brief survey of prior work related to this theorem. A general upper bound on on the Waring rank of the permanent follows from the work of Glynn [9], who developed the formula of Ryser [15] and expressed per_d as a sum of 2^{d-1} products of linear forms. As explained by Ilten and Teitler [10], this implies $\mathrm{WR}(\operatorname{per}_3) \leqslant 16$ because the monomials of degree three have Waring ranks

2010 *Mathematics Subject Classification.* 15A03, 15A15, 15A69.
Key words and phrases. Waring rank, permanent, tensor decomposition.

not exceeding 4. In particular, an expression of the 3×3 permanent as a sum of three products of linear forms would improve the upper bound on the Waring rank of per_3 to 12, but Ilten and Teitler [10] showed that such an expression is impossible. So 16 stood as the best known upper bound for $\mathrm{WR}(\mathrm{per}_3)$ until now, and Theorem 1.1 shows that this bound is tight.

Concerning the lower bounds, the inequality

$$\mathrm{WR}(\mathrm{per}_d) \geqslant \frac{1}{2}\binom{2d}{d}$$

was proved by Shafiei [16] using the result of Ranestad and Schreyer [14], and the $d = 3$ case gives $\mathrm{WR}(\mathrm{per}_3) \geqslant 10$. Landsberg and Teitler [13] used a lower bound based on the singularities of the hypersurface of a given polynomial and proved that $\mathrm{WR}(\mathrm{per}_3) \geqslant 12$. Farnsworth [6] improved this bound further to $\mathrm{WR}(\mathrm{per}_3) \geqslant 14$; his method was based on Koszul-Young flattenings and implied the same lower bound for the border Waring rank of per_3 as well. Boij and Teitler [2] showed that the so-called *symmetric cactus rank* of per_3 is at least 14, which gave an alternative proof of the inequality $\mathrm{WR}(\mathrm{per}_3) \geqslant 14$. Conner, Gesmundo, Landsberg, Ventura, see Theorem 2.1 in [5], showed that the tensor corresponding to per_3 has border rank at least 15, which implies $\mathrm{WR}(\mathrm{per}_3) \geqslant 15$ and hence

$$15 \leqslant \mathrm{WR}(\mathrm{per}_3) \leqslant 16.$$

Whether the Waring rank of per_3 is 15 or 16 remained open until now.

Our approach to Theorem 1.1 can be adapted to give the inequality $\mathrm{WR}(\det_3) \geqslant 16$ as well, but we do not give the details because it is already known that

$$17 \leqslant \mathrm{WR}(\det_3) \leqslant 18,$$

where the upper bound was proved by Conner–Gesmundo–Landsberg–Ventura [5], and the lower bound follows from the paper of Conner–Harper–Landsberg [4].

Acknowledgments. The author would like to thank J. M. Landsberg and Zach Teitler for useful comments on the topic of this paper.

2. Partially symmetric tensors

Our approach to Theorem 1.1 employs the natural correspondence between the Waring rank and symmetric tensor decompositions [1]. Since the polynomial per_3 has degree three, we switch to three-dimensional tensors, and we recall that the field $\mathbb{F}$ is assumed to satisfy either $\operatorname{char}\mathbb{F} = 0$ or $\operatorname{char}\mathbb{F} \geqslant 5$. All matrices and tensors considered below are supposed to have entries in $\mathbb{F}$, all linear spaces are assumed to have $\mathbb{F}$ as a ground field, and the notation $\operatorname{span}\Phi$ denotes the $\mathbb{F}$-linear span of a family Φ of vectors in some $\mathbb{F}$-linear space.

A *symmetric tensor* T is an $n \times n \times n$ array of scalars taken in the field $\mathbb{F}$ such that the value of $T(i|j|k)$ remains invariant under a permutation of elements i, j, k in an indexing set of cardinality n. The *symmetric rank* of T is the smallest integer r for which there exist vectors $u_1, \ldots, u_r \in \mathbb{F}^n$ and scalars $c_1, \ldots, c_r \in \mathbb{F}$ such that

$$T = c_1 u_1^{\otimes 3} + \ldots + c_r u_r^{\otimes 3} \tag{2.1}$$

with $v^{\otimes 3}$ being the tensor whose $(i|j|k)$ coordinate equals $v_i v_j v_k$.

To explain the equivalence between (1.1) and (2.1), we consider an homogeneous cubic polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ in which a monomial $x_i x_j x_k$ has coefficient s_{ijk}.

The corresponding symmetric $n \times n \times n$ tensor $T := T(f)$ is defined as

$$T(i|i|i) = s_{iii}, \quad T(i|i|j) = s_{iij}/3, \quad T(i|j|k) = s_{ijk}/6$$

if i, j, k are pairwise distinct. Now (1.1) is equivalent to (2.1) if the linear form $l_i = l_{i1}x_1 + \ldots + l_{in}x_n$ corresponds to the vector $u_i = (l_{i1} \ldots l_{in})$ for all i.

