Solving the 106 years old 3^k Points Problem with the Clockwise-algorithm

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Abstract. In this paper, we present the clockwise-algorithm that solves the extension in k-dimensions of the infamous nine-dot problem, the well known two-dimensional thinking outside the box puzzle. We describe a general strategy that constructively produces minimum length covering trails, for any $k \in \mathbb{N} - \{0\}$, solving the NP-complete (3 X 3 X … X 3)-points problem inside a 3 X 3 X … X 3 hypercube. In particular, using our algorithm, we explicitly draw different covering trails of minimal length $h(k) = \frac{3^k-1}{2}$ for $k = 3$ and $k = 4$, and we also conjecture that, for every $k \geq 1$, it is possible to solve the 3^k-points problem with $h(k)$ lines starting from any of the 3^k nodes, except from the central one.

Keywords: Nine dots puzzle, Nine-dot problem, Clockwise-algorithm, Thinking outside the box, Hypergraph, Lateral thinking, Link-length, Connectivity, Polygonal path, Optimization problem.

2010 Mathematics Subject Classification: Primary 05C85; Secondary 05C57, 68R10.

1 Introduction

The classic *nine dots puzzle* [8,10] is the well known thinking outside the box challenge [3, 11], and it corresponds to the two-dimensional case of the general 3^k-points problem (assuming $k = 2$) [2, 5, 9, 13].

The statement of the 3^k-points problem problem is as follows:
"Given a finite set of 3^k points in $\mathbb{R}^k$, we need to visit all of them (at least once) with a polygonal path that has the minimum number of line segments $h(k)$, and we simply define the aforementioned line segments as *lines*. Let G_k be a 3 X 3 X … X 3 grid in ${\mathbb{N}_0}^k$, we are asked to join all the points of G_k with a minimum (link) length covering trail $C := C(k)$ ($C(k)$ represents any trail consisting of $h(k)$ lines), without letting one single line of C go outside of a 3 X 3 X … X 3 k-dimensional (hyper-)box (i.e., remaining inside a 4 X 4 X … X 4 grid in $\mathbb{Z}^k$, which strictly contains G_k, and we call it *box*)".

It is trivial to note that the formulation of our problem is equivalent to asking:

"Which is the minimum number of turns ($h(k) - 1$) in order to visit (at least once) all the points of the k-dimensional regular grid G_k with a connected series of line segments (i.e., a possibly self-crossing polygonal chain allowed to turn at nodes and at Steiner points)?" [1, 14].

In the present paper, our goal is to definitely solve the 3^k-points problem for any $k \in \mathbb{N} - \{0\}$. We introduce a general algorithm, that we name as the *clockwise-algorithm*, which produces minimum length trails $C(k)$ for the 3^k-points problem. In particular, we show that $C(k)$ has $h(k) = \frac{3^k-1}{2}$ lines, answering to the most spontaneous 106 years old question which arose from the original Loyd's puzzle [10].

The aspect of the 3^k-points problem that most amazed us, when we eventually solved it, is the central role of Loyd's expected solution for the $k = 2$ case. In fact, the clockwise-algorithm, able to solve the main problem in a k-dimensional space, is the natural generalization of the classic solution of the nine dots puzzle.

3k-points problem

The stated 3^k-points optimization problem, especially for $k < 4$, appears to have concrete applications in manufacturing, drone routing, cognitive psychology, and integrated circuits (VLSI design). Many suboptimal bounds have been proved for the NP-complete [4] 3^k-points problem under additional constraints (such as limiting the solutions to Hamiltonian's paths or considering only rectilinear spanning paths [2, 6, 9]), but (to the best of our knowledge) the $3^{k>3}$-points problem remains unsolved to the present day, and this article provides its first exact solution so far [12].

2.1 A tight lower bound

Given the 3^k-points problem as introduced in Section 1, if we remove its constraint on the inside the box solutions, then we have that a lower bound is provided by Theorem 1.

Theorem 1. For any $k \in \mathbb{N} - \{0\}$, $h(k) \geq \frac{3^k-1}{2}$.

Proof. If $k = 1$, then it is necessary to spend (at least) 1 line to join the 3 points.

Given $k = 2$, we already know that the nine points problem cannot be solved with less than 4 lines (see [7], assuming $n = 3$).

Let k be greater than 2. We invoke the proof of Theorem 1 in [12], substituting $n_i = 3$. Thus, equation (4) of [12] can be rewritten as

$$h_l(3_1, 3_2, \ldots, 3_k) = \left\lceil \frac{3^k-1}{2} \right\rceil, \quad (1)$$

which is an integer (since $3^k - 1$ is always even).

Therefore, $h(k) \geq h_l(3_1, 3_2, \ldots, 3_k) = \frac{3^k-1}{2}$ for any (strictly positive) natural number k. □

It is redundant to point out that Theorem 1 provides also a valid lower bound for the standard 3 X 3 X … X 3 box constrained 3^k-points problem. The purpose of Section 2.2 is to show that this bound matches $h(k)$ for any k.

