QUANTUM ISOMETRIES AND NONCOMMUTATIVE GEOMETRY

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Abstract. The space $\mathbb{C}^N$ has no free analogue, but we can talk instead about the free sphere $S^{N-1}_{\mathbb{C},+}$, as the manifold defined by the equations $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$. We discuss here the structure and hierarchy of the submanifolds $X \subset S^{N-1}_{\mathbb{C},+}$, with particular attention to the manifolds having an integration functional $tr : C(X) \to \mathbb{C}$.

Contents

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Introduction

The notion of "quantum space" is as old as quantum mechanics. The foundations of the theory suggest that the quantum spaces X should appear in connection with the algebras $A \subset B(H)$ formed by the bounded linear operators $T : H \to H$ on a separable Hilbert space H. There has been a lot of work on this, in the last 100 years, the 4 main visions of the subject, by the 4 main architects of the theory, being as follows:

(1) Von Neumann looked at the $*$-algebras $A \subset B(H)$ which are closed under the weak topology, now called von Neumann algebras. The commutative such algebras are those of the form $L^\infty(X)$, with X being a measured space. In view of this, we can write any von Neumann algebra as $A = L^\infty(X)$, with X being a "quantum measured space".

(2) Voiculescu's idea was to develop probability theory on such spaces X. This basically leads to the conclusion that the integration functional of X must be a trace $tr : A \to \mathbb{C}$, having the property $tr(TS) = tr(ST)$. In other words, the random variables $S, T \in A$ might be non-commuting, but they must commute under the integration.

(3) Connes' approach is based on the idea that X should be Riemannian. In practice, besides the von Neumann algebra $A = L^\infty(X)$, this asks for the existence of a smaller algebra $\mathcal{A} = C(X)$, which must be a C^*-algebra, in the sense that it must be norm closed, and of an even smaller algebra $C^\infty(X)$ as well, which must be a $*$-algebra.

(4) Jones looked at the inclusions $A \subset B$ of von Neumann algebras, instead of the von Neumann algebras themselves. More specifically, he looked at the inclusions of II_1 factors, with the conclusion that the symmetries of such an inclusion $A \subset B$ are encoded by a kind of "quantum group" G, of the most possible general type.

It is not clear what to conclude from all this. The quantum spaces X definitely exist, in relation with the operator algebras $A \subset B(H)$, or rather with the quantum physics that these operator algebras are supposed to encode, but come in different flavors. It probably does not matter much what kind of flavor of quantum space you use, since we are still in the dark ages of quantum mechanics, and everything is potentially useful.

Our purpose here is to describe certain classes of quantum spaces X, related to the compact quantum groups introduced by Woronowicz in [148], which are related to the above, but do not exactly fit in any particular formalism. Technically speaking, our spaces will be "quantum algebraic manifolds", but we will develop their theory by having in mind the above ideas of von Neumann and Connes, Jones, Voiculescu. In short, we will be doing "noncommutative geometry" in a large sense, with positivity, but with no particular preference between algebra, geometry, analysis and probability.

As a basic example of manifold in our sense, we have the free complex sphere $S^{N-1}_{\mathbb{C},+}$. This sphere is by definition the compact quantum space, meaning dual of a C^*-algebra, whose coordinates $x_1,\ldots,x_N$ are subject to the following relations:

$$\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$$

This sphere is a potentially important object, for the simple reason that it exists. Indeed, the free analogue of $\mathbb{C}^N$ itself does not exist, because the coordinates x_i are unbounded. However, when imposing the above relations, we obtain by positivity:

$$||x_i||^2=||x_ix_i^*||\leq||\sum_i x_ix_i^*||=1$$

Thus the coordinates are bounded, and so $S^{N-1}_{\mathbb{C},+}$ exists. In the lack of an "ideal" space of type $\mathbb{C}^N_+$, at least we have this sphere, and its submanifolds $X\subset S^{N-1}_{\mathbb{C},+}$, which are numerous and interesting, for doing "noncommutative geometry", in a large sense.

The free complex sphere $S=S^{N-1}_{\mathbb{C},+}$ does not come alone, but it rather part of a quadruplet (S,T,U,K), comprising as well the free complex torus $T=\mathbb{T}_N^+$, which is the dual of the free group on N generators F_N, the free unitary quantum group $U=U_N^+$, which is the free analogue of the unitary group U_N, and the free complex reflection group $K=K_N^+$, which is the free analogue of the complex reflection group $K_N=\mathbb{T}\wr S_N$, with these latter objects being constructed via generators and relations, a bit like the sphere itself.

These four objects are intimately related to each other, with a full set of correspondences between them, obtained a bit like in the classical case, as follows:

$$\begin{tikzcd}
S^{N-1}_{\mathbb{C},+} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \mathbb{T}_N^+ \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
U_N^+ \arrow[r, leftrightarrow] & K_N^+
\end{tikzcd}$$

Summarizing, $\mathbb{C}^N_+$ does not exist, but its "basic geometry" exists, in the form of the above diagram. We will actually call this diagram $\mathbb{C}^N_+$, the name being not taken.

Before going further, we should clarify the relation of all this with the above-mentioned 4 main visions in operator algebras. The situation here is as follows:

(1) The torus $\mathbb{T}_N^+$, the unitary group U_N^+ and the relection group K_N^+ are all compact quantum groups, and as such, they have Haar measures. As for the sphere, this has a Haar measure too, appearing as the unique invariant measure under the action $U_N^+\curvearrowright S^{N-1}_{\mathbb{C},+}$. Thus, all our objects are quantum measured spaces, in the sense of von Neumann. The

asociated von Neumann algebras are II_1 factors, typically of the same type as the free group factors $L(F_N)$, although for the sphere this is for the moment conjectural.

(2) It is possible to do free probability on our 4 objects, by using a Weingarten type formula for the associated integration functionals. As an example here, let $\mathcal{NC}_2(k)$ be the set of noncrossing matching pairings of a colored integer $k=k_1\ldots k_p$, with the colors being exponents $k_i\in\{\emptyset,*\}$, for $\pi,\sigma\in\mathcal{NC}_2(k)$ set $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$, and finally let $W_{kN}=G_{kN}^{-1}$. The integration formula for the sphere is then:

$$\int_{S^{N-1}_{\mathbb{C},+}}x_{i_1}^{k_1}\ldots x_{i_p}^{k_p}\,dx=\sum_{\pi\in\mathcal{NC}_2(k)}\sum_{\sigma\leq\ker i}W_{kN}(\pi,\sigma)$$

(3) Regarding differential geometry, what is presently known is that $S^{N-1}_{\mathbb{C},+}$ has a Laplacian filtration, and that eigenvalues for the Laplacian can be constructed as well. However, $S^{N-1}_{\mathbb{C},+}$ is not "fully Riemannian", missing a Dirac operator in the sense of Connes.

(4) In relation with the Jones subfactors, the quantum group U_N^+ produces Temperley-Lieb subfactors, and the quantum group K_N^+ produces Fuss-Catalan subfactors, with these two classes of subfactors being the most basic objects of the theory.

Summarizing, we have here some interesting objects, commonly denoted as $\mathbb{C}^N_+$, and two questions appear. The first question is that of looking for further geometries, of the same type, and the second question is that of developing the geometries that we found.

In connection with the first question, the first remark is that the above constructions have straightforward "real" analogues, obtained by imposing the conditions $x_i=x_i^*$ to the standard coordinates. Thus, we have in fact four main geometries, as follows:

$$\begin{array}{ccc}\mathbb{R}^N_+&\longrightarrow&\mathbb{C}^N_+\\ \uparrow&&\uparrow\\ \mathbb{R}^N&\longrightarrow&\mathbb{C}^N\end{array}$$

In order to solve now the axiomatization question, let us start with arbitrary intermediate objects S,T,U,K, between classical real and free complex, as follows:

$$S^{N-1}_{\mathbb{R}}\subset S\subset S^{N-1}_{\mathbb{C},+}$$
$$T_N\subset T\subset\mathbb{T}_N^+$$
$$O_N\subset U\subset U_N^+$$
$$H_N\subset K\subset K_N^+$$

Observe that we are mixing here real and complex. This comes from subtle fact that we have $PO_N^+=PU_N^+$, which tends to "blur" the usual distinction between $\mathbb{R}$ and $\mathbb{C}$.

The problem is that of working out axioms for the 12 possible correspondences between our objects S,T,U,K, based on what happens in the real/complex, classical/free cases. Skipping the technical details, our axioms in simplified form will be as follows:

$$\begin{matrix} S & = & S_U & & \\ S\cap\mathbb{T}_N^+ & = & T & = & K\cap\mathbb{T}_N^+\\ G^+(S) & = & <O_N,T> & = & U\\ K^+(T) & = & U\cap K_N^+ & = & K \end{matrix}$$

With these axioms in hand, we can start looking for further examples. The conclusion will be that we have some natural intermediate geometries, both on the horizontal and the vertical, and so a 3×3 diagram, refining the above 2×2 one, as follows:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{T}\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{C}^N_* \arrow[u] \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

We will establish as well some classification results, stating that under strong "easiness" axioms these 9 geometries are the only ones, and that when adding a supplementary "uniformity" axiom, the initial 4 geometries are the only ones.

There is of course a lot of flexibility left in all this. As an example here, the basic diagram of 4 geometries has a $q=-1$ twisted counterpart, which is as follows:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\bar{\mathbb{R}}^N \arrow[r] \arrow[u] & \bar{\mathbb{C}}^N \arrow[u]
\end{tikzcd}$$

As a second variation, we can talk about projective geometries, and here the whole 9-diagram above collapses to a very simple 3-diagram, as follows:

$$P^{N-1}_{\mathbb{R}}\subset P^{N-1}_{\mathbb{C}}\subset P^{N-1}_+$$

Here P^{N-1}_+ is the free projective space, which is at the same time real and complex. We can combine also these 2 variations, and talk about twisted projective spaces.

Regarding now the second question, that of developing the geometries that we found, this depends a lot on the geometry in question, the situation being as follows:

(1) Generally speaking, the idea will be that of looking at various homogeneous spaces $X = G/H$, coming from suitable closed subgroups $H \subset G \subset U$. We will develop such a theory for $\mathbb{R}^N_+$, $\mathbb{C}^N_+$, with algebraic and probabilistic results. The question however of axiomatizing the real and complex "free manifolds" will remain open.

(2) For the other geometries, such as the half-classical ones $\mathbb{R}^N_*$, $\mathbb{C}^N_*$, or the twisted ones $\bar{\mathbb{R}}^N$, $\bar{\mathbb{C}}^N$, or the combinations of these, which are closer to the $\mathbb{R}^N$, $\mathbb{C}^N$ world, it is possible to go far beyond the level of the above-mentioned quotient spaces, with the development of a full and broad geometry theory, in analogy with the geometry of $\mathbb{R}^N$, $\mathbb{C}^N$.

Finally, we will mostly insist on algebraic and probabilistic apects. The differential geometric aspects will remain unclear, although we believe that a unification with the Connes theory could come via a noncommutative analogue of the Nash embedding theorem [114]. In short, we believe in the existence of a "Nash-Connes Geometry" (NCG).

This book is organized as follows: 1-4 contain axiomatization and classification work, in 5-8 we develop the real and complex free geometries, in 9-12 we discuss the half-classical and hybrid geometries, and in 13-16 we discuss a number of more specialized topics.

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1. **Spheres and tori**

What is geometry? A naive approach to this question suggests that we should have at least a sphere S, a torus T, a unitary group U, and a reflection group K, as starting objects. These basic objects should have relations between them, as follows:

$$\begin{tikzcd}
S \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
U \arrow[r, leftrightarrow] & K
\end{tikzcd}$$

Our idea here will be that of axiomatizing such quadruplets (S,T,U,K). With this axiomatization in hand, and some classification results as well, we will discuss then the development of each of the geometries that we found. This will be our plan.

Let us first discuss the case of the usual geometry, in $\mathbb{R}^N$. Basic common sense would suggest to add $\mathbb{R}^N$ itself to our list of objects, and with this addition done, why not erasing afterwards all the other objects, which can be reconstructed anyway from $\mathbb{R}^N$.

Unfortunately, this is something that we cannot do, in view of our noncommutative geometry goals and motivations. To be more precise, it is well-known that $\mathbb{R}^N$ has no interesting noncommutative analogues. Technically speaking, the problem comes from the fact that $\mathbb{R}^N$ is not compact. We will be back later to this issue.

So, let us go ahead, and construct our quadruplet (S,T,U,K). We have:

Definition 1.1. *The real sphere, torus, unitary group and reflection group are:*

$$\begin{aligned}
S^{N-1}_\mathbb{R} &= \left\{x\in\mathbb{R}^N\Big|\sum_i x_i^2=1\right\}\\
T_N &= \left\{x\in\mathbb{R}^N\Big|x_i=\pm\frac{1}{\sqrt{N}}\right\}\\
O_N &= \left\{U\in M_N(\mathbb{R})\Big|U^t=U^{-1}\right\}\\
H_N &= \left\{U\in M_N(-1,0,1)\Big|U^t=U^{-1}\right\}
\end{aligned}$$

These are the usual sphere, cube, orthogonal group, and hyperoctahedral group.

Here the superscript $N-1$ for the sphere, which is very standard, stands for the real dimension as manifold, which is $N-1$. Also, the $1/\sqrt{N}$ normalization for the cube/torus is there in order to have an embedding $T_N\subset S^{N-1}_\mathbb{R}$, this being convenient for us.

Regarding the correspondences between our objects, there are many ways of establishing them, depending on knowledge and taste, but this is not crucial for us. We just need a statement here, in order to get started, so let us formulate things as follows:

Theorem 1.2. *We have a full set of correspondences, as follows,*

$$\begin{tikzcd}
S^{N-1}_\mathbb{R} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T_N \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
O_N \arrow[r, leftrightarrow] & H_N
\end{tikzcd}$$

obtained via various results from basic geometry and group theory.

Proof. As already mentioned, there are several possible solutions to the problem, and all this is not crucial for us. Here is a way of constructing these correspondences:

(1) $S^{N-1}_\mathbb{R} \leftrightarrow T_N$. Here T_N comes from $S^{N-1}_\mathbb{R}$ via $|x_1| = \ldots = |x_N|$, while $S^{N-1}_\mathbb{R}$ appears from $T_N \subset \mathbb{R}^N$ by "deleting" this relation, while still keeping $\sum_i x_i^2 = 1$.

(2) $S^{N-1}_\mathbb{R} \leftrightarrow O_N$. This comes from the fact that O_N is the isometry group of $S^{N-1}_\mathbb{R}$, and that, conversely, $S^{N-1}_\mathbb{R}$ appears as $\{Ux|U \in O_N\}$, where $x = (1, 0, \ldots, 0)$.

(3) $S^{N-1}_\mathbb{R} \leftrightarrow H_N$. This is something trickier, but the passage can definitely be obtained, for instance via T_N, by using the constructions in (1) above and (5) below.

(4) $T_N \leftrightarrow O_N$. Here $T_N \simeq \mathbb{Z}_2^N$ is a maximal torus of O_N, and the group O_N itself can be reconstructed from this maximal torus, by using various methods.

(5) $T_N \leftrightarrow H_N$. Here, similarly, $T_N \simeq \mathbb{Z}_2^N$ is a maximal torus of H_N, and the group H_N itself can be reconstructed from this torus as a wreath product, $H_N = T_N \wr S_N$.

(6) $O_N \leftrightarrow H_N$. This is once again something trickier, but the passage can definitely be obtained, for instance via T_N, by using the constructions in (4) and (5) above. $\square$

The above result is of course something quite non-trivial, and having it understood properly would take some time. However, as already said, we will technically not need all this. Our purpose for the moment is just to explain our (S, T, U, K) philosophy.

As a second basic example of geometry, we have the usual geometry of $\mathbb{C}^N$. Here, as before, we cannot include the space $\mathbb{C}^N$ itself in our formalism, because this space is not compact, and as already said, we would like to deal with compact spaces only. The corresponding quadruplet (S, T, U, K) can be constructed as follows:

Definition 1.3. *The complex sphere, torus, unitary group and reflection group are:*

$$\begin{aligned} S^{N-1}_\mathbb{C} &= \left\{x\in\mathbb{C}^N\Big|\sum_i|x_i|^2=1\right\}\\ \mathbb{T}_N &= \left\{x\in\mathbb{C}^N\Big|\ |x_i|=\frac{1}{\sqrt{N}}\right\}\\ U_N &= \left\{U\in M_N(\mathbb{C})\Big|U^*=U^{-1}\right\}\\ K_N &= \left\{U\in M_N(\mathbb{T}\cup\{0\})\Big|U^*=U^{-1}\right\}\end{aligned}$$

These are the usual complex sphere, torus, unitary group, and complex reflection group.

As before, the superscript $N-1$ for the sphere does not fit with the rest, but is quite standard, somewhat coming from dimension considerations. We will use it as such. Also, the $1/\sqrt{N}$ factor is there in order to have an embedding $\mathbb{T}_N\subset S^{N-1}_\mathbb{C}$.

Also as before, in what regards the correspondences between our objects, there are many ways of establishing them, will all this being not crucial for us. In analogy with Theorem 1.2, let us formulate a second informal statement, as follows:

Theorem 1.4. *We have a full set of correspondences, as follows,*

$$\begin{tikzcd} S^{N-1}_\mathbb{C} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \mathbb{T}_N \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ U_N \arrow[r, leftrightarrow] & K_N \end{tikzcd}$$

obtained via various results from basic geometry and group theory.

Proof. We follow the proof in the real case, by making adjustments where needed, and with of course the reiterated comment that all this is not crucial for us:

(1) $S^{N-1}_\mathbb{C}\leftrightarrow\mathbb{T}_N$. Same proof as before, using $|x_1|=\ldots=|x_N|$.

(2) $S^{N-1}_\mathbb{C}\leftrightarrow U_N$. Here "isometry" must be taken in an affine complex sense.

(3) $S^{N-1}_\mathbb{C}\leftrightarrow K_N$. Trickier as before, best viewed by passing via $\mathbb{T}_N$.

(4) $\mathbb{T}_N\leftrightarrow U_N$. Coming from the fact that $\mathbb{T}_N\simeq\mathbb{T}^N$ is a maximal torus of U_N.

(5) $\mathbb{T}_N\leftrightarrow K_N$. Once again, maximal torus argument, and $K_N=\mathbb{T}_N\wr S_N$.

(6) $U_N\leftrightarrow K_N$. Trickier as before, best viewed by passing via $\mathbb{T}_N$. □

As a conclusion, our (S,T,U,K) philosophy seems to work, in the sense that these 4 objects, and the relations between them, encode interesting facts about $\mathbb{R}^N,\mathbb{C}^N$. Our plan in what follows will be that of leaving aside the complete understanding of what has been said above, and going directly for the noncommutative case. We will see that in the noncommutative setting things are more rigid, and therefore, simpler.

In order to talk about noncommutative geometry, the idea will be that of defining our quantum spaces X as being abstract manifolds, whose coordinates $x_1,\ldots,x_N$ do not necessarily commute. Thus, we are in need of some good algebraic geometry correspondence, between such abstract spaces X, and the corresponding algebras of coordinates A. Following Heisenberg, von Neumann and many others, we will use here the correspondence coming from operator algebras. A first idea is that of using "continuous coordinates", with each quantum space X corresponding to a certain C^*-algebra, via:

$$A=C(X)$$

With this idea in mind, getting back to our (S,T,U,K) philosophy, we would like to have C^*-algebras with correspondences between them, as follows:

$$\begin{tikzcd}
C(S) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & C(T) \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
C(U) \arrow[r, leftrightarrow] & C(K)
\end{tikzcd}$$

A second idea, which is viable as well, and is probably more far-reaching, in view of the loads of uncertainty and probability theory involved with quantum mechanics, but which is technically more complicated to develop, is that of using L^∞ coordinates for our manifolds, according to a formula of the following type:

$$A''=L^\infty(X)$$

With this second idea in mind, in connection with our (S,T,U,K) program, we would like to have von Neumann algebras with correspondences between them, as follows:

$$\begin{tikzcd}
L^\infty(S) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & L^\infty(T) \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
L^\infty(U) \arrow[r, leftrightarrow] & L^\infty(K)
\end{tikzcd}$$

In what follows we will use both the above ideas, which are both fruitful. To be more precise, our plan will be that of developing first the continuous theory, and leaving the more advanced aspects, involving von Neumann algebras and probability, for later.

In order to get started now, we will need a number of preliminaries on operators and operator algebras. Let us begin with the following standard definition:

Definition 1.5. *A Hilbert space is a complex vector space H, given with a scalar product $< x, y >$, satisfying the following conditions:*

1. *$< x, y >$ is linear in x, and antilinear in y.*
2. *$\overline{< x, y >} =< y, x >$, for any x, y.*
3. *$< x, x >> 0$, for any $x \neq 0$.*
4. *H is complete with respect to the norm $||x|| = \sqrt{< x, x >}$.*

Here the fact that $||.||$ is indeed a norm comes from the Cauchy-Schwarz inequality, $| < x, y > | \leq ||x|| \cdot ||y||$, which comes from the fact that the following degree 2 polynomial, with $t \in \mathbb{R}$ and $w \in \mathbb{T}$, being positive, its discriminant must be negative:

$$f(t) = ||x + wty||^2$$

In finite dimensions, any algebraic basis $\{f_1, \ldots, f_N\}$ can be turned into an orthonormal basis $\{e_1, \ldots, e_N\}$, by using the Gram-Schmidt procedure. Thus, we have $H \simeq \mathbb{C}^N$, with this latter space being endowed with its usual scalar product:

$$< x, y >= \sum_i x_i \bar{y}_i$$

The same happens in infinite dimensions, once again by Gram-Schmidt, coupled if needed with the Zorn lemma, in case our space is really very big. In other words, any Hilbert space has an orthonormal basis $\{e_i\}_{i \in I}$, and we have:

$$H \simeq l^2(I)$$

Of particular interest is the "separable" case, where I is countable. According to the above, there is up to isomorphism only one Hilbert space here, namely:

$$H = l^2(\mathbb{N})$$

All this is, however, quite tricky, and can be a bit misleading. Consider for instance the space $H = L^2[0, 1]$ of square-summable functions $f : [0, 1] \to \mathbb{C}$, with:

$$< f, g >= \int_0^1 f(x)\overline{g(x)}dx$$

This space is of course separable, because we can use the basis $f_n = x^n$ with $n \in \mathbb{N}$, orthogonalized by Gram-Schmidt. However, the orthogonalization procedure is something non-trivial, and so the isomorphism $H \simeq l^2(\mathbb{N})$ that we obtain is something non-trivial as well. Doing some computations here is actually a very good exercise.

In what follows we will be interested in the linear operators $T : H \to H$ which are bounded. Regarding such operators, we have the following result:

Theorem 1.6. *Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that*

$$||T|| = \sup_{||x||\leq 1} ||Tx||$$

is finite, form a complex algebra $B(H)$, having the following properties:

(1) *$B(H)$ is complete with respect to $||.||$, so we have a Banach algebra.*

(2) *$B(H)$ has an involution $T \to T^*$, given by $< Tx, y >=< x, T^*y >$.*

In addition, the norm and involution are related by the formula $||TT^*|| = ||T||^2$.

Proof. The fact that we have indeed an algebra follows from:

$$||S + T|| \leq ||S|| + ||T|| \quad , \quad ||\lambda T|| = |\lambda| \cdot ||T|| \quad , \quad ||ST|| \leq ||S|| \cdot ||T||$$

(1) Assuming that $\{T_n\} \subset B(H)$ is Cauchy then $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define indeed the limit $T = \lim_{n\to\infty} T_n$ by setting:

$$Tx = \lim_{n\to\infty} T_n x$$

(2) Here the existence of T^* comes from the fact that $\varphi(x) =< Tx, y >$ being a linear form $H \to \mathbb{C}$, we must have $\varphi(x) =< x, T^*y >$, for a certain vector $T^*y \in H$. Moreover, since this vector is unique, T^* is unique too, and we have as well:

$$(S + T)^* = S^* + T^* \quad , \quad (\lambda T)^* = \bar{\lambda} T^* \quad , \quad (ST)^* = T^*S^* \quad , \quad (T^*)^* = T$$

Observe also that we have indeed $T^* \in B(H)$, because:

$$||T|| = \sup_{||x||=1} \sup_{||y||=1} < Tx, y >= \sup_{||y||=1} \sup_{||x||=1} < x, T^*y >= ||T^*||$$

Regarding now the last assertion, we have:

$$||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$$

We have as well the following estimate:

$$||T||^2 = \sup_{||x||=1} | < Tx, Tx > | = \sup_{||x||=1} | < x, T^*Tx > | \leq ||T^*T||$$

By replacing $T \to T^*$ we obtain from this:

$$||T||^2 \leq ||TT^*||$$

Thus, we have proved the last equality, and we are done. □

Observe that when H comes with an orthonormal basis $\{e_i\}_{i\in I}$, the linear map $T \to M$ given by $M_{ij} =< Te_j, e_i >$ produces an embedding as follows:

$$B(H) \subset M_I(\mathbb{C})$$

Moreover, in this picture the operation $T \to T^*$ takes a very simple form, namely:

$$(M^*)_{ij} = \overline{M}_{ji}$$

The conditions found in Theorem 1.6 suggest the following definition:

Definition 1.7. *A C^*-algebra is a complex algebra A, having:*

1. *A norm $a \to ||a||$, making it a Banach algebra.*
2. *An involution $a \to a^*$, satisfying $||aa^*|| = ||a||^2$.*

Generally speaking, the elements $a \in A$ are best thought of as being some kind of "generalized operators", on some Hilbert space which is not present. By using this idea, one can emulate spectral theory in this setting, as follows:

Proposition 1.8. *Given $a \in A$, define its spectrum as being the set*

$$\sigma(a) = \left\{\lambda \in \mathbb{C}\Big| a - \lambda \notin A^{-1}\right\}$$

and its spectral radius $\rho(a)$ as the radius of the smallest centered disk containing $\sigma(a)$.

1. *The spectrum of a norm one element is in the unit disk.*
2. *The spectrum of a unitary element ($a^* = a^{-1}$) is on the unit circle.*
3. *The spectrum of a self-adjoint element ($a = a^*$) consists of real numbers.*
4. *The spectral radius of a normal element ($aa^* = a^*a$) is equal to its norm.*

Proof. Our first claim is that for any polynomial $f \in \mathbb{C}[X]$, and more generally for any rational function $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$, we have:

$$\sigma(f(a)) = f(\sigma(a))$$

This indeed something well-known for the usual matrices. In the general case, assume first that we have a polynomial, $f \in \mathbb{C}[X]$. If we pick an arbitrary number $\lambda \in \mathbb{C}$, and write $f(X) - \lambda = c(X - r_1) \ldots (X - r_k)$, we have then, as desired:

$$\begin{aligned}
\lambda \notin \sigma(f(a)) &\iff f(a) - \lambda \in A^{-1} \\
&\iff c(a - r_1) \ldots (a - r_k) \in A^{-1} \\
&\iff a - r_1, \ldots, a - r_k \in A^{-1} \\
&\iff r_1, \ldots, r_k \notin \sigma(a) \\
&\iff \lambda \notin f(\sigma(a))
\end{aligned}$$

Assume now that we are in the general case, $f \in \mathbb{C}(X)$. We pick $\lambda \in \mathbb{C}$, we write $f = P/Q$, and we set $F = P - \lambda Q$. By using the above finding, we obtain, as desired:

$$\begin{aligned}
\lambda \in \sigma(f(a)) &\iff F(a) \notin A^{-1} \\
&\iff 0 \in \sigma(F(a)) \\
&\iff 0 \in F(\sigma(a)) \\
&\iff \exists \mu \in \sigma(a), F(\mu) = 0 \\
&\iff \lambda \in f(\sigma(a))
\end{aligned}$$

Regarding now the assertions in the statement, these basically follows from this:

(1) This comes from the following formula, valid when $||a||<1$:

$$\frac{1}{1-a}=1+a+a^2+\ldots$$

(2) Assuming $a^*=a^{-1}$, we have the following norm computations:

$$||a||=\sqrt{||aa^*||}=\sqrt{1}=1$$

$$||a^{-1}||=||a^*||=||a||=1$$

If we denote by D the unit disk, we obtain from this, by using (1):

$$||a||=1\implies\sigma(a)\subset D$$

$$||a^{-1}||=1\implies\sigma(a^{-1})\subset D$$

On the other hand, by using the rational function $f(z)=z^{-1}$, we have:

$$\sigma(a^{-1})\subset D\implies\sigma(a)\subset D^{-1}$$

Now by putting everything together we obtain, as desired:

$$\sigma(a)\subset D\cap D^{-1}=\mathbb{T}$$

(3) This follows by using (2), and the rational function $f(z)=(z+it)/(z-it)$, with $t\in\mathbb{R}$. Indeed, for $t>>0$ the element $f(a)$ is well-defined, and we have:

$$\left(\frac{a+it}{a-it}\right)^*=\frac{a-it}{a+it}=\left(\frac{a+it}{a-it}\right)^{-1}$$

Thus $f(a)$ is a unitary, and by (2) its spectrum is contained in $\mathbb{T}$. We conclude that we have $f(\sigma(a))=\sigma(f(a))\subset\mathbb{T}$, and so $\sigma(a)\subset f^{-1}(\mathbb{T})=\mathbb{R}$, as desired.

(4) We have $\rho(a)\leq||a||$ from (1). Conversely, given $\rho>\rho(a)$, we have:

$$\int_{|z|=\rho}\frac{z^n}{z-a}\,dz=\sum_{k=0}^\infty\left(\int_{|z|=\rho}z^{n-k-1}dz\right)a^k=a^{n-1}$$

By applying the norm and taking n-th roots we obtain:

$$\rho\geq\lim_{n\to\infty}||a^n||^{1/n}$$

In the case $a=a^*$ we have $||a^n||=||a||^n$ for any exponent of the form $n=2^k$, and by taking n-th roots we get $\rho\geq||a||$. This gives the missing inequality, namely:

$$\rho(a)\geq||a||$$

In the general normal case, $aa^*=a^*a$, we have $a^n(a^n)^*=(aa^*)^n$, and so:

$$\rho(a)^2=\rho(aa^*)$$

Now since aa^* is self-adjoint, we get $\rho(aa^*)=||a||^2$, and we are done. $\square$

We can now formulate a key theorem, from [88], as follows:

Theorem 1.9 (Gelfand)**.** *If X is a compact space, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a commutative C^*-algebra, with structure as follows:*

(1) *The norm is the usual sup norm, $||f|| = \sup_{x\in X} |f(x)|$.*
(2) *The involution is the usual involution, $f^*(x) = \overline{f(x)}$.*

Conversely, any commutative C^*-algebra is of the form $C(X)$, with its "spectrum" $X = Spec(A)$ appearing as the space of characters $\chi : A \to \mathbb{C}$.

Proof. In what regards the first assertion, almost everything here is trivial. We have indeed a commutative algebra, with norm and involution, the Cauchy sequences inside are well-known to converge, and the condition $||ff^*|| = ||f||^2$ is satisfied. Conversely, given a commutative C^*-algebra A, we can define X to be the set of characters $\chi : A \to \mathbb{C}$, with the topology making continuous all the evaluation maps $ev_a : \chi \to \chi(a)$. Then X is a compact space, and $a \to ev_a$ is a morphism of algebras:

$$ev : A \to C(X)$$

We first prove that ev is involutive. We use the following formula:

$$a = \frac{a + a^*}{2} - i \cdot \frac{i(a - a^*)}{2}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$.

Since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

The Gelfand theorem suggests formulating the following definition:

Definition 1.10. *Given a C^*-algebra A, not necessarily commutative, we write*

$$A = C(X)$$

and call the abstract object X a "compact quantum space".

We will be back to this, with examples, and with some technical comments as well. Let us discuss now the other basic result regarding the C^*-algebras, namely the GNS representation theorem. We will need some more spectral theory, as follows:

Proposition 1.11. *For a normal element $a \in A$, the following are equivalent:*

(1) *a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.*
(2) *$a = b^2$, for some $b \in A$ satisfying $b = b^*$.*
(3) *$a = cc^*$, for some $c \in A$.*

Proof. This is something very standard, as follows:

(1) $\implies$ (2) Since our element a is normal the algebra $< a >$ that is generates is commutative, and by using the Gelfand theorem, we can set $b = \sqrt{a}$.

(2) $\implies$ (3) This is trivial, because we can set $c = b$.

(3) $\implies$ (1) We proceed by contradiction. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. By writing now $d = x + iy$ with $x = x^*, y = y^*$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$. But this contradicts the elementary fact that $\sigma(dd^*), \sigma(d^*d)$ must coincide outside $\{0\}$, which can be checked by explicit inversion. □

Here is now the representation theorem from [89], along with the idea of the proof:

Theorem 1.12 (GNS theorem)**.** *Let A be a C^*-algebra.*

1. *A appears as a closed $*$-subalgebra $A \subset B(H)$, for some Hilbert space H.*
2. *When A is separable (usually the case), H can be chosen to be separable.*
3. *When A is finite dimensional, H can be chosen to be finite dimensional.*

Proof. Let us first discuss the commutative case, $A = C(X)$. Our claim here is that if we pick a probability measure on X, we have an embedding as follows:

$$C(X) \subset B(L^2(X)) \quad , \quad f \to (g \to fg)$$

Indeed, given a function $f \in C(X)$, consider the operator $T_f(g) = fg$, acting on $H = L^2(X)$. Observe that T_f is indeed well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 dx} \leq ||f||_\infty ||g||_2$$

The application $f \to T_f$ being linear, involutive, continuous, and injective as well, we obtain in this way a C^*-algebra embedding $C(X) \subset B(H)$, as claimed.

In general, we can use a similar idea, with the algebraic aspects being fine, and with the positivity issues being taken care of by Proposition 1.8 and Proposition 1.11.

Indeed, assuming that a linear form $\varphi : A \to \mathbb{C}$ has some suitable positivity properties, making it analogous to the integration functionals $\int_X : A \to \mathbb{C}$ from the commutative case, we can define a scalar product on A, by the following formula:

$$< a, b >= \varphi(ab^*)$$

By completing we obtain a Hilbert space H, and we have an embedding as follows:

$$A \subset B(H) \quad , \quad a \to (b \to ab)$$

Thus we obtain the assertion (1), and a careful examination of the construction $A \to H$, outlined above, shows that the assertions (2,3) are in fact proved as well. □

With the above formalism is hand, we can go ahead, and construct two free quadruplets (S,T,U,K), in analogy with those corresponding to the classical real and complex geometries. Let us begin with the spheres. Following [5], [32], we have:

Definition 1.13. *We have free real and complex spheres, defined via*

$$C(S^{N-1}_{\mathbb{R},+})=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_ix_i^2=1\right)$$

$$C(S^{N-1}_{\mathbb{C},+})=C^*\left(x_1,\ldots,x_N\Big|\sum_ix_ix_i^*=\sum_ix_i^*x_i=1\right)$$

where the symbol C^* stands for universal enveloping C^*-algebra.

All this deserves some explanations. Given an integer $N\in\mathbb{N}$, consider the free complex algebra on $2N$ variables, denoted $x_1,\ldots,x_N$ and $x_1^*,\ldots,x_N^*$:

$$A=\left\langle x_1,\ldots,x_N,x_1^*,\ldots,x_N^*\right\rangle$$

In other words, the elements of A are the formal linear combinations, with complex coefficients, of products between our variables x_i,x_i^*, and of the unit 1.

