Additive-Contingent Nonlinearity: On Some Properties Of $X^2+Y^2+Z^2+V^2$=dXYZ, and $X^2+Y^2+Z^2+V^2+U^2$=dXYZ, And The *Markoff Equation* $X^2+Y^2+Z^2$=dXYZ.

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Abstract.
Some properties of the equations $X^2+Y^2+Z^2+V^2$=dXYZ, and $X^2+Y^2+Z^2+V^2+U^2$=dXYZ, and the Markoff Equation $X^2+Y^2+Z^2=aXYZ$ in real numbers are analyzed in this article, and common elements are highlighted.

Keywords: Markoff Equation; *Nonlinearity*; Prime Numbers; Mathematical Cryptography; *Beal Conjecture*; Dynamical Systems; Ill-posed Problems.

1. Introduction.

The Markoff equation M_a: $X^2+Y^2+Z^2 =aXYZ$ is not new in the literature. In 1779, Euler studied the equation $X^2+Y^2+Z^2$, and derived a solution that was somewhat different from Markoff's solution. However, similar equations such as $x^2+y^2+z^2+v^2$=dXYZ, and $x^2+y^2+z^2+v^2+u^2$=dXYZ, have not been studied in as much detail.

The first novelty in this study of the three equations is that the scope of the solutions is real numbers and not only positive integers, and each of the three equations is an ill-posed problem because their behavior can change drastically over any range of real numbers. The second novelty in this study is that taken together the three equations $X^2+Y^2+Z^2 =aXYZ$, $x^2+y^2+z^2+v^2$=dXYZ, and $x^2+y^2+z^2+v^2+u^2$=dXYZ exhibit or can exhibit:

i) *Super-Additive Nonlinearity* and *Homomorphisms* – wherein as more variables are added to the left side of each equation, the greater the absolute amount of, and probability of Nonlinearity.

ii) *Contingent Nonlinearity* and *Homomorphisms* – wherein for each equation, the greater the absolute magnitudes of the independent variables (on the left side of each equation), the greater the Nonlinearity of the equation. Absolute Magnitude refers to magnitude of a variable without regard to its sign.

2. Existing Literature.

Silverman (1990) analyzed the Markoff Equation $X^2+Y^2+Z^2=aXYZ$, over quadratic imaginary fields. Jiang, Gao & Cao (2020), Abram, Lapointe & Reutenauer (2020), Bourgain, Gamburd & Sarnak (2016), Chen & Chen (2013), Gaire (2018), González-Jiménez & Tornero (2013), McGinn (2015), and Togbe, Kafle & Srinivasan (2020) analyzed the *Markoff Equation* $X^2 + Y^2 + Z^2$ = aXYZ.

On Homomorphisms, see: Wang & Chin (2012). Chu (2008) and Lu & Wu (2016) studied dynamical systems pertaining to Diophantine equations (and equations such as $x^2+y^2+z^2+v^2$=dXYZ, and $x^2+y^2+z^2+v^2+u^2$=dXYZ, and the *Markoff Equation* $X^2 + Y^2 + Z^2$ = aXYZ can approximate Dynamical Systems). Luca, Moree & Weger (2011) discussed *Group Theory* as it relates to Diophantine Equations. Elia (2005), Jones, Sato, et. al. (1976) and Matijasevič (1981) noted that primes can also be represented as Diophantine equations or as polynomials (ie. each of the equations $x^2+y^2+z^2+v^2$=dXYZ, and $x^2+y^2+z^2+v^2+u^2$=dXYZ can represent a prime). On uses of Diophantine Equations in Cryptography, see: Ding, Kudo, et. al. (2018), Okumura (2015), and Ogura (2012) (ie. the equations $x^2+y^2+z^2+v^2$=dXYZ, and $x^2+y^2+z^2+v^2+u^2$=dXYZ can be used in cryptoanalysis, and in the creation of public-keys). Zadeh (2019) notes that Diophantine equations have been used in analytic functions.