2.2 The clockwise-algorithm

In order to introduce the clockwise-algorithm, let we begin from the trivial case $k = 1$. This means that we have to visit 3 collinear points with a single line, remaining inside a unidimensional box which is 3 units long.

One solution is shown in Figure 1.

Figure 1. Solving the 3 X 1 puzzle inside the box (3 units of length), starting from one of the line segment endpoints. The puzzle is solvable with this $C(1)$ path starting from any of the two red points.

Considering the spanning path by Figure 1, it is easy to see that we cannot solve the 3^1-points problem starting from one point of G_1 iff this point is the central one.

Given $k = 2$, we are facing the classic nine dots puzzle considering a 3 X 3 box (9 units of area). The well-known Hamiltonian path shown in Figure 2 proves that we can solve the problem, without allowing any line to exit from the box, if we start from any node of G_2 except from the central one [7].

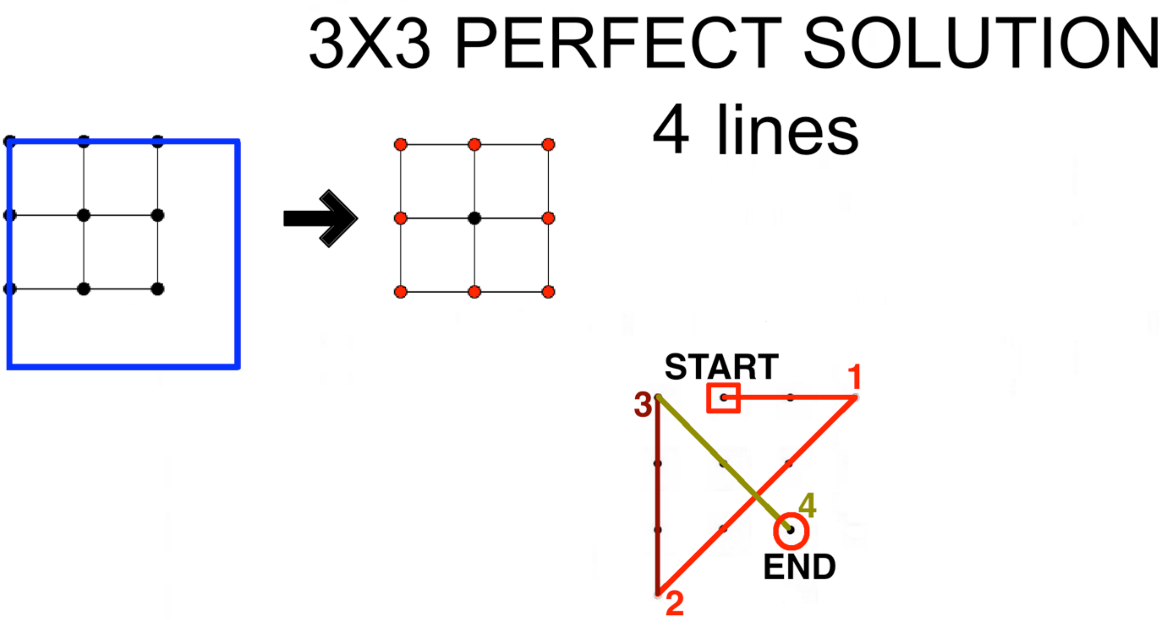

Figure 2. $C(2)$ is a path that consists of $h(2) = \frac{3^2-1}{2}$ lines. In order to solve the 3 X 3 puzzle with 4 lines starting from one node of G_2, it is necessary to avoid to start from the central point of the grid.

Looking carefully at $C(2)$, as shown in Figure 2, we note that line 1 includes $C(1)$ if we simply extend it by one unit backward. Thus, $C(1)$ and the first line of $C(2)$ are essentially the same trail and so they are considering the clockwise-algorithm. Line 2 can be obtained from line 1 going backward when we apply a standard rotation of $\frac{\pi}{4}$ radians: we are just spinning around in a two-dimensional space, forgetting the $3^{2-1}-1$ collinear points that will later be covered by the repetition of $C(1)$ following a different direction. Now, we are able to understand what line 3 really is: it is just a link between the repeated $C(2-1)$ trail backward and the final $C(2-1)$ trail following the new direction. In general, the aforementioned link corresponds to line $2 \cdot h(k-1)+1 = 3^{k-1}$ of any $C(k)$ generated by the clockwise-algorithm.

Definition 1. Let G_3 be the 3 X 3 X 3 regular grid in $\mathbb{N}_0{}^3$. We call "nodes" all the 27 points of G_3, as usual. In particular, we indicate the nodes $V_1 \equiv (0,0,0)$, $V_2 \equiv (2,0,0)$, $V_3 \equiv (0,2,0)$, $V_4 \equiv (0,0,2)$, $V_5 \equiv (2,2,0)$, $V_6 \equiv (2,0,2)$, $V_7 \equiv (0,2,2)$, $V_8 \equiv (2,2,2)$ as "vertices", we indicate the nodes $F_1 \equiv (1,1,0)$, $F_2 \equiv (1,0,1)$, $F_3 \equiv (0,1,1)$, $F_4 \equiv (2,1,1)$, $F_5 \equiv (1,2,1)$, $F_6 \equiv (1,1,2)$ as "face-centers", we call "center" the node $X_3 \equiv (1,1,1)$, and we indicate as "edges" the remaining 12 nodes of G_3.