This algebra has an involution $*:A\to A$, given by:

$$x_i\leftrightarrow x_i^*$$

Now let us consider the following $*$-algebra quotients of our $*$-algebra A:

$$\begin{aligned}A_R&=&A\Big/\left\langle x_i=x_i^*,\sum_ix_i^2=1\right\rangle\\ A_C&=&A\Big/\left\langle\sum_ix_ix_i^*=\sum_ix_i^*x_i=1\right\rangle\end{aligned}$$

Since the first relations imply the second ones, we have quotient maps as follows:

$$A\to A_C\to A_R$$

Our claim now is both A_C,A_R admit enveloping C^*-algebras, in the sense that the biggest C^*-norms on these $*$-algebras are bounded. We only have to check this for the bigger algebra A_C. But here, our claim follows from the following estimate:

$$||x_i||^2=||x_ix_i^*||\leq||\sum_ix_ix_i^*||=1$$

Summarizing, our claim is proved, so we can define $C(S^{N-1}_{\mathbb{R},+}),C(S^{N-1}_{\mathbb{C},+})$ as being the enveloping C^*-algebras of A_R,A_C, and so Definition 1.13 makes sense.

In order to formulate some results, let us introduce as well:

Definition 1.14. *Given a compact quantum space X, its classical version is the usual compact space $X_{class} \subset X$ obtained by dividing $C(X)$ by its commutator ideal:*

$$C(X_{class}) = C(X)/I \quad , \quad I =< [a,b] >$$

In this situation, we also say that X appears as a "liberation" of X.

In other words, the space X_{class} appears as the Gelfand spectrum of the commutative C^*-algebra $C(X)/I$. Observe in particular that X_{class} is indeed a classical space. As a first result now, regarding the above free spheres, we have:

Theorem 1.15. *We have embeddings of compact quantum spaces, as follows,*

$$\begin{tikzcd} S^{N-1}_{\mathbb{R},+} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\ S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C}} \arrow[u] \end{tikzcd}$$

and the spaces on top appear as liberations of the spaces on the bottom.

Proof. The first assertion, regarding the inclusions, comes from the fact that at the level of the associated C^*-algebras, we have surjective maps, as follows:

$$\begin{tikzcd} C(S^{N-1}_{\mathbb{R},+}) \arrow[d] & C(S^{N-1}_{\mathbb{C},+}) \arrow[l] \arrow[d] \\ C(S^{N-1}_{\mathbb{R}}) & C(S^{N-1}_{\mathbb{C}}) \arrow[l] \end{tikzcd}$$

For the second assertion, we must establish the following isomorphisms, where the symbol C^*_{comm} stands for "universal commutative C^*-algebra generated by":

$$C(S^{N-1}_{\mathbb{R}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i = x_i^*, \sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^* = \sum_i x_i^*x_i = 1\right)$$

As a first observation, it is enough to establish the second isomorphism, because the first one will follow from it, by dividing by the relations $x_i = x_i^*$.

So, consider the second universal commutative C^*-algebra A constructed above. Since the standard coordinates on $S^{N-1}_{\mathbb{C}}$ satisfy the defining relations for A, we have a quotient map of as follows, mapping standard coordinates to standard coordinates:

$$A \to C(S^{N-1}_{\mathbb{C}})$$

Conversely, let us write $A = C(S)$, by using the Gelfand theorem. The variables $x_1, \ldots, x_N$ become in this way true coordinates, providing us with an embedding $S \subset \mathbb{C}^N$. Also, the quadratic relations become $\sum_i |x_i|^2 = 1$, so we have $S \subset S^{N-1}_{\mathbb{C}}$. Thus, we have a quotient map $C(S^{N-1}_{\mathbb{C}}) \to A$, as desired, and this gives all the results. □

Summarizing, we are done with the spheres. Before getting into tori, let us talk about algebraic manifolds. By using the free spheres constructed above, we can formulate:

Definition 1.16. *A real algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$ is a closed quantum subspace defined, at the level of the corresponding C^*-algebra, by a formula of type*

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_i(x_1, \ldots, x_N) = 0 \Big\rangle$$

for certain family of noncommutative polynomials, as follows:

$$f_i \in \mathbb{C} < x_1, \ldots, x_N >$$

We denote by $\mathcal{C}(X)$ the $*$-subalgebra of $C(X)$ generated by the coordinates $x_1, \ldots, x_N$.

As a basic example of such a manifold, we have the free real sphere $S^{N-1}_{\mathbb{R},+}$. The classical spheres $S^{N-1}_{\mathbb{C}}, S^{N-1}_{\mathbb{R}}$, and their real submanifolds, are covered as well by this formalism. At the level of the general theory, we have the following version of the Gelfand theorem, which is something very useful, and that we will use many times in what follows:

Theorem 1.17. *If $X \subset S^{N-1}_{\mathbb{C},+}$ is an algebraic manifold, as above, we have*

$$X_{class} = \left\{ x \in S^{N-1}_{\mathbb{C}} \Big| f_i(x_1, \ldots, x_N) = 0 \right\}$$

and X appears as a liberation of X_{class}.

Proof. This is something that already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we denote by X'_{class} the manifold constructed in the statement, then we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class}) \to C(X'_{class})$$

Conversely now, from $X \subset S^{N-1}_{\mathbb{C},+}$ we obtain $X_{class} \subset S^{N-1}_{\mathbb{C}}$. Now since the relations defining X'_{class} are satisfied by X_{class}, we obtain an inclusion $X_{class} \subset X'_{class}$. Thus, at the level of algebras of continuous functions, we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X'_{class}) \to C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and we are done. □

Finally, once again at the level of the general theory, we have:

Definition 1.18. *We agree to identify two real algebraic submanifolds $X,Y\subset S^{N-1}_{\mathbb{C},+}$ when we have a $*$-algebra isomorphism between $*$-algebras of coordinates*

$$f:\mathcal{C}(Y)\to\mathcal{C}(X)$$

mapping standard coordinates to standard coordinates.

Let us go back now to our general (S,T,U,K) program. Now that we are done with the free spheres, we can introduce as well free tori, as follows:

Definition 1.19. *We have free real and complex tori, defined via*

$$C(T_N^+)=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,x_i^2=\frac{1}{N}\right)$$

$$C(\mathbb{T}_N^+)=C^*\left(x_1,\ldots,x_N\Big|x_ix_i^*=x_i^*x_i=\frac{1}{N}\right)$$

where the symbol C^* stands for universal enveloping C^*-algebra.

The fact that these tori are indeed well-defined comes from the fact that they are algebraic manifolds, in the sense of Definition 1.16 above. In fact, we have:

Proposition 1.20. *We have inclusions of algebraic manifolds, as follows:*

$$\begin{tikzcd}
S^{N-1}_{\mathbb{R},+} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\
T_N^+ \arrow[u] \arrow[r] & \mathbb{T}_N^+ \arrow[u]
\end{tikzcd}$$

In addition, this is an intersection diagram, in the sense that $T_N^+=\mathbb{T}_N^+\cap S^{N-1}_{\mathbb{R},+}$.

Proof. All this is clear indeed, by using the equivalence relation in Definition 1.18, in order to get rid of functional analytic issues at the C^*-algebra level. □

In analogy with Theorem 1.15, we have the following result:

Theorem 1.21. *We have inclusions of algebraic manifolds, as follows,*

$$\begin{tikzcd}
T_N^+ \arrow[r] & \mathbb{T}_N^+ \\
T_N \arrow[u] \arrow[r] & \mathbb{T}_N \arrow[u]
\end{tikzcd}$$

and the manifolds on top appear as liberations of those of the bottom.

Proof. This follows exactly as Theorem 1.15, and best here is to invoke Theorem 1.17 above, which is there precisely for dealing with such situations. □

Summarizing, we have free spheres and tori, having quite similar properties. Let us further study the tori. Up to a rescaling, these are given by algebras generated by unitaries, so studying the algebras generated by unitaries will be our next task. The point is that we have many such algebras, coming from the following construction:

Theorem 1.22. *Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by the fact that all group elements are unitaries, $g^* = g^{-1}$.*

(1) *The maximal C^*-seminorm on $\mathbb{C}[\Gamma]$ is a C^*-norm, and the closure of $\mathbb{C}[\Gamma]$ with respect to this norm is a C^*-algebra, denoted $C^*(\Gamma)$.*
(2) *When Γ is abelian, we have an isomorphism $C^*(\Gamma) \simeq C(G)$, where $G = \widehat{\Gamma}$ is its Pontrjagin dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$.*

Proof. All this is very standard, the idea being as follows:

(1) In order to prove the result, we must find a $*$-algebra embedding $\mathbb{C}[\Gamma] \subset B(H)$, with H being a Hilbert space. For this purpose, consider the space $H = l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi : \mathbb{C}[\Gamma] \subset B(H) \quad , \quad \pi(g)(h) = gh$$

Indeed, since $\pi(g)$ maps the basis $\{h\}_{h\in\Gamma}$ into itself, this operator is well-defined, bounded, and is an isometry. It is also clear from the formula $\pi(g)(h) = gh$ that $g \to \pi(g)$ is a morphism of algebras, and since this morphism maps the unitaries $g \in \Gamma$ into isometries, this is a morphism of $*$-algebras. Finally, the faithfulness of π is clear.

(2) Since Γ is abelian, the corresponding group algebra $A = C^*(\Gamma)$ is commutative. Thus, we can apply the Gelfand theorem, and we obtain $A = C(X)$, with:

$$X = Spec(A)$$

But the spectrum $X = Spec(A)$, consisting of the characters $\chi : C^*(\Gamma) \to \mathbb{C}$, can be identified with the Pontrjagin dual $G = \widehat{\Gamma}$, and this gives the result. □

The above result suggests the following definition:

Definition 1.23. *Given a discrete group Γ, the compact quantum space G given by*

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G = \widehat{\Gamma}$.

This is in fact something which is not very satisfactory, in general, due to amenability issues. However, in the case of the finitely generated discrete groups $\Gamma =< g_1, \ldots, g_N >$, which is the one that we are interested in here, the corresponding duals appear as algebraic submanifolds $\widehat{\Gamma} \subset S^{N-1}_{\mathbb{C},+}$, and the notion of equivalence from Definition 1.18 is precisely the one that we need, identifying full and reduced group algebras.

We can now refine our findings about tori, as follows:

Theorem 1.24. *The basic tori are all group duals, as follows,*

$$\begin{tikzcd} T_N^+ \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[u] \arrow[r] & \mathbb{T}_N \arrow[u] \end{tikzcd} \quad = \quad \begin{tikzcd} \widehat{\mathbb{Z}_2^{*N}} \arrow[r] & \widehat{F_N} \\ \mathbb{Z}_2^N \arrow[u] \arrow[r] & \mathbb{T}^N \arrow[u] \end{tikzcd}$$

where F_N is the free group on N generators, and $*$ is a group-theoretical free product.

Proof. The basic tori appear indeed as group duals, and together with the Fourier transform identifications from Theorem 1.22 (2), this gives the result. □

Following [19], let us try now to understand the correspondence between the spheres S and tori T. We first have the following result, summarizing our knowledge so far:

Theorem 1.25. *The four main quantum spheres produce the main quantum tori*

$$\begin{tikzcd} S^{N-1}_{\mathbb{R},+} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\ S^{N-1}_{\mathbb{R}} \arrow[u] \arrow[r] & S^{N-1}_{\mathbb{C}} \arrow[u] \end{tikzcd} \quad \to \quad \begin{tikzcd} T_N^+ \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[u] \arrow[r] & \mathbb{T}_N \arrow[u] \end{tikzcd}$$

via the formula $T = S \cap \mathbb{T}_N^+$, with the intersection being taken inside $S^{N-1}_{\mathbb{C},+}$.

Proof. This comes from the above results, the situation being as follows:

(1) Free complex case. Here the formula in the statement reads $\mathbb{T}_N^+ = S^{N-1}_{\mathbb{C},+} \cap \mathbb{T}_N^+$. But this is something trivial, because we have $\mathbb{T}_N^+ \subset S^{N-1}_{\mathbb{C},+}$.

(2) Free real case. Here the formula in the statement reads $T_N^+ = S^{N-1}_{\mathbb{R},+} \cap \mathbb{T}_N^+$. But this is something that we already know, from Proposition 1.20 above.

(3) Classical complex case. Here the formula in the statement reads $\mathbb{T}_N = S^{N-1}_{\mathbb{C}} \cap \mathbb{T}_N^+$. But this is clear as well, the classical version of $\mathbb{T}_N^+$ being $\mathbb{T}_N$.

(4) Classical real case. Here the formula in the statement reads $T_N = S^{N-1}_{\mathbb{R}} \cap \mathbb{T}_N^+$. But this follows by intersecting the formulae from the proof of (2) and (3). □

The correspondence $S \to T$ found above is not the only one. In order to discuss this, let us start with a general result, as follows:

Theorem 1.26. *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, the category of toral subgroups $T \subset \mathbb{T}_N^+$ acting affinely on X, in the sense that $\Phi(x_i) = x_i \otimes g_i$ defines a morphism*

$$\Phi : C(X) \to C(X) \otimes C(T)$$

has a universal object, denoted $T^+(X)$, and called toral isometry group of X.

Proof. In order to prove the result, assume that $X \subset S^{N-1}_{\mathbb{C},+}$ comes as follows:

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_\alpha(x_1, \ldots, x_N) = 0 \Big\rangle$$

Consider now the following variables:

$$X_i = x_i \otimes g_i \in C(X) \otimes C(\mathbb{T}_N^+)$$

Our claim is that the torus $T = T^+(X)$ in the statement appears as follows:

$$C(T) = C(\mathbb{T}_N^+) \Big/ \Big\langle f_\alpha(X_1, \ldots, X_N) = 0 \Big\rangle$$

In order to prove this claim, we have to clarify how the relations $f_\alpha(X_1, \ldots, X_N) = 0$ are interpreted inside $C(\mathbb{T}_N^+)$, and then show that T is indeed a toral subgroup.

So, pick one of the defining polynomials, $f = f_\alpha$, and write it as follows:

$$f(x_1, \ldots, x_N) = \sum_r \sum_{i_1^r \ldots i_{s_r}^r} \lambda_r \cdot x_{i_1^r} \ldots x_{i_{s_r}^r}$$

With $X_i = x_i \otimes g_i$ as above, we have the following formula:

$$f(X_1, \ldots, X_N) = \sum_r \sum_{i_1^r \ldots i_{s_r}^r} \lambda_r x_{i_1^r} \ldots x_{i_{s_r}^r} \otimes g_{i_1^r} \ldots g_{i_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f(X_1, \ldots, X_N) = 0$ correspond to certain relations between the variables g_i. Thus, we have indeed a subspace $T \subset \mathbb{T}_N^+$, with a universal map, as follows:

$$\Phi : C(X) \to C(X) \otimes C(T)$$

In order to show now that T is a group dual, consider the following elements:

$$g_i' = g_i \otimes g_i \quad , \quad X_i' = x_i \otimes g_i'$$

Then from $f(X_1, \ldots, X_N) = 0$ we deduce that, with $\Delta(g) = g \otimes g$, we have:

$$f(X_1', \ldots, X_N') = (id \otimes \Delta) f(X_1, \ldots, X_N) = 0$$

Thus we can map $g_i \to g_i'$, and it follows that T is a group dual, as desired. □

We will be back to this in section 3 below, with a full discussion of the various types of quantum isometries an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$ can have. Now with the above toral isometry group formalism in hand, we can formulate a second result regarding the spheres and tori, which is complementary to Theorem 1.25, as follows:

Theorem 1.27. *The four main quantum spheres produce via*

$$T = T^+(S)$$

the corresponding four main quantum tori.

Proof. This is something elementary, which can be established as follows:

(1) Free complex case. Here is there is nothing to be proved, because we obviously have an action $\mathbb{T}_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$, and this action can only be universal.

(2) Free real case. Here the situation is similar, because we have an obvious action $T_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, and it is clear that this action can only be universal.

(3) Classical complex case. Once again, we have a similar situation here, with the obvious action, namely $\mathbb{T}_N \curvearrowright S^{N-1}_{\mathbb{C}}$, being easily seen to be universal.

(4) Classical real case. Here the obvious action, namely $T_N \curvearrowright S^{N-1}_{\mathbb{R}}$, is universal as well, the reasons for this coming from (2) and (3) above. □

As a conclusion now, following [19], we can formulate:

Definition 1.28. *A "baby noncommutative geometry" consists of a quantum sphere S and a quantum torus T, which are by definition algebraic manifolds as follows,*

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C}+}$$

$$T_N \subset T \subset \mathbb{T}_N^+$$

which must be subject to the following compatibility conditions,

$$T = S \cap \mathbb{T}_N^+ = T^+(S)$$

with the intersection being taken inside $S^{N-1}_{\mathbb{C},+}$, and T^+ being the toral isometry group.

With this notion in hand, our main results so far can be summarized as follows:

Theorem 1.29. *We have 4 baby noncommutative geometries, as follows,*

$$\begin{array}{ccc} \mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\ \uparrow & & \uparrow \\ \mathbb{R}^N & \longrightarrow & \mathbb{C}^N \end{array}$$

with each symbol $\mathbb{K}^N_\times$ standing for the corresponding pair (S,T).

Proof. This follows indeed from Theorem 1.25 and Theorem 1.27. □

In what follows we will extend our baby theory, with pairs of type (U,K), consisting of unitary and reflection groups. This will lead to a theory which is more advanced.

2. Quantum groups

We have seen so far that the pairs sphere/torus (S,T) corresponding to the real and complex geometries, of $\mathbb{R}^N,\mathbb{C}^N$, have some natural free analogues. Our objective now will be that of adding to the picture a pair of quantum groups (U,K), as to reach to a quadruplet of objects (S,T,U,K), with relations between them, as follows:

$$\begin{tikzcd}
S \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
U \arrow[r, leftrightarrow] & K
\end{tikzcd}$$

The quantum group axioms that we need, coming from [148], are as follows:

Definition 2.1. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u\in M_N(A)$ whose coefficients generate A, such that the formulae*

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}\quad,\quad\varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}^*$$

define morphisms of C^*-algebras as follows,

$$\Delta:A\to A\otimes A\quad,\quad\varepsilon:A\to\mathbb{C}\quad,\quad S:A\to A^{opp}$$

called comultiplication, counit and antipode.

In this definition $A\otimes A$ is the universal C^*-algebraic completion of the usual algebraic tensor product of A with itself, and A^{opp} is the opposite C^*-algebra, with multiplication $a\cdot b=ba$. The reasons for using A^{opp} instead of A itself will become clear later on.

We say that A is cocommutative when $\Sigma\Delta=\Delta$, where $\Sigma(a\otimes b)=b\otimes a$ is the flip. We have the following result, which justifies the terminology and axioms:

Theorem 2.2. *The following are Woronowicz algebras:*

(1) *$C(G)$, with $G\subset U_N$ compact Lie group. Here the structural maps are:*

$$\Delta(\varphi)=(g,h)\to\varphi(gh)$$
$$\varepsilon(\varphi)=\varphi(1)$$
$$S(\varphi)=g\to\varphi(g^{-1})$$

(2) *$C^*(\Gamma)$, with $F_N\to\Gamma$ finitely generated group. Here the structural maps are:*

$$\Delta(g)=g\otimes g$$
$$\varepsilon(g)=1$$
$$S(g)=g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

Proof. In both cases, we have to exhibit a certain matrix u:

(1) Here we can use the matrix $u = (u_{ij})$ formed by matrix coordinates of G:

$$g = \begin{pmatrix} u_{11}(g) & \ldots & u_{1N}(g) \\ \vdots & & \vdots \\ u_{N1}(g) & \ldots & u_{NN}(g) \end{pmatrix}$$

(2) Here we can use the diagonal matrix formed by generators of Γ:

$$u = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Finally, the last assertion follows from the Gelfand theorem, in the commutative case. In the cocommutative case, this is something more technical, to be explained below. □

In general now, the structural maps Δ, ε, S have the following properties:

Proposition 2.3. *Let (A, u) be a Woronowicz algebra.*

(1) *Δ, ε satisfy the usual axioms for a comultiplication and a counit, namely:*

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$$

(2) *S satisfies the antipode axiom, on the $*$-subalgebra generated by entries of u:*

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(.)1$$

(3) *In addition, the square of the antipode is the identity, $S^2 = id$.*

Proof. Observe first that the result holds in the case where A is commutative. Indeed, by using Theorem 2.2 (1) we can write:

$$\Delta = m^T \quad , \quad \varepsilon = u^T \quad , \quad S = i^T$$

The 3 conditions in the statement come then by transposition from the basic 3 group theory conditions satisfied by m, u, i, namely:

$$m(m \times id) = m(id \times m)$$

$$m(id \times u) = m(u \times id) = id$$

$$m(id \times i)\delta = m(i \times id)\delta = 1$$

Here $\delta(g) = (g, g)$. Observe also that the last condition, $S^2 = id$, is satisfied as well, coming from the identity $i^2 = id$, which is a consequence of the group axioms.

Observe also that the result holds as well in the case where A is cocommutative, by using Theorem 2.2 (1). Indeed, the 3 formulae in the statement are all trivial, and the condition $S^2 = id$ follows once again from the group theory formula $(g^{-1})^{-1} = g$.

In the general case now, the proof goes as follows:

(1) We have the following computation:

$$(\Delta\otimes id)\Delta(u_{ij})=\sum_l\Delta(u_{il})\otimes u_{lj}=\sum_{kl}u_{ik}\otimes u_{kl}\otimes u_{lj}$$

We have as well the following computation, which gives the first formula:

$$(id\otimes\Delta)\Delta(u_{ij})=\sum_k u_{ik}\otimes\Delta(u_{kj})=\sum_{kl}u_{ik}\otimes u_{kl}\otimes u_{lj}$$

On the other hand, we have the following computation:

$$(id\otimes\varepsilon)\Delta(u_{ij})=\sum_k u_{ik}\otimes\varepsilon(u_{kj})=u_{ij}$$

We have as well the following computation, which gives the second formula:

$$(\varepsilon\otimes id)\Delta(u_{ij})=\sum_k\varepsilon(u_{ik})\otimes u_{kj}=u_{ij}$$

(2) By using the fact that the matrix $u=(u_{ij})$ is unitary, we obtain:

$$m(id\otimes S)\Delta(u_{ij})=\sum_k u_{ik}S(u_{kj})=\sum_k u_{ik}u_{jk}^*=(uu^*)_{ij}=\delta_{ij}$$

We have as well the following computation, which gives the result:

$$m(S\otimes id)\Delta(u_{ij})=\sum_k S(u_{ik})u_{kj}=\sum_k u_{ki}^*u_{kj}=(u^*u)_{ij}=\delta_{ij}$$

(3) Finally, the formula $S^2=id$ holds as well on the generators, and we are done. $\square$

Let us record as well the following technical result:

Proposition 2.4. *Given a Woronowicz algebra (A,u), we have $u^t=\bar{u}^{-1}$, so u is biunitary, in the sense that it is unitary, with unitary transpose.*

Proof. We have the following computation, based on the fact that u is unitary:

$$\begin{aligned}(uu^*)_{ij}=\delta_{ij}\quad&\Longrightarrow\quad\sum_k S(u_{ik}u_{jk}^*)=\delta_{ij}\\&\Longrightarrow\quad\sum_k u_{kj}u_{ki}^*=\delta_{ij}\\&\Longrightarrow\quad(u^t\bar{u})_{ji}=\delta_{ij}\end{aligned}$$

Similarly, we have the following computation, once agan using the unitarity of u:

$$\begin{aligned}(u^*u)_{ij}=\delta_{ij}\quad&\implies\quad\sum_kS(u_{ki}^*u_{kj})=\delta_{ij}\\&\implies\quad\sum_ku_{jk}^*u_{ik}=\delta_{ij}\\&\implies\quad(\bar{u}u^t)_{ji}=\delta_{ij}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Summarizing, the Woronowicz algebras appear to have nice properties. In view of Theorem 2.2 and of Proposition 2.3, we can formulate the following definition:

Definition 2.5. *Given a Woronowicz algebra A, we formally write*

$$A=C(G)=C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

When A is both commutative and cocommutative, G and Γ are usual abelian groups, dual to each other. In general, we still agree to write $G=\widehat{\Gamma},\Gamma=\widehat{G}$, but in a formal sense. With this picture in mind, let us call now corepresentation of A any unitary matrix $v\in M_n(A)$ satisfying the same conditions are those satisfied by u, namely:

$$\Delta(v_{ij})=\sum_kv_{ik}\otimes v_{kj}\quad,\quad\varepsilon(v_{ij})=\delta_{ij}\quad,\quad S(v_{ij})=v_{ji}^*$$

These corepresentations can be thought of as corresponding to the representations of the underlying compact quantum group G. Following Woronowicz [148], we have:

Theorem 2.6. *Any Woronowicz algebra $A=C(G)$ has a Haar integration functional,*

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

which can be constructed by starting with any faithful positive form $\varphi\in A^*$, and setting

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation $v\in M_n(\mathbb{C})\otimes A$ we have

$$\left(id\otimes\int_G\right)v=P$$

where P is the orthogonal projection onto the corresponding fixed point space:

$$Fix(v)=\left\{\xi\in\mathbb{C}^n\Big|v\xi=\xi\right\}$$

Proof. Following [148], this can be done in 3 steps, as follows:

(1) Given $\varphi \in A^*$, our claim is that the following limit converges, for any $a \in A$:

$$\int_\varphi a = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \varphi^{*k}(a)$$

Indeed, we can assume, by linearity, that a is the coefficient of a corepresentation:

$$a = (\tau \otimes id)v$$

But in this case, an elementary computation shows that we have the following formula, where P_φ is the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi)v$:

$$\left(id \otimes \int_\varphi\right) v = P_\varphi$$

(2) Since $v\xi = \xi$ implies $[(id \otimes \varphi)v]\xi = \xi$, we have $P_\varphi \geq P$, where P is the orthogonal projection onto $Fix(v)$. The point now is that when $\varphi \in A^*$ is faithful, by using a positivity trick, one can prove that we have $P_\varphi = P$. Thus our linear form $\int_\varphi$ is independent of φ, and is given on the coefficients of corepresentations $a = (\tau \otimes id)v$ by:

$$\left(id \otimes \int_\varphi\right) v = P$$

(3) With the above formula in hand, the left and right invariance of $\int_G = \int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [148]. □

Consider the dense $*$-subalgebra $\mathcal{A} \subset A$ generated by the coefficients of the fundamental corepresentation u, and endow it with the following scalar product:

$$< a, b >= \int_G ab^*$$

Once again following [148], we have the following result:

Theorem 2.7. *We have the following Peter-Weyl type results:*

1. *Any corepresentation decomposes as a sum of irreducible corepresentations.*
2. *Each irreducible corepresentation appears inside a certain $u^{\otimes k}$.*
3. *$\mathcal{A} = \bigoplus_{v\in Irr(A)} M_{\dim(v)}(\mathbb{C})$, the summands being pairwise orthogonal.*
4. *The characters of irreducible corepresentations form an orthonormal system.*

Proof. All these results are from [148], the idea being as follows:

(1) Given $v \in M_n(A)$, its intertwiner algebra $End(v) = \{T \in M_n(\mathbb{C}) | Tv = vT\}$ is a finite dimensional C^*-algebra, and so decomposes as $End(v) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C})$. But this gives a decomposition of type $v = v_1 + \ldots + v_r$, as desired.

(2) Consider indeed the Peter-Weyl corepresentations, $u^{\otimes k}$ with k colored integer, defined by $u^{\otimes\emptyset} = 1$, $u^{\otimes\circ} = u$, $u^{\otimes\bullet} = \bar{u}$ and multiplicativity. The coefficients of these corepresentations span the dense algebra $\mathcal{A}$, and by using (1), this gives the result.

(3) Here the direct sum decomposition, which is technically a $*$-coalgebra isomorphism, follows from (2). As for the second assertion, this follows from the fact that $(id \otimes \int_G)v$ is the orthogonal projection P_v onto the space $Fix(v)$, for any corepresentation v.

(4) Let us define indeed the character of $v \in M_n(A)$ to be the matrix trace, $\chi_v = Tr(v)$. Since this character is a coefficient of v, the orthogonality assertion follows from (3). As for the norm 1 claim, this follows once again from $(id \otimes \int_G)v = P_v$. $\square$

We refer to [148] for full details on all the above, and for some applications as well. Let us just record here the fact that in the cocommutative case, we obtain from (4) that the irreducible corepresentations must be all 1-dimensional, and so that we must have $A = C^*(\Gamma)$ for some discrete group Γ, as mentioned in Theorem 2.2 above. At a more technical level now, following [51], we have the following result:

Theorem 2.8. *Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of A_{full} is faithful.*
2. *The projection map $A_{full} \to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red}.*
4. *We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

Proof. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial, and conversely, a counit map $\varepsilon : A_{red} \to \mathbb{C}$ produces an isomorphism $A_{red} \to A_{full}$, via a formula of type $(\varepsilon \otimes id)\Phi$.

(3) $\iff$ (4) Here $\implies$ is clear, coming from $\varepsilon(N - Re(\chi(u))) = 0$, and the converse can be proved by doing some standard functional analysis. $\square$

Yet another important technical result is Tannakian duality, as follows:

Theorem 2.9. *The following operations are inverse to each other:*

1. *The construction $A \to C$, which associates to any Woronowicz algebra A the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
2. *The construction $C \to A$, which associates to any tensor category C the Woronowicz algebra A presented by the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

Proof. This is something quite deep, going back to [149] in a slightly different form, and to [106] in the simplified form presented above. The idea is as follows:

(1) We have indeed a construction $A \to C$ as above, whose output is a tensor C^*-subcategory with duals of the tensor C^*-category of Hilbert spaces.

(2) We have as well a construction $C \to A$ as above, simply by dividing the free $*$-algebra on N^2 variables by the relations in the statement.

Regarding now the bijection claim, some elementary algebra shows that $C = C_{A_C}$ implies $A = A_{C_A}$, and also that $C \subset C_{A_C}$ is automatic. Thus we are left with proving $C_{A_C} \subset C$. But this latter inclusion can be proved indeed, by doing some algebra, and using von Neumann's bicommutant theorem, in finite dimensions. See [106]. □

As a concrete consequence of the above result, we have:

Theorem 2.10. *We have an embedding as follows, using double indices,*

$$G \subset S^{N^2-1}_{\mathbb{C},+} \quad , \quad x_{ij} = \frac{u_{ij}}{\sqrt{N}}$$

making G an algebraic submanifold of the free sphere.

Proof. The fact that we have an embedding as above follows from the fact that $u = (u_{ij})$ is biunitary, that we know from Proposition 2.4. As for the algebricity claim, this follows from Theorem 2.9. Indeed, assuming that $A = C(G)$ is of the form $A = A_C$, it follows that G is algebraic. But this is always the case, because we can take $C = C_A$. □

Let us get back now to our original objective, namely constructing pairs of quantum unitary and reflection groups (O_N^+, H_N^+) and (U_N^+, K_N^+), as to complete the pairs $(S^{N-1}_{\mathbb{R},+}, T_N^+)$ and $(S^{N-1}_{\mathbb{C},+}, \mathbb{T}_N^+)$ that we already have. Following Wang [140], we have:

Theorem 2.11. *The following constructions produce compact quantum groups,*

$$C(O_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u = \bar{u}, u^t = u^{-1}\right)$$
$$C(U_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u^* = u^{-1}, u^t = \bar{u}^{-1}\right)$$

which appear respectively as liberations of the groups O_N and U_N.

Proof. This first assertion follows from the elementary fact that if a matrix $u = (u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$u^\Delta_{ij} = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u^\varepsilon_{ij} = \delta_{ij} \quad , \quad u^S_{ij} = u^*_{ji}$$

Regarding now the matrix $u^\varepsilon = 1_N$, this is clearly biunitary. Finally, regarding the matrix u^S, there is nothing to prove here either, because its unitarity its clear too. Finally, observe that if u is real, then so are the above matrices $u^\Delta, u^\varepsilon, u^S$.

Thus, we can define morphisms Δ,ε,S as in Definition 2.1, by using the universal properties of $C(O_N^+)$, $C(U_N^+)$. As for the second assertion, this follows exactly as for the free spheres, by adapting the sphere proof from section 1 above. $\square$

The basic properties of O_N^+,U_N^+ can be summarized as follows:

Theorem 2.12. *The quantum groups O_N^+,U_N^+ have the following properties:*

1. *The closed subgroups $G\subset U_N^+$ are exactly the $N\times N$ compact quantum groups. As for the closed subgroups $G\subset O_N^+$, these are those satisfying $u=\bar{u}$.*
2. *We have liberation embeddings $O_N\subset O_N^+$ and $U_N\subset U_N^+$, obtained by dividing the algebras $C(O_N^+),C(U_N^+)$ by their respective commutator ideals.*
3. *We have as well embeddings $\widehat{L}_N\subset O_N^+$ and $\widehat{F}_N\subset U_N^+$, where L_N is the free product of N copies of $\mathbb{Z}_2$, and where F_N is the free group on N generators.*

Proof. All these assertions are elementary, as follows:

(1) This is clear from definitions, with the remark that, in the context of Definition 2.1 above, the formula $S(u_{ij})=u_{ji}^*$ shows that the matrix $\bar{u}$ must be unitary too.

(2) This follows from the Gelfand theorem. To be more precise, this shows that we have presentation results for $C(O_N),C(U_N)$, similar to those in Theorem 2.11, but with the commutativity between the standard coordinates and their adjoints added:

$$\begin{aligned}C(O_N)&=C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},u^t=u^{-1}\right)\\ C(U_N)&=C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right)\end{aligned}$$

Thus, we are led to the conclusion in the statement.

(3) This follows from (1) and from Theorem 2.2 above, with the remark that with $u=diag(g_1,\ldots,g_N)$, the condition $u=\bar{u}$ is equivalent to $g_i^2=1$, for any i. $\square$

The last assertion in Theorem 2.12 suggests the following construction:

Proposition 2.13. *Given a closed subgroup $G\subset U_N^+$, consider its "diagonal torus", which is the closed subgroup $T\subset G$ constructed as follows:*

$$C(T)=C(G)\Big/\left\langle u_{ij}=0\Big|\forall i\neq j\right\rangle$$

This torus is then a group dual, $T=\widehat{\Lambda}$, where $\Lambda=<g_1,\ldots,g_N>$ is the discrete group generated by the elements $g_i=u_{ii}$, which are unitaries inside $C(T)$.

Proof. Since u is unitary, its diagonal entries $g_i=u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i)=g_i\otimes g_i$$

It follows that we have $C(T)=C^*(\Lambda)$, modulo identifying as usual the C^*-completions of the various group algebras, and so that we have $T=\widehat{\Lambda}$, as claimed. $\square$

With this notion in hand, Theorem 2.12 (3) reformulates as follows:

Theorem 2.14. *The diagonal tori of the basic unitary groups are the basic tori:*

$$\begin{matrix} O_N^+ & \to & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \to & U_N \end{matrix} \quad \to \quad \begin{matrix} T_N^+ & \to & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \to & \mathbb{T}_N \end{matrix}$$

In particular, the basic unitary groups are all distinct.

Proof. This is something clear and well-known in the classical case, and in the free case, this is a reformulation of Theorem 2.12 (3) above, which tells us that the diagonal tori of O_N^+, U_N^+, in the sense of Proposition 2.13, are the group duals $\widehat{L}_N, \widehat{F}_N$. $\square$

There is an obvious relation here with the considerations from section 1 above, that we will analyse later on. As a second result now regarding our free quantum groups, relating them this time to the free spheres constructed in section 1, we have:

Proposition 2.15. *We have embeddings of algebraic manifolds as follows, obtained in double indices by rescaling the coordinates, $x_{ij} = u_{ij}/\sqrt{N}$:*

$$\begin{matrix} O_N^+ & \to & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \to & U_N \end{matrix} \quad \to \quad \begin{matrix} S^{N^2-1}_{\mathbb{R},+} & \to & S^{N^2-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N^2-1}_{\mathbb{R}} & \to & S^{N^2-1}_{\mathbb{C}} \end{matrix}$$

Moreover, the quantum groups appear from the quantum spheres via

$$G = S \cap U_N^+$$

with the intersection being computed inside the free sphere $S^{N^2-1}_{\mathbb{C},+}$.