3. The Theorems.

Theorem-1A: **For the equation $X^2+Y^2+Z^2+V^2$= gXYZ in real numbers, if *a, b,j* and *c* are multiplicative components of X, Y, V and Z respectively (each of X, Y, V and Z are derived by multiplying *a, b, j* and *c* respectively by (*n-f*), another real number), and XYZ*g* = (*n-f*), for some real number *g*; then XYZg = (ea)*(fb)*(hc)*(kj), for some real numbers g, e, f , k and h.**
Proof:

XYZg = (n-f)
XYZg = $(n-f)^4$(abcj)g
XYZ = $(n-f)^4$(abcj)
(n-f) = $(n-f)^4$(abcj)g
Thus 1 = $(n-f)^3$(abcj)g

(ea)*(fb)*(hc) (kj)= (efhk)(abcj)
If XYZg = (ea)*(fb)*(hc)*(kj),
Then: (XYZg)/(abcj) = efhk
but: (efhk)(abcj) = $(n-f)^4$(abcj)g
Thus, (efhk) =$(n-f)^4$g = [(XYZ)/(abcj)]g
And: =$(n-f)^4$ = [(XYZ)/(abcj)]
From above, if XYZg = $(n-f)^4$(abcj)g; then XYZ = $(n-f)^4$(abcj) ■

Theorem-1B: For the equation $X^2+Y^2+Z^2$= gXYZ in real numbers, if *a*, *b and c* in positive real numbers are multiplicative components of X, Y and Z respectively (each of X, Y and Z are derived by multiplying *a*, *b* and *c* respectively by (*n-f*), another real number), and XYZ*g* = (*n-f*), for some real number *g*; then XYZg = (ea)*(fb)*(hc), for some real numbers g, e,f and h.
Proof:

XYZg = (n-f)
XYZg = $(n-f)^3$(abc)g
XYZ = $(n-f)^3$(abc)
(n-f) = $(n-f)^3$(abc)g
Thus 1 = $(n-f)^2$(abc)g

(ea)*(fb)*(hc) = (efh)(abc)
If XYZg = (ea)*(fb)*(hc),
Then: (XYZg)/(abc) = efh
but: (efh)(abc) = $(n-f)^3$(abc)g
Thus, (efh) =$(n-f)^3$g = [(XYZ)/(abc)]g
And: =$(n-f)^3$ = [(XYZ)/(abc)]
From above, if XYZg = $(n-f)^3$(abc)g; then XYZ = $(n-f)^3$(abc). ■

Theorem-1C: **For the equation $X^2+Y^2+Z^2+V^2+U^2$=gXYZ in real numbers, if *a, b,c, j and m* are multiplicative components of X, Y, Z, V and U respectively (each of X, Y, V and Z are derived by**

multiplying each of *a*, *b*, *c*,*j* and *m* respectively by (*n-f*), another real number), and XYZ*g* = (*n-f*), for some real number *g*; then XYZg = (ea)*(fb)*(hc)*(kj)*(rm), for some real numbers g, e,f , h, k and r.
Proof:

$XYZg = (n-f)$
$XYZg = (n-f)^5(abcjm)g$
$XYZ = (n-f)^5(abcjm)$
$(n-f) = (n-f)^5(abcjm)g$
Thus, $1 = (n-f)^4(abcjm)g$

(ea)*(fb)*(hc)*(kj)*(rm)= (efhkr)(abcjm)
If XYZg = (ea)*(fb)*(hc)*(kj)*(rm),
Then: (XYZg)/(abcjm) = efhkr
but: $(efhkr)(abcjm) = (n-f)^5(abcjm)g$
Thus, $(efhkr) =(n-f)^5g = [(XYZ)/(abcjm)]g$
And: $=(n-f)^5 = [(XYZ)/(abcjm)]$
From above, if $XYZg = (n-f)^5(abcjm)g$; then $XYZ = (n-f)^5(abcjm)$ ∎

Theorem-2A: For the equation $X^2+Y^2+Z^2+V^2= gXYZ$, and given **Theorem-1a**, for all values of X, Y, V, Z, g,n and f that are real numbers, the upper bound and the lower bound of *g* can be defined.
Proof:
$XYZg = (n-f)$
$XYZg = (n-f)^4(abcj)g$
$XYZ = (n-f)^4(abcj)$
$(n-f) = (n-f)^4(abcj)g$
$1= (n-f)^3(abcj)g$

In $X^2+Y^2+Z^2 +V^2= gXYZ$, g varies vary primarily with the magnitudes (and to a lesser extent, the signs) of X,Y, V and Z.

Thus, $g = 1/((n-f)^3(abcj))$,
As $(n-f) \rightarrow +\infty$, $(abcj)g \rightarrow -\infty$;

(n-f) = gXYZ, and (n-f) is a multiplicative component of each of X,Y and Z:
(n-f)a = X
(n-f)b = Y
(n-f)c = Z
(n-f)j=V
a = 1/YZg; and b =1/XZg; and c=1/XYg; and j= V/XYZg

Given X, Y, Z and V; then a,b, c and j can be determined by substituting a = 1/YZg, b = 1/XZg and c = 1/XYg, and j= V/XYZ, into X/a = Y/b = Z/c = V/j = (n-f) = XYZg

In $X^2+Y^2+Z^2 +V^2= gXYZ$, both *n* and *f* vary primarily with the magnitudes (and to a lesser extent, the signs) of X,Y and Z.

