Nonlinearity, *m-Stability* And Properties Of $x^a+y^b+z^c=m$.

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Abstract.
The equation $x^a+y^b+z^c=m$ in real numbers is an ill-posed problem and can be used in physical sciences and social sciences. This article introduces some properties of $x^a+y^b+z^c=m$ in real numbers, and also defines "***m-Stability***", a new measure of stability of nonlinear equations.

Keywords: *Nonlinearity*; *Stability*; Number Theory; Dynamical Systems, Prime Numbers; Mathematical Cryptography; Ill-posed Problems.

1. Introduction.

While the concept of "stability" has been widely analyzed in Math and Mathematical Physics, the properties and nature of Stability have often been generalized. This article develops properties of the equation $x^a+y^b+z^c=m$ that either pertain to *Stability*, or can help in developing additional *Stability* properties, or illustrate *Nonlinearity* (and conditions for non-Homomorphisms) in the equation. On Homomorphisms, see: Wang & Chin (2012).

Chu (2008) and Lu & Wu (2016) studied dynamical systems pertaining to Diophantine equations (and $x^a+y^b+z^c=m$ can approximate a Dynamical System). Luca, Moree & Weger (2011) discussed *Group Theory*. Elia (2005), Jones, Sato, et. al. (1976) and Matijasevič (1981) noted that primes can also be represented as Diophantine Equations or as polynomials (ie. the equation $x^a+y^b+z^c=m$ can represent a prime). On uses of Diophantine Equations in Cryptography, see: Ding, Kudo, et. al. (2018), Okumura (2015), and Ogura (2012) (ie. the equation $x^a+y^b+z^c=m$ can be used to create public-keys and in cryptoanalysis). Zadeh (2019) notes that Diophantine equations have been used in analytic functions.

On approaches to solving Diophantine Equations, see: Rahmawati, Sugandha, et. al. (2019), and Ibarra & Dang (2006).

Definition-1: The equation $x^a+y^b+z^c=m$ in positive integers is said to be "**m-stable**" *iff*:

i) Small changes in the bases of any of the exponential variables (ie. Δx, Δy, Δz) cause relatively small changes in the dependent variable *m* (Δm).

ii) Changes in the exponents (in the exponential variables - ie. Δa, Δb, Δc) cause proportionally less corresponding changes in the dependent variable *m*, than similar/proportional percentage changes in the exponential variables (Δx^a, Δy^b, Δz^c).

iii) As *x*, *y* and *z* tend to positive infinity, corresponding changes in the absolute values of *m* decrease; and vice-versa.

Theorem-1: For all a,b,c,x, m, y and z in positive integers, and d,e,g in real numbers, if $x^a+y^b+z^c = m$, then $m \geq (x+y+z)n$, where $1\leq n \leq \mathrm{Max}([(x^a+y^b+z^c)/(x+y+z)], \max\{x^a,y^b,z^c\})$ and *n* is some real number.

Proof:

n has to be greater than zero otherwise the condition $m \geq (x+y+z)n$, will not exist.

If $m \geq (x+y+z)n$, then *n* is a multiplicative component of m (ie. $n \varepsilon m$), and each of x,y and z can be defined as some multiple of *n*.

Thus, let:
m = (nxd), and x=m/nd
m= (nye), and y= m/ne
m = (nzg), and z = m/ng
Where d,e,g are real numbers.

Thus by substitution, $[(m/nd)^a+(m/ne)^b+(m/ng)^c] = m$; and $m^2/n^2d^2 =x^2$; and $m^2/n^2e^2= y^2$; and $m^2/n^2g^2 = z^2$.

The condition (n,d,e,g >0), must exist in order for (m,x,y,z>0) to be positive integers. Thus, the lower-bound of *n* is one (1).

If $x^a+y^b+z^c = m$, and x,y,z,=1,1,1, then the lower-bound of *n* is one (1).

If $x^a+y^b+z^c = m$, and x,y,z,=1,1,1, and m ≥ (x+y+z)n, then n ε (1, $Max\{x^a,y^b,z^c\}$); and an upper-bound of *n* is $Max\{x^a,y^b,z^c\}$.