Proof. As explained in Theorem 2.10 above, the biunitarity of the matrix $u = (u_{ij})$ gives an embedding of algebraic manifolds, as follows:

$$U_N^+ \subset S^{N^2-1}_{\mathbb{C},+}$$

Now since the relations defining $O_N, O_N^+, U_N \subset U_N^+$ are the same as those defining $S^{N^2-1}_{\mathbb{R}}, S^{N^2-1}_{\mathbb{R},+}, S^{N^2-1}_{\mathbb{C}} \subset S^{N^2-1}_{\mathbb{C},+}$, this gives the result. $\square$

Summarizing, in connection with our (S,T,U,K) program, we have so far triples of type (S,T,U), along with some correspondences between S,T,U. In order to introduce now the reflection groups K, things are more tricky, involving quantum permutation groups. Following Wang [141], these quantum groups are introduced as follows:

Theorem 2.16. *The following construction, where "magic" means formed of projections, which sum up to* 1 *on each row and column,*

$$C(S_N^+)=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\text{magic}\right)$$

produces a quantum group liberation of S_N. Moreover, the inclusion

$$S_N\subset S_N^+$$

is an isomorphism at $N\leq 3$, but not at $N\geq 4$, where S_N^+ is not classical, nor finite.

Proof. The quantum group assertion follows by using the same arguments as those in the proof of Theorem 2.11. Consider indeed the following matrix:

$$U_{ij}=\sum_k u_{ik}\otimes u_{kj}$$

As a first observation, the entries of this matrix are self-adjoint:

$$U_{ij}=U_{ij}^*$$

In fact the entries U_{ij} are orthogonal projections, because we have as well:

$$\begin{aligned}U_{ij}^2&=\sum_{kl}u_{ik}u_{il}\otimes u_{kj}u_{lj}\\&=\sum_k u_{ik}\otimes u_{kj}\\&=U_{ij}\end{aligned}$$

In order to prove now that the matrix $U=(U_{ij})$ is magic, it remains to verify that the sums on the rows and columns are 1. For the rows, this can be checked as follows:

$$\begin{aligned}\sum_j U_{ij}&=\sum_{jk}u_{ik}\otimes u_{kj}\\&=\sum_k u_{ik}\otimes 1\\&=1\otimes 1\end{aligned}$$

For the columns the computation is similar, as follows:

$$\begin{aligned}\sum_i U_{ij}&=\sum_{ik}u_{ik}\otimes u_{kj}\\&=\sum_k 1\otimes u_{kj}\\&=1\otimes 1\end{aligned}$$

Thus the $U=(U_{ij})$ is magic, and so we can define a comultiplication map by using the universality property of $C(S_N^+)$, by setting $\Delta(u_{ij})=U_{ij}$. By using a similar reasoning, we can define as well a counit map by $\varepsilon(u_{ij}) = \delta_{ij}$, and an antipode map by $S(u_{ij}) = u_{ji}$. Thus the Woronowicz algebra axioms from Definition 2.1 are satisfied, and this finishes the proof of the first assertion, stating that S_N^+ is indeed a compact quantum group.

Observe now that we have an embedding of compact quantum groups $S_N \subset S_N^+$, obtained by using the standard coordinates of S_N, viewed as an algebraic group:

$$u_{ij} = \chi\left(\sigma \in S_N \middle| \sigma(j) = i\right)$$

By using the Gelfand theorem and working out the details, as we did with the free spheres are free unitary groups, the embedding $S_N \subset S_N^+$ is indeed a liberation.

Finally, regarding the last assertion, the study here is as follows:

Case $N = 2$. The result here is trivial, the 2×2 magic matrices being by definition as follows, with p being a projection:

$$U = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Indeed, this shows that the entries of a 2×2 magic matrix must pairwise commute, and so the algebra $C(S_2^+)$ follows to be commutative, which gives the result.

Case $N = 3$. This is more tricky, and we present here a short proof from [105]. By using the same abstract argument as in the $N = 2$ case, and by permuting rows and columns, it is enough to check that u_{11}, u_{22} commute. But this follows from:

$$\begin{aligned} u_{11}u_{22} &= u_{11}u_{22}(u_{11} + u_{12} + u_{13}) \\ &= u_{11}u_{22}u_{11} + u_{11}u_{22}u_{13} \\ &= u_{11}u_{22}u_{11} + u_{11}(1 - u_{21} - u_{23})u_{13} \\ &= u_{11}u_{22}u_{11} \end{aligned}$$

Indeed, by applying the involution to this formula, we obtain from this:

$$u_{22}u_{11} = u_{11}u_{22}u_{11}$$

Thus we obtain $u_{11}u_{22} = u_{22}u_{11}$, as desired.

Case $N = 4$. In order to prove our various claims about S_4^+, consider the following matrix, with p, q being projections, on some infinite dimensional Hilbert space:

$$U = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

This matrix is magic, and if we choose p, q as for the algebra $< p, q >$ to be not commutative, and infinite dimensional, we conclude that $C(S_4^+)$ is not commutative and infinite dimensional as well, and in particular is not isomorphic to $C(S_4)$.

Case $N \geq 5$. Here we can use the standard embedding $S_4^+ \subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u \to \begin{pmatrix} u & 0 \\ 0 & 1_{N-4} \end{pmatrix}$$

Indeed, with this embedding in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N \geq 5$ has these two properties as well. $\square$

With the above result in hand, we can now introduce the quantum reflections:

Theorem 2.17. *The following constructions produce compact quantum groups,*

$$\begin{aligned} C(H_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u_{ij} = u_{ij}^*,\ (u_{ij}^2) = \text{magic}\right) \\ C(K_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|[u_{ij}, u_{ij}^*] = 0,\ (u_{ij}u_{ij}^*) = \text{magic}\right) \end{aligned}$$

which appear respectively as liberations of the reflection groups H_N and K_N.

Proof. This can be proved in the usual way, with the first assertion coming from the fact that if u satisfies the relations in the statement, then so do the matrices $u^\Delta, u^\varepsilon, u^S$, and with the second assertion coming as in the sphere case. See [15], [20]. $\square$

Summarizing, we are done with our construction task for the quadruplets (S, T, U, K), in the free real and complex cases, and we can now formulate:

Proposition 2.18. *We have a quadruplet as follows, called free real,*

$$\begin{tikzcd} S^{N-1}_{\mathbb{R},+} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & T_N^+ \arrow[d, dash] \arrow[dl, dash] \\ O_N^+ \arrow[r, dash] & H_N^+ \end{tikzcd}$$

and a quadruplet as follows, called free complex:

$$\begin{tikzcd} S^{N-1}_{\mathbb{C},+} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbb{T}_N^+ \arrow[d, dash] \arrow[dl, dash] \\ U_N^+ \arrow[r, dash] & K_N^+ \end{tikzcd}$$

Proof. This is more of an empty statement, coming from the constructions above. $\square$

Going ahead now, we must construct correspondences between our objects (S,T,U,K), completing the work for the pairs (S,T) started in section 1 above. This will take some time, and we will need some preliminaries. To start with, let us record the following result, which refines the various liberation statements formulated above:

Theorem 2.19. *The quantum unitary and reflection groups are as follows,*

$$\begin{tikzcd}
& K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
& K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[ur] \arrow[uu] & & O_N \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

and in this diagram, any face $P\subset Q, R\subset S$ has the property $P=Q\cap R$.

Proof. The fact that we have inclusions as in the statement follows from the definition of the various quantum groups involved. As for the various intersection claims, these follow as well from definitions. For some further details on all this, we refer to [14]. □

In order to efficiently deal with the above quantum groups, we will need Tannakian duality results, in the spirit of the Brauer theorem [54]. Following [37], we have:

Definition 2.20. *Associated to any partition $\pi\in P(k,l)$ between an upper row of k points and a lower row of l points is the linear map $T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$ given by*

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not.

To be more precise, we agree to put the two multi-indices on the two rows of points, in the obvious way. The Kronecker symbols are then defined by $\delta_\pi=1$ when all the strings of π join equal indices, and by $\delta_\pi=0$ otherwise. This construction is motivated by:

Proposition 2.21. *The assignement $\pi\to T_\pi$ is categorical, in the sense that we have*

$$T_\pi\otimes T_\sigma=T_{[\pi\sigma]}$$

$$T_\pi T_\sigma=N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}$$

$$T_\pi^*=T_{\pi^*}$$

where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.

Proof. This follows from some routine computations, as follows:

(1) The concatenation axiom follows from the following computation:

$$\begin{aligned}
&(T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\ T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})
\end{aligned}$$

(2) The composition axiom follows from the following computation:

$$\begin{aligned}
&T_\pi T_\sigma(e_{i_1}\otimes\ldots\otimes e_{i_p})\\
=&\sum_{j_1\ldots j_q}\delta_\sigma\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\sum_{k_1\ldots k_r}\delta_\pi\begin{pmatrix}j_1&\ldots&j_q\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
=&\sum_{k_1\ldots k_r}N^{c(\pi,\sigma)}\delta_{[^\sigma_\pi]}\begin{pmatrix}i_1&\ldots&i_p\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
=&\ N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}(e_{i_1}\otimes\ldots\otimes e_{i_p})
\end{aligned}$$

(3) Finally, the involution axiom follows from the following computation:

$$\begin{aligned}
&T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q})\\
=&\sum_{i_1\ldots i_p}<T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q}),e_{i_1}\otimes\ldots\otimes e_{i_p}>e_{i_1}\otimes\ldots\otimes e_{i_p}\\
=&\sum_{i_1\ldots i_p}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}e_{i_1}\otimes\ldots\otimes e_{i_p}\\
=&\ T_{\pi^*}(e_{j_1}\otimes\ldots\otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. See [37]. □

We have the following notion, from [37], [127]:

Definition 2.22. *A collection of sets $D=\bigsqcup_{k,l}D(k,l)$ with $D(k,l)\subset P(k,l)$ is called a category of partitions when it has the following properties:*

1. *Stability under the horizontal concatenation, $(\pi,\sigma)\to[\pi\sigma]$.*
2. *Stability under vertical concatenation $(\pi,\sigma)\to[^\sigma_\pi]$, with matching middle symbols.*
3. *Stability under the upside-down turning $*$, with switching of colors, $\circ\leftrightarrow\bullet$.*
4. *Each set $P(k,k)$ contains the identity partition $||\ldots||$.*
5. *The sets $P(\emptyset,\circ\bullet)$ and $P(\emptyset,\bullet\circ)$ both contain the semicircle $\cap$.*

We can now formulate a key result, from [37], as follows:

Theorem 2.23. *Each category of partitions $D=(D(k,l))$ produces a family of compact quantum groups $G=(G_N)$, one for each $N\in\mathbb N$, via the formula*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

which produces a Tannakian category, and therefore a closed subgroup $G_N\subset U_N^+$. The quantum groups which appear in this way are called "easy".

Proof. This follows indeed from Woronowicz's Tannakian duality, in its "soft" form from [106], as explained in Theorem 2.9 above. Indeed, let us set:

$$C(k,l)=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

By using the axioms in Definition 2.22, and the categorical properties of the operation $\pi\to T_\pi$, from Proposition 2.21, we deduce that $C=(C(k,l))$ is a Tannakian category. Thus the Tannakian duality result applies, and gives the result. □

We can now formulate a general Brauer theorem, as follows:

Theorem 2.24. *The basic quantum unitary and quantum reflection groups, namely*

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[ur] \arrow[uu] & & O_N \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

are all easy. The corresponding categories of partitions form an intersection diagram.

Proof. This is well-known, the corresponding categories being as follows, with P_{even} being the category of partitions having even blocks, and with $\mathcal P_{even}(k,l)\subset P_{even}(k,l)$ consisting of the partitions satisfying $\#\circ=\#\bullet$ in each block, when flattening the partition:

$$\begin{tikzcd}
 & \mathcal{NC}_{even} \arrow[dl] \arrow[dd] & & \mathcal{NC}_2 \arrow[ll] \arrow[dl] \arrow[dd] \\
NC_{even} \arrow[dd] & & NC_2 \arrow[ll] \arrow[dd] & \\
 & \mathcal P_{even} \arrow[dl] & & \mathcal P_2 \arrow[ll] \arrow[dl] \\
P_{even} & & P_2 \arrow[ll] &
\end{tikzcd}$$

To be more precise, the proof goes as follows:

(1) The quantum group U_N^+ is defined via the following relations:

$$u^* = u^{-1}$$

$$u^t = \bar{u}^{-1}$$

But these relations tell us precisely that the following two operators must be in the associated Tannakian category C:

$$T_\pi \quad , \quad \pi = {}^{\,\cap}_{\circ\bullet}$$

$$T_\pi \quad , \quad \pi = {}^{\,\cap}_{\bullet\circ}$$

Thus the associated Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D = < {}^{\,\cap}_{\circ\bullet} \ , \ {}^{\,\cap}_{\bullet\circ} > = \mathcal{N}C_2$$

Thus, we are led to the conclusion in the statement.

(2) The quantum group $O_N^+ \subset U_N^+$ is defined by imposing the following relations:

$$u_{ij} = \bar{u}_{ij}$$

But these relations tell us that the following operators must be in the associated Tannakian category C:

$$T_\pi \quad , \quad \pi = \substack{\circ \\ | \\ \bullet}$$

$$T_\pi \quad , \quad \pi = \substack{\bullet \\ | \\ \circ}$$

Thus the associated Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D = < \mathcal{NC}_2, \substack{\circ \\ | \\ \bullet}, \substack{\bullet \\ | \\ \circ} > = NC_2$$

Thus, we are led to the conclusion in the statement.

(3) The group $U_N \subset U_N^+$ is defined via the following relations:

$$[u_{ij}, u_{kl}] = 0$$

$$[u_{ij}, \bar{u}_{kl}] = 0$$

But these relations tell us that the following operators must be in the associated Tannakian category C:

$$T_\pi \quad , \quad \pi = \substack{\circ\circ \\ \times \\ \circ\circ}$$

$$T_\pi \quad , \quad \pi = \substack{\circ\bullet \\ \times \\ \bullet\circ}$$

Thus the associated Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D = < \mathcal{NC}_2, \substack{\circ\circ \\ \times \\ \circ\circ}, \substack{\circ\bullet \\ \times \\ \bullet\circ} > = \mathcal{P}_2$$

Thus, we are led to the conclusion in the statement.

(4) In order to deal now with O_N, we can simply use the following formula:

$$O_N = O_N^+ \cap U_N$$

At the categorical level, this tells us indeed that the associated Tannakian category is given by $C = span(T_\pi|\pi \in D)$, with:

$$D =< NC_2, \mathcal{P}_2 >= P_2$$

Thus, we are led to the conclusion in the statement.

(5) The proof for the reflection groups is similar, by adding and interpreting the reflection relations. We refer here to [20] and [15], for full details.

(6) As for the second assertion, which will be of use later on, this is something well-known and standard too. We refer here to [15], [20], [23], and to [14], [37] as well. $\square$

Getting back now to our axiomatization questions, we must establish correspondences between our objects (S, T, U, K), as a continuation of the work started in section 1, for the pairs (S, T). Let us start by discussing the following correspondences:

$$U \to K \to T$$

We know from Theorem 2.14 that the correspondences $U \to T$ appear by taking the diagonal tori. In fact, the correspondences $K \to T$ appear by taking the diagonal tori as well, and the correspondences $U \to K$ are something elementary too, obtained by taking the "reflection subgroup". The complete statement here is as follows:

Theorem 2.25. *For the basic quadruplets (S, T, U, K), the correspondences*

$$\begin{tikzcd} O_N^+ \arrow[r] & U_N^+ & & H_N^+ \arrow[r] & K_N^+ & & T_N^+ \arrow[r] & \mathbb{T}_N^+ \\ & & \to & & & \to & & \\ O_N \arrow[r] \arrow[uu] & U_N \arrow[uu] & & H_N \arrow[r] \arrow[uu] & K_N \arrow[uu] & & T_N \arrow[r] \arrow[uu] & \mathbb{T}_N \arrow[uu] \end{tikzcd}$$

appear in the following way:

1. *$U \to K$ appears by taking the reflection subgroup, $K = U \cap K_N^+$.*
2. *$U \to T$ appears by taking the diagonal torus, $T = U \cap \mathbb{T}_N^+$.*
3. *$K \to T$ appears as well by taking the diagonal torus, $T = K \cap \mathbb{T}_N^+$.*

Proof. This follows from the results that we already have, as follows:

(1) This follows from Theorem 2.24, because the left face of the cube diagram there appears by intersecting the right face with the quantum group K_N^+.

(2) This is something that we already know, from Theorem 2.14 above.

(3) This follows exactly as in the unitary case, via the proof of Theorem 2.14. $\square$

As a conclusion now, with respect to the "baby theory" developed in section 1 above, concerning the pairs (S,T), we have some advances. First, we have completed the pairs (S,T) there into quadruplets (S,T,U,K). And second, we have established some correspondences between our objects, the situation here being as follows:

$$\begin{tikzcd} S \arrow[rr] & & T \\ & & \\ U \arrow[rr] \arrow[uurr] & & K \arrow[uu] \end{tikzcd}$$

There is still a long way to go, in order to establish a full set of correspondences, and to reach to an axiomatization, the idea being that the correspondences $S \leftrightarrow U$ can be established by using quantum isometries, and that the correspondences $T \to K \to U$ can be established by using advanced quantum group theory, and with all this heavily relying on the easiness theory developed above. We will discuss this in sections 3-4 below.

3. Affine isometries

We have seen so far that we have quadruplets (S,T,U,K) consisting of a sphere S, a torus T, a unitary group U and a reflection group K, corresponding to the four main geometries, namely real and complex, classical and free, which are as follows:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

We have to work out now the various correspondences between our objects (S,T,U,K). We know from sections 1-2 that we already have 4 such correspondences, namely $S\to T$ and $U\to K\to T$. In this section we discuss 3 more correspondences, namely $S\leftrightarrow U$ and $T\to K$, as to reach to a total of 7 correspondences, as follows:

$$\begin{tikzcd}
S \arrow[r] \arrow[d, leftrightarrow] & T \arrow[d, leftrightarrow] \\
U \arrow[r] \arrow[ur] & K
\end{tikzcd}$$

In order to connect the spheres and tori (S,T) to the quantum groups (U,K), the idea will be that of using quantum isometry groups. Let us start with:

Proposition 3.1. *Given an algebraic manifold $X\subset S^{N-1}_\mathbb{C}$, the formula*

$$G(X)=\left\{U\in U_N\Big|U(X)=X\right\}$$

defines a compact group of unitary matrices, or isometries, called affine isometry group of X. For the spheres $S^{N-1}_\mathbb{R},S^{N-1}_\mathbb{C}$ we obtain in this way the groups O_N,U_N.

Proof. The fact that $G(X)$ as defined above is indeed a group is clear, its compactness is clear as well, and finally the last assertion is clear as well. In fact, all this works for any closed subset $X\subset\mathbb{C}^N$, but we are not interested here in such general spaces. $\square$

Observe that in the case of the spheres, the affine isometry group $G(X)$ leaves invariant the Riemannian metric, because this metric is equivalent to the one inherited from $\mathbb{C}^N$, which is preserved by our isometries $U\in U_N$. Thus, we could have constructed as well $G(X)$ as being the group of metric isometries of X, with of course some extra care in relation with the complex structure, as for the complex sphere $X=S^{N-1}_\mathbb{C}$ to produce $G(X)=U_N$ instead of $G(X)=O_{2N}$. However, in the noncommutative setting, all this becomes considerably more complicated, and we prefer to use the above construction, as such. We will be back later to metric aspects, at the end of the present section.

We have the following quantum analogue of Proposition 3.1:

Theorem 3.2. *Given an algebraic manifold $X\subset S^{N-1}_{\mathbb C,+}$, the category of the closed subgroups $G\subset U_N^+$ acting affinely on X, in the sense that the formula*

$$\Phi(x_i)=\sum_j x_j\otimes u_{ji}$$

defines a morphism of C^*-algebras

$$\Phi:C(X)\to C(X)\otimes C(G)$$

has a universal object, denoted $G^+(X)$, and called affine quantum isometry group of X.

Proof. Observe first that in the case where the above morphism Φ exists, this morphism is automatically a coaction, in the sense that it satisfies the following conditions:

$$(\Phi\otimes id)\Phi=(id\otimes\Delta)\Phi$$

$$(id\otimes\varepsilon)\Phi=id$$

In order to prove now the result, assume that $X\subset S^{N-1}_{\mathbb C,+}$ comes as follows:

$$C(X)=C(S^{N-1}_{\mathbb C,+})\Big/\Big\langle f_\alpha(x_1,\ldots,x_N)=0\Big\rangle$$

Consider now the following variables:

$$X_i=\sum_j x_j\otimes u_{ji}\in C(X)\otimes C(U_N^+)$$

Our claim is that the quantum group in the statement $G=G^+(X)$ appears as:

$$C(G)=C(U_N^+)\Big/\Big\langle f_\alpha(X_1,\ldots,X_N)=0\Big\rangle$$

In order to prove this claim, we have to clarify how the relations $f_\alpha(X_1,\ldots,X_N)=0$ are interpreted inside $C(U_N^+)$, and then show that G is indeed a quantum group.

So, pick one of the defining polynomials, $f=f_\alpha$, and write it as follows:

$$f(x_1,\ldots,x_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\cdot x_{i_1^r}\ldots x_{i_{s_r}^r}$$

With $X_i=\sum_j x_j\otimes u_{ji}$ as above, we have the following formula:

$$f(X_1,\ldots,X_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\sum_{j_1^r\ldots j_{s_r}^r}x_{j_1^r}\ldots x_{j_{s_r}^r}\otimes u_{j_1^ri_1^r}\ldots u_{j_{s_r}^ri_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f(X_1,\ldots,X_N)=0$ correspond to certain relations between the variables u_{ij}. Thus, we have indeed a subspace $G\subset U_N^+$, with a universal map:

$$\Phi:C(X)\to C(X)\otimes C(G)$$

In order to show now that G is a quantum group, consider the following elements:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}^*$$

Consider as well the following elements, with $\gamma\in\{\Delta,\varepsilon,S\}$:

$$X_i^\gamma=\sum_j x_j\otimes u_{ji}^\gamma$$

From the relations $f(X_1,\ldots,X_N)=0$ we deduce that we have:

$$f(X_1^\gamma,\ldots,X_N^\gamma)=(id\otimes\gamma)f(X_1,\ldots,X_N)=0$$

Thus we can map $u_{ij}\to u_{ij}^\gamma$ for any $\gamma\in\{\Delta,\varepsilon,S\}$, and we are done. $\square$

Before getting further, we should clarify the relation between Proposition 3.1, Theorem 3.2, and the "toral isometry" constructions from section 1 above. By adding as well into the picture the reflection groups, we are led to the following statement:

Theorem 3.3. *Given an algebraic manifold $X\subset S^{N-1}_{\mathbb{C},+}$, the category of the closed subgroups $G\subset\mathcal{G}$ acting affinely on X, with $\mathcal{G}$ being one of the following quantum groups,*

$$\begin{tikzcd}
\mathbb{T}_N^+ \arrow[r] & K_N^+ \arrow[r] & U_N^+ \\
\mathbb{T}_N \arrow[r] \arrow[u] & K_N \arrow[r] \arrow[u] & U_N \arrow[u]
\end{tikzcd}$$

has a universal object, denoted respectively as follows,

$$\begin{tikzcd}
T^+(X) \arrow[r] & K^+(X) \arrow[r] & G^+(X) \\
T(X) \arrow[r] \arrow[u] & K(X) \arrow[r] \arrow[u] & G(X) \arrow[u]
\end{tikzcd}$$

which appears by intersecting $G^+(X)$ and $\mathcal{G}$, inside U_N^+.

Proof. Here the assertion regarding $G^+(X)$ is something that we know, from Theorem 3.2, and all the other assertions follow from this, by intersecting with $\mathcal{G}$. $\square$

In connection with our axiomatization questions for the quadruplets (S, T, U, K), we can construct now the correspondences $S \to U$, in the following way:

Theorem 3.4. *The quantum isometry groups of the basic spheres,*

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{array}$$

are respectively the basic unitary quantum groups, namely

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

modulo identifying, as usual, the various C^*-algebraic completions.

Proof. We have 4 results to be proved, and we can proceed as follows:

$\underline{S^{N-1}_{\mathbb{C},+}}$. Let us first construct an action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$. We must prove here that the variables $X_i = \sum_j x_j \otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb{C},+}$, namely:

$$\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$$

By using the biunitarity of u, we have the following computation:

$$\begin{aligned} \sum_i X_i X_i^* &= \sum_{ijk} x_j x_k^* \otimes u_{ji} u_{ki}^* \\ &= \sum_j x_j x_j^* \otimes 1 \\ &= 1 \otimes 1 \end{aligned}$$

Once again by using the biunitarity of u, we have as well:

$$\begin{aligned} \sum_i X_i^* X_i &= \sum_{ijk} x_j^* x_k \otimes u_{ji}^* u_{ki} \\ &= \sum_j x_j^* x_j \otimes 1 \\ &= 1 \otimes 1 \end{aligned}$$

Thus we have an action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$, which gives $G^+(S^{N-1}_{\mathbb{C},+}) = U_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{R},+}}$. Let us first construct an action $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$. We already know that the variables $X_i = \sum_j x_j \otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb{C},+}$, so we just have to check that these variables are self-adjoint. But this follows from $u = \bar{u}$, as follows:

$$X_i^* = \sum_j x_j^* \otimes u_{ji}^* = \sum_j x_j \otimes u_{ji} = X_i$$

Conversely, assume that we have an action $G \curvearrowright S^{N-1}_{\mathbb{R},+}$, with $G \subset U_N^+$. The variables $X_i = \sum_j x_j \otimes u_{ji}$ must be then self-adjoint, and the above computation shows that we must have $u = \bar{u}$. Thus our quantum group must satisfy $G \subset O_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{C}}}$. The fact that we have an action $U_N \curvearrowright S^{N-1}_{\mathbb{C}}$ is clear. Conversely, assume that we have an action $G \curvearrowright S^{N-1}_{\mathbb{C}}$, with $G \subset U_N^+$. We must prove that this implies $G \subset U_N$, and we will use a trick from [44]. We have:

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

By multiplying this formula with itself we obtain:

$$\Phi(x_i x_k) = \sum_{jl} x_j x_l \otimes u_{ji} u_{lk}$$

$$\Phi(x_k x_i) = \sum_{jl} x_l x_j \otimes u_{lk} u_{ji}$$

Since the variables x_i commute, these formulae can be written as:

$$\Phi(x_i x_k) = \sum_{j<l} x_j x_l \otimes (u_{ji} u_{lk} + u_{li} u_{jk}) + \sum_j x_j^2 \otimes u_{ji} u_{jk}$$

$$\Phi(x_i x_k) = \sum_{j<l} x_j x_l \otimes (u_{lk} u_{ji} + u_{jk} u_{li}) + \sum_j x_j^2 \otimes u_{jk} u_{ji}$$

Since the tensors at left are linearly independent, we must have:

$$u_{ji} u_{lk} + u_{li} u_{jk} = u_{lk} u_{ji} + u_{jk} u_{li}$$

By applying the antipode to this formula, then applying the involution, and then relabelling the indices, we succesively obtain:

$$u_{kl}^* u_{ij}^* + u_{kj}^* u_{il}^* = u_{ij}^* u_{kl}^* + u_{il}^* u_{kj}^*$$

$$u_{ij} u_{kl} + u_{il} u_{kj} = u_{kl} u_{ij} + u_{kj} u_{il}$$

$$u_{ji} u_{lk} + u_{jk} u_{li} = u_{lk} u_{ji} + u_{li} u_{jk}$$

Now by comparing with the original formula, we obtain from this:

$$u_{li} u_{jk} = u_{jk} u_{li}$$

In order to finish, it remains to prove that the coordinates u_{ij} commute as well with their adjoints. For this purpose, we use a similar method. We have:

$$\Phi(x_i x_k^*) = \sum_{jl} x_j x_l^* \otimes u_{ji} u_{lk}^*$$

$$\Phi(x_k^* x_i) = \sum_{jl} x_l^* x_j \otimes u_{lk}^* u_{ji}$$

Since the variables on the left are equal, we deduce from this that we have:

$$\sum_{jl} x_j x_l^* \otimes u_{ji} u_{lk}^* = \sum_{jl} x_j x_l^* \otimes u_{lk}^* u_{ji}$$

Thus we have $u_{ji} u_{lk}^* = u_{lk}^* u_{ji}$, and so $G \subset U_N$, as claimed.

$\underline{S^{N-1}_{\mathbb{R}}}$. The fact that we have an action $O_N \curvearrowright S^{N-1}_{\mathbb{R}}$ is clear. In what regards the converse, this follows by combining the results that we already have, as follows:

$$\begin{aligned} G \curvearrowright S^{N-1}_{\mathbb{R}} \quad &\implies \quad G \curvearrowright S^{N-1}_{\mathbb{R},+}, S^{N-1}_{\mathbb{C}} \\ &\implies \quad G \subset O_N^+, U_N \\ &\implies \quad G \subset O_N^+ \cap U_N = O_N \end{aligned}$$

Thus, we conclude that we have $G^+(S^{N-1}_{\mathbb{R}}) = O_N$, as desired. $\square$

Summarizing, in relation with our plan for this section, we are done with the correspondences $S \to U$, modulo the fact, which is of importance, but not directly related to our axiomatization, that we still have to clarify the metric aspects of the actions $U \curvearrowright S$ that we constructed. We will discuss this at the end of this section.

Let us discuss now the construction $U \to S$. In the classical case the situation is very simple, because S appears by rotating the point $x = (1, 0, \ldots, 0)$ by the isometries in U. Equivalently, $S = S^{N-1}$ appears from $U = U_N$ as an homogeneous space, as follows:

$$S^{N-1} = U_N / U_{N-1}$$

In functional analytic terms, all this becomes even simpler, the correspondence $U \to S$ being obtained, at the level of algebras of functions, as follows:

$$C(S^{N-1}) \subset C(U_N) \quad , \quad x_i \to u_{1i}$$

In general now, let us start with the following observation:

Proposition 3.5. *For the basic spheres, we have a diagram as follows,*

$$\begin{tikzcd} C(S) \arrow[r, "\Phi"] \arrow[d, "\alpha"] & C(S)\otimes C(U) \arrow[d, "\alpha\otimes id"] \\ C(U) \arrow[r, "\Delta"] & C(U)\otimes C(U) \end{tikzcd}$$

where the map on top is the affine coaction map,

$$\Phi(x_i)=\sum_j x_j\otimes u_{ji}$$

and the map on the left is given by $\alpha(x_i)=u_{1i}$.

Proof. The diagram in the statement commutes indeed on the standard coordinates, the corresponding arrows being as follows:

$$\begin{tikzcd} x_i \arrow[r] \arrow[d] & \sum_j x_j\otimes u_{ji} \arrow[d] \\ u_{1i} \arrow[r] & \sum_j u_{1j}\otimes u_{ji} \end{tikzcd}$$

Thus by linearity and multiplicativity, the whole the diagram commutes. □

We therefore have the following result:

Theorem 3.6. *We have a quotient map and an inclusion as follows,*

$$U\to S_U\subset S$$

with S_U being the first row space of U, given by

$$C(S_U)=<u_{1i}>\subset C(U)$$

at the level of the corresponding algebras of functions.

Proof. At the algebra level, we have an inclusion and a quotient map as follows:

$$C(S)\to C(S_U)\subset C(U)$$

Thus, we obtain the result, by transposing. □

We will prove in what follows that the inclusion $S_U\subset S$ is an isomorphism. This will produce the correspondence $U\to S$ that we are currently looking for. In order to do so, we will use the uniform integration over S, which can be introduced as follows:

Definition 3.7. *We endow each of the algebras $C(S)$ with its integration functional*

$$\int_S : C(S) \to C(U) \to \mathbb{C}$$

obtained by composing the morphism given by $x_i \to u_{1i}$ with the Haar integral of U.

In order to efficiently integrate over the sphere S, we need to know how to efficiently integrate over the corresponding quantum group U. Following [23], [37], we have:

Theorem 3.8. *Assuming that a compact quantum group $G \subset U_N^+$ is easy, coming from a category of partitions $D \subset P$, we have the Weingarten formula*

$$\int_G u_{i_1 j_1}^{e_1} \ldots u_{i_k j_k}^{e_k} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

for any indices $i_r, j_r \in \{1,\ldots,N\}$ and any exponents $e_r \in \{\emptyset, *\}$, where δ are the usual Kronecker type symbols, and where

$$W_{kN} = G_{kN}^{-1}$$

is the inverse of the matrix $G_{kN}(\pi,\sigma) = N^{|\pi \vee \sigma|}$.

Proof. Let us arrange indeed all the integrals to be computed, at a fixed value of the exponent $k = (e_1 \ldots e_k)$, into a single matrix, of size $N^k \times N^k$, as follows:

$$P_{i_1 \ldots i_k, j_1 \ldots j_k} = \int_G u_{i_1 j_1}^{e_1} \ldots u_{i_k j_k}^{e_k}$$

By [148], this matrix P is the orthogonal projection onto the following space:

$$Fix(u^{\otimes k}) = span\left(\xi_\pi \Big| \pi \in D(k)\right)$$

Consider now the following linear map:

$$E(x) = \sum_{\pi \in D(k)} < x, \xi_\pi > \xi_\pi$$

Consider as well the inverse W of the restriction of E to:

$$span\left(T_\pi \Big| \pi \in D(k)\right)$$

By a standard linear algebra computation, it follows that we have:

$$P = WE$$

But the restriction of E is the linear map corresponding to G_{kN}, so W is the linear map corresponding to W_{kN}, and this gives the result. See [23], [37]. □

Following [4], [32], we can now integrate over the spheres S, as follows:

Proposition 3.9. *The integration over the basic spheres is given by*

$$\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\sum_\pi\sum_{\sigma\leq\ker i}W_{kN}(\pi,\sigma)$$

with $\pi,\sigma\in D(k)$*, where* $W_{kN}=G_{kN}^{-1}$ *is the inverse of* $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

Proof. According to our conventions, the integration over S is a particular case of the integration over U, via $x_i=u_{1i}$. By using the formula in Theorem 3.8, we obtain:

$$\begin{aligned}\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}&=\int_U u_{1i_1}^{e_1}\ldots u_{1i_k}^{e_k}\\&=\sum_{\pi,\sigma\in D(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\&=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Following now [32], we have the following key result:

Theorem 3.10. *The integration functional of S has the ergodicity property*

$$\left(id\otimes\int_U\right)\Phi(x)=\int_S x$$

where $\Phi:C(S)\to C(S)\otimes C(U)$ *is the universal affine coaction map.*

Proof. In the real case, $x_i=x_i^*$, it is enough to check the equality in the statement on an arbitrary product of coordinates, $x_{i_1}\ldots x_{i_k}$. The left term is as follows:

$$\begin{aligned}\left(id\otimes\int_U\right)\Phi(x_{i_1}\ldots x_{i_k})&=\sum_{j_1\ldots j_k}x_{j_1}\ldots x_{j_k}\int_U u_{j_1i_1}\ldots u_{j_ki_k}\\&=\sum_{j_1\ldots j_k}\sum_{\pi,\sigma\in D(k)}\delta_\pi(j)\delta_\sigma(i)W_{kN}(\pi,\sigma)x_{j_1}\ldots x_{j_k}\\&=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\sum_{j_1\ldots j_k}\delta_\pi(j)x_{j_1}\ldots x_{j_k}\end{aligned}$$

Let us look now at the last sum on the right. The situation is as follows:

(1) In the free case we have to sum quantities of type $x_{j_1}\ldots x_{j_k}$, over all choices of multi-indices $j=(j_1,\ldots,j_k)$ which fit into our given noncrossing pairing π, and just by using the condition $\sum_i x_i^2=1$, we conclude that the sum is 1.