$XYZg = (n-f)^4(abc)g$
n = XYZg+f
$n = [XYZ/(abc)]^{1/4}+f$
Thus $XYZg = [XYZ/(abc)]^{1/4}$
$g = \{[XYZ/(abc)]^{1/4}\}/XYZ$ (referred to as “LB”)
but also $g = 1/[(n-f)^3(abc)]$ (referred to as “UB”)

As the denominator in UB tends to zero, *g* in UB can become greater than one and significant – that can occur if $0<a$, or b or $c<1$, and or if $0<(n-f)<1$.
As the denominator in UB tends to minus infinity from zero, *g* in UB becomes smaller – that can occur if (a, or b or c)<0.

On the contrary, as the denominator in LB tends to zero, *g* in LB can become much smaller (unless $0<abc<1$) – that can occur if $0<X$, or Y or $Z<1$, and or if $0<a,b,c$, or if $(XYZ)<(abc)$.
As the denominator in LB tends to minus infinity from zero, g in LB can become smaller or bigger depending on the magnitude of abc.
Thus, its more likely that LB defines the lower bound of *g*, while UB defines upper bound of *g*. ■

Theorem-2B: **For the equation $X^2+Y^2+Z^2= gXYZ$ in real numbers, and given Theorem-1b, for all values of X, Y and Z that are real numbers, the upper bound and the lower bound of *g* can be defined.**
Proof:
$XYZg = (n-f)$
$XYZg = (n-f)^3(abc)g$
$XYZ = (n-f)^3(abc)$
$(n-f) = (n-f)^3(abc)g$
$1 = (n-f)^2(abc)g$

In $X^2+Y^2+Z^2 = gXYZ$, g varies with the magnitudes (and not the signs) of X,Y and Z.

Thus, $g = 1/((n-f)^2(abc))$,
As $(n-f) \rightarrow +\infty$, $(abc)g \rightarrow -\infty$;

$(n-f) = gXYZ$, and (n-f) is a multiplicative component of each of X,Y and Z:
$(n-f)a = X$
$(n-f)b = Y$
$(n-f)c = Z$
$a = 1/YZg$; and $b = 1/XZg$; and $c = 1/XYg$

Given that, X, Y and Z, a,b, and c can be determined by substituting $a = 1/YZg$, $b = 1/XZg$ and $c = 1/XYg$, into $X/a = Y/b = Z/c = (n-f) = XYZg$

In $X^2+Y^2+Z^2 = gXYZ$, both n and f vary primarily with the magnitudes (and to a much lesser extent, the signs) of X,Y and Z.

$XYZg = (n-f)^3(abc)g$
$(n-f) = XYZg$
$n = XYZg+f$
$n = [XYZ/(abc)]^{1/3}+f$
Thus: $XYZg = [XYZ/(abc)]^{1/3}$
$g = \{[XYZ/(abc)]^{1/3}\}/XYZ$ or $g = \{[1/(abc)]^{1/3}\}$ (referred to as “LB”)
but also $g = 1/[(n-f)^2(abc)]$ (referred to as “UB”)
As the denominator in UB tends to zero, *g* in UB can become greater than one and significant – that can occur if $0<a$, or b or $c<1$, and or if $0<(n-f)<1$.
As the denominator in UB tends to minus infinity from zero, *g* in UB becomes smaller – that can occur if (a, or b or c)<0.

On the contrary, as the denominator in LB tends to zero, *g* in LB can become much smaller (unless $0<abc<1$) – that can occur if $0<X$, or Y or $Z<1$, and or if $0<a,b,c$, or if $(XYZ)<(abc)$.

As the denominator in LB tends to minus infinity from zero, g in LB can become smaller or bigger depending on the magnitude of abc.
Thus, its more likely that LB defines the lower bound of *g*, while UB defines upper bound of *g*. ■

Theorem-2C: For the equation $\mathbf{X^2+Y^2+Z^2+V^2+U^2}$= gXYZ, and given **Theorem-1c**, for all values of X, Y, Z, V and U that are real numbers, the upper bound and the lower bound of *g* can be defined.
Proof:
XYZg = (n-f)
XYZg = $(n-f)^5$(abcjm)g
XYZ = $(n-f)^5$(abcjm)
(n-f) = $(n-f)^5$(abcjm)g

In $\mathbf{X^2+Y^2+Z^2+V^2+U^2}$= gXYZ, g varies primarily with the magnitudes (and to a lesser extent, the signs) of X,Y, V and Z.