Now we are ready to write down the tensor corresponding to the polynomial in Theorem 1.1. An ith *slice* of a symmetric $n \times n \times n$ tensor T is the $n \times n$ symmetric matrix whose (j, k) entry equals $T(i|j|k)$. The linear space spanned by the slices of the $9 \times 9 \times 9$ tensor corresponding to per_3 is

$$\mathcal{L} = \left(\begin{array}{c|ccc|ccc|ccc}
 & x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} & x_{31} & x_{32} & x_{33} \\
\hline
x_{11} & 0 & 0 & 0 & 0 & k & h & 0 & f & e \\
x_{12} & 0 & 0 & 0 & k & 0 & g & f & 0 & d \\
x_{13} & 0 & 0 & 0 & h & g & 0 & e & d & 0 \\
\hline
x_{21} & 0 & k & h & 0 & 0 & 0 & 0 & c & b \\
x_{22} & k & 0 & g & 0 & 0 & 0 & c & 0 & a \\
x_{23} & h & g & 0 & 0 & 0 & 0 & b & a & 0 \\
\hline
x_{31} & 0 & f & e & 0 & c & b & 0 & 0 & 0 \\
x_{32} & f & 0 & d & c & 0 & a & 0 & 0 & 0 \\
x_{33} & e & d & 0 & b & a & 0 & 0 & 0 & 0
\end{array} \right) \tag{2.2}$$

with the first row and first column indicating the labels in the indexing set. The x_{11} slice of per_3 is obtained by taking the variable a equal to $1/6$ and all other variables equal to 0 in (2.2). Similarly, the variable b corresponds to the x_{12} slice, the variable c indicates the x_{13} slice, and so on.

Definition 2.1. (See [3].) Let L be a linear space spanned by a family of symmetric $n \times n$ matrices. The *partially symmetric rank* of L is the smallest cardinality of a family Φ of symmetric rank-one matrices such that $L \subseteq \mathrm{span}\,\Phi$. The partially symmetric rank of L is denoted by $\mathrm{psr}(L)$.

Every slice of a tensor T satisfying (2.1) belongs to $\mathrm{span}\{u_1 \otimes u_1, \ldots, u_r \otimes u_r\}$, so the symmetric rank of a tensor is greater than or equal to the corresponding partially symmetric rank. We are going to prove the following result.

Theorem 2.2. *We have* $\mathrm{psr}(\mathcal{L}) \geqslant 16$*, where* $\mathcal{L}$ *is the linear space in* (2.2).

The rest of this paper is devoted to the proof of Theorem 2.2, which implies the desired lower bound in Theorem 1.1 by the above discussion.

3. Our notation and general observations

We are going to prove Theorem 2.2 by contradiction. Assuming the converse, we adopt the following conventions.

Assumption 3.1. Taking $\rho = 15$, we consider a family $\alpha_1, \ldots, \alpha_\rho \in \mathbb{F}^9$ such that the linear space

$$\Lambda = \mathrm{span}\{\alpha_1 \otimes \alpha_1, \ldots, \alpha_\rho \otimes \alpha_\rho\} \tag{3.1}$$

has dimension ρ and contains the space $\mathcal{L}$ as in (2.2).

Remark 3.2. The assumption $\rho = 15$ instead of $\rho \leqslant 15$ does not cause a loss of generality because a pair of matrices $\alpha_i \otimes \alpha_i$ and $\alpha_j \otimes \alpha_j$ in a spanning set of Λ can be replaced by a triple of generic matrices of the form $(\xi_i\alpha_i + \xi_j\alpha_j) \otimes (\xi_i\alpha_i + \xi_j\alpha_j)$

without breaking the property $L \subseteq \Lambda$. In what follows, we do not use the notation ρ and simply use the number 15 instead.

Notation 3.3. We denote by U the three-dimensional subspace of $\mathbb{F}^9$ corresponding to the (x_{11}, x_{12}, x_{13}) coordinates, which corresponds to taking the three leftmost columns with respect to the block partition of (2.2). Similarly, we define M as the six-dimensional subspace of $\mathbb{F}^9$ corresponding to the $(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})$ coordinates, which corresponds to taking the six leftmost columns. We denote by u_i the projection of α_i onto U, and we define m_i as the projection of α_i onto M.

The following result is proved with the standard *substitution method*, see Subsection 1.2 in [19] or Proposition 3.1 in [12] for recent account. See also Lemma 2 in [7] for an early appearance of a related result.

Claim 3.4. *Let P be a subspace of the space Λ as in* (3.1)*. If $\dim P \geqslant 7$, then*

$$\dim \mathcal{L} \cap P \geqslant \dim P - 6,$$

where $\mathcal{L}$ is the space (2.2)*.*

Proof. We have $\dim(\mathcal{L} + P) \leqslant 15$ because every matrix in $\mathcal{L}$ or P belongs to the linear span of the 15 matrices $\alpha_i \otimes \alpha_i$ as in Assumption 3.1. We get $15 \geqslant \dim(\mathcal{L} + P) = \dim \mathcal{L} + \dim P - \dim \mathcal{L} \cap P$, and the result follows because $\dim \mathcal{L} = 9$. □

We proceed with three easy general observations.