And because (m,n,d,e,g,x,y,z>0), and $(m/nd)^a =x^a$; and $(m/ne)^b = y^b$; and $(m/ng)^c = z^c$; then: m>x,y,z.
If a,b,c>0, then m≥ (x+y+z) and: m≥ [m/nd) +(m/ne)+(m/ng)]

Because: n=m/xd = m/ye = m/zg, and substituting n into m≥ (x+y+z)n,
m≥ (x+y+z)n is equivalent to: m≥ [(m/d)(xd/m)] +[(m/e)(ye/m)]+[(m/g)(zg/m)]
which is equivalent to: m≥ (mxd/dm) +(mye/em)+(mzg/gm)
which is equivalent to: m≥ [(x) +(y)+(z)]
Thus, m≥ (x+y+z)n can be valid.

If: $x^a+y^b+z^c = m$, and m ≥ (x+y+z)n, then:
$x^a+y^b+z^c \geq (x+y+z)n$, and:
$[(x^a+y^b+z^c)/(x+y+z)] \geq n$, is an upper bound for n.

As a,b,c→ +∞, $[(x^a+y^b+z^c)/(x+y+z)] \to +\infty$; and $x^a/(x+y+z)$, $y^b/(x+y+z)$, $z^c/(x+y+z) \to +\infty$, regardless of the magnitudes of x,y and z; and for increasing and higher values of Max(a,b,c) and an upper-bound for *n* is $Max(x^a,y^b, z^c)$. ■

Theorem-2: **For all a, b, c, m, x, y and z that are positive integers, and *n* is a real number, if $x^a+y^b+z^c = m$; and a=b=c, then $(m^{1/a}= m^{1/b}=m^{1/c}) \geq x,y,z$.**
Proof:
$x^a+y^b+z^c = m = x^a+y^a+z^a$ because a=b=c.
$m^{1/a} = (x^a+y^a+z^a)^{1/a}$

Let: $m^{1/a}=m^{1/b}=m^{1/c} = n$.
Thus, $n^a = (x^a+y^a+z^a)$

Because each of x, y and z are raised to the same power as *n*, each of *x*, *y* and *z* is implicitly represented in *n*, and none of *x*,*y* and *z* can be greater than *n*. That is, *n* is said to be an implicit "*Exponential Averaging*" of x, y and z. The main effect of *Exponential Averaging* (in equation $n^a=[x^a+y^a+z^a]$) is that *n* → Max(x,y,z) for all x,y, and z in positive integers. However, as x,y,z → +∞, and as (a=b=c) → +∞, n → Max(x,y,z) and n≥ x,y,z. ■

Theorem-3: **Where a,b,c,g,m,x, y and z are positive integers, if $x^a+y^b+z^c = m$; and x=y=z; then there can be a positive integer g, where $m = x^g=y^g=z^g$, and g≥ a,b,c.**
Proof:
$x^a+y^b+z^c = m = x^g$

$x^a+x^b+x^c = x^g$
and $a+b+c=g$
Thus, $g\geq a,b,c$. ■

Theorem-4: **Where a,b,c,j,n,m,x, y and z are positive integers, if $x^a+y^b+z^c = m$; and $x=y=z=a=b=c=n$; then:**

- **i) xyz<m; and**
- **ii) for some real number j>n, $(xyz)^{j-n} = m$.**

Proof:
$x^a+y^b+z^c = m = x^x+y^x+z^x$
$= [(x^{2a}y^bz^c+ x^ay^{2b}z^c+x^ay^bz^{2c})/(x^ay^bz^c)] = [(x^{2a}y^bz^c+ x^ay^{2b}z^c+x^ay^bz^{2c})/(xyz)^a] = [(x^{4a}+y^{4a}+z^{4a})/(xyz)^a] =$
$[(x^{4a}+x^{4a}+x^{4a})/x^{3a}] = [(3x^{4a})/x^{3a}] = 3x^a = 3xyz = m$
Therefore: xyz<m.