Thus, g = $1/((n-f)^4(abcjm))$,
As (n-f) $\rightarrow+\infty$, (abcjm)g$\rightarrow-\infty$;

(n-f) = gXYZ, and (n-f) is a multiplicative component of each of X,Y, Z, V and U:
(n-f)a = X
(n-f)b = Y
(n-f)c = Z
(n-f)j= V
(n-f)m= U
a = 1/YZg; and b = 1/XZg; and c = 1/XYg; and j= V/XYZg; and m= U/XYZg

Given X, Y, Z, V and U; then a,b, and c can be determined by substituting a = 1/YZg, b = 1/XZg and c = 1/XYg, into X/a = Y/b = Z/c = V/j = U/m = (n-f) = XYZg

In $\mathbf{X^2+Y^2+Z^2+V^2+U^2}$= gXYZ, both *n* and *f* vary primarily with the magnitudes (and to a lesser extent, the signs) of X,Y and Z.

XYZg = $(n-f)^5$(abc)g
n = XYZg+f
n = $[XYZ/(abc)]^{1/5}$+f
Thus XYZg = $[XYZ/(abc)]^{1/5}$
g = $\{[XYZ/(abc)]^{1/5}\}$/XYZ (referred to as “LB”)
but also g = $1/[(n-f)^4(abc)]$ (referred to as “UB”)
As the denominator in UB tends to zero, *g* in UB can become greater than one and significant – that can occur if 0<a, or b or c<1, and or if 0<(n-f)<1.
As the denominator in UB tends to minus infinity from zero, *g* in UB becomes smaller – that can occur if (a, or b or c)<0.

On the contrary, as the denominator in LB tends to zero, *g* in LB can become much smaller (unless 0<abc<1) – that can occur if 0<X, or Y or Z<1, and or if 0<a,b,c, or if (XYZ)<(abc).
As the denominator in LB tends to minus infinity from zero, g in LB can become smaller or bigger depending on the magnitude of abc.
Thus, its more likely that LB defines the lower bound of *g*, while UB defines upper bound of *g*. ■

Theorem-3A: **For the equation $X^2+Y^2+Z^2+V^2= gXYZ$ in real numbers, and given Theorem-1 above, and for all values of X, Y, V and Z that are real numbers, if (n-f)=gXYZ, and (n-f) is a multiplicative component of each of X,Y, V and Z, then there exists a real number *d* such that $X^2+Y^2+Z^2+V^2= dXYZ$; where for all *g* that are real numbers, g ϵ d.**
Proof:

(n-f) = gXYZ, and (n-f) is a multiplicative component of each of X,Y and Z, and as stated herein and above:
(n-f)a = X
(n-f)b = Y
(n-f)c = Z
(n-f)j=V
a = 1/YZg; and b = 1/XZg; and c = 1/XYg; and j= V/XYZg

Thus: X=(gXYZ)(a); and Y=(gXYZ)(b); and Z=(gXYZ)(c); and V = (gXYZ)(j)

If: $X^2+Y^2+Z^2 +V^2= dXYZ$;
Then by substitution: $[(g^2X^2Y^2Z^2)(a^2)]+[(g^2X^2Y^2Z^2)(b^2)]+[(g^2X^2Y^2Z^2)(c^2)] + [(g^2X^2Y^2Z^2)(j^2)]= dXYZ$

Then by dividing both sides of the equation by dXYZ and substituting a=(1/YZg), b=(1/XZg), c=(1/XYg), and j= V/XYZg, the result is:
$\{[(g^2X^2Y^2Z^2)(1/(Y^2Z^2g^2))]/dXYZ\}+\{[(g^2X^2Y^2Z^2)(1/(X^2Z^2g^2))]/dXYZ\}+ \{[(g^2X^2Y^2Z^2)(1/(X^2Y^2g^2))]/dXYZ\} +$
$\{[(g^2X^2Y^2Z^2)(V^2/(X^2Y^2Z^2g^2))]/dXYZ\}= 1$;
and thus: $[(X/dYZ)+(Y/dXZ)+(Z/dXY)] +(V^2/dXYZ)]= 1$

By taking a common denominator dXYZ for the left-hand side of the equation, the result is:
$[(X^2+Y^2+V^2+Z^2)/dXYZ] = 1$;
and by multiplying both sides of the equation by dXYZ, the result is: $X^2+Y^2+Z^2+V^2= dXYZ$

d can be expressed solely in terms of g, X, Y, Z and V as follows:
$\{[((g^2X^2Y^2Z^2)(1/(Y^2Z^2g^2)))]+[((g^2X^2Y^2Z^2)(1/(X^2Z^2g^2)))]+[((g^2X^2Y^2Z^2)(1/(X^2Y^2g^2)))]$
$+[((g^2X^2Y^2Z^2)(V^2/(X^2Y^2Z^2g^2)))]\}/XYZ = d$;