(2) The same happens in the classical case. Indeed, our pairing π can now be crossing, but we can use the commutation relations $x_ix_j=x_jx_i$, and the sum is again 1.

Thus the sum on the right is 1, in all cases, and we obtain:

$$\left(id\otimes\int_U\right)\Phi(x_{i_1}\ldots x_{i_k})=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)$$

On the other hand, another application of the Weingarten formula gives:

$$\begin{aligned}\int_S x_{i_1}\ldots x_{i_k}&=\int_U u_{1i_1}\ldots u_{1i_k}\\&=\sum_{\pi,\sigma\in D(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\&=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\end{aligned}$$

Thus, we are done with the proof of the result, in the real case. In the complex case the proof is similar, by adding exponents everywhere. □

We can now deduce an abstract characterization of the integration, as follows:

Theorem 3.11. *There is a unique positive unital trace $tr:C(S)\to\mathbb C$ satisfying*

$$(tr\otimes id)\Phi(x)=tr(x)1$$

where Φ is the coaction map of the corresponding quantum isometry group,

$$\Phi:C(S)\to C(S)\otimes C(U)$$

and this is the canonical integration, as constructed in Definition 3.7.

Proof. First of all, it follows from the Haar integral invariance condition for U that the canonical integration has indeed the invariance property in the statement, namely:

$$(tr\otimes id)\Phi(x)=tr(x)1$$

In order to prove now the uniqueness, let tr be as in the statement. We have:

$$\begin{aligned}tr\left(id\otimes\int_U\right)\Phi(x)&=\int_U(tr\otimes id)\Phi(x)\\&=\int_U(tr(x)1)\\&=tr(x)\end{aligned}$$

On the other hand, according to Theorem 3.10, we have as well:

$$tr\left(id\otimes\int_U\right)\Phi(x)=tr\left(\int_S x\right)=\int_S x$$

We therefore conclude that tr equals the standard integration, as claimed. □

Getting back now to our axiomatization questions, we have:

Theorem 3.12. *We have correspondences between the basic unitary groups*

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

and the basic noncommutative spheres,

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{array}$$

obtained via the operation $U \to S_U$.

Proof. We use the ergodicity formula from Theorem 3.10, namely:

$$\left(id \otimes \int_U\right)\Phi = \int_S$$

We know that $\int_U$ is faithful on $\mathcal{C}(U)$, and that we have:

$$(id \otimes \varepsilon)\Phi = id$$

The coaction map Φ follows to be faithful as well. Thus for any $x \in \mathcal{C}(S)$ we have:

$$\int_S xx^* = 0 \implies x = 0$$

Thus $\int_S$ is faithful on $\mathcal{C}(S)$. But this shows that we have:

$$S = S_U$$

Thus, we are led to the conclusion in the statement. □

Summarizing, in relation with our plan for this section, we have satisfactory correspondences $S \leftrightarrow U$. It remains to discuss the correspondence $T \to K$. Normally this can be obtained as well via affine isometries, because in the classical case, we have:

$$K = G(T)$$

In the free case, however, things are quite tricky, with the naive formula $K = G^+(T)$ being wrong. In order to discuss this, and find the fix, we must compute the quantum isometry groups of the tori that we have. We will need the following construction:

Theorem 3.13. *The following constructions produce compact quantum groups,*

$$\begin{aligned} C(\bar{O}_N) &= C(O_N^+)\Big/\Big\langle u_{ij}u_{kl} = \pm u_{kl}u_{ij}\Big\rangle \\ C(\bar{U}_N) &= C(U_N^+)\Big/\Big\langle u_{ij}\dot{u}_{kl} = \pm \dot{u}_{kl}u_{ij}\Big\rangle \end{aligned}$$

with the signs corresponding to anticommutation of different entries on same rows or same columns, and commutation otherwise, and where $\dot{u}$ stands for u or for $\bar{u}$.

Proof. This is something well-known, coming from [20] and subsequent papers, where these quantum groups were first introduced, the idea being as follows:

(1) First of all, the operations $O_N \to \bar{O}_N$ and $U_N \to \bar{U}_N$ in the statement, obtained by replacing the commutation between the standard coordinates by some appropriate commutation/anticommutation, should be thought of as being $q = -1$ twistings.

(2) However, this is not exactly the $q = -1$ twisting in the sense of Drinfeld [83] and Jimbo [94], which produces non-semisimple objects, and so the result must be checked as such, independently of the $q = -1$ twisting literature related to [83], [94].

(3) But this is something elementary, which follows in the usual way, by considering the matrices $u^\Delta, u^\varepsilon, u^S$, and proving that these matrices satisfy the same relations as u. We will be back later to all this, in section 11 below, with full details. □

Now back to our axiomatization questions, the quantum isometry groups of the main tori that we have are given by a quite surprising result, as follows:

Theorem 3.14. *The quantum isometry groups of the basic tori*

$$\begin{tikzcd} T_N^+ \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[u] \arrow[r] & \mathbb{T}_N \arrow[u] \end{tikzcd}$$

are the following quantum groups,

$$\begin{tikzcd} H_N^+ \arrow[r] \arrow[d, dash, dotted] & K_N^+ \arrow[d, dash, dotted] \\ \bar{O}_N \arrow[r] & \bar{U}_N \end{tikzcd}$$

where $\bar{O}_N, \bar{U}_N$ are the standard $q = -1$ twists of O_N, U_N.

Proof. In all cases we must find the conditions on a closed subgroup $G \subset O_N^+$ such that the following formula defines a coaction:

$$g_i \to \sum_j g_j \otimes u_{ji}$$

Since the coassociativity of such a map is automatic, we are left with checking that the map itself exists, and this is the same as checking that the following variables satisfy the same relations as the generators $g_i \in G$:

$$G_i = \sum_j g_j \otimes u_{ji}$$

(1) For $\Gamma = \mathbb{Z}_2^N$ the relations to be checked are as follows:

$$G_i^2 = 1 \quad , \quad G_i G_j = G_j G_i$$

We have the following formula, for the squares:

$$\begin{aligned} G_i^2 &= \sum_{kl} g_k g_l \otimes u_{ki} u_{li} \\ &= 1 + \sum_{k<l} g_k g_l \otimes (u_{ki} u_{li} + u_{li} u_{ki}) \end{aligned}$$

We have as well the following formula, for the commutants:

$$\begin{aligned} [G_i, G_j] &= \sum_{kl} g_k g_l \otimes (u_{ki} u_{lj} - u_{kj} u_{li}) \\ &= \sum_{k<l} g_k g_l \otimes (u_{ki} u_{lj} - u_{kj} u_{li} + u_{li} u_{kj} - u_{lj} u_{ki}) \end{aligned}$$

From the first relation we obtain $ab = -ba$ for $a \neq b$ on the same column of u, and by using the antipode, the same happens for rows. From the second relation we obtain:

$$[u_{ki}, u_{lj}] = [u_{kj}, u_{li}] \quad , \quad \forall k \neq l$$

Now by applying the antipode we obtain from this:

$$[u_{ik}, u_{jl}] = [u_{jk}, u_{il}] \quad , \quad \forall k \neq l$$

By relabelling, this gives the following formula:

$$[u_{ki}, u_{lj}] = [u_{li}, u_{kj}] \quad , \quad \forall i \neq j$$

Summing up, we are therefore led to the following conclusion:

$$[u_{ki}, u_{lj}] = [u_{kj}, u_{li}] = 0 \quad , \quad \forall i \neq j, k \neq l$$

Thus we must have $G \subset \bar{O}_N$, and this finishes the proof.

(2) For $\Gamma = \mathbb{Z}^N$ the proof is similar, as explained in [8].

(3) For $\Gamma=\mathbb{Z}_2^{*N}$ the only relations to be checked are $G_i^2=1$. We have:

$$\begin{aligned}G_i^2 &= \sum_{kl} g_kg_l\otimes u_{ki}u_{li}\\ &= 1+\sum_{k\neq l} g_kg_l\otimes u_{ki}u_{li}\end{aligned}$$

Thus we obtain $G\subset H_N^+$, as claimed.

(4) For $\Gamma=F_N$ the proof is similar, as explained in [8]. □

The above result is not exactly what we want, but we can "recycle" it, as follows:

Theorem 3.15. *The basic noncommutative tori, namely*

$$\begin{tikzcd} T_N^+ \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[u] \arrow[r] & \mathbb{T}_N \arrow[u] \end{tikzcd}$$

produce the basic quantum reflection groups, namely

$$\begin{tikzcd} H_N^+ \arrow[r] & K_N^+ \\ H_N \arrow[u] \arrow[r] & K_N \arrow[u] \end{tikzcd}$$

via the operation $T\to G^+(T)\cap K_N^+$.

Proof. The operation in the statement produces the following intersections:

$$\begin{tikzcd} H_N^+ \arrow[r] & K_N^+ \\ \bar{O}_N\cap H_N^+ \arrow[u] \arrow[r] & \bar{U}_N\cap K_N^+ \arrow[u] \end{tikzcd}$$

But a routine computation, coming from the fact that commutation + anticommutation means vanishing, gives the quantum groups in the statement. See [8]. □

As a conclusion to all this, we have now correspondences between the pairs (S,T) constructed in section 1, and the pairs (U,K) constructed in section 2, and together with the correspondences already established in sections 1-2, our diagram looks as follows:

$$\begin{tikzcd} S \arrow[r] \arrow[d, leftrightarrow] & T \arrow[d, leftrightarrow] \\ U \arrow[r] \arrow[ur] & K \end{tikzcd}$$

We will be back to this in the next section, with the construction of the correspondences which are left, and with the axiomatization of the quadruplets of type (S,T,U,K).

Following [32] and subsequent papers, let us comment now on the "metric" aspects of our quantum isometry group constructions. To start with, we have:

Definition 3.16. *Given a compact Riemannian manifold X, we denote by $\Omega^1(X)$ the space of smooth 1-forms on X, with scalar product given by*

$$<\omega,\eta>=\int_X<\omega(x),\eta(x)>dx$$

and we construct the Hodge Laplacian $\Delta:L^2(X)\to L^2(X)$ by setting

$$\Delta=d^*d$$

where $d:C^\infty(X)\to\Omega^1(X)$ is the usual differential map, and d^* is its adjoint.

According to a standard differential geometry result, whose proof is elementary, the classical isometry group $\mathcal{G}(X)$ of our Riemannian manifold X is then the group of diffeomorphisms $\varphi:X\to X$ whose induced action on $C^\infty(X)$ commutes with Δ.

In view of this, following now Goswami [90], we can formulate:

Definition 3.17. *Associated to a compact Riemannian manifold X are:*

(1) $\mathcal{D}^+(X)$*: the category of compact quantum groups acting on X.*
(2) $\mathcal{G}^+(X)\in\mathcal{D}^+(X)$*: the universal object with a coaction commuting with Δ.*

Here the coactions maps $\Phi:C(X)\to C(G)\otimes C(X)$ that we consider in (1) must satisfy by definition the usual axioms for the algebraic coactions, namely:

$$(\Phi\otimes id)\Phi=(id\otimes\Delta)\Phi$$

$$(id\otimes\varepsilon)\Phi=id$$

In addition, these are subject as well to the following smoothness assumption:

$$\Phi(C^\infty(X))\subset C(G)\otimes C^\infty(X)$$

As for the commutation condition with Δ in (2) above, this regards the canonical extension of the action to the space $L^2(X)$. For details here, see [90].

Before getting further, we should mention that Definition 3.17 above does not really bring new examples of compact quantum groups, and this due to a non-trivial result of Goswami, stating that when the compact Riemannian manifold X is connected we have $\mathcal{G}^+(X) = \mathcal{G}(X)$. We refer here to [90], [91], [92] and subsequent papers.

Let us discuss now the case of the noncommutative Riemannian manifolds. We will use in what follows some very light axioms, inspired from Connes' ones from [66]:

Definition 3.18. *A baby spectral triple $X = (A, H, D)$ consists of the following:*
1. *A is a unital C^*-algebra.*
2. *H is a Hilbert space, on which A acts.*
3. *D is an unbounded self-adjoint operator on H, with compact resolvents, such that $[D, \phi]$ has a bounded extension, for any ϕ in a dense $*$-subalgebra $\mathcal{A} \subset A$.*

The guiding examples come from the compact Riemannian manifolds X. Indeed, associated to any such manifold X are several triples (A, H, D), with the dense $*$-subalgebra $\mathcal{A} \subset A$ being the algebra $C^\infty(X) \subset C(X)$, as follows:

1. H is the space of square-integrable spinors, and D is the Dirac operator.
2. H is the space of forms on X, and D is the Hodge-Dirac operator $d + d^*$.
3. $H = L^2(X, dv)$, dv being the Riemannian volume, and $D = d^*d$.

In this list the first example is the most interesting one, and by far, and this because under a number of supplementary axioms, a reconstruction theorem for X holds, in terms of the associated triple (A, H, D). We refer to Connes' paper [68] for a proof of this fact, and for the definition of the "true" spectral triples as well.

In view of Definition 3.17 (2), however, the last example in the above list will be in fact the one that we will be interested in. Once again following Goswami [90], we have:

Definition 3.19. *Associated to a baby spectral triple $X = (A, H, D)$ are:*
1. *$\mathcal{D}^+(X)$: the category of compact quantum groups acting on (A, H).*
2. *$\mathcal{G}^+(X) \in \mathcal{D}^+(X)$: the universal object with a coaction commuting with D.*

In other words, $\mathcal{G}^+(X)$ must have a unitary representation U on H which commutes with D, satisfies $U1_A = 1 \otimes 1_A$, and is such that ad_U maps A'' into itself.

Now back to our spheres, we will construct a baby spectral triple, in the sense of Definition 3.18, and then compute the corresponding quantum isometry group, in the sense of Definition 3.19, with the result that this is in fact the affine isometry one.

The idea is to use the inclusion $S^{N-1}_\mathbb{R} \subset S^{N-1}_{\mathbb{R},\times}$, and to construct the Laplacian filtration as the pullback of the Laplacian filtration for $S^{N-1}_\mathbb{R}$, as follows:

Proposition 3.20. *Associated to any real sphere $S^{N-1}_{\mathbb{R},\times}$ is the baby triple (A,H,D), where $A=C(S^{N-1}_{\mathbb{R},\times})$, and where D acting on $H=L^2(A,tr)$ is defined as follows:*

(1) *Consider the following linear space:*
$$H_k=span\left(x_{i_1}\ldots x_{i_r}\Big|r\leq k\right)$$
(2) *Define $E_k=H_k\cap H_{k-1}^\perp$, so that we have:*
$$H=\bigoplus_{k=0}^\infty E_k$$
(3) *Finally, set $Dx=\lambda_kx$, for any $x\in E_k$, where λ_k are distinct numbers.*

Proof. We have to show that the operator $[D,T_i]$ is bounded, where T_i is the left multiplication by x_i. Since $x_i\in A$ is self-adjoint, so is the corresponding operator T_i. Now since we have $T_i(H_k)\subset H_{k+1}$, by self-adjointness we get:
$$T_i(H_k^\perp)\subset H_{k-1}^\perp$$
Thus we have inclusions as follows:
$$T_i(E_k)\subset E_{k-1}\oplus E_k\oplus E_{k+1}$$
This gives a decomposition of the following type:
$$T_i=T_i^{-1}+T_i^0+T_i^1$$
We have then $[D,T_i^\alpha]=\alpha T_i^\alpha$ for any $\alpha\in\{-1,0,1\}$, and this gives the result. □

Summarizing, what we constructed above is some kind of algebraic structure on our spheres, coming from the eigenspaces of the Laplacian. This structure misses however the fine geometric aspects, coming from the eigenvalues, at least in the above formulation.

However, with our formalism, we can now prove, following [32]:

Theorem 3.21. *We have the quantum isometry group formula*
$$\mathcal{G}^+(S^{N-1}_{\mathbb{R},\times})=O_N^\times$$
with respect to the baby spectral triple structure constructed above.

Proof. Consider the universal affine coaction map on our sphere:
$$\Phi:C(S^{N-1}_{\mathbb{R},\times})\to C(O_N^\times)\otimes C(S^{N-1}_{\mathbb{R},\times})$$
This coaction map extends to a unitary representation on the GNS space H, that we denote by U. We have then an inclusion, as follows:
$$\Phi(H_k)\subset C(O_N^\times)\otimes H_k$$

Now observe that this formula reads $U(H_k)\subset H_k$. By unitarity we obtain as well $U(H_k^\perp)\subset H_k^\perp$, so each space E_k is U-invariant, and U,D must commute. We conclude that Φ is isometric with respect to D. Finally, the universality of $O_N^\times$ is clear. $\square$

There are several interesting questions in relation with the above. First, we have the question of understanding what happens for the complex spheres, and also for the tori, real or complex. Also, we have the question of understanding what the eigenvalues of the Laplacian are, and whether this resulting Laplacian can be used in order to formulate basic PDE over our spheres. We refer here to [75] and related papers, for a discussion.

4. Axiomatization

We finish here our axiomatization work. We recall that our goal is that of axiomatizing the quadruplets (S,T,U,K) consisting of a quantum sphere, torus, unitary group and reflection group, with a full set of correspondences between them, as follows:

$$\begin{tikzcd} S \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ U \arrow[r, leftrightarrow] & K \end{tikzcd}$$

In order to discuss all this, we first need precise definitions for all the objects involved. So, let us start with the following general definition:

Definition 4.1. *We call quantum sphere, quantum torus, quantum unitary group and quantum reflection group the intermediate objects as follows,*

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C},+}$$

$$T_N \subset T \subset \mathbb{T}_N^+$$

$$O_N \subset U \subset U_N^+$$

$$H_N \subset K \subset K_N^+$$

with S being an algebraic manifold, and T,U,K being compact quantum groups.

Here, as usual, all the objects are taken up to the standard equivalence relation for the noncommutative algebraic manifolds, discussed in section 1 above.

As a first observation, the above definition, with intermediate objects ranging between classical real and free complex, brings us into the "hybrid" zone, between real and complex. One reason for doing so is that we would like to deal at the same time with the real and complex cases, in order to simplify our axiomatization work.

Also, and importantly, at a more advanced level, we will see later on that we have an isomorphism between the free real and complex projective spaces, as follows:

$$P^{N-1}_{\mathbb{R},+} = P^{N-1}_{\mathbb{C},+}$$

This isomorphism is something quite interesting, the conclusion being that in the free setting, the usual $\mathbb{R}/\mathbb{C}$ dichotomy tends to become "blurred". Thus, it is a good idea to forget about this dichotomy, and formulate things as above.

We will be back to projective geometry questions in section 15 below, with more explanations regarding the above isomorphism, and with other results as well.

At the level of the basic examples, the situation is as follows:

Proposition 4.2. *We have "basic" quadruplets (S,T,U,K) as follows:*

(1) *A classical real and a classical complex quadruplet, as follows:*

$$\begin{array}{ccc}S^{N-1}_\mathbb{R}&-&T_N\\ |&\times&|\\ O_N&-&H_N\end{array}\qquad\qquad\begin{array}{ccc}S^{N-1}_\mathbb{C}&-&\mathbb{T}_N\\ |&\times&|\\ U_N&-&K_N\end{array}$$

(2) *A free real and a free complex quadruplet, as follows:*

$$\begin{array}{ccc}S^{N-1}_{\mathbb{R},+}&-&T_N^+\\ |&\times&|\\ O_N^+&-&H_N^+\end{array}\qquad\qquad\begin{array}{ccc}S^{N-1}_{\mathbb{C},+}&-&\mathbb{T}_N^+\\ |&\times&|\\ U_N^+&-&K_N^+\end{array}$$

Proof. This is more or less an empty statement, with the various objects appearing in the above diagrams being those constructed in sections 1 and 2 above. ☐

Regarding now the correspondences between our objects (S,T,U,K), we would like to have all 12 of them axiomatized. There is still quite some work to be done here, and in order to get started, let us begin with a summary of what we have so far:

Theorem 4.3. *For the basic quadruplets, we have correspondences as follows,*

$$\begin{tikzcd}S \arrow[r] \arrow[d, leftrightarrow] & T \arrow[d, leftrightarrow]\\ U \arrow[r] \arrow[ur] & K\end{tikzcd}$$

constructed via the following formulae:

(1) $S=S_U$.
(2) $T=S\cap\mathbb{T}_N^+=U\cap\mathbb{T}_N^+=K\cap\mathbb{T}_N^+$.
(3) $U=G^+(S)$.
(4) $K=U\cap K_N^+=K^+(T)$.

Proof. This is a summary of what we already have, with the fact that the 7 correspondences in the statement work well for the 4 basic quadruplets, from Proposition 4.2, coming from the various results established in sections 1-3 above:

(1) The formula $S=S_U$ is from section 3, with the proof being based on an ergodicity result, ultimately coming from easiness, and the Weingarten formula.

(2) The formula $T=S\cap\mathbb{T}_N^+$ is from section 1, and this is something elementary, coming from definitions.

(3) The formula $T=U\cap\mathbb{T}_N^+$ is from section 2, and this is once again something elementary, coming from definitions.

(4) The formula $T=K\cap\mathbb{T}_N^+$ is once again from section 2, coming as before essentially from the definitions.

(5) The formula $U=G^+(S)$ is from section 3, with the proof being something quite standard, based on the tricks from [32].

(6) The formula $K=U\cap K_N^+$ is from section 2, and this is something elementary, coming from definitions.

(7) The formula $K=K^+(T)$ is from section 3, and this is definitely something quite tricky, involving $q=-1$ twists. □

Our goal is that of having a full set of correspondences between our objects (S,T,U,K). In view of the above result, a key problem is that of passing from the discrete objects (T,K) to the continuous objects (S,U). We will solve this by doing some work at the quantum group level, in relation with the quantum groups T,K,U. To be more precise, what we would like to have are correspondences as follows:

$$T\to K\to U$$

In order to discuss this, we need some preliminaries, in relation with the intersection and generation operations for the compact quantum groups. Let us start with:

Proposition 4.4. *The closed subgroups of U_N^+ are subject to operations as follows:*

1. *Intersection: $H\cap K$ is the biggest quantum subgroup of H,K.*
2. *Generation: $<H,K>$ is the smallest quantum group containing H,K.*

Proof. We must prove that the universal quantum groups in the statement exist indeed. For this purpose, let us pick writings as follows, with I,J being Hopf ideals:

$$C(H)=C(U_N^+)/I$$
$$C(K)=C(U_N^+)/J$$

We can then construct our two universal quantum groups, as follows:

$$C(H\cap K)=C(U_N^+)/<I,J>$$
$$C(<H,K>)=C(U_N^+)/(I\cap J)$$

Thus, we obtain the result. □

In practice, the operation $\cap$ can be usually computed by using:

Proposition 4.5. *Assuming $H,K\subset G$, the intersection $H\cap K$ is given by*
$$C(H\cap K)=C(G)/\{\mathcal R,\mathcal P\}$$
whenever we have formulae of type
$$C(H)=C(G)/\mathcal R$$
$$C(K)=C(G)/\mathcal P$$
with $\mathcal R,\mathcal P$ being sets of polynomial $*$-relations between the standard coordinates.

Proof. This follows from Proposition 4.4 above, or rather from its proof, and from the following trivial fact, regarding relations and ideals:
$$I=<\mathcal R>,J=<\mathcal P>\implies <I,J>=<\mathcal R,\mathcal P>$$

Thus, we are led to the conclusion in the statement. □

In relation with the generation operation, let us call Hopf image of a representation $C(G)\to A$ the smallest Hopf algebra quotient $C(L)$ producing a factorization:
$$C(G)\to C(L)\to A$$

The fact that such a quotient exists indeed is routine, by dividing by a suitable ideal. This notion can be generalized to families of representations, and we have:

Proposition 4.6. *Assuming $H,K\subset G$, the quantum group $<H,K>$ is such that*
$$C(G)\to C(H\cap K)\to C(H),C(K)$$
is the joint Hopf image of the following quotient maps:
$$C(G)\to C(H),C(K)$$

Proof. In the particular case from the statement, the joint Hopf image appears as the smallest Hopf algebra quotient $C(L)$ producing factorizations as follows:
$$C(G)\to C(L)\to C(H),C(K)$$

Thus $L=<H,K>$, which leads to the conclusion in the statement. See [59]. □

In the Tannakian setting now, we have the following result:

Theorem 4.7. *The intersection and generation operations $\cap$ and $<,>$ can be constructed via the Tannakian correspondence $G\to C_G$, as follows:*

1. *Intersection: defined via $C_{G\cap H}=<C_G,C_H>$.*
2. *Generation: defined via $C_{<G,H>}=C_G\cap C_H$.*

Proof. This follows from Proposition 4.4, or rather from its proof, by taking I,J to be the ideals coming from Tannakian duality, in its soft form, from section 2 above. □

In relation now with easiness, we first have the following result:

Proposition 4.8. *Assuming that H,K are easy, then so is $H\cap K$, and we have*

$$D_{H\cap K}=<D_H,D_K>$$

at the level of the corresponding categories of partitions.

Proof. We have indeed the following computation:

$$\begin{aligned}C_{H\cap K}&=<C_H,C_K>\\&=<span(D_H),span(D_K)>\\&=span(<D_H,D_K>)\end{aligned}$$

Thus, by Tannakian duality we obtain the result. □

Regarding the generation operation, the situation is more complicated, as follows:

Proposition 4.9. *Assuming that H,K are easy, we have an inclusion*

$$<H,K>\subset\{H,K\}$$

coming from an inclusion of Tannakian categories as follows,

$$C_H\cap C_K\supset span(D_H\cap D_K)$$

where $\{H,K\}$ is the easy quantum group having as category of partitions $D_H\cap D_K$.

Proof. This follows from the general properties of the generation operation, and from:

$$\begin{aligned}C_{<H,K>}&=C_H\cap C_K\\&=span(D_H)\cap span(D_K)\\&\supset span(D_H\cap D_K)\end{aligned}$$

Indeed, by Tannakian duality we obtain from this all the assertions. □

Summarizing, we have some problems here, and we must proceed as follows:

Theorem 4.10. *The intersection and easy generation operations $\cap$ and $\{\,,\}$ can be constructed via the Tannakian correspondence $G\to D_G$, as follows:*

1. *Intersection: defined via $D_{G\cap H}=<D_G,D_H>$.*
2. *Easy generation: defined via $D_{\{G,H\}}=D_G\cap D_H$.*

Proof. Here (1) is an result coming from Proposition 4.8, and (2) is more of an empty statement, related to the difficulties that we met in Proposition 4.9. □

With the above notions in hand, let us go back to the various quantum groups of type T,U,K that we are interested in. We have the following summary of the results that we have so far, regarding these quantum groups, established in sections 2-3 above, along with a few new things, in relation with the intersection and generation operations:

Theorem 4.11. *The basic quantum unitary and reflection groups,*

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[ur] \arrow[rr] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[uu] \arrow[rr] & & U_N \arrow[uu] \\
H_N \arrow[uu] \arrow[ur] \arrow[rr] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

form an intersection and easy generation diagram, and their diagonal tori

$$\begin{tikzcd}
 & \mathbb{T}_N^+ \arrow[rr] & & \mathbb{T}_N^+ \\
T_N^+ \arrow[ur] \arrow[rr] & & T_N^+ \arrow[ur] & \\
 & \mathbb{T}_N \arrow[uu] \arrow[rr] & & \mathbb{T}_N \arrow[uu] \\
T_N \arrow[uu] \arrow[ur] \arrow[rr] & & T_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

form an intersection and generation diagram.

Proof. We have two assertions here, the idea being as follows:

(1) We know from section 2 above that the quantum unitary and reflection groups are all easy, the corresponding categories of partitions being as follows:

$$\begin{tikzcd}
 & \mathcal{NC}_{even} \arrow[dl] \arrow[dd] & & \mathcal{NC}_2 \arrow[ll] \arrow[dl] \arrow[dd] \\
NC_{even} \arrow[dd] & & NC_2 \arrow[ll] \arrow[dd] & \\
 & \mathcal{P}_{even} \arrow[dl] & & \mathcal{P}_2 \arrow[ll] \arrow[dl] \\
P_{even} & & P_2 \arrow[ll] &
\end{tikzcd}$$

Now since these categories form an intersection and generation diagram, it follows that the quantum groups form an intersection and easy generation diagram, as claimed.

(2) Regarding now the corresponding diagonal tori, we know from section 2 that these are indeed the tori in the statement. As for the fact that these tori form an intersection and generation diagram, this is something well-known, and elementary. □

It is conjectured that the above quantum group diagram should be actually a plain generation diagram. We will be back to this later.

As a first consequence of the above result, in connection with our axiomatization questions for the quadruplets (S,T,U,K), we have the following result:

Proposition 4.12. *The unitary quantum groups appear from their classical versions*

$$\begin{tikzcd} O_N^+ \arrow[r] & U_N^+ \\ O_N \arrow[u, dotted] \arrow[r] & U_N \arrow[u, dotted] \end{tikzcd}$$

via the easy liberation formula

$$G=\left\{G_{class},K\right\}$$

where $K\subset G$ is the quantum reflection group, given by $K=G\cap K_N^+$.

Proof. We have two formulae to be established, the idea being as follows:

(1) For the quantum group O_N^+ the classical version is O_N, and the corresponding reflection group is H_N^+, and from the fact that the front face of the quantum group diagram in Theorem 4.11 is an easy generation diagram we obtain, as desired:

$$O_N^+=\left\{O_N,H_N^+\right\}$$

(2) For the quantum group U_N^+ the classical version is U_N, and the corresponding reflection group is K_N^+, and from the fact that the rear face of the quantum group diagram in Theorem 4.11 is an easy generation diagram we obtain, as desired:

$$U_N^+=\left\{U_N,K_N^+\right\}$$

Thus, we are led to the conclusion in the statement. □

We can further reformulate the above result, in the following way:

Proposition 4.13. *The unitary quantum groups appear from their reflection subgroups*

$$\begin{matrix} H_N^+ & \to & K_N^+ \\ \uparrow & & \uparrow \\ H_N & \to & K_N \end{matrix} \quad \to \quad \begin{matrix} O_N^+ & \to & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \to & U_N \end{matrix}$$

via the following easy generation formula

$$U = \{O_N, K\}$$

computed inside the quantum group U_N^+.

Proof. This is a reformulation of Proposition 4.12, as follows:

(1) In the classical orthogonal case the formula to be proved is trivial, namely:

$$O_N = \{O_N, H_N\}$$

(2) In the free orthogonal case the formula etablished in Proposition 4.12 is precisely the one that we need, namely:

$$O_N^+ = \{O_N, H_N^+\}$$

(3) In the classical unitary case now, the formula in the statement is as follows, coming from the fact that the bottom face of the quantum group diagram in Theorem 4.11 is an easy generation diagram:

$$U_N = \{O_N, K_N\}$$

(4) In the free unitary case, we have the following computation, based on the unitary formula established in Proposition 4.12, and on the formula in (3) above:

$$\begin{aligned} U_N^+ &= \{U_N, K_N^+\} \\ &= \{\{O_N, K_N\}, K_N^+\} \\ &= \{O_N, \{K_N, K_N^+\}\} \\ &= \{O_N, K_N^+\} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

We can now update our main result so far, as follows:

Theorem 4.14. *For the basic quadruplets, we have correspondences as follows,*

$$\begin{tikzcd} S \arrow[r] \arrow[d, leftrightarrow] & T \arrow[d, leftrightarrow] \\ U \arrow[ur] \arrow[r, leftrightarrow] & K \end{tikzcd}$$

constructed via the following formulae:

(1) $S = S_U$.
(2) $T = S \cap \mathbb{T}_N^+ = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+$.
(3) $U = G^+(S) = \{O_N, K\}$.
(4) $K = U \cap K_N^+ = K^+(T)$.

Proof. This is an update of Theorem 4.3, taking into account Proposition 4.13. $\square$

Regarding the missing correspondences, namely $T \to S, U$ and $S \leftrightarrow K$, the situation here is more complicated, and we will discuss this later. We can however compose the correspondences that we have, and formulate, as a conclusion to what we did so far:

Definition 4.15. *A quadruplet (S, T, U, K) is said to produce an easy geometry when U, K are easy, and one can pass from each object to all the other objects, as follows,*

$$\begin{array}{ccccccc} S & = & S_{\{O_N, K^+(T)\}} & = & S_U & = & S_{\{O_N, K\}} \\ \\ S \cap \mathbb{T}_N^+ & = & T & = & U \cap \mathbb{T}_N^+ & = & K \cap \mathbb{T}_N^+ \\ \\ G^+(S) & = & \{O_N, K^+(T)\} & = & U & = & \{O_N, K\} \\ \\ K^+(S) & = & K^+(T) & = & U \cap K_N^+ & = & K \end{array}$$

with the usual convention that all this is up to the equivalence relation.

As a first remark, if we plug the data from any axiom line into the 3 other lines, we obtain axiomatizations in terms of S, T, U, K alone, that we can try to simplify afterwards. It is of course possible to axiomatize everything in terms of ST, SU, SK, TU, TK, UK as well, and also in terms of STU, STK, SUK, TUK, and try to simplify afterwards.

In what follows we will not bother much with this, and use Definition 4.15 as it is. We will need that 12 correspondences, as results, and whether we call such results "verifications of the axioms" or "basic properties of our geometry" is irrelevant.

Regarding now the basic examples, these are of course the classical and free, real and complex geometries. To be more precise, we have the following result:

Theorem 4.16. *We have 4 basic easy geometries, denoted*

$$\begin{array}{ccc}\mathbb{R}^N_+&\longrightarrow&\mathbb{C}^N_+\\ \uparrow&&\uparrow\\ \mathbb{R}^N&\longrightarrow&\mathbb{C}^N\end{array}$$

which appear from quadruplets as above, as follows:

1. *Classical real: produced by* $(S^{N-1}_\mathbb{R},T_N,O_N,H_N)$.
2. *Classical complex: produced by* $(S^{N-1}_\mathbb{C},\mathbb{T}_N,U_N,K_N)$.
3. *Free real: produced by* $(S^{N-1}_{\mathbb{R},+},T_N^+,O_N^+,H_N^+)$.
4. *Free complex: produced by* $(S^{N-1}_{\mathbb{C},+},\mathbb{T}_N^+,U_N^+,K_N^+)$.

Proof. This is something that we already know, which follows from Theorem 4.14, as explained in the discussion preceding Definition 4.15. □

It is possible to construct some further easy geometries in the above sense, and also to work out some structure and classification results. We will be back to this.

Moving ahead now, if we want to improve all the above, we have two problems which are still in need to be solved, namely: (1) understanding the operation $K\to U$, without reference to easiness, and (2) understanding the operation $T\to U$. In short, we are back to the problem mentioned after Theorem 4.3, namely understanding the following operations, and this time without reference to easiness:

$$T\to K\to U$$

This is something quite subtle, which will take us into advanced quantum group theory. Let us start our discussion with the following definition:

Definition 4.17. *Consider a closed subgroup* $G\subset U_N^+$*, and let*

$$T\subset K\subset G$$

be its diagonal torus, and its reflection subgroup. The inclusion $G_{class}\subset G$ *is called:*

1. *A soft liberation, when* $G=<G_{class},K>$.
2. *A hard liberation, when* $G=<G_{class},T>$.

As a first remark, given $G \subset U_N^+$, we have a diagram as follows, which is an intersection diagram, in the sense that any subsquare $P \subset Q, R \subset S$ satisfies $P = Q \cap R$:

$$\begin{tikzcd} T \arrow[r] & K \arrow[r] & G \\ T_{class} \arrow[r] \arrow[u] & K_{class} \arrow[r] \arrow[u] & G_{class} \arrow[u] \end{tikzcd}$$

With this picture in mind, the soft liberation condition states that the square on the right $P \subset Q, R \subset S$ is a generation diagram, $< Q, R >= S$. As for the hard liberation condition, which is stronger, this states that the whole rectangle has this property.