Observation 3.5. *If A, B, C are subspaces of some finite dimensional linear space, then $\dim(A + B) \cap C \leqslant \dim(A \cap C) + \dim B$.*

Proof. Using the formula $\dim U + \dim V = \dim(U \cap V) + \dim(U + V)$, we get $\dim(A+B) \cap C - \dim(A \cap C) - \dim B = \dim(A+B) - \dim(A+B+C) + \dim(A+C) - \dim A - \dim B = \dim(A+C) - \dim(A+B+C) - \dim(A \cap B) \leqslant 0$. □

Observation 3.6. *Assume $V \subset W$ are linear spaces, let w be a vector in $W \setminus V$. If m is a matrix in $V \otimes V$, then $\operatorname{rank}(m + w \otimes w) = \operatorname{rank} m + 1$.*

Proof. The rank of a block-diagonal matrix equals the sum of the ranks of the diagonal blocks. □

Observation 3.7. *Let $c_1, \ldots, c_n$ be a family of non-zero scalars. If vectors $v_1, \ldots, v_n$ taken from some linear space satisfy*

$$c_1(v_1 \otimes v_1) + \ldots + c_k(v_n \otimes v_n) = 0, \tag{3.2}$$

then $\dim \operatorname{span}\{v_1, \ldots, v_n\} \leqslant n/2$.

Proof. If vectors $v_{i_1}, \ldots, v_{i_t}$ are linearly independent, then the total of the corresponding t summands in (3.2) has rank t by the previous observation. The remaining summands have rank at most $n - t$, which implies $t \leqslant n - t$. □

4. The upper left 3×3 block of $\mathcal{L}$

We need several technical statements concerning the vectors $u_1, \ldots, u_{15}$ as in Notation 3.3. We recall that they belong to the subspace called U, which consists of all vectors with zero coordinates outside x_{11}, x_{12}, x_{13} as in (2.2).

Claim 4.1. *Let ε be a non-zero linear form involving the variables with indexes (x_{11}, x_{12}, x_{13}). Consider the subspace L_ε consisting of all matrices of the form* (2.2) *whose row spaces lie in $\ker \varepsilon$. Then $\dim L_\varepsilon \leqslant 5$.*

Proof. If we have $d \neq 0$, $e \neq 0$, $f \neq 0$ for a matrix in (2.2), then the bottom-left block of it is non-singular, so this matrix cannot belong to L_ε. This means that one of these variables should be zero; we assume that $d = 0$ without loss of generality. The value of (e, f) cannot be arbitrary as well, because the sum of the row spaces of all the 3×3 matrices of the form

$$\begin{pmatrix} 0 & f & e \\ f & 0 & 0 \\ e & 0 & 0 \end{pmatrix}$$

is the whole three-dimensional space. Therefore, the description of L_ε in $\mathcal{L}$ should involve two non-collinear equations involving d, e, f, and the same argument applied to g, h, k shows that $\operatorname{codim} L_\varepsilon \geqslant 4$. $\square$

Claim 4.2. *If $\mu = (\mu_1, \mu_2, \mu_3)$ is a basis of U, then at least 5 vectors among $u_1, \ldots, u_{15}$ have a non-zero μ_3 coordinate over μ.*

Proof. Let A' be the linear span of those $\alpha_i \otimes \alpha_i$ for which the corresponding u_i has a non-zero μ_3 coordinate. Assuming that the statement is false, we get $\dim A' \leqslant 4$. Since every $\alpha_i \otimes \alpha_i \in A'$ has a non-zero $\mu_3 \otimes \mu_3$ position, the subspace $A^\circ \subseteq A$ of matrices with zeros at the $\mu_3 \otimes \mu_3$ position satisfies $\dim A^\circ \leqslant 3$. We define A'' as the linear span of those $\alpha_j \otimes \alpha_j$ for which the corresponding vector u_j has a zero μ_3 coordinate, and we get $\Lambda = A' + A''$ for the space Λ as in (3.1). Since both $\mathcal{L}$ and A'' have zeros at the $\mu_3 \otimes \mu_3$ position, the inclusion $\mathcal{L} \subseteq A' + A''$ implies $\mathcal{L} \subseteq A^\circ + A''$. According to Claim 4.1, we have $\dim(A'' \cap \mathcal{L}) \leqslant 5$, which implies

$$\dim \mathcal{L} = \dim(A^\circ + A'') \cap \mathcal{L} \leqslant \dim(A'' \cap \mathcal{L}) + \dim A^\circ \leqslant 5 + 3,$$

in which the middle inequality is valid by Observation 3.5. The proof is complete because we get a contradiction to $\dim \mathcal{L} = 9$. $\square$

The main technical result of this section is a lower bound of 5 on the dimension of the linear span of the matrices $u_i \otimes u_i$. This is proved in Claim 4.5 below, and we record the corresponding space for ease of reference.

Notation 4.3. Let Φ_U be the space $\operatorname{span}\{u_1 \otimes u_1, \ldots, u_{15} \otimes u_{15}\}$.

Claim 4.4. *For the space Φ_U as in Notation 4.3, we have $\dim \Phi_U \geqslant 4$.*

Proof. According to Claim 4.2, we have $\operatorname{span}\{u_1, \ldots, u_{15}\} = U$, so we can assume that u_1, u_2, u_3 are a basis of U up to relabeling the matrices of the decomposition. So we have $\dim \Phi_U \geqslant 3$, and if it was the case that $\dim \Phi_U = 3$, then every matrix $u_i \otimes u_i$ would be collinear to one of $u_1 \otimes u_1$, $u_2 \otimes u_2$, $u_3 \otimes u_3$. According to Claim 4.2, at least five of the $u_i \otimes u_i$ matrices should be non-zero and collinear to $u_3 \otimes u_3$, and we can assume that these matrices have indexes $3, 4, 5, 6, 7$.