Now, we are ready to describe the generalization of the original Loyd's covering trail to a higher number of dimensions. Given $k=3$, a minimum length covering trail has already been shown in [12], but this time we need to solve the problem inside a 3 X 3 X 3 box. Our strategy is to follow the optimal two-dimensional covering trail (see Figure 2) swirling in one more dimension, according to the 3-steps scheme given by lines 1 to 3 of $C(2)$, and beginning from a congruent starting point.

Thus, we take one vertex of G_3 and, while we rotate in the space at every turn (as observed for $k=2$), it is possible to repeat twice (forward and backward) the whole $C(2)$ or, alternatively (Figure 3), we can follow $\frac{8}{3}$ times the scheme provided by its lines 1 to 3. In both cases, at the end of the process, $3^{3-2}-\frac{1}{3}$ gyratories have been performed, so we spend the (3^{3-1})-th line to close the subtour ($C(3)$ can never be a cycle plus we avoided to extend its first line backwards, but we have already seen that this fact does not really matter), joining $3-1$ new points. In this way, we reach the "starting vertex" again, and the last 3^3-1 unvisited nodes belong only to $G_{k-1}=G_2$ (choosing the right direction). Therefore, we can finally paste $C(2)$ (Figure 2) by extending one unit backward its first line (the new $(2 \cdot h(3-1)+2)$-th line) in order to visit every 3^2 nodes of G_{3-1}.

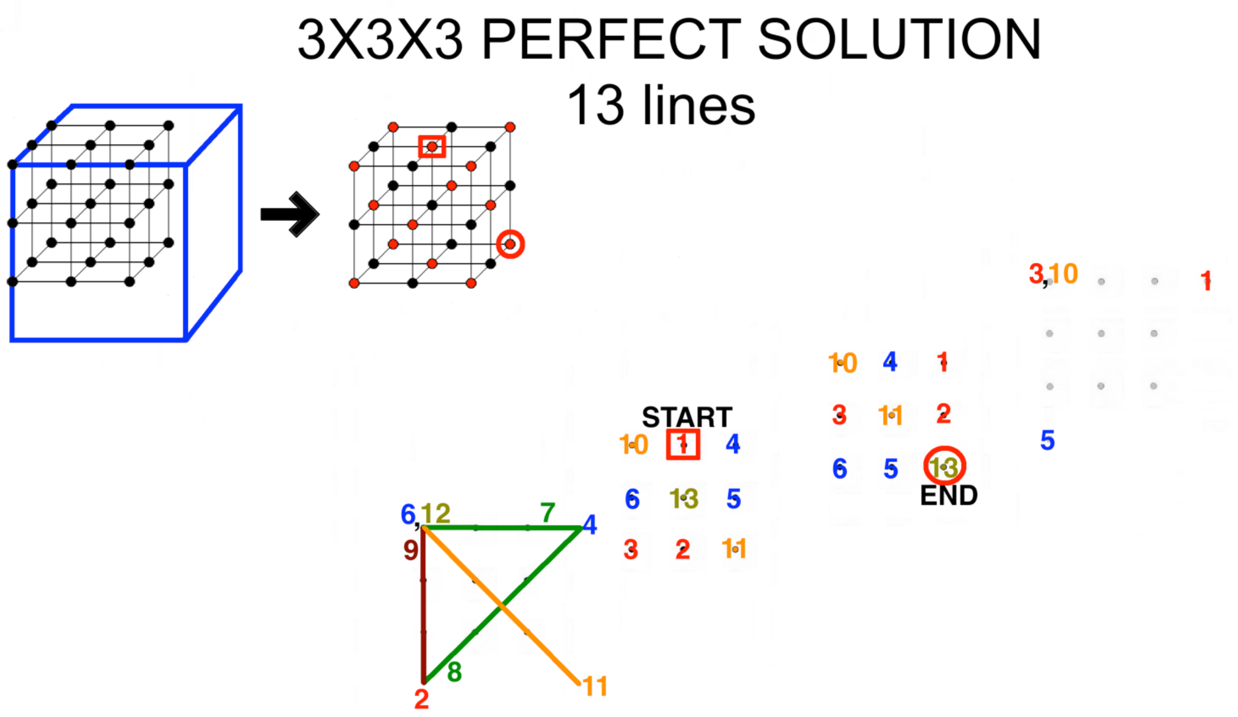

Figure 3. $C(3)$ solves the 3 X 3 X 3 puzzle inside a 3 X 3 X 3 box (27 cubic units of volume), starting from face-centers or vertices, thanks to the clockwise-algorithm.