We have the following key result, coming from [53], [58], [60]:

Theorem 4.18. *The following happen:*
(1) O_N^+, U_N^+ *appear as soft liberations of* O_N, U_N*.*
(2) O_N^+, U_N^+ *appear as well as hard liberations of* O_N, U_N*.*
(3) H_N^+, K_N^+ *appear as soft liberations of* H_N, K_N*.*
(4) H_N^+, K_N^+ *do not appear as hard liberations of* H_N, K_N*.*

Proof. This result, while being fundamental for us, is something quite technical. In the lack of a simple proof for all this, here is the idea:

(1) This simply follows from (2) below. Normally there should be a simpler proof for this, by using Tannakian duality, but this is something which is not known yet.

(2) A key result from [58], [60], whose proof is quite technical, not to be explained here, states that we have the following generation formula, valid at any $N \geq 3$:

$$O_N^+ =< O_N, O_{N-1}^+ >$$

With this in hand, the hard liberation formula $O_N^+ =< O_N, T_N^+ >$ can be proved by recurrence on N. Indeed, at $N = 1$ there is nothing to prove, at $N = 2$ this is something well-known, and elementary, as explained for instance in [58], [60], and in general, the recurrence step $N - 1 \to N$ can be established as follows:

$$\begin{aligned} O_N^+ &= < O_N, O_{N-1}^+ > \\ &= < O_N, O_{N-1}, T_{N-1}^+ > \\ &= < O_N, T_{N-1}^+ > \\ &= < O_N, T_N, T_{N-1}^+ > \\ &= < O_N, T_N^+ > \end{aligned}$$

Regarding now the hard liberation formula $U_N^+ =< U_N, \mathbb{T}_N^+ >$, this basically follows from $O_N^+ =< O_N, T_N^+ >$, via the following standard isomorphism:

$$PO_N^+ = PU_N^+$$

Indeed, as explained for instance in [58], [60], this latter isomorphism shows that we have an isomorphism as follows:

$$U_N^+ =< U_N, O_N^+ >$$

By using this isomorphism, we obtain:

$$\begin{aligned} U_N^+ &= < U_N, O_N^+ > \\ &= < U_N, O_N, T_N^+ > \\ &= < U_N, T_N^+ > \\ &= < U_N, \mathbb{T}_N^+ > \end{aligned}$$

(3) This is something trivial, because H_N^+, K_N^+ equal their reflection subgroups.

(4) This result, which is something quite surprising, is well-known, coming from the fact that the quantum group $H_N^{[\infty]} \subset H_N^+$ constructed in [120], and its unitary counterpart $K_N^{[\infty]} \subset K_N^+$, have the same diagonal subgroups as H_N^+, K_N^+. Thus, the hard liberation procedure stops at $H_N^{[\infty]}, K_N^{[\infty]}$, and cannot reach H_N^+, K_N^+. $\square$

Before going further, let us make some comments on all this. As a first comment, in constrast to what happens in the classical case, the correspondence $T \to K$ cannot be constructed via the hard generation formula $K =< H_N, T >$, because this formula is wrong in the free case, due to the negative result from Theorem 4.18 (4), and more specifically to the intermediate quantum reflection groups $H_N^{[\infty]}, K_N^{[\infty]}$ used there, in the proof. Thus, our formula $K = K^+(T)$ is the only solution to the $T \to K$ probem.

As a second comment, the above is interesting in connection with the cube formed by the quantum unitary and reflection groups. Let us recall from Theorem 4.11 that these quantum groups form an intersection and easy generation diagram, as follows:

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[uu] \arrow[ur] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

It is conjectured that this diagram should be a generation diagram too, and the above results prove this conjecture for 5 of the faces. For the remaining face, namely the one on the left, the corresponding formula $K_N^+=<K_N,H_N^+>$ is not proved yet.

As yet another comment, the material in Theorem 4.18 is definitely waiting for more study. Indeed, we have the following Tannakian formulae:

$$C_{H\cap K}=<C_H,C_K>$$

$$C_{<H,K>}=C_H\cap C_K$$

Thus, from a Tannakian viewpoint, all the above results ultimately correspond to doing some combinatorics. To be more precise, the soft and hard generation properties in Definition 4.17 amount respectively in proving the following formulae:

$$C_G=<C_G,C_{U_N}>\cap C_K$$

$$C_G=<C_G,C_{U_N}>\cap C_T$$

In the easy case now, where $C_G=span(D)$, which is the case for the various quantum groups from Theorem 4.18, these two equalities reformulate as follows:

$$span(D)=span(D,\slash\!\!\!\backslash)\cap C_K$$

$$span(D)=span(D,\slash\!\!\!\backslash)\cap C_T$$

Thus, we are led into some combinatorics, which remains to be understood, in a direct way, without reference to algebra and recurrence methods.

Getting back now to our axiomatization questions, the soft and hard liberation can be thought of as being refinements of the easy liberation, and Theorem 4.18 can be regarded as being a refinement of Proposition 4.12. With this idea in mind, we have the following refinement of Proposition 4.13, dealing this time with hard liberation:

Proposition 4.19. *The unitary quantum groups appear from their diagonal subgroups*

$$\begin{tikzcd}
T_N^+ \arrow[r] & \mathbb{T}_N^+ & & O_N^+ \arrow[r] & U_N^+ \\
T_N \arrow[u] \arrow[r] & \mathbb{T}_N \arrow[u] & \rightarrow & O_N \arrow[u] \arrow[r] & U_N \arrow[u]
\end{tikzcd}$$

via the following hard generation formula

$$U=<O_N,T>$$

computed inside the quantum group U_N^+.

Proof. This comes from the results in Theorem 4.18, as follows:

(1) In the classical real case the condition is $O_N =< O_N, T_N >$, which is trivial.

(2) In the free real case the condition is $O_N^+ =< O_N, T_N^+ >$, which is exactly the hard liberation property of $O_N \subset O_N^+$, as explained in Theorem 4.18.

(3) In the classical complex case the condition is $U_N =< O_N, \mathbb{T}_N >$. But this is something well-known, coming for instance from the fact that the inclusion of compact Lie groups $\mathbb{T}O_N \subset U_N$ is maximal. For more details here, we refer to [21].

(4) In the free complex case the condition is $U_N^+ =< O_N, \mathbb{T}_N^+ >$. But this comes from the hard liberation formula $U_N^+ =< U_N, T_N^+ >$ from Theorem 4.18, as follows:

$$\begin{aligned} U_N^+ &= < U_N, \mathbb{T}_N^+ > \\ &= < O_N, \mathbb{T}_N, \mathbb{T}_N^+ > \\ &= < O_N, \mathbb{T}_N^+ > \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Generally speaking, the same comments as those after Theorem 4.18 apply. In Tannakian formulation, the equalities to be proved are as follows:

$$C_U = span(P_2) \cap C_K$$

$$C_U = span(P_2) \cap C_T$$

Thus, we are led into some combinatorics, of basically the same type as the combinatorics needed for Theorem 4.18, which remains to be understood.

We can now update our main result from the general, non-easy case, as follows:

Theorem 4.20. *For the basic quadruplets, we have correspondences as follows,*

$$\begin{tikzcd} S \arrow[r] \arrow[d, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ U \arrow[r, leftrightarrow] & K \end{tikzcd}$$

constructed via the following formulae:

(1) $S = S_U$.
(2) $T = S \cap \mathbb{T}_N^+ = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+$.
(3) $U = G^+(S) =< O_N, T >=< O_N, K >$.
(4) $K = U \cap K_N^+ = K^+(T)$.

Proof. This is an update of Theorem 4.3, taking into account Proposition 4.19. □

As already mentioned before, in section 1 and afterwards, in what regards the missing correspondences, $T \to S$ and $S \leftrightarrow K$, the situation here is quite complicated. In short, we have to give up now with our general principle of constructing all the correspondences independently of each other, and compose what we have. We are led to:

Definition 4.21. *A quadruplet (S,T,U,K) is said to produce a noncommutative geometry when one can pass from each object to all the other objects, as follows,*

$$\begin{array}{ccccccc} S & = & S_{<O_N,T>} & = & S_U & = & S_{<O_N,K>} \\ S \cap \mathbb{T}_N^+ & = & T & = & U \cap \mathbb{T}_N^+ & = & K \cap \mathbb{T}_N^+ \\ G^+(S) & = & < O_N, T > & = & U & = & < O_N, K > \\ K^+(S) & = & K^+(T) & = & U \cap K_N^+ & = & K \end{array}$$

with the usual convention that all this is up to the equivalence relation.

The same comments as those made after Definition 4.15 apply. To be more precise, if we plug the data from any axiom line into the 3 other lines, we obtain axiomatizations in terms of S,T,U,K alone, that we can try to simplify afterwards. It is of course possible to axiomatize everything in terms of ST,SU,SK,TU,TK,UK as well, and also in terms of STU,STK,SUK,TUK, and try to simplify afterwards. In what follows we will not bother much with this, and use Definition 4.21 as it is. We will need that 12 correspondences, as results, and whether we call such results "verifications of the axioms" or "basic properties of our geometry" is irrelevant.

Observe also that the above definition is independent from Definition 4.15, in the sense that an easy geometry in the sense of Definition 4.15 does not automatically satisfy the above axioms, or vice versa. However, we do not know counterexamples here.

As another technical comment, the previous work in [19] was based on (S,T,U) triples, but as explained there, this formalism, missing a lot of restrictions coming from K, is a bit too broad. As for the subsequent work in [13], this was based on sextuplets $(S,\bar{S},T,U,\bar{U},K)$, with the bars standing for twists, which is perhaps something quite natural, but which leads to too many correspondences between objects, namely 30.

Regarding now the basic examples, these are of course the classical and free, real and complex geometries. To be more precise, we have the following result:

Theorem 4.22. *We have 4 basic easy geometries, denoted*

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

which appear from quadruplets as above, as follows:

1. *Classical real: produced by* $(S^{N-1}_\mathbb{R}, T_N, O_N, H_N)$.
2. *Classical complex: produced by* $(S^{N-1}_\mathbb{C}, \mathbb{T}_N, U_N, K_N)$.
3. *Free real: produced by* $(S^{N-1}_{\mathbb{R},+}, T_N^+, O_N^+, H_N^+)$.
4. *Free complex: produced by* $(S^{N-1}_{\mathbb{C},+}, \mathbb{T}_N^+, U_N^+, K_N^+)$.

Proof. This is something that we already know, which follows from Theorem 4.20, as explained in the discussion preceding Definition 4.21. □

We will be back to more examples in sections 9-12 below, and with some classification results as well, the idea being that of looking for intermediate geometries on the horizontal, and on the vertical of the above diagram, and then combining these constructions.

The conclusion there will be that the 4-diagram of geometries from Theorem 4.22 can be extended into a 9-diagram of geometries, as follows:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{T}\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{C}^N_* \arrow[u] \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

Getting back to abstract things, and to the axioms from Definition 4.21 above, let us recall that the correspondences there were partly obtained by composing.

Here is an equivalent formulation of our axioms, which is more convenient, and that we will use in what follows, cutting some trivial redundancies:

Theorem 4.23. *A quadruplet (S,T,U,K) produces a noncommutative geometry when*

$$\begin{array}{ccccc} S & = & S_U & & \\ S\cap\mathbb{T}_N^+ & = & T & = & K\cap\mathbb{T}_N^+ \\ G^+(S) & = & <O_N,T> & = & U \\ K^+(T) & = & U\cap K_N^+ & = & K \end{array}$$

with the usual convention that all this is up to the equivalence relation.

Proof. This follows indeed by examining the axioms in Definition 4.21 above, by cutting some trivial redundancies, and then by rescaling the whole table. □

We will use many times the above result, in what follows, so let us comment now, a bit informally, on the 7 axioms that we have, arranged in increasing order of complexity, based on the 4 computations that we have already:

(1) $T=S\cap\mathbb{T}_N^+$ is usually something quite trivial, and easy to check.

(2) $T=K\cap\mathbb{T}_N^+$ is once again something quite trivial, and easy to check.

(3) $K=U\cap K_N^+$ is of the same nature, usually some trivial algebra.

(4) $U=G^+(S)$ is something more subtle, of algebraic geometric nature, and which usually requires some tricks, in the spirit of [44]. These tricks can actually get very complicated, and for many examples of quantum spheres S, the corresponding quantum isometry groups $G^+(S)$ are not known yet.

(5) $K=K^+(T)$ is something in the same spirit, but more complicated, with even the simplest possible non-trivial cases, namely the free real and complex ones, requiring subtle ingredients, such as a good knowledge of the $q=-1$ twisting.

(6) $S=S_U$ is something fairly heavy, requiring a good knowledge of the advanced representation theory and probability theory of compact quantum groups. Note that this is our only way here of getting to the sphere S.

(7) $U=<O_N,T>$ is something heavy too, requiring an excellent knowledge of the advanced representation theory of compact quantum groups. In fact, this is the key axiom, beating in complexity all the previous axioms, taken altogether.

Regarding now further work on these axioms, with new examples of geometries, and will classification results, we will discuss this later, in sections 9-12 below. We will see there, among others, that under strong supplementary axioms, called "purity" and "uniformity", the 4 main geometries, from Theorem 4.22 above, are the only ones.

In view of this, the question of developing the real and complex free geometries, which are the “main” non-classical geometries, appears. We will discuss this in sections 5-8 below, with the construction of various “free homogeneous spaces”, and we will come back to this later as well, in sections 13-16 below, with more advanced results.

5. **Free integration**

We have seen so far that the two basic geometries, namely those of $\mathbb{R}^N, \mathbb{C}^N$, have free analogues, namely those of $\mathbb{R}^N_+, \mathbb{C}^N_+$. The question of "developing" these new geometries appears. To be more precise, each of our free geometries consists so far of 4 objects, namely a sphere S, a torus T, a unitary group U, and a reflection group K. We must on one hand study S, T, U, K, from a geometric perspective, and on the other hand construct other "free manifolds", like suitable homogeneous spaces, and study them as well.

Following the operator algebra tradition, coming from von Neumann, and then Connes, Jones, Voiculescu, we will primarily regard our various manifolds X as "quantum measured spaces", corresponding to von Neumann algebras $L^\infty(X)$. From this perspective, the main question to be investigated is the computation of the Haar functional:

$$tr : L^\infty(X) \to \mathbb{C}$$

We will investigate this question in this section, for S, T, U, K. Later on, in sections 6-8 below, we will introduce other manifolds, such as quotient spaces $X = G/H$ coming from quantum groups $H \subset G \subset U$, and compute their integration functional as well.

In practice now, our first task will be that of explaining how to integrate over S, T, U, K. In order to integrate over U, K, we can use the Weingarten formula [63], [143], whose quantum group formulation, from [23], [37], is as follows:

Theorem 5.1. *Assuming that a compact quantum group $G \subset U_N^+$ is easy, coming from a category of partitions $D \subset P$, we have the Weingarten formula*

$$\int_G u_{i_1j_1}^{e_1} \ldots u_{i_kj_k}^{e_k} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi, \sigma)$$

for any indices $i_r, j_r \in \{1, \ldots, N\}$ and exponents $e_r \in \{\emptyset, *\}$, where δ are Kronecker type symbols, and where the Weingarten matrix

$$W_{kN} = G_{kN}^{-1}$$

is the inverse of the Gram matrix $G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$.

Proof. This is a formula that we know from section 3, the idea being that the matrix formed by the integrals in the statement is the projection on the following space:

$$Fix(u^{\otimes k}) = span\left(\xi_\pi \Big| \pi \in D(k)\right)$$

By doing the linear algebra, this gives the result, as explained in section 3. $\square$

Regarding now the integration over the torus T, this is something very simple, because we can use here the following fact, coming again from [148]:

Theorem 5.2. *Given a finitely generated discrete group $\Gamma=<g_1,\ldots,g_N>$, the integrals over the corresponding torus $T=\widehat{\Gamma}$ are given by*

$$\int_T g_{i_1}^{e_1}\ldots g_{i_k}^{e_k}=\delta_{g_{i_1}^{e_1}\ldots g_{i_k}^{e_k},1}$$

for any indices $i_r\in\{1,\ldots,N\}$ and any exponents $e_r\in\{\emptyset,*\}$, with the Kronecker symbol on the right being a usual one, computed inside the group Γ.

Proof. This is something standard, coming from the fact that the Haar integration over the torus $T=\widehat{\Gamma}$ is given by the following formula:

$$\int_T g=\delta_{g1}$$

Indeed, this formula defines a functional on the algebra $C(T)=C^*(\Gamma)$, which is left and right invariant. For details on all this, we refer to [148]. □

Finally, regarding the associated spheres S, here the integrals appear as particular cases of the integrals over the corresponding unitary groups U, as explained in section 3 above, and in the easy case, we have a Weingarten formula, as follows:

Theorem 5.3. *The integration over a noncommutative sphere S, coming from a category of pairings D, is given by the Weingarten formula*

$$\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\sum_\pi\sum_{\sigma\leq\ker i}W_{kN}(\pi,\sigma)$$

with $\pi,\sigma\in D(k)$, where $W_{kN}=G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

Proof. This follows from the definition of the integration functional over S, as being the composition of the morphism $C(S)\to C(U)$ with the Haar integration over U:

$$\int_S:C(S)\to C(U)\to\mathbb{C}$$

Indeed, with this description of the integration functional in mind, we can compute this functional via the Weingarten formula for U, from Theorem 5.1, as follows:

$$\begin{aligned}\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}&=\int_U u_{1i_1}^{e_1}\ldots u_{1i_k}^{e_k}\\&=\sum_{\pi,\sigma\in D(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\&=\sum_\pi\sum_{\sigma\leq\ker i}W_{kN}(\pi,\sigma)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Let us discuss now the explicit computation of the various integrals over our manifolds, with respect to the uniform measure. In order to formulate our results in a conceptual form, we use the modern measure theory language, namely probability theory. In the noncommutative setting, the starting definition is as follows:

Definition 5.4. *Let A be a C^*-algebra, given with a trace tr.*

(1) *The elements $a \in A$ are called random variables.*
(2) *The moments of such a variable are the numbers $M_k(a) = tr(a^k)$.*
(3) *The law of such a variable is the functional $\mu : P \to tr(P(a))$.*

Here $k = \circ\bullet\bullet\circ\ldots$ is as usual a colored integer, and the powers a^k are defined by the usual formulae, namely $a^\emptyset = 1, a^\circ = a, a^\bullet = a^*$ and multiplicativity. As for the polynomial P, this is by definition a noncommuting $*$-polynomial in one variable:

$$P \in \mathbb{C} < X, X^* >$$

Observe that the law is uniquely determined by the moments, because:

$$P(X) = \sum_k \lambda_k X^k \implies \mu(P) = \sum_k \lambda_k M_k(a)$$

In the self-adjoint case, the law is a usual probability measure, supported by the spectrum of a. This follows indeed from the Gelfand theorem, and the Riesz theorem.

There are many things that can be said, at this general level, so as a more concrete objective, let us try to understand how the main result in classical probability, namely the Central Limit Theorem (CLT), can be extended in the noncommutative setting.

Let us start with the usual formulation of the CLT, which is as follows:

Theorem 5.5 (CLT). *Given real random variables $x_1, x_2, x_3, \ldots$, which are i.i.d., centered, and with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim g_t$$

where g_t is the Gaussian law of parameter t, having as density:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

Proof. This is something standard, the proof being in three steps, as follows:

(1) Linearization of the convolution. It well-known that the log of the Fourier transform $F_x(\xi) = \mathbb{E}(e^{i\xi x})$ does the job, in the sense that if x, y are independent, then:

$$F_{x+y} = F_x F_y$$

(2) Study of the limit. We have the following formula for a general Fourier transform $F_x(\xi)=\mathbb{E}(e^{i\xi x})$, in terms of moments:

$$F_x(\xi)=\sum_{k=0}^\infty\frac{i^kM_k(x)}{k!}\,\xi^k$$

It follows that the Fourier transform of the variable in the statement is:

$$\begin{aligned}F(\xi)&=&\left[F_x\left(\frac{\xi}{\sqrt{n}}\right)\right]^n\\&=&\left[1-\frac{t\xi^2}{2n}+O(n^{-2})\right]^n\\&\simeq&e^{-t\xi^2/2}\end{aligned}$$

(3) Gaussian laws. The Fourier transform of the Gaussian law is given by:

$$\begin{aligned}F_{g_t}(x)&=&\frac{1}{\sqrt{2\pi t}}\int_\mathbb{R}e^{-y^2/2t+ixy}dy\\&=&\frac{1}{\sqrt{2\pi t}}\int_\mathbb{R}e^{-(y/\sqrt{2t}-\sqrt{t/2}ix)^2-tx^2/2}dy\\&=&\frac{1}{\sqrt{2\pi t}}\int_\mathbb{R}e^{-z^2-tx^2/2}\sqrt{2t}dz\\&=&\frac{1}{\sqrt{\pi}}e^{-tx^2/2}\int_\mathbb{R}e^{-z^2}dz\\&=&\frac{1}{\sqrt{\pi}}e^{-tx^2/2}\cdot\sqrt{\pi}\\&=&e^{-tx^2/2}\end{aligned}$$

Thus the variables on the left and on the right in the statement have the same Fourier transform, and we obtain the result. □

Following Voiculescu [132], [133], in order to extend the CLT to the free setting, our starting point will be the following definition:

Definition 5.6. *Given a pair (A,tr), two subalgebras $B,C\subset A$ are called free when the following condition is satisfied, for any $x_i\in B$ and $y_i\in C$:*

$$tr(x_i)=tr(y_i)=0\implies tr(x_1y_1x_2y_2\ldots)=0$$

Also, two noncommutative random variables $b,c\in A$ are called free when the C^*-algebras $B=<b>$, $C=<c>$ that they generate inside A are free, in this sense.

As a first observation, there is a similarity here with the classical notion of independence. Indeed, modulo some standard identifications, two subalgebras $B,C\subset L^\infty(X)$ are independent when the following condition is satisfied, for any $x \in B$ and $y \in C$:

$$tr(x) = tr(y) = 0 \implies tr(xy) = 0$$

Thus, freeness appears as a kind of "free analogue" of independence. As a basic result now regarding the notion of freeness, which provides us with a useful class of examples, which can be used for various modelling purposes, we have:

Theorem 5.7. *We have the following results, valid for group algebras:*
(1) $C^*(\Gamma), C^*(\Lambda)$ *are independent inside* $C^*(\Gamma \times \Lambda)$.
(2) $C^*(\Gamma), C^*(\Lambda)$ *are free inside* $C^*(\Gamma * \Lambda)$.

Proof. In order to prove these results, we can use the fact that each group algebra is spanned by the corresponding group elements. Thus, it is enough to check the independence and freeness formulae on group elements, which is something clear. □

There are many things that can be said about the analogy between independence and freeness. We have in particular the following result, due to Voiculescu [133]:

Theorem 5.8. *Given a real probability measure* μ, *consider its Cauchy transform*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t}$$

and define its R-transform as being the solution of the following equation:

$$G_\mu\left(R_\mu(\xi) + \frac{1}{\xi}\right) = \xi$$

The operation $\mu \to R_\mu$ *linearizes then the free convolution operation.*

Proof. In order to prove this, we need a good model for the free convolution. The best here is to use the semigroup algebra of the free semigroup on two generators:

$$A = C^*(\mathbb{N} * \mathbb{N})$$

Indeed, we have some freeness in the semigroup setting, a bit in the same way as for the group algebras $C^*(\Gamma * \Lambda)$, from Theorem 5.7 (2), and in addition to this fact, and to what happens in the group algebra case, the following two key things happen:

(1) The variables of type $S^* + f(S)$, with $S \in C^*(\mathbb{N})$ being the shift, and with $f \in \mathbb{C}[X]$ being a polynomial, model in moments all the distributions $\mu : \mathbb{C}[X] \to \mathbb{C}$. This is indeed something elementary, which can be checked via a direct algebraic computation.

(2) Given $f, g \in \mathbb{C}[X]$, the variables $S^* + f(S)$ and $T^* + g(T)$, where $S, T \in C^*(\mathbb{N} * \mathbb{N})$ are the shifts corresponding to the generators of $\mathbb{N} * \mathbb{N}$, are free, and their sum has the same law as $S^* + (f + g)(S)$. This follows indeed by using a 45° argument.

With these results in hand, we can see that the operation $\mu \to f$ linearizes the free convolution. We are therefore left with a computation inside $C^*(\mathbb{N})$, whose conclusion is that $R_\mu = f$ can be recaptured from μ via the Cauchy transform G_μ, as stated. □

We can now state and prove a free analogue of the CLT, from [132], as follows:

Theorem 5.9 (FCLT)**.** *Given self-adjoint variables $x_1, x_2, x_3, \ldots$, which are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i \sim \gamma_t$$

where γ_t is the Wigner semicircle law of parameter t, having density:

$$\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$$

Proof. At $t = 1$, the R-transform of the variable in the statement can be computed by using the linearization property with respect to the free convolution, and is given by:

$$R(\xi) = nR_x\left(\frac{\xi}{\sqrt{n}}\right) \simeq \xi$$

On the other hand, some elementary computations show that the Cauchy transform of the Wigner law γ_1 satisfies the following equation:

$$G_{\gamma_1}\left(\xi + \frac{1}{\xi}\right) = \xi$$

Thus we have $R_{\gamma_1}(\xi) = \xi$, which by the way follows as well from:

$$S^* + S \sim \gamma_1$$

But this gives the result. The passage to the general case, $t > 0$, is routine. □

Let us discuss now the complex versions of the main limiting theorems. In the classical case, we recall that the complex Gaussian law of parameter $t > 0$ is defined as follows, with a, b being independent, each following the law g_t:

$$G_t = law\left(\frac{1}{\sqrt{2}}(a + ib)\right)$$

With this convention, we have the following result:

Theorem 5.10 (CCLT)**.** *Given complex classical random variables $x_1, x_2, x_3, \ldots$, which are i.i.d., centered, and with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i \sim G_t$$

where G_t is the complex Gaussian law of parameter t.

Proof. This follows indeed from the real CLT, without new computations needed, just by taking real and imaginary parts. □

In the free case, the Voiculescu circular law of parameter $t>0$ is defined as follows, with α,β being independent, each following the law γ_t:

$$\Gamma_t = law\left(\frac{1}{\sqrt{2}}(\alpha+i\beta)\right)$$

With this convention, we have the following result:

Theorem 5.11 (FCCLT)**.** *Given noncommutative random variables $x_1,x_2,x_3,\ldots$, which are f.i.d., centered, and with variance $t>0$, we have, with $n\to\infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i \sim \Gamma_t$$

where Γ_t is the Voiculescu circular law of parameter t.

Proof. This follows indeed from the free real CLT, without new computations needed, just by taking real and imaginary parts. $\square$

With these ingredients in hand, let us go back now to our quantum groups. We can compute the character laws for the unitary groups, as follows:

Theorem 5.12. *With $N\to\infty$, the main characters*

$$\chi = \sum_{i=1}^N u_{ii}$$

for the basic unitary quantum groups are as follows:

1. *O_N: real Gaussian, following g_1.*
2. *O_N^+: semicircular, following γ_1.*
3. *U_N: complex Gaussian, following G_1.*
4. *U_N^+: circular, following Γ_1.*

Proof. Following [1], [23], we use the moment method. For an arbitrary closed subgroup $G_N\subset U_N^+$, we have, according to the general Peter-Weyl type results from [148]:

$$\int_{G_N}\chi^k = \dim(Fix(u^{\otimes k}))$$

In the easy case now, where $G=(G_N)$ comes from a certain category of partitions D, the fixed point space on the right is spanned by the vectors T_π with $\pi\in D(k)$. Now since by [104] these vectors are linearly independent with $N\to\infty$, we have:

$$\lim_{N\to\infty}\int_{G_N}\chi^k = |D(k)|$$

Thus, we are led into some combinatorics, and the continuation is as follows:

(1) For O_N we have $D=P_2$, so we obtain as even asymptotic moments the numbers $|P_2(2k)|=k!!$, which are well-known to be the moments of the Gaussian law.

(2) For O_N^+ we have $D=NC_2$, so we obtain as even asymptotic moments the Catalan numbers $|NC_2(2k)|=C_k$, which are the moments of the Wigner semicircle law.

(3) For U_N we have $D=\mathcal{P}_2$, and we can conclude as in the real case, involving O_N, by using this time moments with respect to colored integers, as in Definition 5.4.

(4) For U_N^+ we have $D=\mathcal{NC}_2$, and once again we can conclude as in the real case, involving O_N^+, by using moments with respect to colored integers, as in Definition 5.4. □

Summarizing, we have seen so far that for O_N,O_N^+,U_N,U_N^+, the asymptotic laws of the main characters are the laws $g_1,\gamma_1,G_1,\Gamma_1$ coming from the various CLT. This is certainly nice, but there is still one conceptual problem, coming from:

Proposition 5.13. *The above convergences $law(\chi_u)\to g_1,\gamma_1,G_1,\Gamma_1$ are as follows:*

1. *They are non-stationary in the classical case.*
2. *They are stationary in the free case, starting from $N=2$.*

Proof. This is something quite subtle, which can be proved as follows:

(1) Here we can use an amenability argument, based on the Kesten criterion. Indeed, O_N,U_N being coamenable, the upper bound of the support of the law of $Re(\chi_u)$ is precisely N, and we obtain from this that the law of χ_u itself depends on $N\in\mathbb{N}$.

(2) Here the result follows from the fact that the linear maps T_π associated to the noncrossing pairings are linearly independent, at any $N\geq 2$. □

Fortunately, the solution to the convergence question is quite simple. The idea will be that of improving our $g_1,\gamma_1,G_1,\Gamma_1$ results with certain $g_t,\gamma_t,G_t,\Gamma_t$ results, which will require $N\to\infty$ in both the classical and free cases, in order to hold at any t. Following [23], the definition that we will need is as follows:

Definition 5.14. *Given a Woronowicz algebra (A,u), the variable*

$$\chi_t=\sum_{i=1}^{[tN]}u_{ii}$$

is called truncation of the main character, with parameter $t\in(0,1]$.

Our purpose in what follows will be that of proving that for O_N,O_N^+,U_N,U_N^+, the asymptotic laws of the truncated characters χ_t with $t\in(0,1]$ are the laws $g_t,\gamma_t,G_t,\Gamma_t$. This is something quite technical, but natural, motivated by the findings in Proposition 5.13 above, and also by a number of more advanced considerations, to become clear later on. In order to study the truncated characters, we can use:

Theorem 5.15. *The moments of the truncated characters are given by*

$$\int_G (u_{11}+\ldots+u_{ss})^k = Tr(W_{kN}G_{ks})$$

and with $N\to\infty$ this quantity equals $(s/N)^k|D(k)|$.

Proof. The first assertion follows from the following computation:

$$\begin{aligned}\int_G (u_{11}+\ldots+u_{ss})^k &= \sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\int u_{i_1i_1}\ldots u_{i_ki_k}\\ &= \sum_{\pi,\sigma\in D(k)} W_{kN}(\pi,\sigma)\sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\delta_\pi(i)\delta_\sigma(i)\\ &= \sum_{\pi,\sigma\in D(k)} W_{kN}(\pi,\sigma)G_{ks}(\sigma,\pi)\\ &= Tr(W_{kN}G_{ks})\end{aligned}$$

We have $G_{kN}(\pi,\sigma)=N^k$ for $\pi=\sigma$, and $G_{kN}(\pi,\sigma)\leq N^{k-1}$ for $\pi\neq\sigma$. Thus with $N\to\infty$ we have $G_{kN}\sim N^k1$, which gives:

$$\begin{aligned}\int_G (u_{11}+\ldots+u_{ss})^k &= Tr(G_{kN}^{-1}G_{ks})\\ &\sim Tr((N^k1)^{-1}G_{ks})\\ &= N^{-k}Tr(G_{ks})\\ &= N^{-k}s^k|D(k)|\end{aligned}$$

Thus, we have obtained the formula in the statement. See [23]. □

In order to process the above moment formula, we will need some more probability theory. Following [124], [125], given a random variable a, we write:

$$\log F_a(\xi)=\sum_n k_n(a)\xi^n$$

$$R_a(\xi)=\sum_n \kappa_n(a)\xi^n$$

We call the above coefficients $k_n(a),\kappa_n(a)$ the cumulants, respectively free cumulants of our variable a. With this notion in hand, we can define then more general quantities $k_\pi(a),\kappa_\pi(a)$, depending on arbitrary partitions $\pi\in P(k)$, which coincide with the above ones for the 1-block partitions, and then by multiplicativity over the blocks.

With these conventions, we have the following result, from [124]:

Theorem 5.16. *We have the classical and free moment-cumulant formulae*

$$M_k(a) = \sum_{\pi \in P(k)} k_\pi(a)$$

$$M_k(a) = \sum_{\pi \in NC(k)} \kappa_\pi(a)$$

where $k_\pi(a), \kappa_\pi(a)$ are the generalized cumulants and free cumulants of a.

Proof. This is standard, either by using the formulae of F_a, R_a, or by doing some direct combinatorics, based on the Möbius inversion formula. □

Following [23], we can now improve our results about characters, as follows:

Theorem 5.17. *With $N \to \infty$, the laws of truncated characters are as follows:*
(1) *For O_N we obtain the Gaussian law g_t.*
(2) *For O_N^+ we obtain the Wigner semicircle law γ_t.*
(3) *For U_N we obtain the complex Gaussian law G_t.*
(4) *For U_N^+ we obtain the Voiculescu circular law Γ_t.*

Proof. With $s = [tN]$ and $N \to \infty$, the general moment formula in Theorem 5.15 above gives the following estimate:

$$\lim_{N \to \infty} \int_{G_N} \chi_t^k = \sum_{\pi \in D(k)} t^{|\pi|}$$

By using now the formulae in Theorem 5.16, and doing a number of standard computations, this gives the results. See [23]. □

As an interesting consequence, related to [40], let us formulate as well:

Theorem 5.18. *The asymptotic laws of truncated characters for the liberation operations*

$$O_N \to O_N^+$$

$$U_N \to U_N^+$$

are in Bercovici-Pata bijection, in the sense that the classical cumulants in the classical case equal the free cumulants in the free case.

Proof. This follows indeed from the computations in the proof of Theorem 5.17, and from the combinatorial interpretation of the Bercovici-Pata bijection [40]. □

Let us discuss now the integration over the spheres. A basic probabilistic question regarding the spheres concerns the computation of the associated hyperspherical laws. We have here the following result, from [8], [32]:

Theorem 5.19. *With $N \to \infty$, the rescaled coordinates of the various spheres*

$$\sqrt{N} x_i \in C(S^{N-1}_\times)$$

are as follows, with respect to the uniform integration:

(1) $S^{N-1}_\mathbb{R}$: *real Gaussian.*
(2) $S^{N-1}_{\mathbb{R},+}$: *semicircular.*
(3) $S^{N-1}_\mathbb{C}$: *complex Gaussian.*
(4) $S^{N-1}_{\mathbb{C},+}$: *circular.*

Proof. This follows from Theorem 5.17, but we can use as well the Weingarten formula for the spheres, from Theorem 5.3 above. Indeed, we have the following estimate:

$$\int_{S^{N-1}_\times} x_{i_1} \ldots x_{i_k} \, dx \simeq N^{-k/2} \sum_{\sigma \in P_2^\times(k)} \delta_\sigma(i)$$

With this formula in hand, we can compute the asymptotic moments of each coordinate x_i. Indeed, by setting $i_1 = \ldots = i_k = i$, all Kronecker symbols are 1, and we obtain:

$$\int_{S^{N-1}_\times} x_i^k \, dx \simeq N^{-k/2} |P_2^\times(k)|$$

But this gives the results, modulo the same combinatorics as before. See [4], [32]. □

In order to discuss now the quantum reflection groups, we will need some more theory, namely Poisson limit theorems. In the classical case, we have the following result:

Theorem 5.20 (PLT)**.** *We have the following convergence, in moments,*

$$\left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\delta_1\right)^{*n} \to p_t$$

the limiting measure being

$$p_t = \frac{1}{e^t} \sum_{k=0}^\infty \frac{t^k \delta_k}{k!}$$

which is the Poisson law of parameter $t > 0$.