According to Claim 3.4, we can express some non-zero matrix in $\mathcal{L}$ as

$$\ell = s_1(\alpha_1 \otimes \alpha_1) + \ldots + s_7(\alpha_7 \otimes \alpha_7),$$

but a comparison of the $U \otimes U$ blocks shows that $s_1 = s_2 = 0$. So the projection of the row space of ℓ onto U is a subspace of $\operatorname{span}\{u_3\}$, and hence it is zero because the matrices in (2.2) cannot have such a projection of dimension one. So we see that ℓ has zeros outside the two 3×3 blocks corresponding to the variables a, b, c, and we have $\operatorname{rank} \ell \geqslant 4$. Assuming without loss of generality that $s_3 \neq 0$, we get

$$\ell - s_3(\alpha_3 \otimes \alpha_3) = s_4(\alpha_4 \otimes \alpha_4) + \ldots + s_7(\alpha_7 \otimes \alpha_7),$$

which is a contradiction because the matrix on the left has rank at least five by Observation 3.6 while the matrix on the right is the sum of at most four rank-one matrices. □

Claim 4.5. *For the space Φ_U as in Notation 4.3, we have* $\dim \Phi_U \geqslant 5$.

Proof. Assuming the converse, we have $\dim \Phi_U = 4$ by Claim 4.4. Up to relabeling the matrices of the decomposition, we can assume that u_1, u_2, u_3 are a basis of U, and we can take $u_{15} = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ with at least two non-zero coefficients among $\lambda_1, \lambda_2, \lambda_3$. If $\lambda_3 = 0$, then the condition $\dim \Phi_U \leqslant 4$ shows that every vector $u_i \notin \operatorname{span}\{u_1, u_2\}$ is collinear to u_3. Arguing as in Claim 4.4, we find five non-zero vectors u_i that are collinear to u_3, and we assume that their indexes are $3, 4, 5, 6, 7$. We conclude the argument as in Claim 4.4 by showing that

$$\operatorname{span}\{\alpha_1 \otimes \alpha_1, \ldots, \alpha_7 \otimes \alpha_7\} \cap \mathcal{L} = \{0\},$$

which is a contradiction to Claim 3.4.

If $\lambda_1, \lambda_2, \lambda_3$ are all non-zero, then no rank-one matrix is spanned by $u_1 \otimes u_1$, $u_2 \otimes u_2$, $u_3 \otimes u_3$, $u_{15} \otimes u_{15}$ except the scalar multiples of these matrices. By the pigeonhole principle, there are 4 indexes j for which the corresponding u_j's are all collinear to the same vector among u_1, u_2, u_3, u_{15}. We assume without loss of generality that u_3, u_4, u_5, u_6 are collinear, and we again conclude the argument similarly to Claim 4.4 by showing that

$$\operatorname{span}\{\alpha_1 \otimes \alpha_1, \ldots, \alpha_6 \otimes \alpha_6, \alpha_{15} \otimes \alpha_{15}\} \cap \mathcal{L} = \{0\},$$

which is a contradiction to Claim 3.4. □

Claim 4.6. *If i_1, i_2, i_3 are distinct indexes, then* $\dim \operatorname{span}\{u_{i_1}, u_{i_2}, u_{i_3}\} \geqslant 2$.

Proof. If the statement is false, we can use Claim 4.5 and find four additional indexes i_4, i_5, i_6, i_7 such that the matrices

$$u_{i_4} \otimes u_{i_4}, u_{i_5} \otimes u_{i_5}, u_{i_6} \otimes u_{i_6}, u_{i_7} \otimes u_{i_7}$$

are linearly independent and do not belong to the linear span of $\{u_{i_1} \otimes u_{i_1}, u_{i_2} \otimes u_{i_2}, u_{i_3} \otimes u_{i_3}\}$. According to Claim 3.4, we can express some non-zero matrix in $\mathcal{L}$ as

$$\ell = s_1(\alpha_{i_1} \otimes \alpha_{i_1}) + \ldots + s_7(\alpha_{i_7} \otimes \alpha_{i_7}),$$

and since the $U \otimes U$ block in $\mathcal{L}$ is zero, we get $s_4 = s_5 = s_6 = s_7 = 0$. This means that ℓ has rank at most three, which is a contradiction because, in fact, every non-zero matrix in $\mathcal{L}$ has rank at least four. □

5. The upper left 6×6 block of $\mathcal{L}$

Now we switch to a consideration of the vectors $m_1, \ldots, m_{15}$ as in Notation 3.3. We recall that they belong to the subspace called M, which consists of all vectors with zero coordinates outside $x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{33}$ as in (2.2). The following two statements are similar to Claims 4.1 and 4.2.

Claim 5.1. *Let ζ be a non-zero linear form involving the variables with indexes $(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{33})$. Consider the subspace L_ζ consisting of all matrices of the form* (2.2) *whose row spaces lie in* $\ker \zeta$. *Then* $\dim L_\zeta \leqslant 6$.