Similarly, *d* can also be expressed solely in terms of X, Y, Z and V as follows:
$\{[((X^2Y^2Z^2)(1/(Y^2Z^2g^2)))/XYZ]+[((g^2X^2Y^2Z^2)(1/(X^2Z^2g^2)))/XYZ]+[((g^2X^2Y^2Z^2)(1/(X^2Y^2g^2)))/XYZ] +$
$[((g^2X^2Y^2Z^2) (V^2/(X^2Y^2Z^2g^2)))/XYZ]\}= d = [(X/YZ)+(Y/XZ)+(Z/XY)+ (V^2/(XYZ))]$

Given the foregoing and since X=(gXYZ)(a); and Y=(gXYZ)(b); and Z=(gXYZ)(c) and V=(gXYZ)(j); and $X^2+Y^2+Z^2+ V^2= dXYZ$, for all X, Y, Z, V and *g* that are real numbers, $g<d$; and *g* ϵ *d*. ■

Theorem-3B: **For the equation $X^2+Y^2+Z^2=dXYZ$ in real numbers, and given Theorem-1 above, and for all values of X, Y, Z, *n* and *f* that are real numbers, if (n-f)=gXYZ, and (n-f) is a multiplicative component of each of X,Y and Z, then there exists a real number *d* such that $X^2+Y^2+Z^2= dXYZ$; where for all *g* that are real numbers, g ϵ d.**
Proof:

(n-f) = gXYZ, and (n-f) is a multiplicative component of each of X,Y and Z, and as stated herein and above:
(n-f)a = X
(n-f)b = Y
(n-f)c = Z
a = 1/YZg
b = 1/XZg
c = 1/XYg

Thus, gXYZ is a multiplicative component of each of X, Y and Z. That is:
X=(gXYZ)(a); and Y=(gXYZ)(b); and Z=(gXYZ)(c)

If: $X^2+Y^2+Z^2= dXYZ$;
Then by substitution: $[(g^2X^2Y^2Z^2)(a^2)]+[(g^2X^2Y^2Z^2)(b^2)]+[(g^2X^2Y^2Z^2)(c^2)] = dXYZ$

Then by dividing both sides of the equation by dXYZ and substituting a=(1/YZg), b=(1/XZg) and c=(1/XYg), the result is: $\{[(g^2X^2Y^2Z^2)(1/(Y^2Z^2g^2))]/dXYZ\}+\{[(g^2X^2Y^2Z^2)(1/(X^2Z^2g^2))]/dXYZ\}+\{[(g^2X^2Y^2Z^2)(1/(X^2Y^2g^2))]/dXYZ\} = 1$;
and thus: [(X/dYZ)+(Y/dXZ)+(Z/dXY)] = 1

By taking a common denominator dXYZ for the left-hand side of the equation, the result is:
$[(X^2+Y^2+Z^2)/dXYZ] = 1$;
and by multiplying both sides of the equation by dXYZ, the result is: $X^2+Y^2+Z^2 = dXYZ$

d can be expressed solely in terms of g, X, Y and Z as follows:
$\{[((g^2X^2Y^2Z^2)(1/(Y^2Z^2g^2)))]+[((g^2X^2Y^2Z^2)(1/(X^2Z^2g^2)))]+[((g^2X^2Y^2Z^2)(1/(X^2Y^2g^2)))]\}/XYZ = d$;

Similarly, *d* can also be expressed solely in terms of X, Y and Z as follows:
$\{[((X^2Y^2Z^2)(1/(Y^2Z^2g^2)))/XYZ]+[((g^2X^2Y^2Z^2)(1/(X^2Z^2g^2)))/XYZ]+[((g^2X^2Y^2Z^2)(1/(X^2Y^2g^2)))/XYZ]\}= d =$
[(X/YZ)+(Y/XZ)+(Z/XY)]

Given the foregoing and since X=(gXYZ)(a); and Y=(gXYZ)(b); and Z=(gXYZ)(c); and $X^2+Y^2+Z^2 = dXYZ$, for all X, Y, Z and *g* that are real numbers, $g<d$; and $g \in d$. ■

Theorem-3C: **For the equation $X^2+Y^2+Z^2+V^2+U^2 = gXYZ$ in real numbers, and given Theorem-1c above, and for all values of X, Y, V, Z and U that are real numbers, if (n-f)=gXYZ, and (n-f) is a multiplicative component of each of X,Y, V, Z and U, then there exists a real number *d* such that $X^2+Y^2+Z^2+V^2+U^2 =dXYZ$; where for all *g* that are real numbers, g ϵ d.**
Proof:

(n-f) = gXYZ, and (n-f) is a multiplicative component of each of X,Y and Z, where a,b,c,d,j and m are some real numbers:
(n-f)a = X
(n-f)b = Y
(n-f)c = Z
(n-f)j=V
(n-f)m= U
a = 1/YZg; and b = 1/XZg; and c = 1/XYg; and j = V/XYZg; and m = U/XYZg

Thus, gXYZ is a multiplicative component of each of X, Y, V and Z:
X=(gXYZ)(a); and Y=(gXYZ)(b); and Z=(gXYZ)(c) and V = (gXYZ)(j) and U= (gXYZ)(m)

If: $\mathbf{X^2+Y^2+Z^2+V^2+U^2} = dXYZ$;
Then by substitution: $[(g^2X^2Y^2Z^2)(a^2)]+[(g^2X^2Y^2Z^2)(b^2)]+[(g^2X^2Y^2Z^2)(c^2)] + [(g^2X^2Y^2Z^2)(j^2) + [(g^2X^2Y^2Z^2)(m^2)]= dXYZ$

Then by dividing both sides of the equation by dXYZ and substituting a=(1/YZg), b=(1/XZg), c=(1/XYg), and j= V/XYZg, and m= U/XYZg, the result is:
$\{[(g^2X^2Y^2Z^2)(1/(Y^2Z^2g^2))]/dXYZ\}+\{[(g^2X^2Y^2Z^2)(1/(X^2Z^2g^2))]/dXYZ\}+ \{[(g^2X^2Y^2Z^2)(1/(X^2Y^2g^2))]/dXYZ\} + \{[(g^2X^2Y^2Z^2)(V^2/(X^2Y^2Z^2g^2))]/dXYZ\}+ \{[(g^2X^2Y^2Z^2)(U^2/(X^2Y^2Z^2g^2))]/dXYZ\}= 1$;

and thus: $[(X/dYZ)+(Y/dXZ)+(Z/dXY)] +(V^2/dXYZ) +(U^2/dXYZ)]= 1$

By taking a common denominator dXYZ for the left-hand side of the equation, the result is:
$[(\mathbf{X^2+Y^2+Z^2+V^2 +U^2})/dXYZ] = 1$;
and by multiplying both sides of the equation by dXYZ, the result is: $\mathbf{X^2+Y^2+Z^2+V^2 +U^2} = dXYZ$

d can be expressed solely in terms of g, X, Y,V and Z as follows:
$\{[((g^2X^2Y^2Z^2)(1/(Y^2Z^2g^2)))]+[((g^2X^2Y^2Z^2)(1/(X^2Z^2g^2)))]+[((g^2X^2Y^2Z^2)(1/(X^2Y^2g^2)))]$
$+[((g^2X^2Y^2Z^2)(V^2/(X^2Y^2Z^2g^2)))] +[((g^2X^2Y^2Z^2)(U^2/(X^2Y^2Z^2g^2)))]]\}/XYZ = d$;

Similarly, *d* can also be expressed solely in terms of X, Y, Z, V and U as follows:
$\{[((X^2Y^2Z^2)(1/(Y^2Z^2g^2)))/XYZ]+[((g^2X^2Y^2Z^2)(1/(X^2Z^2g^2)))/XYZ]+[((g^2X^2Y^2Z^2)(1/(X^2Y^2g^2)))/XYZ] + [((g^2X^2Y^2Z^2) (V^2/(X^2Y^2Z^2g^2)))/XYZ] + [((g^2X^2Y^2Z^2) (U^2/(X^2Y^2Z^2g^2)))/XYZ]\}= d = [(X/YZ)+(Y/XZ)+(Z/XY)+ (V^2/(XYZ))+(U^2/(XYZ))]$

Given the foregoing and since X=(gXYZ)(a); and Y=(gXYZ)(b); and Z=(gXYZ)(c); and V=(gXYZ)(j); and U=(gXYZ)(m); and $\mathbf{X^2+Y^2+Z^2+V^2 +U^2}= dXYZ$, for all X, Y, Z, V, U and *g* that are real numbers, $g<d$; and $g \in d$. ■

Theorem-4A: For the equation $X^2+Y^2+Z^2+V^2= XYZg$ in real numbers, if (*n-f*) is a multiplicative component of each of X, Y, V and Z (each of X,Y, V and Z are derived by multiplying (*n-f*) by another real number), then XYZ*g*=(*n-f*), for some real numbers *g, n and f*.
Proof:

If (n-f) is a multiplicative component of each of X, Y, V and Z (each of X,Y, V and Z are derived by multiplying (n-f) by another real number), then:
X = (n-f)a
Y = (n-f)b
Z = (n-f)c
V = (n-f)j
Let: X/a = Y/b = Z/c =V/j= (n-f) = XYZVg
Thus: a=1/YZg; and b=1/XZg; and c=1/XYg; and j=V/XYZg

Where $-\infty<n,f,a,b,c,j< +\infty$; and n,f,a,b, j and c are real numbers.