Proof. We recall that the Fourier transform is given by:

$$F_f(x) = \mathbb{E}(e^{ixf})$$

We therefore obtain the following formula:

$$\begin{aligned}
F_{p_t}(x) &= e^{-t}\sum_k \frac{t^k}{k!}F_{\delta_k}(x)\\
&= e^{-t}\sum_k \frac{t^k}{k!}\,e^{ikx}\\
&= e^{-t}\sum_k \frac{(e^{ix}t)^k}{k!}\\
&= \exp(-t)\exp(e^{ix}t)\\
&= \exp\left((e^{ix}-1)t\right)
\end{aligned}$$

Let us denote by μ_n the measure under the convolution sign:

$$\mu_n = \left(1-\frac{t}{n}\right)\delta_0 + \frac{t}{n}\delta_1$$

We have the following computation:

$$\begin{aligned}
F_{\delta_r}(x) = e^{irx} \quad &\implies \quad F_{\mu_n}(x) = \left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\\
&\implies \quad F_{\mu_n^{*n}}(x) = \left(\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\right)^n\\
&\implies \quad F_{\mu_n^{*n}}(x) = \left(1+\frac{(e^{ix}-1)t}{n}\right)^n\\
&\implies \quad F(x) = \exp\left((e^{ix}-1)t\right)
\end{aligned}$$

Thus, we obtain the Fourier transform of p_t, as desired. $\square$

In the free case, the result is as follows:

Theorem 5.21 (FPLT)**.** *We have the following convergence, in moments,*

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{\boxplus n} \to \pi_t$$

the limiting measure being the Marchenko-Pastur law of parameter $t>0$,

$$\pi_t = \max(1-t,0)\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter $t>0$.

Proof. Consider the measure in the statement, appearing under the convolution sign:

$$\mu = \left(1-\frac{t}{n}\right)\delta_0 + \frac{t}{n}\delta_1$$

The Cauchy transform of this measure is elementary to compute, given by:

$$G_\mu(\xi) = \left(1 - \frac{t}{n}\right)\frac{1}{\xi} + \frac{t}{n} \cdot \frac{1}{\xi - 1}$$

By using the linearization results for the free convolution explained above, we want to compute the following R-transform:

$$R = R_{\mu^{\boxplus n}}(y) = nR_\mu(y)$$

The equation for this function R is as follows:

$$\left(1 - \frac{t}{n}\right)\frac{1}{y^{-1} + R/n} + \frac{t}{n} \cdot \frac{1}{y^{-1} + R/n - 1} = y$$

By multiplying by n/y, this equation can be written as:

$$\frac{t + yR}{1 + yR/n} = \frac{t}{1 + yR/n - y}$$

With $n \to \infty$ we obtain the following formula:

$$t + yR = \frac{t}{1 - y}$$

Thus $R = \frac{t}{1-y} = R_{\pi_t}$, which gives the result. □

In order to get beyond this, let us introduce the following notions:

Definition 5.22. *Associated to any compactly supported positive measure ρ on $\mathbb{C}$ are the probability measures*

$$p_\rho = \lim_{n\to\infty}\left(\left(1 - \frac{c}{n}\right)\delta_0 + \frac{1}{n}\rho\right)^{*n}$$

$$\pi_\rho = \lim_{n\to\infty}\left(\left(1 - \frac{c}{n}\right)\delta_0 + \frac{1}{n}\rho\right)^{\boxplus n}$$

where $c = mass(\rho)$, called compound Poisson and compound free Poisson laws.

In what follows we will be interested in the case where ρ is discrete, as is for instance the case for $\rho = t\delta_1$ with $t > 0$, which produces the Poisson and free Poisson laws. The following result allows one to detect compound Poisson/free Poisson laws:

Theorem 5.23. *For a discrete measure, written as $\rho = \sum_{i=1}^s c_i\delta_{z_i}$ with $c_i > 0$ and $z_i \in \mathbb{R}$, we have the formulae*

$$F_{p_\rho}(y) = \exp\left(\sum_{i=1}^s c_i(e^{iyz_i} - 1)\right)$$

$$R_{\pi_\rho}(y) = \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i}$$

where F, R denote respectively the Fourier transform, and Voiculescu's R-transform.

Proof. Let μ_n be the measure appearing in Definition 5.22, under the convolution signs:

$$\mu_n=\left(1-\frac{c}{n}\right)\delta_0+\frac{1}{n}\rho$$

In the classical case, we have the following computation:

$$\begin{aligned}
&& F_{\mu_n}(y)&=\left(1-\frac{c}{n}\right)+\frac{1}{n}\sum_{i=1}^sc_ie^{iyz_i}\\
&\implies& F_{\mu_n^{*n}}(y)&=\left(\left(1-\frac{c}{n}\right)+\frac{1}{n}\sum_{i=1}^sc_ie^{iyz_i}\right)^n\\
&\implies& F_{p_\rho}(y)&=\exp\left(\sum_{i=1}^sc_i(e^{iyz_i}-1)\right)
\end{aligned}$$

In the free case now, we use a similar method. The Cauchy transform of μ_n is:

$$G_{\mu_n}(\xi)=\left(1-\frac{c}{n}\right)\frac{1}{\xi}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{\xi-z_i}$$

Consider now the R-transform of the measure $\mu_n^{\boxplus n}$, which is given by:

$$R_{\mu_n^{\boxplus n}}(y)=nR_{\mu_n}(y)$$

The above formula of G_{μ_n} shows that the equation for $R=R_{\mu_n^{\boxplus n}}$ is as follows:

$$\begin{aligned}
&&\left(1-\frac{c}{n}\right)\frac{1}{y^{-1}+R/n}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{y^{-1}+R/n-z_i}&=y\\
&\implies&\left(1-\frac{c}{n}\right)\frac{1}{1+yR/n}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{1+yR/n-yz_i}&=1
\end{aligned}$$

Now multiplying by n, rearranging the terms, and letting $n\to\infty$, we get:

$$\begin{aligned}
&&\frac{c+yR}{1+yR/n}&=\sum_{i=1}^s\frac{c_i}{1+yR/n-yz_i}\\
&\implies& c+yR_{\pi_\rho}(y)&=\sum_{i=1}^s\frac{c_i}{1-yz_i}\\
&\implies& R_{\pi_\rho}(y)&=\sum_{i=1}^s\frac{c_iz_i}{1-yz_i}
\end{aligned}$$

This finishes the proof in the free case, and we are done. □

We also have the following technical result, providing a useful alternative to Definition 5.22, in order to detect the classical and free compound Poisson laws:

Theorem 5.24. *For a discrete measure, written as $\rho = \sum_{i=1}^s c_i\delta_{z_i}$ with $c_i > 0$ and $z_i \in \mathbb{R}$, we have the classical/free formulae*

$$p_\rho/\pi_\rho = \mathrm{law}\left(\sum_{i=1}^s z_i\alpha_i\right)$$

where the variables α_i are Poisson/free Poisson(c_i), independent/free.

Proof. Let α be the sum of Poisson/free Poisson variables in the statement:

$$\alpha = \sum_{i=1}^s z_i\alpha_i$$

By using some well-known Fourier transform formulae, we have:

$$\begin{aligned} F_{\alpha_i}(y) = \exp(c_i(e^{iy}-1)) \quad &\implies \quad F_{z_i\alpha_i}(y) = \exp(c_i(e^{iyz_i}-1)) \\ &\implies \quad F_\alpha(y) = \exp\left(\sum_{i=1}^s c_i(e^{iyz_i}-1)\right) \end{aligned}$$

Also, by using some well-known R-transform formulae, we have:

$$\begin{aligned} R_{\alpha_i}(y) = \frac{c_i}{1-y} \quad &\implies \quad R_{z_i\alpha_i}(y) = \frac{c_iz_i}{1-yz_i} \\ &\implies \quad R_\alpha(y) = \sum_{i=1}^s \frac{c_iz_i}{1-yz_i} \end{aligned}$$

Thus we have indeed the same formulae as those which are needed. □

We refer to [40], [133], [136] for the general theory here, to [23], [27], [63] for representation theory aspects, and to [109], [135], [145] for random matrix aspects. In what follows we will only need the main examples of classical and free compound Poisson laws, which are the classical and free Bessel laws. These laws are constructed as follows:

Definition 5.25. *The Bessel and free Bessel laws are the compound Poisson laws*

$$b_t^s = p_{t\varepsilon_s} \quad , \quad \beta_t^s = \pi_{t\varepsilon_s}$$

where ε_s is the uniform measure on the s-th roots unity. In particular:

1. *At $s = 1$ we obtain the usual Poisson and free Poisson laws, p_t, π_t.*
2. *At $s = 2$ we obtain the "real" Bessel and free Bessel laws, denoted b_t, β_t.*
3. *At $s = \infty$ we obtain the "complex" Bessel and free Bessel laws, denoted $B_t, \mathfrak{B}_t$.*

There is a lot of theory regarding these laws, involving classical and quantum reflection groups, subfactors and planar algebras, and free probability and random matrices. We refer here to [15], where these laws were introduced. Let us just record here:

Theorem 5.26. *The moments of the various central limiting measures, namely*

$$\begin{tikzcd}
& \mathfrak{B}_t \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & \Gamma_t \arrow[dl, dash] \arrow[dd, dash] \\
\beta_t \arrow[rr, dash] \arrow[dd, dash] & & \gamma_t \arrow[dd, dash] & \\
& B_t \arrow[rr, dash] \arrow[dl, dash] & & G_t \arrow[dl, dash] \\
b_t \arrow[rr, dash] & & g_t &
\end{tikzcd}$$

are always given by the same formula, involving partitions, namely

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

with the sets of partitions $D(k)$ in question being respectively

$$\begin{tikzcd}
& \mathcal{NC}_{even} \arrow[dl] \arrow[dd] & & \mathcal{NC}_2 \arrow[ll] \arrow[dl] \arrow[dd] \\
NC_{even} \arrow[dd] & & NC_2 \arrow[ll] \arrow[dd] & \\
& \mathcal{P}_{even} \arrow[dl] & & \mathcal{P}_2 \arrow[ll] \arrow[dl] \\
P_{even} & & P_2 \arrow[ll] &
\end{tikzcd}$$

and with $|.|$ being the number of blocks.

Proof. This follows by putting together the various moment results that we have. We refer here to [15]. □

Getting back now to our quantum reflection groups, we first have:

Theorem 5.27. *With $N \to \infty$, the laws of characters are as follows:*

(1) *For H_N we obtain the Bessel law b_1.*
(2) *For H_N^+ we obtain the free Bessel law β_1.*
(3) *For K_N we obtain the complex Bessel law B_1.*
(4) *For K_N^+ we obtain the complex free Bessel law $\mathfrak{B}_1$.*

Proof. This is routine indeed, by counting the partitions, a bit as in the continuous case, in the proof of Theorem 5.12 above. For the full proof here, we refer to [15]. □

At the level of truncated characters, we have:

Theorem 5.28. *With $N\to\infty$, the laws of truncated characters are as follows:*

1. *For H_N we obtain the Bessel law b_t.*
2. *For H_N^+ we obtain the free Bessel law β_t.*
3. *For K_N we obtain the complex Bessel law B_t.*
4. *For K_N^+ we obtain the complex free Bessel law $\mathfrak{B}_t$.*

Also, we have the Bercovici-Pata bijection for truncated characters.

Proof. Once again this is routine, by using the Weingarten formula, as in the continuous case, in the proof of Theorem 5.17 above. For the full proof here, we refer to [15]. □

The results that we have so far, for the quantum unitary and refelection groups, are quite interesting, from a theoretical probability perspective, because we have:

Theorem 5.29. *The laws of the truncated characters for the basic quantum groups,*

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[uu] \arrow[ur] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

and the various classical and free central limiting measures, namely

$$\begin{tikzcd}[arrows=dash]
 & \mathfrak{B}_t \arrow[rr] & & \Gamma_t \\
\beta_t \arrow[rr] \arrow[ur] & & \gamma_t \arrow[ur] & \\
 & B_t \arrow[rr] \arrow[uu] & & G_t \arrow[uu] \\
b_t \arrow[rr] \arrow[uu] \arrow[ur] & & g_t \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

in the $N\to\infty$ limit.

Proof. This follows indeed by putting together the various results obtained above. □

Regarding now the tori, the situation here is more complicated, no longer involving the Bercovici-Pata bijection. Let us recall indeed that the basic tori are as follows:

$$\begin{tikzcd} T_N^+ \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[u] \arrow[r] & \mathbb{T}_N \arrow[u] \end{tikzcd}$$

These tori appear by definiton as duals of the following discrete groups:

$$\begin{tikzcd} \mathbb{Z}_2^{*N} \arrow[d] & F_N \arrow[l] \arrow[d] \\ \mathbb{Z}_2^N & \mathbb{Z}^N \arrow[l] \end{tikzcd}$$

We are interested in the computation of the laws of the associated truncated characters, which are the following variables:

$$\chi_t = g_1 + g_2 + \ldots + g_{[tN]}$$

By dilation we can assume $t = 1$. For the complex tori, $\mathbb{T}_N \subset \mathbb{T}_N^+$, we are led into the computation of the Kesten measures for $F_N \to \mathbb{Z}^N$, and so into the Meixner/free Meixner correspondence. As for the real tori, $T_N \subset T_N^+$, here we are led into the computation of the Kesten measures for $\mathbb{Z}_2^{*N} \to \mathbb{Z}_2^N$, and so into a real version of this correspondence.

Summarizing, we have some nice liberation results for S, T, U, K, with a technical problem, however, coming from the fact that those for S, U, K come from the Bercovici-Pata bijection, while those for T come from the Meixner/free Meixner correspondence.

6. Quotient spaces

In this section and in the next two ones we develop the real and complex free geometry. We will extend the family of objects (S,T,U,K) that we have, first by unifying S,U via a homogeneous space construction, involving row algebras for $C(U)$, and then by further building on this construction, first with more general explicit homogeneous spaces, of "quantum partial isometries", and then with even more abstract manifolds, emerging from our study, that we will call "affine homogeneous spaces". We will also discuss, at the end of section 8, the axiomatization problem for the "free manifolds".

The present section and the next two ones are relatively independent and self-contained, based respectively on the papers [36], then [10], then [11]. However, as explained above, the constructions that we will present will generalize each other, and often in a quite substantial and abstract way, so typically for understanding the main examples of the "higher" constructions, you have to go back to the "lower" constructions.

Let us begin with some generalities regarding the quotient spaces, and more general homogeneous spaces. Regarding the quotients, we have the following construction:

Proposition 6.1. *Given a quantum subgroup $H\subset G$, with associated quotient map $\rho:C(G)\to C(H)$, if we define the quotient space $X=G/H$ by setting*

$$C(X)=\left\{f\in C(G)\Big|(\rho\otimes id)\Delta f=1\otimes f\right\}$$

then we have a coaction $\Phi:C(X)\to C(X)\otimes C(G)$, obtained as the restriction of the comultiplication of $C(G)$. In the classical case, we obtain the usual space $X=G/H$.

Proof. Observe that $C(X)\subset C(G)$ is indeed a subalgebra, because it is defined via a relation of type $\varphi(f)=\psi(f)$, with φ,ψ morphisms. Observe also that in the classical case we obtain the algebra of continuous functions on $X=G/H$, because:

$$\begin{aligned}(\rho\otimes id)\Delta f=1\otimes f &\iff (\rho\otimes id)\Delta f(h,g)=(1\otimes f)(h,g),\forall h\in H,\forall g\in G\\ &\iff f(hg)=f(g),\forall h\in H,\forall g\in G\\ &\iff f(hg)=f(kg),\forall h,k\in H,\forall g\in G\end{aligned}$$

Regarding now the construction of Φ, observe that for $f\in C(X)$ we have:

$$\begin{aligned}(\rho\otimes id\otimes id)(\Delta\otimes id)\Delta f &= (\rho\otimes id\otimes id)(id\otimes\Delta)\Delta f\\ &= (id\otimes\Delta)(\rho\otimes id)\Delta f\\ &= (id\otimes\Delta)(1\otimes f)\\ &= 1\otimes\Delta f\end{aligned}$$

Thus $f\in C(X)$ implies $\Delta f\in C(X)\otimes C(G)$, and this gives the existence of Φ. Finally, the other assertions are clear. □

As an illustration, in the group dual case we have:

Proposition 6.2. *Assume that $G=\widehat{\Gamma}$ is a discrete group dual.*

1. *The quantum subgroups of G are $H=\widehat{\Lambda}$, with $\Gamma\to\Lambda$ being a quotient group.*
2. *For such a quantum subgroup $\widehat{\Lambda}\subset\widehat{\Gamma}$, we have $\widehat{\Gamma}/\widehat{\Lambda}=\widehat{\Theta}$, where $\Theta=\ker(\Gamma\to\Lambda)$.*

Proof. This is well-known, the idea being as follows:

(1) In one sense, this is clear. Conversely, since the algebra $C(G)=C^*(\Gamma)$ is cocommutative, so are all its quotients, and this gives the result.

(2) Consider a quotient map $r:\Gamma\to\Lambda$, and denote by $\rho:C^*(\Gamma)\to C^*(\Lambda)$ its extension. With $f=\sum_{g\in\Gamma}\lambda_g\cdot g\in C^*(\Gamma)$ we have:

$$\begin{aligned}
f\in C(\widehat{\Gamma}/\widehat{\Lambda}) &\iff (\rho\otimes id)\Delta(f)=1\otimes f\\
&\iff \sum_{g\in\Gamma}\lambda_g\cdot r(g)\otimes g=\sum_{g\in\Gamma}\lambda_g\cdot 1\otimes g\\
&\iff \lambda_g\cdot r(g)=\lambda_g\cdot 1,\forall g\in\Gamma\\
&\iff supp(f)\subset\ker(r)
\end{aligned}$$

But this means $\widehat{\Gamma}/\widehat{\Lambda}=\widehat{\Theta}$, with $\Theta=\ker(\Gamma\to\Lambda)$, as claimed. □

Given two noncommutative compact spaces X,Y, we say that X is a quotient space of Y when we have an embedding of C^*-algebras $\alpha:C(X)\subset C(Y)$. We have:

Definition 6.3. *We call a quotient space $G\to X$ homogeneous when*

$$\Delta(C(X))\subset C(X)\otimes C(G)$$

where $\Delta:C(G)\to C(G)\otimes C(G)$ is the comultiplication map.

In other words, an homogeneous quotient space $G\to X$ is a noncommutative space coming from a subalgebra $C(X)\subset C(G)$, which is stable under the comultiplication.

The relation with the quotient spaces from Proposition 6.1 is as follows:

Theorem 6.4. *The following results hold:*

1. *The quotient spaces $X=G/H$ are homogeneous.*
2. *In the classical case, any homogeneous space is of type G/H.*
3. *In general, there are homogeneous spaces which are not of type G/H.*

Proof. Once again these results are well-known, the proof being as follows:

(1) This is clear from Proposition 6.1 above.

(2) Consider a quotient map $p:G\to X$. The invariance condition in the statement tells us that we must have an action $G\curvearrowright X$, given by $g(p(g'))=p(gg')$. Thus:

$$p(g')=p(g'')\implies p(gg')=p(gg''),\ \forall g\in G$$

Now observe that the following subset $H \subset G$ is a subgroup:

$$H = \left\{ g \in G \middle| p(g) = p(1) \right\}$$

Indeed, $g, h \in H$ implies $p(gh) = p(g) = p(1)$, so $gh \in H$, and the other axioms are satisfied as well. Our claim is that we have $X = G/H$, via:

$$p(g) \to Hg$$

Indeed, the map $p(g) \to Hg$ is well-defined and bijective, because $p(g) = p(g')$ is equivalent to $p(g^{-1}g') = p(1)$, and so to $Hg = Hg'$, as desired.

(3) Given a discrete group Γ and an arbitrary subgroup $\Theta \subset \Gamma$, the quotient space $\widehat{\Gamma} \to \widehat{\Theta}$ is homogeneous. Now by using Proposition 6.2 above, we can see that if $\Theta \subset \Gamma$ is not normal, the quotient space $\widehat{\Gamma} \to \widehat{\Theta}$ is not of the form G/H. □

Let us try now to understand the general properties of the homogeneous spaces $G \to X$, in the sense of Theorem 6.4. We recall that any compact quantum group G has a Haar integration functional $\int : C(G) \to \mathbb{C}$, having the following invariance properties:

$$\left(\int \otimes id\right)\Delta = \left(id \otimes \int\right)\Delta = \int (.)1$$

We have the following result, which is once again well-known:

Proposition 6.5. *Assume that a quotient space $G \to X$ is homogeneous.*

1. *The restriction $\Phi : C(X) \to C(X) \otimes C(G)$ of Δ is a coaction.*
2. *We have $\Phi(f) = f \otimes 1 \implies f \in \mathbb{C}1$, and $(id \otimes \int)\Phi f = \int f$.*
3. *The restriction of $\int$ is the unique unital form satisfying $(\tau \otimes id)\Phi = \tau(.)1$.*

Proof. These results are all elementary, the proof being as follows:

(1) This is clear from definitions, because Δ itself is a coaction.

(2) If $f \in C(G)$ is such that $\Delta(f) = f \otimes 1$, then by applying the counit we obtain:

$$(\varepsilon \otimes id)\Delta f = (\varepsilon \otimes id)(f \otimes 1)$$

We conclude from this that we have, as desired:

$$f = \varepsilon(f)1$$

As for the second assertion, $(id \otimes \int)\Phi f = \int f$, this follows from the left invariance property $(id \otimes \int)\Delta f = \int f$ of the Haar functional of $C(G)$, by restriction to $C(X)$.

(3) By using the right invariance property $(\int \otimes id)\Delta f = \int f$ of the Haar functional of $C(G)$, we obtain that $tr = \int_{|C(X)}$ is G-invariant, in the sense that:

$$(tr \otimes id)\Phi f = tr(f)1$$

Conversely, assuming that $\tau : C(X) \to \mathbb{C}$ satisfies $(\tau \otimes id)\Phi f = \tau(f)1$, we have:

$$\left(\tau \otimes \int\right)\Phi(f) = \int(\tau \otimes id)\Phi(f) = \int(\tau(f)1) = \tau(f)$$

On the other hand, we can compute the same quantity as follows:

$$\left(\tau \otimes \int\right)\Phi(f) = \tau\left(id \otimes \int\right)\Phi(f) = \tau(tr(f)1) = tr(f)$$

Thus we have $\tau(f) = tr(f)$ for any $f \in C(X)$, and this finishes the proof. $\square$

Summarizing, we have a notion of noncommutative homogeneous space, which perfectly covers the classical case. In general, however, the group dual case shows that our formalism is more general than that of the quotient spaces G/H.

Let us discuss now an extra issue, of analytic nature. The point is that for one of the most basic examples of actions, $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, the associated morphism $\alpha : C(X) \to C(G)$ is not injective. In order to include such examples, we must relax our axioms:

Definition 6.6. *An extended homogeneous space consists of a morphism of C^*-algebras $\alpha : C(X) \to C(G)$, and a coaction map $\Phi : C(X) \to C(X) \otimes C(G)$, such that*

$$\begin{array}{ccc} C(X) & \xrightarrow{\Phi} & C(X) \otimes C(G) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha\otimes id} \\ C(G) & \xrightarrow{\Delta} & C(G) \otimes C(G) \end{array}$$

commutes, and such that

$$\begin{array}{ccc} C(X) & \xrightarrow{\Phi} & C(X) \otimes C(G) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle id\otimes\int} \\ C(G) & \xrightarrow{\int(.)1} & C(X) \end{array}$$

commutes as well, where $\int$ is the Haar integration over G. We write then $G \to X$.

When α is injective we obtain an homogeneous space in the previous sense. The examples with α not injective, which motivate the above formalism, include the standard action $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, and the standard action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$.

Here are a few general remarks on the above axioms:

Proposition 6.7. *Assume that we have morphisms of C^*-algebras $\alpha : C(X) \to C(G)$ and $\Phi : C(X) \to C(X) \otimes C(G)$, satisfying $(\alpha \otimes id)\Phi = \Delta\alpha$.*

1. *If α is injective on a dense $*$-subalgebra $A \subset C(X)$, and $\Phi(A) \subset A \otimes C(G)$, then Φ is automatically a coaction map, and is unique.*
2. *The ergodicity type condition $(id \otimes \int)\Phi = \int \alpha(.)1$ is equivalent to the existence of a linear form $\lambda : C(X) \to \mathbb{C}$ such that $(id \otimes \int)\Phi = \lambda(.)1$.*

Proof. This is something elementary, the idea being as follows:

(1) Assuming that we have a dense $*$-subalgebra $A \subset C(X)$ as in the statement, satisiying $\Phi(A) \subset A \otimes C(G)$, the restriction $\Phi_{|A}$ is given by:

$$\Phi_{|A} = (\alpha_{|A} \otimes id)^{-1}\Delta\alpha_{|A}$$

This restriction and is therefore coassociative, and unique. By continuity, Φ itself follows to be coassociative and unique, as desired.

(2) Assuming $(id \otimes \int)\Phi = \lambda(.)1$, we have $(\alpha \otimes \int)\Phi = \lambda(.)1$. On the other hand, we have as well the following formula:

$$\left(\alpha \otimes \int\right)\Phi = \left(id \otimes \int\right)\Delta\alpha = \int \alpha(.)1$$

Thus we obtain $\lambda = \int \alpha$, as claimed. $\square$

Given an extended homogeneous space $G \to X$, with associated map $\alpha : C(X) \to C(G)$, we can consider the image of this latter map:

$$\alpha : C(X) \to C(Y) \subset C(G)$$

Equivalently, at the level of the associated noncommutative spaces, we can factorize the corresponding quotient map $G \to Y \subset X$. With these conventions, we have:

Proposition 6.8. *Consider an extended homogeneous space $G \to X$.*

1. $\Phi(f) = f \otimes 1 \implies f \in \mathbb{C}1$.
2. $tr = \int \alpha$ *is the unique unital G-invariant form on $C(X)$.*
3. *The image space obtained by factorizing, $G \to Y$, is homogeneous.*

Proof. We have several assertions to be proved, the idea being as follows:

(1) This follows indeed from $(id \otimes \int)\Phi(f) = \int \alpha(f)1$, which gives:

$$f = \int \alpha(f)1$$

(2) The fact that $tr = \int \alpha$ is indeed G-invariant can be checked as follows:

$$\begin{aligned}(tr \otimes id)\Phi f &= (\textstyle\int \alpha \otimes id)\Phi f \\ &= (\textstyle\int \otimes id)\Delta\alpha f \\ &= \textstyle\int \alpha(f)1 \\ &= tr(f)1\end{aligned}$$

As for the uniqueness assertion, this follows as before.

(3) The condition $(\alpha \otimes id)\Phi = \Delta\alpha$, together with the fact that i is injective, allows us to factorize Δ into a morphism Ψ, as follows:

$$\begin{tikzcd} C(X) \arrow[r, "\Phi"] \arrow[d, "\alpha"'] & C(X) \otimes C(G) \arrow[d, "\alpha\otimes id"] \\ C(Y) \arrow[r, dotted, "\Psi"] \arrow[d, "i"'] & C(Y) \otimes C(G) \arrow[d, "i\otimes id"] \\ C(G) \arrow[r, "\Delta"] & C(G) \otimes C(G) \end{tikzcd}$$

Thus the image space $G \to Y$ is indeed homogeneous, and we are done. $\square$

Finally, we have the following result:

Theorem 6.9. *Let $G \to X$ be an extended homogeneous space, and construct quotients $X \to X'$, $G \to G'$ by performing the GNS construction with respect to $\int \alpha, \int$. Then α factorizes into an inclusion $\alpha' : C(X') \to C(G')$, and we have an homogeneous space.*

Proof. We factorize $G \to Y \subset X$ as above. By performing the GNS construction with respect to $\int i\alpha, \int i, \int$, we obtain a diagram as follows:

$$\begin{tikzcd} C(X) \arrow[r, "p"] \arrow[d, "\alpha"'] & C(X') \arrow[d, "\alpha'"] \arrow[dr, "tr'"] & \\ C(Y) \arrow[r, "q"] \arrow[d, "i"'] & C(Y') \arrow[d, "i'"] & \mathbb{C} \\ C(G) \arrow[r, "r"] & C(G') \arrow[ur, "\int'"'] & \end{tikzcd}$$

Indeed, with $tr = \int \alpha$, the GNS quotient maps p, q, r are defined respectively by:

$$\begin{aligned}
\ker p &= \left\{ f \in C(X) \Big| tr(f^*f) = 0 \right\} \\
\ker q &= \left\{ f \in C(Y) \Big| \textstyle\int (f^*f) = 0 \right\} \\
\ker r &= \left\{ f \in C(G) \Big| \textstyle\int (f^*f) = 0 \right\}
\end{aligned}$$

Next, we can define factorizations i', α' as above. Observe that i' is injective, and that α' is surjective. Our claim now is that α' is injective as well. Indeed:

$$\begin{aligned}
\alpha' p(f) = 0 &\implies q\alpha(f) = 0 \\
&\implies \int \alpha(f^*f) = 0 \\
&\implies tr(f^*f) = 0 \\
&\implies p(f) = 0
\end{aligned}$$

We conclude that we have $X' = Y'$, and this gives the result. □

Summarizing, the basic homogeneous space theory from the classical case extends to the quantum group setting, with a few twists, both of algebraic and analytic nature.

Following [22], let us discuss now some basic examples of homogeneous spaces, which unify the spheres S with the unitary quantum groups U they come form.

We first discuss the construction in the classical case. Given a closed subgroup $G \subset U_N$ and a number $k \leq N$, we can consider the compact group $H = G \cap U_k$, computed inside U_N, where the embedding $U_k \subset U_N$ that we use is given by:

$$g \to \begin{pmatrix} g & 0 \\ 0 & 1_{N-k} \end{pmatrix}$$

We can form the homogeneous space $X = G/H$, and we have the following result:

Proposition 6.10. *Let $G \subset U_N$ be a closed subgroup, and construct as above the group $H = G \cap U_k$. Then the subalgebra*

$$C(G/H) \subset C(G)$$

that we obtain is generated by the last $N - k$ rows of coordinates on G.

Proof. Let $u_{ij} \in C(G)$ be the standard coordinates on G, given by $u_{ij}(g) = g_{ij}$, and consider the following subalgebra of $C(G)$:

$$A = \left\langle u_{ij} \Big| i > k, j > 0 \right\rangle$$

Since each coordinate function u_{ij} with $i > k$ is constant on each coset $Hg \in G/H$, we have an inclusion as follows, between subalgebras of $C(G)$:

$$A \subset C(G/H)$$

In order to prove that this inclusion in a isomorphism, we use the Stone-Weierstrass theorem. Indeed t is enough to show that the functions $\{u_{ij}|i > k, j > 0\}$ separate the cosets $\{Hg|g \in G\}$. But this is the same as saying that $Hg \neq Hh$ implies that $g_{ij} \neq h_{ij}$ for some $i > k, j > 0$, or, equivalently, that $g_{ij} = h_{ij}$ for any $i > k, j > 0$ implies that we have $Hg = Hh$. Now since $Hg = Hh$ is equivalent to $gh^{-1} \in H$, the result follows from the usual matrix formula of gh^{-1}, and from the fact that g, h are unitary. □

In the quantum case now, let $k \leq N$, and consider the embedding $U_k^+ \subset U_N^+$ given by the same formula as before, namely:

$$g \to \begin{pmatrix} g & 0 \\ 0 & 1_{N-k} \end{pmatrix}$$

That is, at the level of algebras, we use the quotient map $C(U_N^+) \to C(U_k^+)$ given by the following formula, where v is the fundamental corepresentation of U_k^+:

$$u \to \begin{pmatrix} v & 0 \\ 0 & 1_{N-k} \end{pmatrix}$$

With this convention, we have the foloowing definition:

Definition 6.11. *Associated to any quantum subgroup $G \subset U_N^+$ and any $k \leq N$ are:*

1. *The compact quantum group $H = G \cap U_k^+$.*
2. *The algebra $C(G/H) \subset C(G)$ constructed above.*
3. *The algebra $C_\times(G/H) \subset C(G/H)$ generated by $\{u_{ij}|i > k, j > 0\}$.*

Regarding (3), let u, v be the fundamental corepresentations of G, H, so that the quotient map $\pi : C(G) \to C(H)$ is given by $u \to diag(v, 1_{N-k})$. We have then:

$$(\pi \otimes id)\Delta(u_{ij}) = \sum_s \pi(u_{is}) \otimes u_{sj} = \begin{cases} \sum_{s \leq k} v_{is} \otimes u_{sj} & i \leq k \\ 1 \otimes u_{ij} & i > k \end{cases}$$

In particular we see that the equality $(\pi \otimes id)\Delta f = 1 \otimes f$ defining $C(G/H)$ holds on all the coefficients $f = u_{ij}$ with $i > k$, and this justifies the inclusion appearing in (3).

Let us first try to understand what happens in the group dual case. We will do our study here in two steps, first in the "diagonal" case, and then in the general case.

We recall that given a discrete group $\Gamma =< g_1, \ldots, g_N >$, the matrix $D = diag(g_i)$ is biunitary, and produces a surjective morphism $C(U_N^+) \to C^*(\Gamma)$. This morphism can be viewed as corresponding to a quantum embedding $\widehat{\Gamma} \subset U_N^+$, that we call "diagonal".

We recall also that the normal closure of a subgroup $\Lambda \subset \Gamma$ is the biggest subgroup $\Lambda' \subset \Gamma$ containing Λ as a normal subgroup. Note that Λ' can be different from the normalizer $N(\Lambda)$. With these conventions, we have the following result:

Proposition 6.12. *Assume that $G = \widehat{\Gamma}$, with $\Gamma =< g_1, \ldots, g_N >$, diagonally embedded, and let $H = G \cap U_k^+$.*

(1) *$H = \widehat{\Theta}$, where $\Theta = \Gamma / < g_{k+1} = 1, \ldots, g_N = 1 >$.*
(2) *$C_\times(G/H) = C^*(\Lambda)$, where $\Lambda =< g_{k+1}, \ldots, g_N >$.*
(3) *$C(G/H) = C^*(\Lambda')$, where "prime" is the normal closure.*
(4) *$C_\times(G/H) = C(G/H)$ if and only if $\Lambda \triangleleft \Gamma$.*

Proof. We use the standard fact that for any group $\Gamma =< a_i, b_j >$, the kernel of the quotient map $\Gamma \to \Gamma / < a_i = 1 >$ is the normal closure of the subgroup $< a_i > \subset \Gamma$.

(1) Since the map $C(U_N^+) \to C(U_k^+)$ is given on diagonal coordinates by $u_{ii} \to v_{ii}$ for $i \leq k$ and $u_{ii} \to 1$ for $i > k$, the result follows from definitions.

(2) Once again, this assertion follows from definitions.

(3) From the above and from (1) we get $G/H = \widehat{\Lambda'}$, where $\Lambda' = \ker(\Gamma \to \Theta)$. By the above observation, this kernel is exactly the normal closure of Λ.