Proof. The bottom-left 3×6 blocks of the matrices in $\mathcal{L}$ have the form

$$\left(\begin{array}{ccc|ccc} 0 & f & e & 0 & c & b \\ f & 0 & d & c & 0 & a \\ e & d & 0 & b & a & 0 \end{array}\right),$$

so the sum of their row spaces over all possible a, b, c, d, e, f is the whole space $\mathbb{F}^6$. This shows that we have a non-trivial linear equation involving a, b, c, d, e, f that needs to be satisfied for the corresponding matrix to belong to L_ζ. The same argument applied to the upper-left 6×6 block gives two additional independent linear equations involving g, h, k, which shows that $\operatorname{codim} L_\zeta \geqslant 3$. □

Claim 5.2. *If $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6)$ is a basis of M, then at least 3 vectors among $m_1, \ldots, m_{15}$ have a non-zero μ_6 coordinate over μ.*

Proof. Let A' be the linear span of those $\alpha_i \otimes \alpha_i$ for which the corresponding m_i has a non-zero μ_6 coordinate. Assuming that the statement is false, we get $\dim A' \leqslant 2$. Similarly, we define A'' as the linear span of those $\alpha_j \otimes \alpha_j$ for which the corresponding vector m_j has a zero μ_6 coordinate, and we get $\Lambda = A' + A''$ for the space Λ as in (3.1). Using Claim 5.1, we get $\dim(A'' \cap \mathcal{L}) \leqslant 6$, which implies

$$\dim \mathcal{L} = \dim(A' + A'') \cap \mathcal{L} \leqslant \dim(A'' \cap \mathcal{L}) + \dim A' \leqslant 6 + 2,$$

in which the middle inequality is valid by Observation 3.5. The proof is complete because we get a contradiction to $\dim \mathcal{L} = 9$. □

The following two claims give lower bounds on the dimension spanned by a family of the (m_i) vectors as a function of the size of a family. Their formulations are similar to Claim 4.6.

Claim 5.3. *If i_1, i_2, i_3 are distinct indexes, then* $\dim \operatorname{span}\{m_{i_1}, m_{i_2}, m_{i_3}\} \geqslant 2$.

Proof. Follows from Claim 4.6 because every vector u_i is the projection of m_i onto the first three coordinates. □

Claim 5.4. *If $i_1, \ldots, i_6$ are distinct indexes, then* $\dim \operatorname{span}\{m_{i_1}, \ldots, m_{i_6}\} \geqslant 3$.

Proof. If the statement is false, then the vectors $m_{i_1}, \ldots, m_{i_6}$ belong to the linear span of two vectors μ_1, μ_2, and then the linear space

$$W = \operatorname{span}\{m_{i_1} \otimes m_{i_1}, \ldots, m_{i_6} \otimes m_{i_6}\}$$

lies in the linear span of $\mu_1 \otimes \mu_1$, $\mu_1 \otimes \mu_2 + \mu_2 \otimes \mu_1$, $\mu_2 \otimes \mu_2$, so we have $\dim W \leqslant 3$. This means that there exists a 3-dimensional subspace A' in

$$\operatorname{span}\{\alpha_{i_1} \otimes \alpha_{i_1}, \ldots, \alpha_{i_6} \otimes \alpha_{i_6}\}$$

consisting of matrices with a zero $M \otimes M$ block, which is, in other words, the upper-left 6×6 block in (2.2).

Claim 4.5 allows one to find indexes j_1, j_2, j_3, j_4, j_5 for which the space

$$A'' = \operatorname{span}\{\alpha_{j_1} \otimes \alpha_{j_1}, \ldots, \alpha_{j_5} \otimes \alpha_{j_5}\}$$

has a five-dimensional projection onto $U \otimes U$, and hence a five-dimensional projection onto $M \otimes M$. Since the spaces A' and A'' intersect trivially, we get $\dim(A' + A'') \geqslant 3 + 5$, and Claim 3.4 shows that $\dim(A' + A'') \cap \mathcal{L} \geqslant 2$. Since the $U \otimes U$ blocks of the matrices in $\mathcal{L}$ are zero, we have $(A' + A'') \cap \mathcal{L} = A' \cap \mathcal{L}$ and

$$\dim A' \cap \mathcal{L} \geqslant 2. \tag{5.1}$$

Now let us recall that the projections of the row spaces of the matrices in W onto M belong to $\operatorname{span}\{\mu_1, \mu_2\}$, and let us deduce a contradiction. An examination of (2.2) shows that there are no two-dimensional subspaces $L' \subset \mathcal{L}$ such that the submatrix obtained by the first 6 columns of any non-zero matrix in L' would have the row space in a two-dimensional subspace of M fixed in advance. This implies $\dim W \cap \mathcal{L} \leqslant 1$ and contradicts to (5.1) because $A' \subset W$. $\square$

6. The proof

In this section, we put together the auxiliary results from Sections 3–5 and complete the proof of Theorem 2.2.

Notation 6.1. We say that an index $i \in \{1, \ldots, 15\}$ is a *twin* if there exists $j \neq i$ such that m_i and m_j are non-zero collinear vectors. The non-zero non-twin vectors are called *essential*.