If: $(x^{2a}y^bz^c+ x^ay^{2b}z^c+x^ay^bz^{2c})/(xyz)^n = m$; and
If: $(x^{2a}y^bz^c+ x^ay^{2b}z^c+x^ay^bz^{2c}) = (x^{2a}y^az^a+ x^ay^{2a}z^a+x^ay^az^{2a}) = (x^{4a}+y^{4a}+z^{4a})$
Then let: $(x^{4a}+y^{4a}+z^{4a}) = (xyz)^j$
Where: $(x^{4a}+y^{4a}+z^{4a}) = (xyz)^j$, and $(xyz)^j = x^{3j}$
Then: $(x^{4a}+x^{4a}+x^{4a}) = 3x^{4a} = x^{3j}$
Thus: $3x^{4a}/(xyz)^n = 3x^{4a}/x^{3n} = m$; and $x^{3j}/(xyz)^n = x^{3j}/x^{3n} = x^{3(j-n)} = m$
Thus, $x^{3(j-n)} = m = (xyz)^{j-n}$; and its easy to see that *j* must be greater than *n* in order for *m* to be a positive integer.

Also, $3x^{4a}/x^{3n} = 3(x^{4a-3n}) = 3(x^{4a-3a}) =3(x^a) = m$; and again xyz<m. ■

Theorem-5: **For all a,b,c,f,n,x, y and z in positive integers, if $x^a+y^b+z^c = m = n^f$; and a=b=c, then:**

i) if $n \leq x,y,z$; then $f\geq a,b,c$; and
ii) if $n \geq x,y,z$; then $f\geq a,b,c$.

Proof:
$x^a+y^b+z^c = m = n^f$;
Thus, $[x^a+y^a+z^a](x^ay^az^a) = n^f(x^ay^az^a)$
Assume that f = ae = be = ce, where *e* is some real number.
$x^a+y^a+z^a = n^{(ae)}$
if e=1, then *n* must be equal to or greater than x,y and z (see **Theorem-2** above).
If n<x,y,z, then e>1 due to a different type of *Exponential Averaging* wherein as $(x,y,z>n) \rightarrow +\infty$, and as $(a=b=c) \rightarrow +\infty$, $e \rightarrow +\infty$. ■

Theorem-6: Where g,a,b,c,m,x, y and z are positive integers each of which may or may not be equal to the other, if $x^a+y^b+z^c = m$, then:

- **i) (xyz) > m, *iff*:**
  - **1) $[x^a/(xyz)]$, $[y^b/(xyz)]$, $[z^c/(xyz)]$ <1; and**
  - **2) $x\leq \sqrt[a]{[xyz-y^b-z^c]}$, and $y\leq \sqrt[b]{[xyz-x^a-z^c]}$, and $z\leq \sqrt[c]{[xyz-x^a-y^b]}$; and**
  - **3) $x^{(a-1)}\leq zy$, and $y^{(b-1)}\leq xz$, and $z^{(c-1)}\leq xy$.**
- **ii) there can be a positive integer *h* such that $xyz = m^h$, *iff*:**
  - **1) $m\geq 1$; and**
  - **2) $[x^{[a-(1/h)]}/(y^{(1/h)}z^{(1/h)})]$, $[y^{[b-(1/h)]}/(x^{(1/h)}z^{(1/h)})]$, $[z^{[c-(1/h)]}/(x^{(1/h)}y^{(1/h)})]$ <1; and**
  - **3) $[a-(1/h)]$, $[b-(1/h)]$, $[c-(1/h)] \leq 0$.**

Proof:
if $x^a+y^b+z^c = m$; and x,y,z=1,1,1, then the lower bound of (xyz) is one (1).

if $x^a+y^b+z^c = m$; and a,b,c=1,1,1, then the lower bound of (xyz) is (xyz) which can be greater than m (thus this is a lower-bound condition for **(xyz) > m**).

if $x^a+y^b+z^c < xyz$, then:

$(x^a+y^b+z^c)/(xyz) < 1$,

$[x^a/(xyz)], [y^b/(xyz)], [z^c/(xyz)] < 1$ (Condition-1A)