(4) This follows from (2) and (3). □

Let us try now to understand the general group dual case. We recall that the subgroups $\widehat{\Gamma} \subset U_N^+$ appear by taking a discrete group $\Gamma =< g_1, \ldots, g_N >$ and a unitary matrix $J \in U_N$, and constructing the morphism $C(U_N^+) \to C^*(\Gamma)$ given by $u \to JDJ^*$, where $D = diag(g_i)$. With this in hand, Proposition 6.12 generalizes as follows:

Theorem 6.13. *Assume that $G = \widehat{\Gamma}$, with $\Gamma =< g_1, \ldots, g_N >$, embedded via $u \to JDJ^*$, and let $H = G \cap U_k^+$.*

(1) *$H = \widehat{\Theta}$, where $\Theta = \Gamma / < g_r = 1 | \exists\, i > k, J_{ir} \neq 0 >$, embedded $u_{ij} \to (JDJ^*)_{ij}$.*
(2) *$C_\times(G/H) = C^*(\Lambda)$, where $\Lambda =< g_r | \exists\, i > k, J_{ir} \neq 0 >$.*
(3) *$C(G/H) = C^*(\Lambda')$, where "prime" is the normal closure.*
(4) *$C_\times(G/H) = C(G/H)$ if and only if $\Lambda \triangleleft \Gamma$.*

Proof. We basically follow the proof of Proposition 6.12 above:

(1) Let $\Lambda =< g_1, \ldots, g_N >$, let $J \in U_N$, and consider the embedding $\widehat{\Lambda} \subset U_N^+$ corresponding to the morphism $C(U_N^+) \to C^*(\Lambda)$ given by $u \to JDJ^*$, where $D = diag(g_i)$.

Let $G = \widehat{\Lambda} \cap U_k^+$. Since we have $G \subset \widehat{\Lambda}$, the algebra $C(G)$ is cocommutative, so we have $G = \widehat{\Theta}$ for a certain discrete group Θ. Moreover, the inclusion $\widehat{\Theta} \subset \widehat{\Lambda}$ must come from a group morphism $\varphi : \Lambda \to \Theta$. Also, since $\widehat{\Theta} \subset U_k^+$, we have a morphism $C(U_k^+) \to C^*(\Theta)$ given by $v \to V$, where V is a certain $k \times k$ biunitary over $C^*(\Theta)$.

With these observations in hand, let us look now at the intersection operation. We must have a group morphism $\varphi:\Lambda\to\Theta$ such that the following diagram commutes:

$$\begin{array}{ccc}
C(U_N^+) & \longrightarrow & C(U_k^+)\\
\downarrow & & \downarrow\\
C^*(\Lambda) & \longrightarrow & C^*(\Theta)
\end{array}$$

Thus we must have $(id\otimes\varphi)(JDJ^*)=diag(V,1_{N-k})$, and with $f_i=\varphi(g_i)$, we get:

$$\sum_r J_{ir}\bar{J}_{jr}f_r=\begin{cases}V_{ij}&\text{if } i,j\leq k\\ \delta_{ij}&\text{otherwise}\end{cases}$$

Now since J is unitary, the second part of the above condition is equivalent to "$f_r=1$ whenever there exists $i>k$ such that $J_{ir}\neq 0$". Indeed, this condition is easily seen to be equivalent to the "$=1$" conditions, and implies the "$=0$" conditions. We claim that:

$$\Theta=\Lambda\Big/\Big\langle g_r=1\Big|\exists\, i>k,J_{ir}\neq 0\Big\rangle$$

Indeed, the above discussion shows that Θ must be a quotient of the group on the right, say Θ_0. On the other hand, since in $C^*(\Theta_0)$ we have $J_{ir}g_r=J_{ir}1$ for any $i>k$, we obtain that $(JDJ^*)_{ij}=\delta_{ij}$ unless $i,j\leq k$, so we have $JDJ^*=diag(V,1_{N-k})$, for a certain matrix V. But V must be a biunitary, so we have a morphism $C(U_k^+)\to C^*(\Theta_0)$ mapping $v\to V$, which completes the proof of our claim.

(2) Let $A_{ij}=\sum_r J_{ir}\bar{J}_{jr}g_r$ with $i>k,j>0$ be the standard generators of $C_\times(G/H)$. Since $\sum_j A_{ij}J_{jm}=J_{im}g_m$ we conclude that $C_\times(G/H)$ contains any g_r such that there exists $i>k$ with $J_{ir}\neq 0$, i.e. contains any $g_r\in\Lambda$. Conversely, if $g_r\in\Gamma-\Lambda$ then $J_{ir}g_r=0$ for any $i>k$, so g_r doesn't appear in the formula of any of the generators A_{ij}.

(3,4) The proof here is similar to the proof of Proposition 6.12 (3,4). □

Going now towards the easy case, and the examples of quotient spaces that are interested in, we will need the following key result, coming from [29], [36], [37]:

Theorem 6.14. *For an easy quantum group $G_N\subset U_N^+$, the following are equivalent:*

1. *$G=(G_N)$ is uniform, in the sense that $G_N\cap U_k^+=G_k$, for any $k\leq N$.*
2. *The corresponding category $D=(D(k,l))$ is stable under removing blocks.*

Proof. We will prove that $G_N\cap U_k^+=G_k'$, where $G'=(G_N')$ is the easy quantum group associated to the category D' generated by all subpartitions of the partitions in D.

We know that the correspondence between categories of partitions and easy quantum groups comes from Woronowicz's Tannakian duality in [148]. More precisely, the quantum group $G_N \subset O_N^+$ associated to a category of partitions $D = (D(s))$ is obtained by imposing to the fundamental representation of O_N^+ the fact that its s-th tensor power must fix ξ_π, for any $s \in \mathbb{N}$ and $\pi \in D(s)$. Thus, we have the following presentation result:

$$C(G_N) = C(O_N^+) \Big/ \Big\langle \xi_\pi \in Fix(u^{\otimes s}), \forall s,\ \forall \pi \in D(s) \Big\rangle$$

Now since $\xi_\pi \in Fix(u^{\otimes s})$ means $u^{\otimes s}(\xi_\pi \otimes 1) = \xi_\pi \otimes 1$, this condition is equivalent to the following collection of equalities, one for each multi-index $i \in \{1, \ldots, N\}^s$:

$$\sum_{j_1 \ldots j_s} \delta_\pi(j) u_{i_1 j_1} \ldots u_{i_s j_s} = \delta_\pi(i) 1$$

Summarizing, we have the following presentation result:

$$C(G_N) = C(O_N^+) \Big/ \left\langle \sum_{j_1 \ldots j_s} \delta_\pi(j) u_{i_1 j_1} \ldots u_{i_s j_s} = \delta_\pi(i) 1, \forall s,\ \forall \pi \in D(s),\ \forall i \right\rangle$$

Let now $k \leq N$, assume that we have a compact quantum group $K \subset O_k^+$, with fundamental representation denoted u, and consider the $N \times N$ matrix $\tilde{u} = diag(u, 1_{N-k})$. Our claim is that for any $s \in \mathbb{N}$ and any $\pi \in P(s)$, we have:

$$\xi_\pi \in Fix(\tilde{u}^{\otimes s}) \iff \xi_{\pi'} \in Fix(u^{\otimes s'}),\ \forall \pi' \subset \pi$$

Here $\pi' \subset \pi$ means that $\pi' \in P(s')$ is obtained from $\pi \in P(s)$ by removing some of its blocks. The proof of this claim is standard. Indeed, when making the replacement $u \to \tilde{u}$ and trying to check the condition $\xi_\pi \in Fix(\tilde{u}^{\otimes s})$, we have two cases:

– $\delta_\pi(i) = 1$. Here the $> k$ entries of i must be joined by certain blocks of π, and we can consider the partition $\pi' \in D(s')$ obtained by removing these blocks. The point now is that the collection of $\delta_\pi(i) = 1$ equalities to be checked coincides with the collection of $\delta_\pi(i) = 1$ equalities expressing the fact that we have $\xi_\pi \in Fix(u^{\otimes s'})$, for any $\pi' \subset \pi$.

– $\delta_\pi(i) = 0$. In this case the situation is quite similar. Indeed, the collection of $\delta_\pi(i) = 0$ equalities to be checked coincides, modulo some $0 = 0$ identities, with the collection of $\delta_\pi(i) = 0$ equalities expressing the fact that we have $\xi_\pi \in Fix(u^{\otimes s'})$, for any $\pi' \subset \pi$.

Our second claim is that given a quantum group $K \subset O_N^+$, with fundamental representation denoted v, the algebra of functions on $H = K \cap O_k^+$ is given by:

$$C(H) = C(O_k^+) \Big/ \Big\langle \xi \in Fix(\tilde{u}^{\otimes s}),\ \forall \xi \in Fix(v^{\otimes s}) \Big\rangle$$

This follows indeed from Woronowicz's results in [148], because the algebra on the right comes from the Tannakian formulation of the intersection operation.

Now with the above two claims in hand, we can conclude that we have $G_N \cap U_k^+ = G'_k$, where $G' = (G'_N)$ is the easy quantum group associated to the category D' generated by all the subpartitions of the partitions in D. In particular we see that the condition $G_N \cap U_k^+ = G_k^+$ for any $k \leq N$ is equivalent to $D = D'$, and this gives the result. $\square$

Let us study now the following inclusions of algebras, constructed in Definition 6.11 above, where $G=(G_n)$ is a uniform easy quantum group:

$$C_\times(G_N/G_k)\subset C(G_N/G_k)$$

We recall from [37] that the basic examples are the classical groups S,O,H,B, and their free analogues S^+,O^+,H^+,B^+. In addition, it is known that in the free case the list of such quantum groups is precisely S^+,O^+,H^+,B^+. See [37]. We have:

Proposition 6.15. *The defining relations for $C(G)$ are as follows, in terms of the standard generators u_{ij}:*

1. *$G=O_N^+$: u is orthogonal, i.e. u_{ij} are self-adjoint, and $u^t=u^{-1}$.*
2. *$G=S_N^+$: u is magic, i.e. orthogonal, with u_{ij} being projections.*
3. *$G=H_N^+$: u is cubic, i.e. orthogonal, with $xy=0$ on rows and columns.*
4. *$G=B_N^+$: u is bistochastic, i.e. orthogonal, with sum 1 on rows and columns.*

Proof. We refer to [37] for a full discussion of these relations. □

Observe that we have "magic = cubic + bistochastic", which follows from definitions, by using basic C^*-algebra tricks. This shows that we have inclusions as follows:

$$\begin{tikzcd}
H_N^+ \arrow[r] & O_N^+ \\
S_N^+ \arrow[u] \arrow[r] & B_N^+ \arrow[u]
\end{tikzcd}$$

Let us go back now to the inclusions $C_\times(G_N/G_k)\subset C(G_N/G_k)$. We first work out a few simple cases, where these inclusions are isomorphisms:

Proposition 6.16. *For the basic easy quantum groups, the inclusion of algebras*

$$C_\times(G_N/G_k)\subset C(G_N/G_k)$$

is an isomorphism at $N=1$, at $k=0$, at $k=N$, as well as in the following special cases:

1. *$G=B^+$: at $k=1$.*
2. *$G=S^+$: at $k=1$, and at $k=2,N=3$.*

Proof. First, the results at $N=1$, at $k=0$, and at $k=N$ are clear from definitions. Regarding now the special cases, the situation here is as follows:

(1) Since the coordinates of B_N^+ sum up to 1 on each column, we have the formula $u_{1j}=1-\sum_{i>1}u_{ij}$, and so the inclusion $C_\times(B_N^+/B_1^+)\subset C(B_N^+)$ is an isomorphism. Thus the inclusion $C_\times(B_N^+/B_1^+)\subset C(B_N^+/B_1^+)$ must be as well an isomorphism.

(2) By using the same argument we obtain that the inclusion $C_\times(S_N^+/S_1^+)\subset C(S_N^+/S_1^+)$ is as well an isomorphism. In the remaining case $k=2,N=3$, or more generally at any

$k \in \mathbb{N}$ and $N < 4$, it is known from Wang [141] that we have $S_N = S_N^+$, so the inclusion in the statement is $C(S_N/S_k) \subset C(S_N/S_k)$, and we are done again. □

The axiomatization of the algebras $C_\times(G_N/G_k)$ is a quite tricky task, because these algebras have a rectangular matrix of generators, which is a transposed isometry, but not much is known about the remaining conditions to be satisfied by the generators.

However, we can axiomatize some bigger algebras, as follows:

Definition 6.17. *Associated to $k \leq N$ is the universal C^*-algebra $C_+(G_N/G_k)$ generated by the entries of a rectangular matrix*

$$p = (p_{ij})_{i>k,j>0}$$

ubject to the following conditions:

(1) *$G = O_N^+$: p is a transposed "orthogonal isometry", in the sense that its entries p_{ij} are self-adjoint, and $pp^t = 1$.*
(2) *$G = S_N^+$: p is a transposed "magic isometry", in the sense that p^t is an orthogonal isometry, and p_{ij} are projections, orthogonal on columns.*
(3) *$G = H_N^+$: p is a transposed "cubic isometry", in the sense that p^t is an orthogonal isometry, with $xy = 0$ for any $x \neq y$ on the same row of p*
(4) *$G = B_N^+$: p is a transposed "stochastic isometry", in the sense that p^t is an orthogonal isometry, with sum 1 on rows.*

Observe that, since the entries p_{ij} of our various rectangular matrices are assumed to be self-adjoint, we have $p^* = p^t$. Thus the condition $pp^t = 1$ reads $(p^t)^*p^t = 1$, so the transposed matrix $q = p^t$ must indeed satisfy the isometry condition $q^*q = 1$.

Observe also that the cubic condition on transposed orthogonal isometry p is equivalent to the fact that the entries $x = p_{ij}$ satisfy the "cubic" condition $x^3 = x$.

Note also that we have by definition surjective maps, as follows:

$$C_+(G_N/G_k) \to C_\times(G_N/G_k)$$

Finally, observe that in the case $G = O^+$ and $k = N - 1$ we obtain the algebra of functions on the free sphere. This will be actually our guiding example. We will need:

Proposition 6.18. *For a transposed orthogonal isometry p, the following are equivalent:*

(1) *p is magic.*
(2) *p is cubic and stochastic.*

Proof. At $k = N$ this result is well-known. In the general case the proof is similar, by using some basic C^*-algebra tricks:

(1) $\implies$ (2). Assume indeed that p is magic. The transposed isometry condition $pp^t=1$ tells us that we have $\sum_j p_{ij}p_{kj}=\delta_{ik}$. At $i=k$ we get $\sum_j p_{ij}^2=1$, and since the elements p_{ij} are projections, this condition becomes $\sum_j p_{ij}=1$. Thus p is stochastic.

With this observation in hand, and since projections summing up to 1 must commute, we conclude that the elements p_{ij} mutually commute on rows, so p is cubic as well.

(2) $\implies$ (1). Assume that p is cubic and stochastic. Since the elements $p_{i1},\ldots,p_{iN}$ are self-adjoint, satisfy $xy=0$, and sum up to 1, they are projections, and we are done. $\square$

We have the following result:

Theorem 6.19. *The algebras $C_+(G_N/G_k)$ and $C_\times(G_N/G_k)$ have the following properties:*

1. *They have coactions of G_N, given by $\alpha(p_{ij})=\sum_s p_{is}\otimes u_{sj}$.*
2. *They have unique G_N-invariant states, which are tracial.*
3. *Their reduced algebra versions are isomorphic.*
4. *Their abelianized versions are isomorphic.*

Proof. We follow the proof in [32], where the above result was proved for $G=O^+$ and $k=N-1$. The only problems, requiring some new ideas, will appear in (4) for $G=S^+,H^+$, and we will follow here the proof in [22]. In practice now:

(1) For $C_\times(G_N/G_k)$ this is clear, because this algebra is "embeddable", and the coaction of G_N is simply the restriction of the comultiplication map.

For the algebra $C_+(G_N/G_k)$, consider the following elements:

$$P_{ij}=\sum_{s=1}^N p_{is}\otimes u_{sj}$$

We have to check that these elements satisfy the same relations as those in Definition 6.17, presenting the algebra $C_+(G_n/G_k)$, and the proof here goes as follows:

<u>O^+ case</u>. First, since p_{ij},u_{ij} are self-adjoint, so is P_{ij}. Also, we have:

$$\begin{aligned}\sum_j P_{ij}P_{rj}&=\sum_{jst}p_{is}p_{rt}\otimes u_{sj}u_{tj}\\&=\sum_{st}p_{is}p_{rt}\otimes\delta_{st}\\&=\sum_s p_{is}p_{rs}\otimes 1\\&=\delta_{ir}\end{aligned}$$

<u>H^+ case</u>. The condition $xy=0$ on rows is checked as follows ($j\neq r$):

$$P_{ij}P_{ir}=\sum_{st}p_{is}p_{it}\otimes u_{sj}u_{tr}=\sum_s p_{is}\otimes u_{sj}u_{sr}=0$$

<u>B^+ case</u>. The sum 1 condition on rows is checked as follows:

$$\sum_j P_{ij} = \sum_{js} p_{is} \otimes u_{sj} = \sum_s p_{is} \otimes 1 = 1$$

<u>S^+ case</u>. Since P^t is cubic and stochastic, we just check the projection condition:

$$P_{ij}^2 = \sum_{st} p_{is} p_{it} \otimes u_{sj} u_{tj} = \sum_s p_{is} \otimes u_{sj} = P_{ij}$$

Summmarizing, P satisfies the same conditions as p, so we can define a morphism of C^*-algebras, as follows:

$$\alpha : C_+(G_N/G_k) \to C_+(G_N/G_k) \otimes C(G_N)$$

$$\alpha(p_{ij}) = P_{ij}$$

We have the following computations:

$$\begin{aligned}
(\alpha \otimes id)\alpha(p_{ij}) &= \sum_s \alpha(p_{is}) \otimes u_{sj} = \sum_{st} p_{it} \otimes u_{ts} \otimes u_{sj} \\
(id \otimes \Delta)\alpha(p_{ij}) &= \sum_t p_{it} \otimes \Delta(u_{ij}) = \sum_{st} p_{it} \otimes u_{ts} \otimes u_{sj}
\end{aligned}$$

Thus our map α is coassociative. The density conditions can be checked by using dense subalgebras generated by p_{ij} and u_{st}, and we are done.

(2) For the existence part we can use the following composition, where the first two maps are the canonical ones, and the map on the right is the integration over G_N:

$$C_+(G_N/G_k) \to C_\times(G_N/G_k) \subset C(G_N) \to \mathbb{C}$$

Also, the uniqueness part is clear for the algebra $C_\times(G_N/G_k)$, as a particular case of the general properties of "embeddable" coactions, i.e. those coactions that can be realized as coactions on subalgebras of $C(G)$, via the restriction of the comultiplication.

Regarding now the uniqueness for $C_+(G_N/G_k)$, let $\int$ be the Haar state on G_N, and φ be the G_N-invariant state constructed above. We claim that α is ergodic:

$$\left(id \otimes \int\right)\alpha = \varphi(.)1$$

Indeed, let us recall that the Haar state is given by the following Weingarten formula, where $W_{sN} = G_{sN}^{-1}$, with $G_{sN}(\pi,\sigma) = N^{|\pi\vee\sigma|}$:

$$\int u_{i_1 j_1} \ldots u_{i_s j_s} = \sum_{\pi,\sigma \in D(s)} \delta_\pi(i)\delta_\sigma(j) W_{sN}(\pi,\sigma)$$

Now, let us go back now to our claim. By linearity it is enough to check the above equality on a product of basic generators $p_{i_1j_1}\ldots p_{i_sj_s}$. The left term is as follows:

$$\begin{aligned}\left(id\otimes\int\right)\alpha(p_{i_1j_1}\ldots p_{i_sj_s}) &= \sum_{l_1\ldots l_s}p_{i_1l_1}\ldots p_{i_sl_s}\int u_{l_1j_1}\ldots u_{l_sj_s}\\ &= \sum_{l_1\ldots l_s}p_{i_1l_1}\ldots p_{i_sl_s}\sum_{\pi,\sigma\in D(s)}\delta_\pi(l)\delta_\sigma(j)W_{sN}(\pi,\sigma)\\ &= \sum_{\pi,\sigma\in D(s)}\delta_\sigma(j)W_{sN}(\pi,\sigma)\sum_{l_1\ldots l_s}\delta_\pi(l)p_{i_1l_1}\ldots p_{i_sl_s}\end{aligned}$$

Let us look now at the sum on the right. We have to sum the elements of type $p_{i_1l_1}\ldots p_{i_sl_s}$, over all multi-indices $l=(l_1,\ldots,l_s)$ which fit into our partition $\pi\in D(s)$. In the case of a one-block partition this sum is simply $\sum_l p_{i_1l}\ldots p_{i_sl}$, and we claim that:

$$\sum_l p_{i_1l}\ldots p_{i_sl}=\delta_\pi(i)$$

Indeed, by using the explicit description of the sets of diagrams $D(s)$ given above, the proof of this formula goes as follows:

<u>O^+ case</u>. Here our one-block partition must be a semicircle, $\pi=\cap$, and the formula to be proved, namely $\sum_l p_{il}p_{jl}=\delta_{ij}$, follows from $pp^t=1$.

<u>S^+ case</u>. Here our one-block partition can be any s-block, $1_s\in P(s)$, and the formula to be proved, namely $\sum_l p_{i_1l}\ldots p_{i_sl}=\delta_{i_1,\ldots,i_s}$, follows from orthogonality on columns, and from the fact that the sum is 1 on rows.

<u>B^+ case</u>. Here our one-block partition must be a semicircle or a singleton. We are already done with the semicircle, and for the singleton the formula to be proved, namely $\sum_l p_{il}=1$, follows from the fact that the sum is 1 on rows.

<u>H^+ case</u>. Here our one-block partition must have an even number of legs, $s=2r$, and due to the cubic condition the formula to be proved reduces to $\sum_l p_{il}^{2r}=1$. But since $p_{il}^{2r}=p_{il}^2$, independently on r, the result follows from the orthogonality on rows.

In the general case now, since π noncrossing, the computations over the blocks will not interfere, and we will obtain the same result, namely:

$$\sum_l p_{i_1l}\ldots p_{i_sl}=\delta_\pi(i)$$

Now by plugging this formula into the computation that we have started, we get:

$$\begin{aligned}\left(id\otimes\int\right)\alpha(p_{i_1j_1}\ldots p_{i_sj_s}) &= \sum_{\pi,\sigma\in D(s)}\delta_\pi(i)\delta_\sigma(j)W_{sN}(\pi,\sigma)\\ &= \int u_{i_1j_1}\ldots u_{i_sj_s}\\ &= \varphi(p_{i_1j_1}\ldots p_{i_sj_s})\end{aligned}$$

This finishes the proof of our claim. So, let us get back now to the original question. Let $\tau:C_+(G_N/G_k)\to\mathbb{C}$ be a linear form as in the statement. We have:

$$\begin{aligned}\tau\left(id\otimes\int\right)\alpha(x) &= \left(\tau\otimes\int\right)\alpha(x)\\ &= \int(\tau\otimes id)\alpha(x)\\ &= \int(\tau(x)1)\\ &= \tau(x)\end{aligned}$$

On the other hand, according to our above claim, we have as well:

$$\tau\left(id\otimes\int\right)\alpha(x)=\tau(\varphi(x)1)=\varphi(x)$$

Thus we get $\tau=\varphi$, which finishes the proof of the uniqueness assertion.

(3) This follows from the uniqueness assertions in (2), and from some standard facts regarding the reduced versions with respect to Haar states, from [148].

(4) We denote by G^- the classical version of G, given by $G^-=O,S,H,B$ in the cases $G=O^+,S^+,H^+,B^+$. We have surjective morphisms of algebras, as follows:

$$C_+(G_N/G_k)\to C_\times(G_k/G_k)\to C_\times(G_N^-/G_k^-)=C(G_N^-/G_k^-)$$

Thus at the level of abelianized versions, we have surjective morphisms as follows:

$$C_+(G_N/G_k)_{comm}\to C_\times(G_N/G_k)_{comm}\to C(G_N^-/G_k^-)$$

In order to prove our claim, namely that the first surjective morphism is an isomorphism, it is enough to prove that the above composition is an isomorphism.

Let $r=N-k$, and denote by $A_{N,r}$ the algebra on the left. This is by definition the algebra generated by the entries of a transposed $N\times r$ isometry, whose entries commute, and which is respectively orthogonal, magic, cubic, bistochastic. We have a surjective morphism $A_{N,r}\to C(G_N^-/G_k^-)$, and we must prove that this is an isomorphism.

<u>S^+ case</u>. Since $\#(S_N/S_k)=N!/k!$, it is enough to prove that $\dim(A_{N,r})=N!/k!$. Let p_{ij} be the standard generators of $A_{N,r}$. By using the Gelfand theorem, we can write

$p_{ij} = \chi(X_{ij})$, where $X_{ij} \subset X$ are certain subets of a given set X. Now at the level of sets the magic isometry condition on (p_{ij}) tells us that the matrix of sets (X_{ij}) has the property that its entries are disjoint on columns, and form partitions of X on rows.

So, let us try to understand this property for N fixed, and $r = 1, 2, 3, \ldots$

– At $r = 1$ we simply have a partition $X = X_1 \sqcup \ldots \sqcup X_N$. So, the universal model can be any such partition, with $X_i \neq 0$ for any i.

– At $r = 2$ the universal model is best described as follows: X is the $N \times N$ square in $\mathbb{R}^2$, regarded as a union of N^2 unit tiles, minus the diagonal, the sets X_{1i} are the disjoint unions on rows, and the sets X_{2i} are the disjoint unions on columns.

– At $r \geq 3$, the universal solution is similar: we can take X to be the N^r cube in $\mathbb{R}^r$, with all tiles having pairs of equal coordinates removed, and say that the sets X_{si} for s fixed are the various "slices" of X in the direction of the s-th coordinate of $\mathbb{R}^r$.

Summarizing, the above discussion tells us that $\dim(A_{N,r})$ equals the number of tiles in the above set $X \subset \mathbb{R}^r$. But these tiles correspond by definition to the various r-tuples $(i_1, \ldots, i_r) \in \{1, \ldots, N\}^r$ with all i_k different, and since there are exactly $N!/k!$ such r-tuples, we obtain $\dim(A_{N,r}) = N!/k!$, and we are done.

<u>H^+ case</u>. We can use here the same method as for S_N^+. This time the functions p_{ij} take values in $\{-1, 0, 1\}$, and the algebra generated by their squares p_{ij}^2 coincides with the one computed above for S_N^+, having dimension $N!/k!$. Now by taking into account the $N - k$ possible signs we obtain $\dim(A_{N,r}) \leq 2^{N-k} N!/k! = \#(H_N/H_k)$, and we are done.

<u>O^+ case</u>. We can use the same method, namely a straightforward application of the Gelfand theorem. However, instead of performing a dimension count, which is no longer possible, we have to complete here any transposed $N \times r$ isometry whose entries commute to a $N \times N$ orthogonal matrix. But this is the same as completing a system of r orthogonal norm 1 vectors in $\mathbb{R}^N$ into an orthonormal basis of $\mathbb{R}^N$, which is of course possible.

<u>B^+ case</u>. Since we have a surjective map $C(O_N^+) \to C(B_N^+)$, we obtain a surjective map $C_+(O_N^+/O_k^+) \to A_{N,r}$, and hence surjective maps as follows:

$$C(O_N/O_k) \to A_{N,r} \to C(B_N/B_k)$$

Now since this composition is the canonical map $C(O_N/O_k) \to C(B_N/B_k)$, by looking at the column vector $\xi = (1, \ldots, 1)^t$, which is fixed by the stochastic matrices, we conclude that the map on the right is an isomorphism, and we are done. $\square$

7. Partial isometries

In what follows we discuss the formalism in [10], which is quite broad, while remaining not very abstract. We will study the spaces of the following type:

$$X = (G_M \times G_N)\big/(G_L \times G_{M-L} \times G_{N-L})$$

These spaces cover indeed the quantum groups and the spheres. And also, they are quite concrete and useful objects, consisting of certain classes of "partial isometries".

We begin with a study in the classical case. Our starting point will be:

Definition 7.1. *Associated to any integers $L \leq M, N$ are the spaces*

$$O_{MN}^L = \left\{T : E \to F \text{ isometry}\Big| E \subset \mathbb{R}^N, F \subset \mathbb{R}^M, \dim_{\mathbb{R}} E = L\right\}$$

$$U_{MN}^L = \left\{T : E \to F \text{ isometry}\Big| E \subset \mathbb{C}^N, F \subset \mathbb{C}^M, \dim_{\mathbb{C}} E = L\right\}$$

where the notion of isometry is with respect to the usual real/complex scalar products.

As a first observation, at $L = M = N$ we obtain the groups O_N, U_N:

$$O_{NN}^N = O_N \quad , \quad U_{NN}^N = U_N$$

Another interesting specialization is $L = M = 1$. Here the elements of O_{1N}^1 are the isometries $T : E \to \mathbb{R}$, with $E \subset \mathbb{R}^N$ one-dimensional. But such an isometry is uniquely determined by $T^{-1}(1) \in \mathbb{R}^N$, which must belong to $S^{N-1}_{\mathbb{R}}$. Thus, we have $O_{1N}^1 = S^{N-1}_{\mathbb{R}}$. Similarly, in the complex case we have $U_{1N}^1 = S^{N-1}_{\mathbb{C}}$, and so our results here are:

$$O_{1N}^1 = S^{N-1}_{\mathbb{R}} \quad , \quad U_{1N}^1 = S^{N-1}_{\mathbb{C}}$$

Yet another interesting specialization is $L = N = 1$. Here the elements of O_{1N}^1 are the isometries $T : \mathbb{R} \to F$, with $F \subset \mathbb{R}^M$ one-dimensional. But such an isometry is uniquely determined by $T(1) \in \mathbb{R}^M$, whichmust belong to $S^{M-1}_{\mathbb{R}}$. Thus, we have $O_{M1}^1 = S^{M-1}_{\mathbb{R}}$. Similarly, in the complex case we have $U_{M1}^1 = S^{M-1}_{\mathbb{C}}$, and so our results here are:

$$O_{M1}^1 = S^{M-1}_{\mathbb{R}} \quad , \quad U_{M1}^1 = S^{M-1}_{\mathbb{C}}$$

In general, the most convenient is to view the elements of O_{MN}^L, U_{MN}^L as rectangular matrices, and to use matrix calculus for their study. We have indeed:

Proposition 7.2. *We have identifications of compact spaces*

$$O_{MN}^L \simeq \left\{U \in M_{M\times N}(\mathbb{R})\Big| UU^t = \text{projection of trace } L\right\}$$

$$U_{MN}^L \simeq \left\{U \in M_{M\times N}(\mathbb{C})\Big| UU^* = \text{projection of trace } L\right\}$$

with each partial isometry being identified with the corresponding rectangular matrix.

Proof. We can indeed identify the partial isometries $T : E \to F$ with their corresponding extensions $U : \mathbb{R}^N \to \mathbb{R}^M$, $U : \mathbb{C}^N \to \mathbb{C}^M$, obtained by setting $U_{E^\perp} = 0$. Then, we can identify these latter linear maps U with the corresponding rectangular matrices. $\square$

As an illustration, at $L = M = N$ we recover in this way the usual matrix description of O_N, U_N. Also, at $L = M = 1$ we obtain the usual description of $S^{N-1}_\mathbb{R}, S^{N-1}_\mathbb{C}$, as row spaces over the corresponding groups O_N, U_N. Finally, at $L = N = 1$ we obtain the usual description of $S^{N-1}_\mathbb{R}, S^{N-1}_\mathbb{C}$, as column spaces over the corresponding groups O_N, U_N.

Now back to the general case, observe that the isometries $T : E \to F$, or rather their extensions $U : \mathbb{K}^N \to \mathbb{K}^M$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, obtained by setting $U_{E^\perp} = 0$, can be composed with the isometries of $\mathbb{K}^M, \mathbb{K}^N$, according to the following scheme:

$$\begin{tikzcd}
\mathbb{K}^N \arrow[r, "B^*"] & \mathbb{K}^N \arrow[r, dotted, "U"] & \mathbb{K}^M \arrow[r, "A"] & \mathbb{K}^M \\
B(E) \arrow[u] \arrow[r, dotted] & E \arrow[u] \arrow[r, "T"] & F \arrow[u] \arrow[r, dotted] & A(F) \arrow[u]
\end{tikzcd}$$

With the identifications in Proposition 7.2 made, the precise statement here is:

Proposition 7.3. *We have an action map as follows, which is transitive,*

$$O_M \times O_N \curvearrowright O^L_{MN}$$

$$(A, B)U = AUB^t$$

as well as an action map as follows, transitive as well,

$$U_M \times U_N \curvearrowright U^L_{MN}$$

$$(A, B)U = AUB^*$$

whose stabilizers are respectively:

$$O_L \times O_{M-L} \times O_{N-L}$$

$$U_L \times U_{M-L} \times U_{N-L}$$

Proof. We have indeed action maps as in the statement, which are transitive. Let us compute now the stabilizer G of the following point:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $(A, B) \in G$ satisfy $AU = UB$, their components must be of the following form:

$$A = \begin{pmatrix} x & * \\ 0 & a \end{pmatrix} \quad , \quad B = \begin{pmatrix} x & 0 \\ * & b \end{pmatrix}$$

Now since A, B are both unitaries, these matrices follow to be block-diagonal, and so:

$$G = \left\{ (A,B) \middle| A = \begin{pmatrix} x & 0 \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} x & 0 \\ 0 & b \end{pmatrix} \right\}$$

The stabilizer of U is then parametrized by triples (x, a, b) belonging respectively to:

$$O_L \times O_{M-L} \times O_{N-L}$$

$$U_L \times U_{M-L} \times U_{N-L}$$

Thus, we are led to the conclusion in the statement. □

Finally, let us work out the quotient space description of O_{MN}^L, U_{MN}^L. We have here:

Theorem 7.4. *We have isomorphisms of homogeneous spaces as follows,*

$$\begin{aligned} O_{MN}^L &= (O_M \times O_N)/(O_L \times O_{M-L} \times O_{N-L}) \\ U_{MN}^L &= (U_M \times U_N)/(U_L \times U_{M-L} \times U_{N-L}) \end{aligned}$$

with the quotient maps being given by $(A, B) \to AUB^*$, *where:*

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof. This is just a reformulation of Proposition 7.3 above, by taking into account the fact that the fixed point used in the proof there was $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. □

Once again, the basic examples here come from the cases $L = M = N$ and $L = M = 1$. At $L = M = N$ the quotient spaces at right are respectively:

$$O_N, U_N$$

At $L = M = 1$ the quotient spaces at right are respectively:

$$O_N/O_{N-1} \quad , \quad U_N/U_{N-1}$$

In fact, in the general orthogonal $L = M$ case we obtain the following spaces:

$$\begin{aligned} O_{MN}^M &= (O_M \times O_N)/(O_M \times O_{N-M}) \\ &= O_N/O_{N-M} \end{aligned}$$

Also, in the general unitary $L = M$ case we obtain the following spaces:

$$\begin{aligned} U_{MN}^M &= (U_M \times U_N)/(U_M \times U_{N-M}) \\ &= U_N/U_{N-M} \end{aligned}$$

Similarly, the examples coming from the cases $L = M = N$ and $L = N = 1$ are particular cases of the general $L = N$ case, where we obtain the following spaces:

$$\begin{aligned} O_{MN}^N &= (O_M \times O_N)/(O_M \times O_{M-N}) \\ &= O_N/O_{M-N} \end{aligned}$$

In the unitary case, we obtain the following spaces:

$$\begin{aligned} U_{MN}^N &= (U_M \times U_N)/(U_M \times U_{M-N}) \\ &= U_N/U_{M-N} \end{aligned}$$

We can liberate the spaces O_{MN}^L, U_{MN}^L, as follows:

Definition 7.5. *Associated to any integers $L \leq M, N$ are the algebras*

$$\begin{aligned} C(O_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big| u = \bar{u}, uu^t = \text{projection of trace } L\right) \\ C(U_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big| uu^*, \bar{u}u^t = \text{projections of trace } L\right) \end{aligned}$$

with the trace being by definition the sum of the diagonal entries.