Remark 6.2. According to Claim 5.3, there cannot be a family of more than two collinear twins. Also, at most one vector among $m_1, \ldots, m_{15}$ can be zero, and this zero vector can exist only if there are no twins.

Claim 6.3. *If we have at most two twin pairs, then we can enumerate the vectors such that*

(1) $m_1, \ldots, m_6$ *are linearly independent,*
(2) $\{m_1, \ldots, m_6\}$ *intersects every twin pair, and*
(3) $m_7, \ldots, m_{10}$ *are essential and linearly independent.*

Proof. Denoting the number of twin pairs by a, we take a total of a representatives of every twin pair as the first a vectors. Since $a \leqslant 2$, these are linearly independent, and, due to Claim 5.2, we can complete them to a basis $m_1, \ldots, m_6$ of M. Now the conditions (1) and (2) of the conclusion of the claim are satisfied; since $a \leqslant 2$, there are at least seven essential vectors in $m_7, \ldots, m_{15}$, and their linear span E should satisfy $\dim E \geqslant 3$ by Claim 5.4. If $\dim E \geqslant 4$, then we are done, so it suffices to consider the case $\dim E = 3$. Then we can find an essential index $i \in \{1, \ldots, 6\}$ such that $m_i \notin E$, which is possible because

$$\dim \operatorname{span}\{m_1, \ldots, m_{15}\} = 6,$$

and the addition of the twins can increase the dimension by at most $a \leqslant 2$.

Further, we use Claim 5.2 to find an index $j \in \{7, \ldots, 15\}$ such that m_j has a non-zero m_i coordinate over the basis $m_1, \ldots, m_6$. It remains to swap the indexes of m_i and m_j to get the desired enumeration of $m_1, \ldots, m_6$, because even if m_j is essential, its removal from the family of the essential vectors in $m_7, \ldots, m_{15}$ does not decrease the dimension of their linear span again by Claim 5.4. So we see that the linear span of the essential vectors in $m_7, \ldots, m_{15}$ remains equal to E even after m_j gets removed, and then the addition of the essential vector $m_i \notin E$ increases the dimension. It remains to choose m_7, m_8, m_9, m_{10} as a basis of E. $\square$

We need two further technical claims.

Claim 6.4. *Let* $\mu_1 = \{m_1, \ldots, m_6\}$ *and* $\mu_2 = \{m_7, \ldots, m_{10}\}$ *be two vector families, and suppose that* μ_1 *is linearly independent. Then there exists a family* P *with* a *vectors in* μ_1 *and* b *vectors in* μ_2 *such that* $\dim \operatorname{span} P \leqslant a/2 + b/2$*, and the pair* (a, b) *is one of* $(0,4)$*,* $(1,3)$*,* $(2,2)$*,* $(2,4)$*,* $(3,3)$*,* $(4,4)$*. If the vectors in* μ_2 *are linearly independent, then the options* $(0,4)$*,* $(1,3)$*,* $(2,4)$ *are impossible.*

Proof. We apply Claim 3.4 to the space

$$M^\circ = \mathrm{span}\{\alpha_1 \otimes \alpha_1, \ldots, \alpha_{10} \otimes \alpha_{10}\},$$

and we conclude that $\dim M^\circ \cap \mathcal{L} \geqslant 4$. From (2.2) we see that the restriction of $\mathcal{L}$ to the $M \otimes M$ block has dimension three, which means that M° contains a non-zero matrix $\ell \in \mathcal{L}$ which has a zero $M \otimes M$ block. In a decomposition

$$\ell = c_1(\alpha_1 \otimes \alpha_1) + \ldots + c_{10}(\alpha_{10} \otimes \alpha_{10}), \tag{6.1}$$

we pass to the $M \otimes M$ block to get

$$c_1(m_1 \otimes m_1) + \ldots + c_{10}(m_{10} \otimes m_{10}) = 0.$$

We define the family P as the set of vectors corresponding to the indexes j with $c_j \neq 0$, and, if the number of such indexes is odd, we remove an arbitrary vector from P. Now the result follows from Observation 3.7. In fact, the linear independence of $m_1, \ldots, m_6$ implies $a \leqslant b$, and if, additionally, the vectors m_7, m_8, m_9, m_{10} are linearly independent as well, we should have $a = b$. It remains to check that the case $a+b \leqslant 2$ is impossible because then at most three of the c_j's would be non-zero in (6.1), but every non-zero matrix in (2.2) has rank at least four. □

Claim 6.5. *One cannot write M as a direct sum of three two-dimensional subspaces M_1, M_2, M_3 such that the restriction of every matrix in $\mathcal{L}$ onto $M \otimes M$ is a block diagonal matrix formed with the $M_1 \otimes M_1$, $M_2 \otimes M_2$, $M_3 \otimes M_3$ blocks.*

Proof. Assuming that this is possible, we take $x \in \{g, h, k\}$ and $j \in \{1, 2, 3\}$, and we denote by $\rho(x, j)$ the rank of the $M_j \otimes M_j$ block of the matrix obtained from (2.2) by taking $x = 1$ and all the other variables zero. The $M \otimes M$ blocks of such matrices have rank 4, which means that

$$\rho(x, 1) + \rho(x, 2) + \rho(x, 3) = 4. \tag{6.2}$$

We have two possible cases up to symmetries.