Before moving on $k = 4$, we wish to prove that the 3^3-points problem is solvable starting from any node of G_3 if we exclude the center of the grid (as we have previously seen for $k \in \{1, 2\}$). This result immediately follows by symmetry when we combine the trails shown in Figures 3&4.

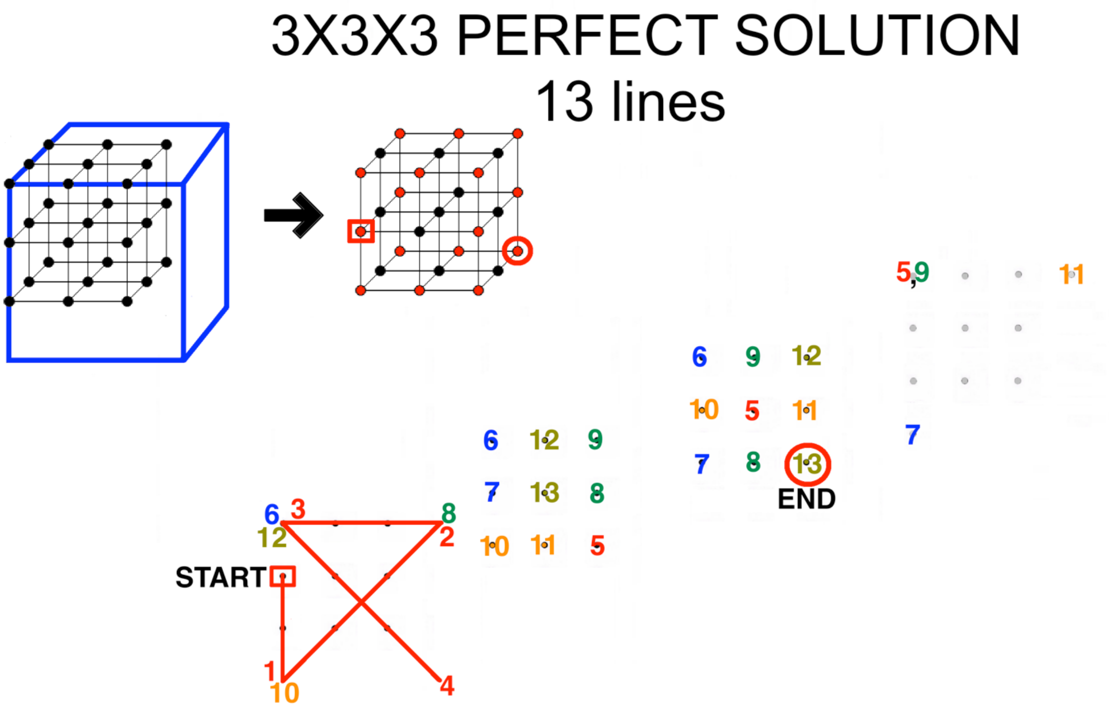

Figure 4. Solving the 3 X 3 X 3 puzzle inside a 3 X 3 X 3 box (27 cubic units of volume), starting from edges or vertices.

The number of $\frac{3^k-1}{2}$ lines solutions increases as k grows. Moreover, if we remove the box constraint, we are able to find new minimal covering trails [12], including those that reproduce (on a given 3 X 3 subgrid of G_3) the endpoints by Figure 2, as shown in Figure 5.