$X^2 = (n-f)a*(n-f)a = (n-f)(n-f)a^2 = (n^2-nf-nf+f^2)a^2 = n^2a^2-2nf(a^2)+f^2a^2$
$Y^2 = (n-f)b*(n-f)b = (n-f)(n-f)b^2 = (n^2-nf-nf+f^2)b^2 = n^2b^2-2nf(b^2)+f^2b^2$
$Z^2 = (n-f)c*(n-f)c = (n-f)(n-f)c^2 = (n^2-nf-nf+f^2)c^2 = n^2c^2-2nf(c^2)+f^2c^2$
$V^2 = (n-f)j*(n-f)j = (n-f)(n-f)j^2 = (n^2-nf-nf+f^2)j^2 = n^2j^2-2nf(j^2)+f^2j^2$

Thus:
$X^2+Y^2+Z^2+V^2 = n^2(a^2+b^2+c^2+j^2)-2nf(a^2+b^2+c^2+j^2)-f^2(a^2+b^2+c^2+j^2) = (n^2-2nf-f^2)(a^2+b^2+c^2+j^2)$

$X^2+Y^2+Z^2+V^2 = (n^2-2nf-f^2)(a^2+b^2+c^2+j^2)$
$= (n-f)(n-f)(a^2+b^2+c^2+j^2)$
$= (XYZg)^2(a^2+b^2+c^2+j^2)$
$= [(XYZg)^2(1/(YZg)^2)]+[(XYZg)^2(1/(XZg^2))]+[(XYZg)^2(1/(XYg)^2)]+[(XYZg)^2(V/XYZg)^2)]$
$= [(XYZg)^2/(YZg)^2]+[(XYZg)^2/XZg^2)]+[(XYZg)^2/(XYg)^2)]+[V^2(XYZg)^2/(XYZg)^2)]$
$= X^2+Y^2+Z^2+V^2$ ■

Theorem-4B: For the equation $X^2+Y^2+Z^2= XYZg$ in real numbers, if (*n-f*) is a multiplicative component of each of X, Y and Z (each of X,Y and Z are derived by multiplying (*n-f*) by another real number), then $XYZg = (n-f)$, for some real numbers *g, n and f*.

Proof:

If (*n-f*) is a multiplicative component of each of X, Y, V and Z (each of X,Y, V and Z are derived by multiplying (*n-f*) by another real number), then:

$X = (n-f)a$
$Y = (n-f)b$
$Z = (n-f)c$

Let: $X/a = Y/b = Z/c = (n-f) = XYZVg$
Then: $a = 1/YZg$; and $b = 1/XZg$; and $c = 1/XYg$;
Where $-\infty<n,f,a,b,c,< +\infty$, are real numbers.

$X^2 = (n-f)a*(n-f)a = (n-f)(n-f)a^2 = (n^2-nf-nf+f^2)a^2 = n^2a^2-2nf(a^2)+f^2a^2$
$Y^2 = (n-f)b*(n-f)b = (n-f)(n-f)b^2 = (n^2-nf-nf+f^2)b^2 = n^2b^2-2nf(b^2)+f^2b^2$
$Z^2 = (n-f)c*(n-f)c = (n-f)(n-f)c^2 = (n^2-nf-nf+f^2)c^2 = n^2c^2-2nf(c^2)+f^2c^2$

Thus:
$X^2+Y^2+Z^2= n^2a^2-2nf(a^2)+f^2a^2 + n^2b^2-2nf(b^2)+f^2b^2 + n^2c^2-2nf(c^2)+f^2c^2$
$= n^2a^2 + n^2b^2 + n^2c^2-2nf(a^2) -2nf(b^2) -2nf(c^2)+f^2a^2 +f^2b^2 +f^2c^2$
$= n^2(a^2+b^2+c^2)-2nf(a^2+b^2+c^2)-f^2(a^2+b^2+c^2) = (n^2-2nf-f^2)(a^2+b^2+c^2)$