Case $\rho(g, 1) = \rho(g, 2) = 2$, $\rho(g, 3) = 0$. This implies $\rho(h, 3) = 0$ because otherwise a matrix obtained by taking g, h generically and all other variables zero would have a non-zero $M_3 \otimes M_3$ block, so the restriction of such matrix to $M \otimes M$ would have rank at least $2 + 2 + 1$, but, in fact, this rank is 4. We have $\rho(k, 3) = 0$ for a similar reason, which implies that the $M_3 \otimes M_3$ block is zero on $\mathcal{L}$, so any matrix in $\mathcal{L}$ has the restriction to $M \otimes M$ of rank at most $2 + 2 + 0$, but, in fact, a generic element of $\mathcal{L}$ has such a restriction of rank six.

Case $\rho(g, 1) = \rho(g, 2) = 1$, $\rho(g, 3) = 2$. The considerations similar to the previous case imply that the restrictions of the matrices in $\mathcal{L}$ to the $M_1 \otimes M_1$ block should all be collinear, and so a generic element of $\mathcal{L}$ has the restriction to $M \otimes M$ of rank at most $1+2+2$, which leads to the contradiction as in the previous paragraph. □

Now we are ready to prove that the assumption of Claim 6.3 does not realize, which means that we should have at least three pairs of twins.

Claim 6.6. *There are at least three pairs of twins.*

Proof. Assuming the converse, we can find two families $\mu_1 = \{m_1, \ldots, m_6\}$ and $\mu_2 = \{m_7, \ldots, m_{10}\}$ satisfying the conclusions of Claim 6.3. Let us say that the *support* of a vector m_j is the set of those $i \in \{1, 2, 3, 4, 5, 6\}$ for which m_j has a non-zero m_i coordinate over μ_1. Using Claim 5.2, we assume without loss of generality that the union of the supports of the vectors in μ_2 equals $\{1, 2, 3, 4, 5, 6\}$. The families μ_1, μ_2 satisfy the assumptions of Claim 6.4; since μ_2 is linearly independent, the conclusion of Claim 6.4 should realize with $a = b$.

Case $a = b = 4$. This means that μ_2 is spanned by four vectors in μ_1, which is not possible because the union of all the supports over μ_2 equals $\{1, 2, 3, 4, 5, 6\}$.

Case $a = b = 3$. This means that there exist three vectors in μ_2 which are spanned by three vectors in μ_1. We can assume without loss of generality that

$$m_7 = s_1 m_1 + s_2 m_2 + s_3 m_3 + s_4 m_4 + s_5 m_5 + s_6 m_6$$

with s_1, s_2, s_3 non-zero, and

$$\begin{cases} m_8 = a_4 m_4 + a_5 m_5 + a_6 m_6, \\ m_9 = b_4 m_4 + b_5 m_5 + b_6 m_6, \\ m_{10} = c_4 m_4 + c_5 m_5 + c_6 m_6. \end{cases}$$

We cannot have $a_4 = b_4 = c_4 = 0$ because μ_2 is linearly independent, and we cannot have $a_4 = a_5 = 0$ because m_8 is essential. The symmetry allows us to assume that each of the pairs $\{a_4, b_4\}$, $\{a_5, b_5\}$, $\{a_6, b_6\}$ contains a non-zero number. According to Claim 5.2, one of the vectors $m_{11}, \ldots, m_{15}$ should lie outside $\operatorname{span}\{m_4, m_5, m_6\}$, so it remains to replace m_{10} by this vector in μ_2 and to check that the conclusion of Claim 6.4 cannot be satisfied with the updated families μ_1 and μ_2.

Case $a = b = 2$. This means that there exist two vectors in μ_2 which are spanned by two vectors in μ_1. The symmetry allows us to assume that the supports of m_9 and m_{10} are both equal to $\{5, 6\}$, and the argument in the first paragraph tells that the union of the supports of m_7 and m_8 contains $\{1, 2, 3, 4\}$. The projections of m_7 and m_8 onto the $\{1, 2, 3, 4\}$ coordinates are linearly independent because μ_2 is linearly independent, and the supports of both m_7 and m_8 contain at least two numbers in $\{1, 2, 3, 4\}$ because otherwise the assumption of the already invalidated case $a = b = 3$ applies. Replacing m_{10} in μ_2 with an arbitrary vector in $m_0 \in \{m_{11}, \ldots, m_{15}\}$, we see that the conclusion of Claim 6.4 can only be valid with $(a, b) = (1, 3)$ or $(a, b) = (2, 2)$ with the new families μ_1 and μ_2.

Subcase 1. Assume that the obstruction $(a, b) = (1, 3)$ is possible, which means that the vectors m_7, m_8, m_0 can become collinear after the removal of one of the coordinates. According to Claim 5.4, this obstruction can arise with at most two choices of m_0. Also, in this case, the vectors m_7, m_8 should have supports of cardinality at least three, and the $(a, b) = (2, 2)$ obstruction can only realize when the support of m_0 is a subset of the support of m_9, which is possible for at most two choices of m_0 for the same reason as above. Since we have five options to choose m_0, we can avoid both of these obstructions.