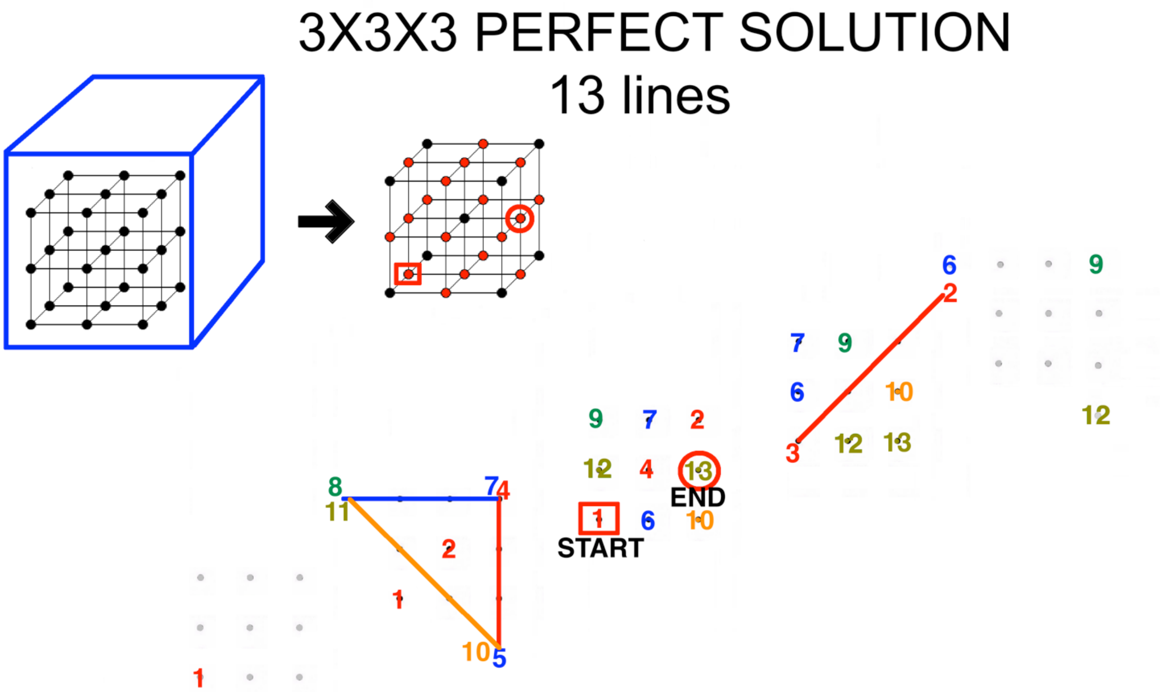

Figure 5. Solving the 3 X 3 X 3 puzzle inside a 3 X 3 X 4 box (36 cubic units of volume).

Finally, we present the solution of the 3^4-points problem. Two examples of minimum length covering trails generated by the clockwise-algorithm are given. The method to find $C(4)$ is basically the same that we have previously discussed for G_3. So, we utilize the standard pattern shown in Figure 3 as we used $C(2)$ in order to solve the 3^3-points problem. We apply $C(3)$ forward (while we spin around following the 3-steps gyratory as shown in Figure 6), then backward (Figure 7), subsequently we return to the "starting point" with line 27 (the $(2 \cdot h(4-1)+1)$-th link), and lastly we join the 3^3-1 unvisited points with $C(3)$ by simply extending backward its first line (corresponding to the 28-th link of $C(4)$ - see Figure 8).

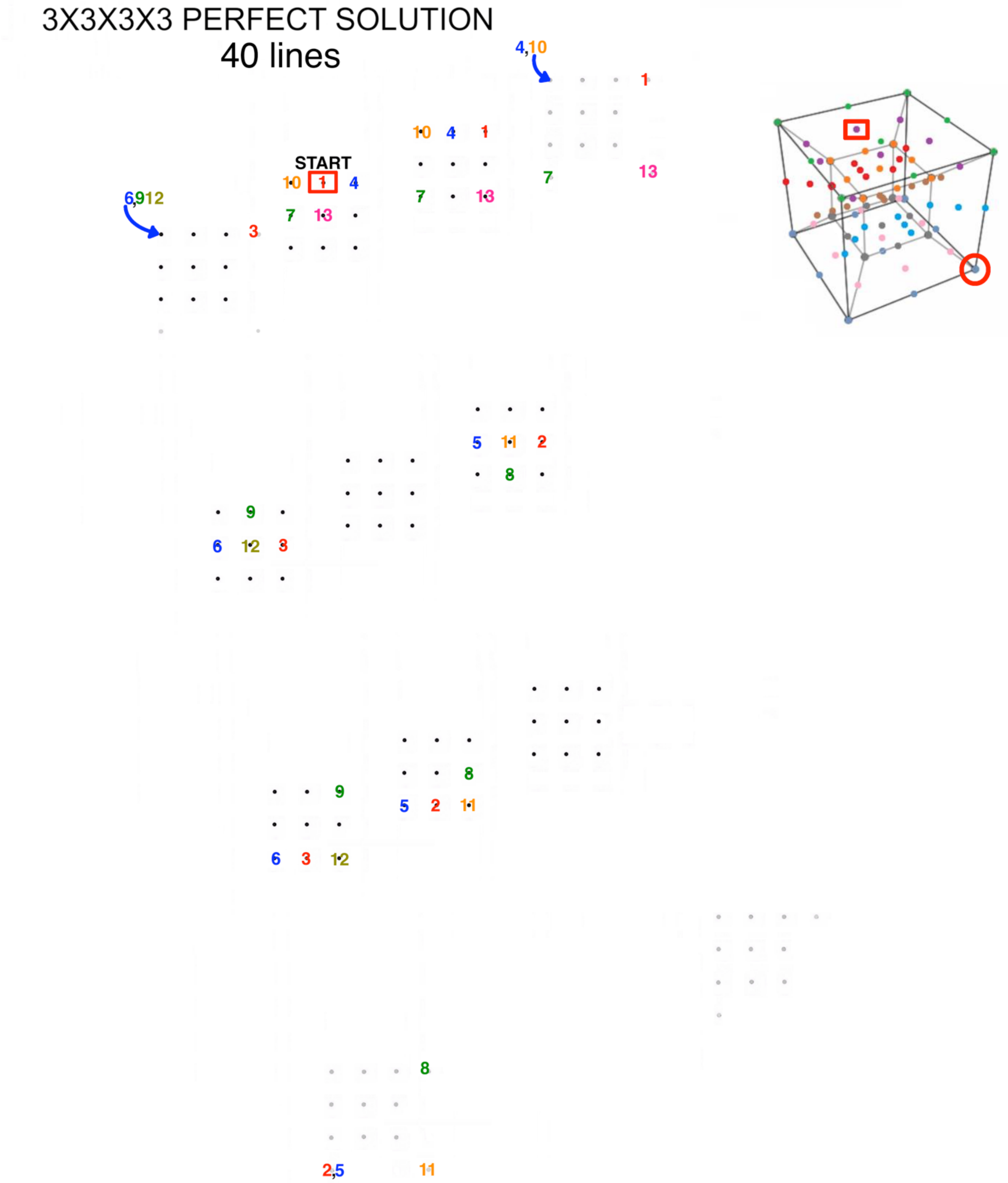

Figure 6. Lines 1 to 13 of $C(4)$ following $C(3)$, as shown in Figure 3.