$x^a \leq [xyz-y^b-z^c]$, and $x \leq \sqrt[a]{[xyz-y^b-z^c]}$; (Condition-1B)

$y^b \leq [xyz-x^a-z^c]$, and $y \leq \sqrt[b]{[xyz-x^a-z^c]}$; (Condition-1B)

$z^c \leq [xyz-x^a-y^b]$, and $z \leq \sqrt[c]{[xyz-x^a-y^b]}$; (Condition-1B)

If: $[x^a/(xyz)], [y^b/(xyz)], [z^c/(xyz)] < 1$, then:

$[x^a/(xyz)] \leq 1$, and thus $[x^a/(x)] \leq zy$, and thus $[x^{(a-1)}] \leq zy$; (Condition-1C)

$[y^b/(xyz)] \leq 1$, and thus $[y^b/(y)] \leq xz$, and thus $[y^{(b-1)}] \leq xz$; (Condition-1C)

$[z^c/(xyz)] \leq 1$, and thus $[z^c/(z)] \leq xy$, and thus $[z^{(c-1)}] \leq xy$; (Condition-1C)

Since all of the variables are positive integers, Conditions 1A and 1B are necessary size constraints for $x^a+y^b+z^c = m = xyz$. Condition-1C ensures that none of the three exponential variables on the left-side of $[x^a/(xyz)], [y^b/(xyz)], [z^c/(xyz)] < 1$, are greater than one.

With regards to **Theorem-6(ii)**:
If: $x^a+y^b+z^c = m$, and $xyz = m^h$, then:

$xyz^{1/h} = m$,

$x^a+y^b+z^c = (xyz)^{1/h}$,

$[x^a/(x^{(1/h)}y^{(1/h)}z^{(1/h)})]+[y^b/(x^{(1/h)}y^{(1/h)}z^{(1/h)})]+[z^c/(x^{(1/h)}y^{(1/h)}z^{(1/h)})] = 1$;

$[x^{[a-(1/h)]}/(y^{(1/h)}z^{(1/h)})]+[y^{[b-(1/h)]}/(x^{(1/h)}z^{(1/h)})]+[z^{[c-(1/h)]}/(x^{(1/h)}y^{(1/h)})]=1$,

Thus: $x^a+y^b+z^c = m$, and $xyz = m^h$ can be valid only if:

$m \geq 1$; (Condition-2A); and

$[x^{[a-(1/h)]}/(y^{(1/h)}z^{(1/h)})], [y^{[b-(1/h)]}/(x^{(1/h)}z^{(1/h)})], [z^{[c-(1/h)]}/(x^{(1/h)}y^{(1/h)})] < 1$; (Condition-2B); and

$[a-(1/h)], [b-(1/h)], [c-(1/h)] \leq 0$; (Condition-2C)

Since the variables are positive integers, Conditions 2A and 2B are necessary to ensure that none of the three exponential variables on the left-side of $[x^{[a-(1/h)]}/(y^{(1/h)}z^{(1/h)})]+[y^{[b-(1/h)]}/(x^{(1/h)}z^{(1/h)})]+[z^{[c-(1/h)]}/(x^{(1/h)}y^{(1/h)})]= 1$, are greater than one. Condition-2C is similar, and ensures that none of the three exponential variables on the left-side of equation $[x^{[a-(1/h)]}/(y^{(1/h)}z^{(1/h)})]+[y^{[b-(1/h)]}/(x^{(1/h)}z^{(1/h)})]+[z^{[c-(1/h)]}/(x^{(1/h)}y^{(1/h)})]= 1$, are negative numbers. ∎

Theorem-7: **Where a,b,c,m,x, y and z are positive integers, each of which may or may not be equal to the other, and if $x^a+y^b+z^c = m$, then:**

i) (xyz) ϵ (1,m) *iff*:

1) $x=y=z \geq 3$, and

2) $a=b=c \leq 2$, and

3) $[x/yz], [y/xz], [z/xy] < 1$, and

4) $(m^a/xy^{(a+1)}z^{(a+1)}), (m^b/yx^{(b+1)}z^{(b+1)}), (m^c/zx^{(c+1)}y^{(c+1)}) < 1$

ii) there can be a positive integer *j* such that $xyz=m^{1/j}$, *iff*:

1) $j \geq 1$; and

2) $[x^a/(x^jy^jz^j)], [y^b/(x^jy^jz^j)], [z^c/(x^jy^jz^j)] < 1$; and

3) $(a-j), (b-j), (c-j) \leq 0$.