$X^2+Y^2+Z^2 = (n^2-2nf-f^2)(a^2+b^2+c^2)$
$= (n-f)(n-f)(a^2+b^2+c^2)$
$= (XYZg)^2(a^2+b^2+c^2)$
$= [(XYZg)^2(1/(YZg)^2)]+[(XYZg)^2(1/(XZg^2))]+[(XYZg)^2(1/(XYg)^2)]$
$= [(XYZg)^2/(YZg)^2]+[(XYZg)^2/XZg^2)]+[(XYZg)^2/(XYg)^2)]$
$= X^2+Y^2+Z^2$ $\blacksquare$

Theorem-4C: For the equation $X^2+Y^2+Z^2+V^2+U^2=XYZg$ in real numbers, if (*n-f*) is a multiplicative component of each of X, Y, Z, V and U (each of X,Y, V, U and Z are derived by multiplying (*n-f*) by another real number), then $XYZg = (n-f)$, for some real numbers *g,n and f*.

Proof:

Let:
$X = (n-f)a$
$Y = (n-f)b$
$Z = (n-f)c$
$V = (n-f)j$
$U = (n-f)m$

Let: $X/a = Y/b = Z/c =V/j=U/m= (n-f) = XYZg$
Then: $a=1/YZg$; and $b=1/XZg$; and $c=1/XYg$; and $j=V/XYZg$; and $m=U/XYZg$
Where $-\infty<n,f,a,b,c,j, m< +\infty$, are real numbers.

$X^2 = (n-f)a*(n-f)a = (n-f)(n-f)a^2 = (n^2-nf-nf+f^2)a^2 = n^2a^2-2nf(a^2)+f^2a^2$
$Y^2 = (n-f)b*(n-f)b = (n-f)(n-f)b^2 = (n^2-nf-nf+f^2)b^2 = n^2b^2-2nf(b^2)+f^2b^2$
$Z^2 = (n-f)c*(n-f)c = (n-f)(n-f)c^2 = (n^2-nf-nf+f^2)c^2 = n^2c^2-2nf(c^2)+f^2c^2$
$V^2 = (n-f)j*(n-f)j = (n-f)(n-f)j^2 = (n^2-nf-nf+f^2)j^2 = n^2j^2-2nf(j^2)+f^2j^2$
$U^2 = (n-f)m*(n-f)m = (n-f)(n-f)m^2 = (n^2-nf-nf+f^2)m^2 = n^2m^2-2nf(m^2)+f^2m^2$

Thus:
$\mathbf{X^2+Y^2+Z^2+V^2+U^2}$= $n^2(a^2+b^2+c^2+j^2+m^2)+2nf(a^2+b^2+c^2+j^2+m^2)-f^2(a^2+b^2+c^2+j^2+m^2)$
$= (n^2-2nf-f^2)(a^2+b^2+c^2+j^2+m^2)$
$= (n-f)(n-f)(a^2+b^2+c^2+j^2+m^2)$
$= (XYZg)^2(a^2+b^2+c^2+j^2+m^2)$
$= [(XYZg)^2(1/(YZg)^2)]+[(XYZg)^2(1/(XZg^2))]+[(XYZg)^2(1/(XYg)^2)]+[(XYZg)^2(V/XYZg)^2)]$
$+[(XYZg)^2(U/XYZg)^2)]$
$= [(XYZg)^2/(YZg)^2]+[(XYZg)^2/XZg^2)]+[(XYZg)^2/(XYg)^2)]+[V^2(XYZg)^2/(XYZg)^2)]+[U^2(XYZg)^2/(XYZg)^2)]$
$= X^2+Y^2+Z^2+V^2+U^2$ ■

Theorem-5: *For the equation $X^i+Y^i+Z^i+V^i$=?, and given Theorems above, and for all values of X, Y, V and Z that are real numbers, if (n-f)=gXYZV, and (n-f) is a multiplicative component of each of X,Y, V and Z, then there exists a real number d such that $X^i+Y^i+Z^i+V^i = dXYZ$; where for all g, X, Y, V and Z that are real numbers, g ϵ d; and d can be expressed as $d = [(X^{(i-1)}/YZ)+(Y^{(i-1)}/XZ)+(Z^{(i-1)}/XY)+(V^{(i-1)}/XYZ)]$.*
Proof: The proof is straightforward and follows from the prior proofs herein and above.

■

Conclusion.
The three equations $X^2+Y^2+Z^2 =aXYZ$, $x^2+y^2+z^2+v^2$=dXYZ, and $x^2+y^2+z^2+v^2+u^2$=dXYZ exhibit patterns of Nonlinearity that have potential applications in Applied Math, Computer Science, Economics and Physics.

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