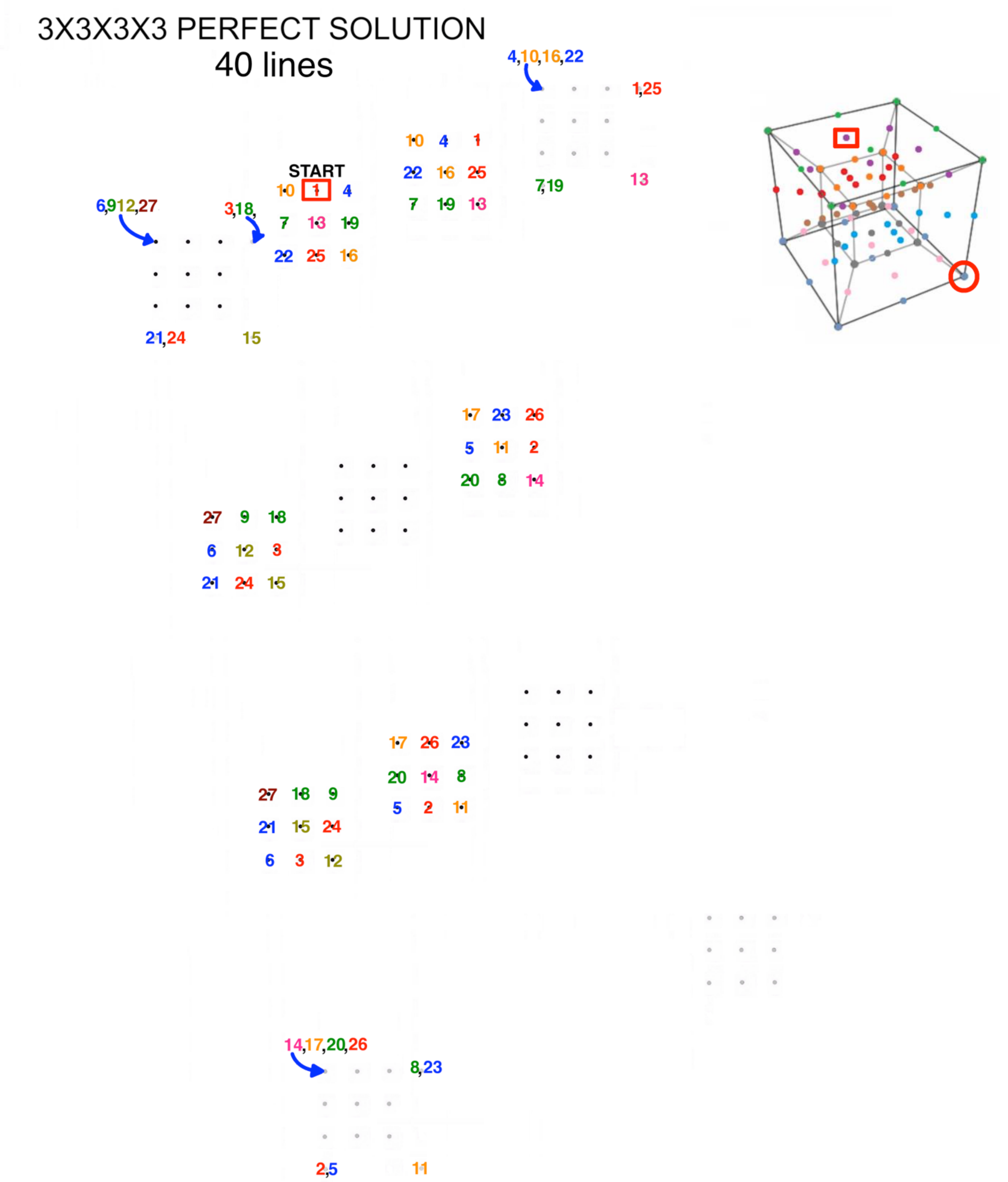

Figure 7. Lines 14 to 27 of $C(4)$ following $C(3)$ backward, the 27-th link to come back to the "starting point" is also included.