Proof:
if $x^a+y^b+z^c = m$; and x,y,z=1,1,1, then the lower bound of (xyz) is one (1).

If $(xyz) \leq m$, then:
$x= m/yz$

$y = m/xz$
$z = m/yx$
Thus: $(m^a/y^az^a) + (m^b/x^bz^b) + (m^c/x^cy^c) = xyz$
$(m^a/xy^{(a+1)}z^{(a+1)})+(m^b/yx^{(b+1)}z^{(b+1)})+(m^c/zx^{(c+1)}y^{(c+1)}) = 1$

Also:
$x^a+y^b+z^c = m= xyz$
$(x^a/xyz)+(y^b/xyz)+(z^c/xyz) = 1$
$(x/yz) + (y/xz) + (z/xy) = 1$,
Thus: $(x/yz) + (y/xz) + (z/xy)= 1$, and $(xyz) \leq m$, and $x^a+y^b+z^c = m$ can be valid *iff*:
$x=y=z \geq 3$ (Condition-1A); and
$a=b=c\leq 2$ (Condition-1B); and
$[x/yz], [y/xz], [z/xy] < 1$ (Condition-1C), and
$(m^a/xy^{(a+1)}z^{(a+1)}), (m^b/yx^{(b+1)}z^{(b+1)}), (m^c/zx^{(c+1)}y^{(c+1)}) < 1$ (Condition-1D).

Since all of the variables are positive integers, Conditions 1A and 1B are necessary size constraints for $x^a+y^b+z^c = m= xyz$.
Condition-1C ensures that none of the three exponential variables on the left-side of $(x/yz)+(y/xz)+(z/xy)= 1$, are greater than one. Condition-1D is similar and ensures that none of the three exponential variables on the left-side of $(m^a/xy^{(a+1)}z^{(a+1)})+(m^b/yx^{(b+1)}z^{(b+1)})+(m^c/zx^{(c+1)}y^{(c+1)})=1$, are greater than one.

With regards to **Theorem-7(ii)**:
If: $x^a+y^b+z^c = m$, and $xyz = m^{1/j}$, then:
$x^a+y^b+z^c = (xyz)^j$,
$[x^a/(x^jy^jz^j)] + [y^b/(x^jy^jz^j)] + [z^c/(x^jy^jz^j)] = 1$;
Thus: $[x^a/(x^jy^jz^j)] + [y^b/(x^jy^jz^j)] + [z^c/(x^jy^jz^j)]=1$, and $x^a+y^b+z^c = m$, and $xyz = m^{1/j}$ can be valid only if:
$j\geq 1$ (Condition-2A); and
$[x^a/(x^jy^jz^j)], [y^b/(x^jy^jz^j)], [z^c/(x^jy^jz^j)] < 1$ (Condition-2B); and
$(a-j), (b-j), (c-j) \leq 0$ (Condition-2C)

Since most of the variables are positive integers, Conditions 2A and 2B are necessary to ensure that none of the three exponential variables on the left-side of $[x^a/(x^jy^jz^j)] + [y^b/(x^jy^jz^j)] + [z^c/(x^jy^jz^j)] = 1$, are greater than one. Condition-2C is similar, and ensures that none of the three exponential variables on the left-side of equation $[x^a/(x^jy^jz^j)]+[y^b/(x^jy^jz^j)]+[z^c/(x^jy^jz^j)]=1$, are negative numbers. ■

Conclusion.

The dynamics of equation $x^a+y^b+z^c=m$ that are introduced herein may be applicable to equations that have similar structure and *Nonlinearity* potential. S*tability* remains a critical issue in various fields of endeavor and the properties discussed herein can help in such research.

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