Subcase 2. If we know that the obstruction $(a, b) = (1, 3)$ does not arise, then it suffices to find an m_0 whose support is not contained in those of the supports of m_7, m_8, m_9 which are of cardinality two. In view of Claim 5.4, the only option when such a choice is impossible is that the supports of m_7, m_8, m_9 are pairwise disjoint and have cardinality two, and every such support contains the supports of exactly five vectors in $m_1, \ldots, m_{15}$. In view of Claim 6.5, this is not possible. $\square$

Now let us go back to the expression (2.2). We recall that the upper-left 6×6 block of the matrices in $\mathcal{L}$ is called the $M \otimes M$ block. Similarly, the upper-left 3×3 block of these matrices is the $U \otimes U$ block. The *bad* positions of (2.2) are those entries which equal zero identically on $\mathcal{L}$.

Claim 6.7. *The space Λ as in* (3.1) *contains a matrix β such that*
(1) β has the zero $M \otimes M$ block,
(2) β has a non-zero bad entry in the bottom-left 3×6 block.

Proof. Assume the converse. Using Claim 6.6 and the symmetry, we can take $m_1 = m_2$, $m_3 = m_4$, $m_5 = m_6$. Then the matrices

$$\beta_1 = \alpha_1 \otimes \alpha_1 - \alpha_2 \otimes \alpha_2, \;\; \beta_3 = \alpha_3 \otimes \alpha_3 - \alpha_4 \otimes \alpha_4, \;\; \beta_5 = \alpha_5 \otimes \alpha_5 - \alpha_6 \otimes \alpha_6$$

have zeros at the $M \otimes M$ block, and the rows of the restriction of β_i to the bottom-left 3×6 block are collinear to m_i, which implies that such restrictions are all rank-one. However, the linear span of a family of three rank-one matrices with zeros at bad positions cannot contain a two-dimensional subspace of

$$\left(\begin{array}{ccc|ccc} 0 & f & e & 0 & c & b \\ f & 0 & d & c & 0 & a \\ e & d & 0 & b & a & 0 \end{array}\right),$$

so we get

$$\dim B \cap \mathcal{L} \leqslant 1, \tag{6.3}$$

where $B = \operatorname{span}\{\beta_1, \beta_3, \beta_5\}$.

Now we use Claim 4.5 to find indexes j_1, j_2, j_3, j_4, j_5 for which the space

$$A = \operatorname{span}\{\alpha_{j_1} \otimes \alpha_{j_1}, \dots, \alpha_{j_5} \otimes \alpha_{j_5}\}$$

has a five-dimensional projection onto $U \otimes U$, and hence a five-dimensional projection onto $M \otimes M$. Since every matrix in B has only zeros in the $M \otimes M$ block, the spaces A and B intersect trivially, and we have $\dim(A + B) \geqslant 3 + 5$. Using Claim 3.4, we get $\dim(A + B) \cap \mathcal{L} \geqslant 2$, but since the $U \otimes U$ blocks of the matrices in $\mathcal{L}$ are zero, we have $(A + B) \cap \mathcal{L} = B \cap \mathcal{L}$ and hence $B \cap \mathcal{L} \geqslant 2$, which contradicts (6.3). □

We are ready to complete the argument.

Theorem 6.8. *Assumption 3.1 is false.*

Proof. Using Claim 6.7 and the symmetry, we can take a matrix $\ell_1 \in \Lambda$ with all zeros in the upper-left 6×6 block and at least one bad non-zero in the bottom-left 3×3 block. Now we apply Claim 6.7 for M equal to the space generated by the $(x_{11}, x_{12}, x_{13}, x_{31}, x_{32}, x_{33})$ coordinates instead of $(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})$, and we can find a matrix ℓ_2 with all zeros in the corner 3×3 blocks and at least one bad non-zero in one of the remaining off-diagonal 3×3 blocks.

Finally, we are going to apply Claim 4.5 as in the last paragraph of the proof of Claim 6.7. We find indexes j_1, j_2, j_3, j_4, j_5 for which the space

$$A = \operatorname{span}\{\alpha_{j_1} \otimes \alpha_{j_1}, \dots, \alpha_{j_5} \otimes \alpha_{j_5}\}$$

has a five-dimensional projection onto $U \otimes U$, which is the upper-left 3×3 block. The matrices ℓ_1 and ℓ_2 have all zeros in the $U \otimes U$ block, and we can check that the restriction of the space

$$\operatorname{span} A \cup \{\ell_1, \ell_2\}$$

onto the bad entries has dimension 7. In particular, this space cannot have a non-zero matrix in $\mathcal{L}$, which contradicts Claim 3.4. □

7. Concluding remarks

We showed that both the symmetric rank and partially symmetric rank of the tensor corresponding to the 3×3 permanent are equal to 16. However, we were unable to generalize our approach to compute the rank or border rank of this tensor.

Question 7.1. What is the rank of the tensor $\mathcal{L}$ in (2.2)?

The above mentioned result of Conner, Gesmundo, Landsberg, Ventura [5] implies that the rank of $\mathcal{L}$ is at least 15. If we have $\operatorname{rk} \mathcal{L} = 15$, then $\mathcal{L}$ gives another counterexample to a recently disproved conjecture of Comon [18] and disproves the partially symmetric version of this conjecture, which remains open [3, 8, 18].

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