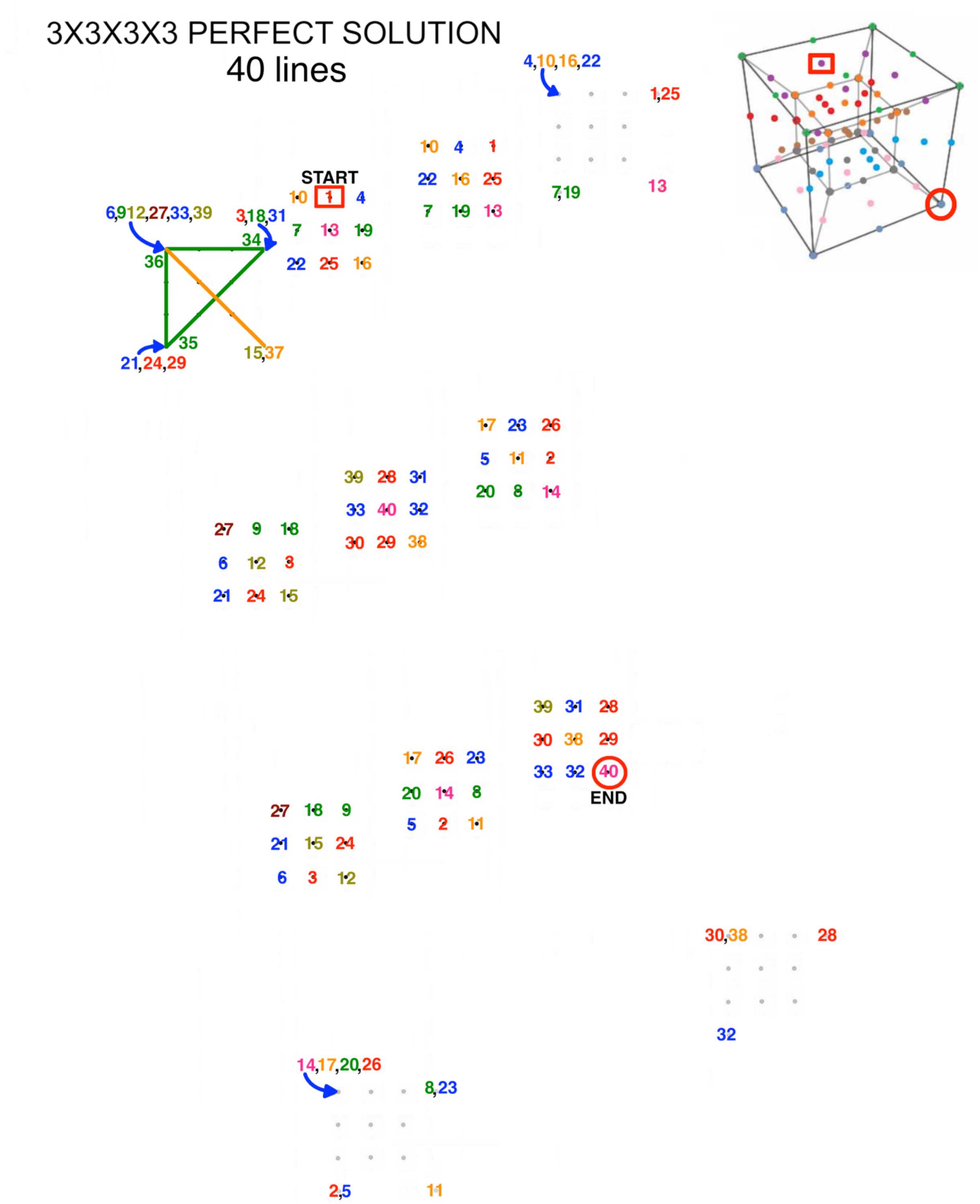

Figure 8. A minimum length covering trail that completely solves the 3 X 3 X 3 X 3 puzzle with 40 lines, inside a 3 X 3 X 3 X 3 box (hyper-volume 81 units^4), thanks to the clockwise-algorithm applied to $C(3)$ from Figure 3.

The clockwise-algorithm reduces the complexity of the 3^k-points problem to the complexity of the 3^{k-1}-points one. A clear example is shown in Figure 9 .

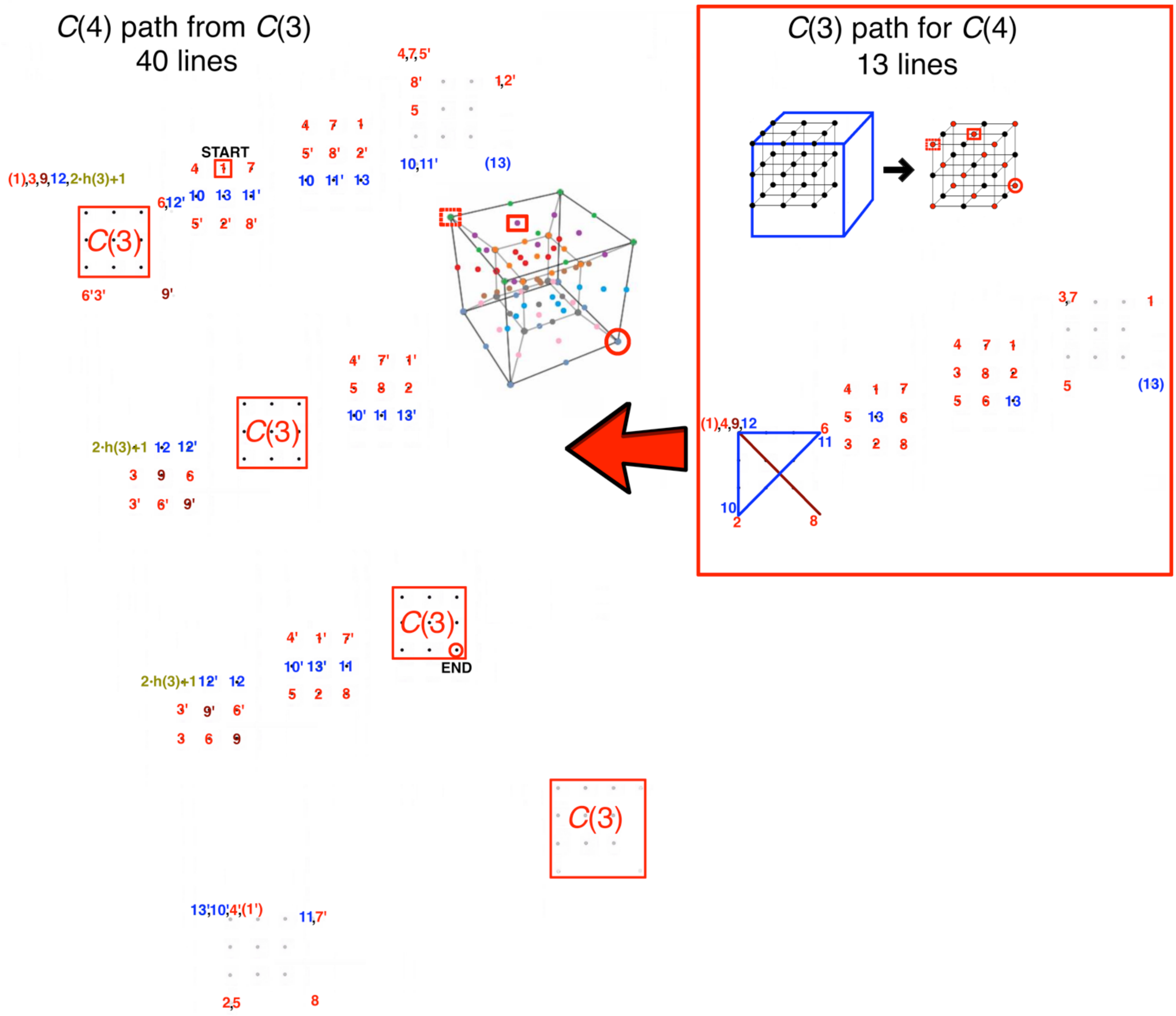

Figure 9. How the clockwise-algorithm concretely works: it takes a minimum length covering trail $C(3)$ as input, and returns $C(4)$. Lines 1-13 belong to the covering trail $C(3)$ (shown in the upper-right quadrant), line 13' follows line 13 and belongs to $C(3)$ backward. $C(3)$ backward ends with line 1': it is extended (by one unit) in order to be connected to the $(2 \cdot h(3^3) + 1)$-th link, and this allows $C(3)$ to be repeated one more time (joining the remaining 26 unvisited points).

Since the clockwise-algorithm takes $C(k-1)$ as input and returns $C(k)$ as its output, it can be applied to any $C(k)$ in order to produce some $C(k+1)$ consisting of $h(k+1) = 3 \cdot h(k) + 1$ lines. In this way, we have shown that the 3^k-points problem can be solved, inside a 3 X 3 X ... X 3 box of hyper-volume 3^k units^k, drawing optimal trails with $3 \cdot h(k-1) + 1$ lines, for any $k > 1$.

Therefore, $\forall k \in \mathbb{N} - \{0\}$,

$$h(k+1) = 3 \cdot h(k) + 1 = \frac{3^{k+1}-1}{2}. \tag{2}$$

3 Conclusion

Given the k-dimensional grid G_k, the clockwise-algorithm let us easily draw different covering trails of $\frac{3^k-1}{2}$ lines, and all of them remain inside the box. After the $(3^k - 1)$-th link, it is possible to switch from the previously applied $C(k-1)$ to another known solution of the 3^{k-1}-points problem, completing a new optimal trial with one different endpoint (e.g., we can take the walk shown in Figure 7 and then apply $C(3)$ from Figure 9).

Let $X_k \equiv (1, 1, \ldots, 1)$ be the central node of G_k (see Definition 1 for the case $k = 3$). We conjecture that, $\forall k \in \mathbb{N} - \{0\}$, the 3^k-points problem is solvable (embracing also every outside the box optimal trail) starting from any node of $G_k - \{X_k\}$ with a covering trail of length $h(k) = \frac{3^k - 1}{2}$, while it is not if we include X_k as an endpoint of $C(k)$.

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