Observe that the above universal algebras are indeed well-defined, as it was previously the case for the free spheres, and this due to the trace conditions, which read:

$$\sum_{ij} u_{ij}u_{ij}^* = \sum_{ij} u_{ij}^*u_{ij} = L$$

We have inclusions between the various spaces constructed so far, as follows:

$$\begin{tikzcd} O_{MN}^{L+} \arrow[r] & U_{MN}^{L+} \\ O_{MN}^L \arrow[r] \arrow[u] & U_{MN}^L \arrow[u] \end{tikzcd}$$

At the level of basic examples now, we first have the following result:

Proposition 7.6. *At $L = M = 1$ we obtain the diagram*

$$\begin{tikzcd} S^{N-1}_{\mathbb{R},+} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\ S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C}} \arrow[u] \end{tikzcd}$$

and at $L = N = 1$ we obtain the diagram:

$$\begin{tikzcd} S^{M-1}_{\mathbb{R},+} \arrow[r] & S^{M-1}_{\mathbb{C},+} \\ S^{M-1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{M-1}_{\mathbb{C}} \arrow[u] \end{tikzcd}$$

Proof. We recall that the various spheres involved are constructed as follows, with the symbol $\times$ standing for "commutative" and "free", respectively:

$$\begin{aligned} C(S^{N-1}_{\mathbb{R},\times}) &= C^*_\times\left(z_1,\ldots,z_N\Big|z_i=z_i^*,\sum_i z_i^2=1\right)\\ C(S^{N-1}_{\mathbb{C},\times}) &= C^*_\times\left(z_1,\ldots,z_N\Big|\sum_i z_iz_i^*=\sum_i z_i^*z_i=1\right)\end{aligned}$$

Now by comparing with the definition of $O_{1N}^{1\times},U_{1N}^{1\times}$, this proves our first claim.

As for the proof of the second claim, this is similar, via standard identifications. $\square$

We have as well the following result:

Proposition 7.7. *At $L=M=N$ we obtain the diagram*

$$\begin{tikzcd} O_N^+ \arrow[r] & U_N^+ \\ O_N \arrow[r] \arrow[u] & U_N \arrow[u] \end{tikzcd}$$

consisting of the groups O_N,U_N, and their liberations.

Proof. We recall that the various quantum groups in the statement are constructed as follows, with the symbol $\times$ standing once again for "commutative" and "free":

$$\begin{aligned} C(O_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},uu^t=u^tu=1\right)\\ C(U_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*=u^*u=1,\bar{u}u^t=u^t\bar{u}=1\right)\end{aligned}$$

On the other hand, according to Proposition 7.2 and to Definition 7.5 above, we have the following presentation results:

$$\begin{aligned} C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},uu^t=\text{projection of trace } N\right)\\ C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,\bar{u}u^t=\text{projections of trace } N\right)\end{aligned}$$

We use now the standard fact that if $p=aa^*$ is a projection then $q=a^*a$ is a projection too. We use as well the following formulae:

$$Tr(uu^*)=Tr(u^t\bar{u})$$

$$Tr(\bar{u}u^t)=Tr(u^*u)$$

We therefore obtain the following formulae:

$$\begin{aligned}C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},\ uu^t,u^tu=\text{projections of trace }N\right)\\ C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,u^*u,\bar{u}u^t,u^t\bar{u}=\text{projections of trace }N\right)\end{aligned}$$

Now observe that, in tensor product notation, and by using the normalized trace, the conditions at right are all of the form:

$$(tr\otimes id)p=1$$

To be more precise, p is a follows, for the above conditions:

$$p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$$

We therefore obtain, for any faithful state φ:

$$(tr\otimes\varphi)(1-p)=0$$

It follows from this that the projections $p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$ must be all equal to the identity, as desired, and this finishes the proof. □

Regarding now the homogeneous space structure of $O_{MN}^{L\times},U_{MN}^{L\times}$, the situation here is more complicated in the free case than in the classical case. We have:

Proposition 7.8. *The spaces $U_{MN}^{L\times}$ have the following properties:*

(1) *We have an action $U_M^\times\times U_N^\times\curvearrowright U_{MN}^{L\times}$, given by:*

$$u_{ij}\to\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

(2) *We have a map $U_M^\times\times U_N^\times\to U_{MN}^{L\times}$, given by:*

$$u_{ij}\to\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

Similar results hold for the spaces $O_{MN}^{L\times}$, with all the $*$ exponents removed.

Proof. In the classical case, consider the action and quotient maps:

$$U_M\times U_N\curvearrowright U_{MN}^L$$

$$U_M\times U_N\to U_{MN}^L$$

The transposes of these two maps are as follows, where $J=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$:

$$\begin{aligned}\varphi &\to ((U,A,B)\to\varphi(AUB^*))\\ \varphi &\to ((A,B)\to\varphi(AJB^*))\end{aligned}$$

But with $\varphi=u_{ij}$ we obtain precisely the formulae in the statement. The proof in the orthogonal case is similar. Regarding now the free case, the proof goes as follows:

(1) Assuming $uu^*u=u$, let us set:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

We have then:

$$\begin{aligned}(UU^*U)_{ij} &= \sum_{pq}\sum_{klmnst}u_{kl}u_{mn}^*u_{st}\otimes a_{ki}a_{mq}^*a_{sq}\otimes b_{lp}^*b_{np}b_{tj}^*\\ &= \sum_{klmt}u_{kl}u_{ml}^*u_{mt}\otimes a_{ki}\otimes b_{tj}^*\\ &= \sum_{kt}u_{kt}\otimes a_{ki}\otimes b_{tj}^*\\ &= U_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij}u_{ij}u_{ij}^*=L$, we obtain:

$$\begin{aligned}\sum_{ij}U_{ij}U_{ij}^* &= \sum_{ij}\sum_{klst}u_{kl}u_{st}^*\otimes a_{ki}a_{si}^*\otimes b_{lj}^*b_{tj}\\ &= \sum_{kl}u_{kl}u_{kl}^*\otimes 1\otimes 1\\ &= L\end{aligned}$$

(2) Assuming $uu^*u=u$, let us set:

$$V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

We have then:

$$\begin{aligned}(VV^*V)_{ij} &= \sum_{pq}\sum_{x,y,z\leq L}a_{xi}a_{yq}^*a_{zq}\otimes b_{xp}^*b_{yp}b_{zj}^*\\ &= \sum_{x\leq L}a_{xi}\otimes b_{xj}^*\\ &= V_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij}u_{ij}u_{ij}^*=L$, we obtain:

$$\begin{aligned}\sum_{ij}V_{ij}V_{ij}^* &= \sum_{ij}\sum_{r,s\leq L}a_{ri}a_{si}^*\otimes b_{rj}^*b_{sj}\\ &= \sum_{l\leq L}1\\ &= L\end{aligned}$$

By removing all the $*$ exponents, we obtain as well the orthogonal results. $\square$

Let us examine now the relation between the above maps. In the classical case, given a quotient space $X=G/H$, the associated action and quotient maps are given by:

$$\begin{cases}a:X\times G\to X & : \quad (Hg,h)\to Hgh\\ p:G\to X & : \quad g\to Hg\end{cases}$$

Thus we have $a(p(g),h)=p(gh)$. In our context, a similar result holds:

Theorem 7.9. *With $G=G_M\times G_N$ and $X=G_{MN}^L$, where $G_N=O_N^\times,U_N^\times$, we have*

$$\begin{tikzcd}
G\times G \arrow[r, "m"] \arrow[d, "p\times id"'] & G \arrow[d, "p"] \\
X\times G \arrow[r, "a"] & X
\end{tikzcd}$$

where a,p are the action map and the map constructed in Proposition 7.8.

Proof. At the level of the associated algebras of functions, we must prove that the following diagram commutes, where Φ,α are morphisms of algebras induced by a,p:

$$\begin{tikzcd}
C(X) \arrow[r, "\Phi"] \arrow[d, "\alpha"'] & C(X\times G) \arrow[d, "\alpha\otimes id"] \\
C(G) \arrow[r, "\Delta"] & C(G\times G)
\end{tikzcd}$$

When going right, and then down, the composition is as follows:

$$\begin{aligned}(\alpha\otimes id)\Phi(u_{ij}) &= (\alpha\otimes id)\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*\\ &= \sum_{kl}\sum_{r\leq L}a_{rk}\otimes b_{rl}^*\otimes a_{ki}\otimes b_{lj}^*\end{aligned}$$

On the other hand, when going down, and then right, the composition is as follows, where F_{23} is the flip between the second and the third components:

$$\begin{aligned}\Delta\pi(u_{ij}) &= F_{23}(\Delta\otimes\Delta)\sum_{r\leq L}a_{ri}\otimes b_{rj}^*\\ &= F_{23}\left(\sum_{r\leq L}\sum_{kl}a_{rk}\otimes a_{ki}\otimes b_{rl}^*\otimes b_{lj}^*\right)\end{aligned}$$

Thus the above diagram commutes indeed, and this gives the result. $\square$

Let us discuss now some extensions of the above constructions. We will be mostly interested in the quantum reflection groups, so let us first discuss, with full details, the case of the quantum groups H_N^s, H_N^{s+}. We use the following notion:

Definition 7.10. *Associated to any partial permutation, $\sigma : I \simeq J$ with $I \subset \{1, \ldots, N\}$ and $J \subset \{1, \ldots, M\}$, is the real/complex partial isometry*

$$T_\sigma : span\left(e_i\Big|i \in I\right) \to span\left(e_j\Big|j \in J\right)$$

given on the standard basis elements by $T_\sigma(e_i) = e_{\sigma(i)}$.

We denote by S_{MN}^L the set of partial permutations $\sigma : I \simeq J$ as above, with range $I \subset \{1, \ldots, N\}$ and target $J \subset \{1, \ldots, M\}$, and with $L = |I| = |J|$. In analogy with the decomposition result $H_N^s = \mathbb{Z}_s \wr S_N$, we have:

Proposition 7.11. *The space of partial permutations signed by elements of $\mathbb{Z}_s$,*

$$H_{MN}^{sL} = \left\{T(e_i) = w_i e_{\sigma(i)}\Big|\sigma \in S_{MN}^L, w_i \in \mathbb{Z}_s\right\}$$

is isomorphic to the quotient space

$$(H_M^s \times H_N^s)/(H_L^s \times H_{M-L}^s \times H_{N-L}^s)$$

via a standard isomorphism.

Proof. This follows by adapting the computations in the proof of Proposition 7.3 above. Indeed, we have an action map as follows, which is transitive:

$$H_M^s \times H_N^s \to H_{MN}^{sL}$$

$$(A, B)U = AUB^*$$

Consider now the following point:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The stabilizer of this point follows to be the following group:

$$H_L^s \times H_{M-L}^s \times H_{N-L}^s$$

To be more precise, this group is embedded via:

$$(x, a, b) \to \left[\begin{pmatrix} x & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & b \end{pmatrix}\right]$$

But this gives the result. □

In the free case now, the idea is similar, by using inspiration from the construction of the quantum group $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$ in [15]. The result here is as follows:

Proposition 7.12. *The compact quantum space H_{MN}^{sL+} associated to the algebra*

$$C(H_{MN}^{sL+})=C(U_{MN}^{L+})\Big/\Big\langle u_{ij}u_{ij}^*=u_{ij}^*u_{ij}=p_{ij}=\text{projections},u_{ij}^s=p_{ij}\Big\rangle$$

has an action map, and is the target of a quotient map, as in Theorem 7.9 above.

Proof. We must show that if the variables u_{ij} satisfy the relations in the statement, then these relations are satisfied as well for the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

$$V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

We use the fact that the standard coordinates a_{ij},b_{ij} on the quantum groups H_M^{s+},H_N^{s+} satisfy the following relations, for any $x\neq y$ on the same row or column of a,b:

$$xy=xy^*=0$$

We obtain, by using these relations:

$$\begin{aligned}U_{ij}U_{ij}^* &= \sum_{klmn}u_{kl}u_{mn}^*\otimes a_{ki}a_{mi}^*\otimes b_{lj}^*b_{mj}\\ &= \sum_{kl}u_{kl}u_{kl}^*\otimes a_{ki}a_{ki}^*\otimes b_{lj}^*b_{lj}\end{aligned}$$

We have as well the following formula:

$$\begin{aligned}V_{ij}V_{ij}^* &= \sum_{r,t\leq L}a_{ri}a_{ti}^*\otimes b_{rj}^*b_{tj}\\ &= \sum_{r\leq L}a_{ri}a_{ri}^*\otimes b_{rj}^*b_{rj}\end{aligned}$$

Consider now the following projections:

$$x_{ij}=a_{ij}a_{ij}^*$$

$$y_{ij}=b_{ij}b_{ij}^*$$

$$p_{ij}=u_{ij}u_{ij}^*$$

In terms of these projections, we have:

$$U_{ij}U_{ij}^*=\sum_{kl}p_{kl}\otimes x_{ki}\otimes y_{lj}$$

$$V_{ij}V_{ij}^*=\sum_{r\leq L}x_{ri}\otimes y_{rj}$$

By repeating the computation, we conclude that these elements are projections. Also, a similar computation shows that $U_{ij}^*U_{ij},V_{ij}^*V_{ij}$ are given by the same formulae.

Finally, once again by using the relations of type $xy = xy^* = 0$, we have:

$$\begin{aligned} U_{ij}^s &= \sum_{k_r l_r} u_{k_1 l_1} \ldots u_{k_s l_s} \otimes a_{k_1 i} \ldots a_{k_s i} \otimes b_{l_1 j}^* \ldots b_{l_s j}^* \\ &= \sum_{kl} u_{kl}^s \otimes a_{ki}^s \otimes (b_{lj}^*)^s \end{aligned}$$

We have as well the following formula:

$$\begin{aligned} V_{ij}^s &= \sum_{r_l \leq L} a_{r_1 i} \ldots a_{r_s i} \otimes b_{r_1 j}^* \ldots b_{r_s j}^* \\ &= \sum_{r \leq L} a_{ri}^s \otimes (b_{rj}^*)^s \end{aligned}$$

Thus the conditions of type $u_{ij}^s = p_{ij}$ are satisfied as well, and we are done. $\square$

Let us discuss now the general case. We have the following result:

Proposition 7.13. *The various spaces G_{MN}^L constructed so far appear by imposing to the standard coordinates of U_{MN}^{L+} the relations*

$$\sum_{i_1 \ldots i_s} \sum_{j_1 \ldots j_s} \delta_\pi(i) \delta_\sigma(j) u_{i_1 j_1}^{e_1} \ldots u_{i_s j_s}^{e_s} = L^{|\pi \vee \sigma|}$$

with $s = (e_1, \ldots, e_s)$ ranging over all the colored integers, and with $\pi, \sigma \in D(0, s)$.

Proof. According to the various constructions above, the relations defining G_{MN}^L can be written as follows, with σ ranging over a family of generators, with no upper legs, of the corresponding category of partitions D:

$$\sum_{j_1 \ldots j_s} \delta_\sigma(j) u_{i_1 j_1}^{e_1} \ldots u_{i_s j_s}^{e_s} = \delta_\sigma(i)$$

We therefore obtain the relations in the statement, as follows:

$$\begin{aligned} \sum_{i_1 \ldots i_s} \sum_{j_1 \ldots j_s} \delta_\pi(i) \delta_\sigma(j) u_{i_1 j_1}^{e_1} \ldots u_{i_s j_s}^{e_s} &= \sum_{i_1 \ldots i_s} \delta_\pi(i) \sum_{j_1 \ldots j_s} \delta_\sigma(j) u_{i_1 j_1}^{e_1} \ldots u_{i_s j_s}^{e_s} \\ &= \sum_{i_1 \ldots i_s} \delta_\pi(i) \delta_\sigma(i) \\ &= L^{|\pi \vee \sigma|} \end{aligned}$$

As for the converse, this follows by using the relations in the statement, by keeping π fixed, and by making σ vary over all the partitions in the category. $\square$

In the general case now, where $G = (G_N)$ is an arbitary uniform easy quantum group, we can construct spaces G_{MN}^L by using the above relations, and we have:

Theorem 7.14. *The spaces $G_{MN}^L\subset U_{MN}^{L+}$ constructed by imposing the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

with π,σ ranging over all the partitions in the associated category, having no upper legs, are subject to an action map/quotient map diagram, as in Theorem 7.9.

Proof. We proceed as in the proof of Proposition 7.8. We must prove that, if the variables u_{ij} satisfy the relations in the statement, then so do the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

$$V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

Regarding the variables U_{ij}, the computation here goes as follows:

$$\begin{aligned}
&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)U_{i_1j_1}^{e_1}\ldots U_{i_sj_s}^{e_s}\\
=&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}\otimes\delta_\pi(i)\delta_\sigma(j)a_{k_1i_1}^{e_1}\ldots a_{k_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\
=&\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\sigma(l)u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}=L^{|\pi\vee\sigma|}
\end{aligned}$$

For the variables V_{ij} the proof is similar, as follows:

$$\begin{aligned}
&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)V_{i_1j_1}^{e_1}\ldots V_{i_sj_s}^{e_s}\\
=&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(i)\delta_\sigma(j)a_{l_1i_1}^{e_1}\ldots a_{l_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\
=&\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(l)\delta_\sigma(l)=L^{|\pi\vee\sigma|}
\end{aligned}$$

Thus we have constructed an action map, and a quotient map, as in Proposition 7.8 above, and the commutation of the diagram in Theorem 7.9 is then trivial. □

Let us discuss now the integration over G_{MN}^L. We have:

Definition 7.15. *The integration functional of G_{MN}^L is the composition*

$$\int_{G_{MN}^L}:C(G_{MN}^L)\to C(G_M\times G_N)\to\mathbb C$$

of the representation $u_{ij}\to\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$ with the Haar functional of $G_M\times G_N$.

Observe that in the case $L=M=N$ we obtain the integration over G_N. Also, at $L=M=1$, or at $L=N=1$, we obtain the integration over the sphere. In the general case now, we first have the following result:

Proposition 7.16. *The integration functional of G_{MN}^L has the invariance property*

$$\left(\int_{G_{MN}^L}\otimes id\right)\Phi(x)=\int_{G_{MN}^L}x$$

with respect to the coaction map:

$$\Phi(u_{ij})=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

Proof. We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must check the following formula:

$$\left(\int_{G_{MN}^L}\otimes id\right)\Phi(u_{i_1j_1}\ldots u_{i_sj_s})=\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}$$

Let us compute the left term. This is given by:

$$\begin{aligned}
X&=\left(\int_{G_{MN}^L}\otimes id\right)\sum_{k_xl_x}u_{k_1l_1}\ldots u_{k_sl_s}\otimes a_{k_1i_1}\ldots a_{k_si_s}\otimes b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\
&=\sum_{k_xl_x}\sum_{r_x\leq L}a_{k_1i_1}\ldots a_{k_si_s}\otimes b_{l_1j_1}^*\ldots b_{l_sj_s}^*\int_{G_M}a_{r_1k_1}\ldots a_{r_sk_s}\int_{G_N}b_{r_1l_1}^*\ldots b_{r_sl_s}^*\\
&=\sum_{r_x\leq L}\sum_{k_x}a_{k_1i_1}\ldots a_{k_si_s}\int_{G_M}a_{r_1k_1}\ldots a_{r_sk_s}\otimes\sum_{l_x}b_{l_1j_1}^*\ldots b_{l_sj_s}^*\int_{G_N}b_{r_1l_1}^*\ldots b_{r_sl_s}^*
\end{aligned}$$

By using now the invariance property of the Haar functionals of G_M,G_N, we obtain:

$$\begin{aligned}
X&=\sum_{r_x\leq L}\left(\int_{G_M}\otimes id\right)\Delta(a_{r_1i_1}\ldots a_{r_si_s})\otimes\left(\int_{G_N}\otimes id\right)\Delta(b_{r_1j_1}^*\ldots b_{r_sj_s}^*)\\
&=\sum_{r_x\leq L}\int_{G_M}a_{r_1i_1}\ldots a_{r_si_s}\int_{G_N}b_{r_1j_1}^*\ldots b_{r_sj_s}^*\\
&=\left(\int_{G_M}\otimes\int_{G_N}\right)\sum_{r_x\leq L}a_{r_1i_1}\ldots a_{r_si_s}\otimes b_{r_1j_1}^*\ldots b_{r_sj_s}^*
\end{aligned}$$

But this gives the formula in the statement, and we are done. □

We will prove now that the above functional is in fact the unique positive unital invariant trace on $C(G_{MN}^L)$. For this purpose, we will need the Weingarten formula:

Theorem 7.17. *We have the Weingarten type formula*

$$\int_{G_{MN}^L} u_{i_1j_1}\ldots u_{i_sj_s}=\sum_{\pi\sigma\tau\nu}L^{|\pi\vee\tau|}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

where the matrices on the right are given by $W_{sM}=G_{sM}^{-1}$, with $G_{sM}(\pi,\sigma)=M^{|\pi\vee\sigma|}$.

Proof. We make use of the usual quantum group Weingarten formula, for which we refer to [23], [37]. By using this formula for G_M,G_N, we obtain:

$$\begin{aligned}
&\int_{G_{MN}^L} u_{i_1j_1}\ldots u_{i_sj_s}\\
=&\sum_{l_1\ldots l_s\leq L}\int_{G_M}a_{l_1i_1}\ldots a_{l_si_s}\int_{G_N}b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\
=&\sum_{l_1\ldots l_s\leq L}\sum_{\pi\sigma}\delta_\pi(l)\delta_\sigma(i)W_{sM}(\pi,\sigma)\sum_{\tau\nu}\delta_\tau(l)\delta_\nu(j)W_{sN}(\tau,\nu)\\
=&\sum_{\pi\sigma\tau\nu}\left(\sum_{l_1\ldots l_s\leq L}\delta_\pi(l)\delta_\tau(l)\right)\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)
\end{aligned}$$

The coefficient being $L^{|\pi\vee\tau|}$, we obtain the formula in the statement. □

We can now derive an abstract characterization of the integration, as follows:

Theorem 7.18. *The integration of G_{MN}^L is the unique positive unital trace*

$$C(G_{MN}^L)\to\mathbb C$$

which is invariant under the action of the quantum group $G_M\times G_N$.

Proof. We use a standard method, from [32], [36], the point being to show that we have the following ergodicity formula:

$$\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)=\int_{G_{MN}^L}x$$

We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must verify that the following holds:

$$\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(u_{i_1j_1}\ldots u_{i_sj_s})=\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}$$

By using the Weingarten formula, the left term can be written as follows:

$$\begin{aligned}
X &= \sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s} u_{k_1l_1}\ldots u_{k_sl_s}\int_{G_M} a_{k_1i_1}\ldots a_{k_si_s}\int_{G_N} b^*_{l_1j_1}\ldots b^*_{l_sj_s}\\
&= \sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s} u_{k_1l_1}\ldots u_{k_sl_s}\sum_{\pi\sigma}\delta_\pi(k)\delta_\sigma(i)W_{sM}(\pi,\sigma)\sum_{\tau\nu}\delta_\tau(l)\delta_\nu(j)W_{sN}(\tau,\nu)\\
&= \sum_{\pi\sigma\tau\nu}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\tau(l)u_{k_1l_1}\ldots u_{k_sl_s}
\end{aligned}$$

By using now the summation formula in Theorem 7.14, we obtain:

$$X = \sum_{\pi\sigma\tau\nu} L^{|\pi\vee\tau|}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

Now by comparing with the Weingarten formula for G_{MN}^L, this proves our claim.

Assume now that $\tau : C(G_{MN}^L) \to \mathbb{C}$ satisfies the invariance condition. We have:

$$\begin{aligned}
\tau\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x) &= \left(\tau\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)(\tau\otimes id)\Phi(x)\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)(\tau(x)1)\\
&= \tau(x)
\end{aligned}$$

On the other hand, according to the formula established above, we have as well:

$$\begin{aligned}
\tau\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x) &= \tau(tr(x)1)\\
&= tr(x)
\end{aligned}$$

Thus we obtain $\tau = tr$, and this finishes the proof. □

As a main application, we have:

Proposition 7.19. *For a sum of coordinates*

$$\chi_E = \sum_{(ij)\in E} u_{ij}$$

which do not overlap on rows and columns we have

$$\int_{G_{MN}^L}\chi_E^s = \sum_{\pi\sigma\tau\nu} K^{|\pi\vee\tau|}L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

where $K = |E|$ is the cardinality of the indexing set.

Proof. With $K=|E|$, we can write $E=\{(\alpha(i),\beta(i))\}$, for certain embeddings:

$$\alpha:\{1,\ldots,K\}\subset\{1,\ldots,M\}$$

$$\beta:\{1,\ldots,K\}\subset\{1,\ldots,N\}$$

In terms of these maps α,β, the moment in the statement is given by:

$$M_s=\int_{G_{MN}^L}\left(\sum_{i\leq K}u_{\alpha(i)\beta(i)}\right)^s$$

By using the Weingarten formula, we can write this quantity as follows:

$$\begin{aligned}
&M_s\\
=&\int_{G_{MN}^L}\sum_{i_1\ldots i_s\leq K}u_{\alpha(i_1)\beta(i_1)}\ldots u_{\alpha(i_s)\beta(i_s)}\\
=&\sum_{i_1\ldots i_s\leq K}\sum_{\pi\sigma\tau\nu}L^{|\sigma\vee\nu|}\delta_\pi(\alpha(i_1),\ldots,\alpha(i_s))\delta_\tau(\beta(i_1),\ldots,\beta(i_s))W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\\
=&\sum_{\pi\sigma\tau\nu}\left(\sum_{i_1\ldots i_s\leq K}\delta_\pi(i)\delta_\tau(i)\right)L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)
\end{aligned}$$

But, as explained before, the coefficient on the left in the last formula is:

$$C=K^{|\pi\vee\tau|}$$

We therefore obtain the formula in the statement. □

We can further advance in the classical/twisted and free cases, where the Weingarten theory for the corresponding quantum groups is available from [15], [23], [23], [37]:

Theorem 7.20. *In the context of the liberation operations*

$$O_{MN}^L\to O_{MN}^{L+}$$

$$U_{MN}^L\to U_{MN}^{L+}$$

$$H_{MN}^{sL}\to H_{MN}^{sL+}$$

the laws of the sums of non-overlapping coordinates,

$$\chi_E=\sum_{(ij)\in E}u_{ij}$$

are in Bercovici-Pata bijection, in the

$$|E|=\kappa N,L=\lambda N,M=\mu N$$

regime and $N\to\infty$ limit.

Proof. We use the general theory in [15], [23], [23], [37]. According to Proposition 7.19, in terms of $K=|E|$, the moments of the variables in the statement are given by:

$$M_s=\sum_{\pi\sigma\tau\nu}K^{|\pi\vee\tau|}L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

We use now two standard facts, namely:

(1) The fact that in the $N\to\infty$ limit the Weingarten matrix W_{sN} is concentrated on the diagonal.

(2) The fact that we have an inequality as follows, with equality precisely when $\pi=\sigma$:

$$|\pi\vee\sigma|\leq\frac{|\pi|+|\sigma|}{2}$$

For details on all this, we refer to [23].

Let us discuss now what happens in the regime from the statement, namely:

$$K=\kappa N,L=\lambda N,M=\mu N,N\to\infty$$

In this regime, we obtain:

$$\begin{aligned}M_s&\simeq\sum_{\pi\tau}K^{|\pi\vee\tau|}L^{|\pi\vee\tau|}M^{-|\pi|}N^{-|\tau|}\\&\simeq\sum_{\pi}K^{|\pi|}L^{|\pi|}M^{-|\pi|}N^{-|\pi|}\\&=\sum_{\pi}\left(\frac{\kappa\lambda}{\mu}\right)^{|\pi|}\end{aligned}$$

In order to interpret this formula, we use general theory from [15], [23], [23]:

(1) For $G_N=O_N,\bar{O}_N/O_N^+$, the above variables χ_E follow to be asymptotically Gaussian/semicircular, of parameter $\frac{\kappa\lambda}{\mu}$, and hence in Bercovici-Pata bijection.

(2) For $G_N=U_N,\bar{U}_N/U_N^+$ the situation is similar, with χ_E being asymptotically complex Gaussian/circular, of parameter $\frac{\kappa\lambda}{\mu}$, and in Bercovici-Pata bijection.

(3) Finally, for $G_N=H_N^s/H_N^{s+}$, the variables χ_E are asymptotically Bessel/free Bessel of parameter $\frac{\kappa\lambda}{\mu}$, and once again in Bercovici-Pata bijection. □

The convergence in the above result is of course in moments, and we do not know whether some stronger convergence results can be formulated. Nor do we know whether one can use linear combinations of coordinates which are more general than the sums χ_E that we consider. These are interesting questions, that we would like to raise here.

Also, there are several possible extensions of the above result, for instance by using quantum reflection groups instead of unitary quantum groups, and by using twisting operations as well. We refer here to [9], and to [36] as well, for a number of supplementary results, which can be obtained by using the stronger formalism there.

8. Higher manifolds

We discuss in this section an abstract extension of the constructions of quantum algebraic manifolds that we have so far. The idea will be that of looking at certain classes of algebraic manifolds $X\subset S^{N-1}_{\mathbb{C},+}$, which are homogeneous spaces, of a special type.

Following [11], [12], let us formulate the following definition:

Definition 8.1. *An affine homogeneous space over a closed subgroup $G\subset U_N^+$ is a closed subset $X\subset S^{N-1}_{\mathbb{C},+}$, such that there exists an index set $I\subset\{1,\ldots,N\}$ such that*

$$\alpha(x_i)=\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}$$

$$\Phi(x_i)=\sum_j x_j\otimes u_{ji}$$

define morphisms of C^*-algebras, satisfying:

$$\left(id\otimes\int_G\right)\Phi=\int_G\alpha(.)1$$

Observe that $U_N^+\to S^{N-1}_{\mathbb{C},+}$ is indeed affine in this sense, with $I=\{1\}$. Also, the $1/\sqrt{|I|}$ constant appearing above is the correct one, because:

$$\begin{aligned}\sum_i\left(\sum_{j\in I}u_{ji}\right)\left(\sum_{k\in I}u_{ki}\right)^* &= \sum_i\sum_{j,k\in I}u_{ji}u_{ki}^*\\ &= \sum_{j,k\in I}(uu^*)_{jk}\\ &= |I|\end{aligned}$$

As a first general result about such spaces, we have:

Theorem 8.2. *Consider an affine homogeneous space X, as above.*
1. *The coaction condition $(\Phi\otimes id)\Phi=(id\otimes\Delta)\Phi$ is satisfied.*
2. *We have as well the formula $(\alpha\otimes id)\Phi=\Delta\alpha$.*

Proof. The coaction condition is clear. For the second formula, we first have:

$$\begin{aligned}(\alpha\otimes id)\Phi(x_i) &= \sum_k\alpha(x_k)\otimes u_{ki}\\ &= \frac{1}{\sqrt{|I|}}\sum_k\sum_{j\in I}u_{jk}\otimes u_{ki}\end{aligned}$$

On the other hand, we have as well:

$$\begin{aligned}\Delta\alpha(x_i) &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\Delta(u_{ji})\\ &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{jk}\otimes u_{ki}\end{aligned}$$

Thus, by linearity, multiplicativity and continuity, we obtain the result. $\square$

Summarizing, the terminology in Definition 8.1 is justified, in the sense that what we have there are indeed certain homogeneous spaces, of very special, "affine" type. As a second result regarding such spaces, which closes the discussion in the case where α is injective, which is something that happens in many cases, we have:

Theorem 8.3. *When α is injective we must have $X = X_{G,I}^{min}$, where:*

$$C(X_{G,I}^{min}) = \left\langle \frac{1}{\sqrt{|I|}}\sum_{j\in I} u_{ji}\Big| i = 1,\ldots,N\right\rangle \subset C(G)$$

Moreover, $X_{G,I}^{min}$ is affine homogeneous, for any $G \subset U_N^+$, and any $I \subset \{1,\ldots,N\}$.

Proof. The first assertion is clear from definitions. Regarding now the second assertion, consider the variables in the statement:

$$X_i = \frac{1}{\sqrt{|I|}}\sum_{j\in I} u_{ji} \in C(G)$$

In order to prove that we have $X_{G,I}^{min} \subset S_{\mathbb{C},+}^{N-1}$, observe first that we have:

$$\begin{aligned}\sum_i X_iX_i^* &= \frac{1}{|I|}\sum_i\sum_{j,k\in I} u_{ji}u_{ki}^*\\ &= \frac{1}{|I|}\sum_{j,k\in I}(uu^*)_{jk}\\ &= 1\end{aligned}$$

We have as well the following computation:

$$\begin{aligned}\sum_i X_i^*X_i &= \frac{1}{|I|}\sum_i\sum_{j,k\in I} u_{ji}^*u_{ki}\\ &= \frac{1}{|I|}\sum_{j,k\in I}(\bar{u}u^t)_{jk}\\ &= 1\end{aligned}$$

Thus $X_{G,I}^{min} \subset S^{N-1}_{\mathbb{C},+}$. Finally, observe that we have:

$$\begin{aligned}\Delta(X_i) &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{jk}\otimes u_{ki}\\ &= \sum_k X_k\otimes u_{ki}\end{aligned}$$

Thus we have a coaction map, given by $\Phi=\Delta$. As for the ergodicity condition, namely $(id\otimes\int_G)\Delta=\int_G(.)1$, this holds as well, by definition of the integration functional $\int_G$. $\square$

Our purpose now will be to show that the affine homogeneous spaces appear as follows, a bit in the same way as the discrete group algebras:

$$X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$$

We make the standard convention that all the tensor exponents k are "colored integers", that is, $k=e_1\ldots e_k$ with $e_i\in\{\circ,\bullet\}$, with $\circ$ corresponding to the usual variables, and with $\bullet$ corresponding to their adjoints. With this convention, we have:

Proposition 8.4. *The ergodicity condition, namely*

$$\left(id\otimes\int_G\right)\Phi=\int_G\alpha(.)1$$

is equivalent to the condition

$$(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\quad,\quad\forall k,\forall i_1,\ldots,i_k$$

where

$$P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$$

and where $(x^{\otimes k})_{i_1\ldots i_k}=x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}$.

Proof. We have the following computation:

$$\begin{aligned}\left(id\otimes\int_G\right)\Phi(x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}) &= \sum_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\ &= \sum_{j_1\ldots j_k}P_{i_1\ldots i_k,j_1\ldots j_k}(x^{\otimes k})_{j_1\ldots j_k}\\ &= (Px^{\otimes k})_{i_1\ldots i_k}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\int_G \alpha(x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}) &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\ &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\end{aligned}$$

But this gives the formula in the statement, and we are done. □

As a consequence, we have the following result:

Theorem 8.5. *We must have $X\subset X_{G,I}^{max}$, as subsets of $S^{N-1}_{\mathbb{C},+}$, where:*

$$C(X_{G,I}^{max})=C(S^{N-1}_{\mathbb{C},+})\Big/\left\langle (Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\Big|\forall k,\forall i_1,\ldots i_k\right\rangle$$

Moreover, $X_{G,I}^{max}$ is affine homogeneous, for any $G\subset U_N^+$, and any $I\subset\{1,\ldots,N\}$.

Proof. Let us first prove that we have an action $G\curvearrowright X_{G,I}^{max}$. We must show here that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $X_{G,I}^{max}$. We have:

$$\begin{aligned}(PX^{\otimes k})_{i_1\ldots i_k} &= \sum_{l_1\ldots l_k}P_{i_1\ldots i_k,l_1\ldots l_k}(X^{\otimes k})_{l_1\ldots l_k}\\ &= \sum_{l_1\ldots l_k}P_{i_1\ldots i_k,l_1\ldots l_k}\sum_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\otimes u_{j_1l_1}^{e_1}\ldots u_{j_kl_k}^{e_k}\\ &= \sum_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\otimes(u^{\otimes k}P^t)_{j_1\ldots j_k,i_1\ldots i_k}\end{aligned}$$

Since by Peter-Weyl the transpose of $P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$ is the orthogonal projection onto $Fix(u^{\otimes k})$, we have $u^{\otimes k}P^t=P^t$. We therefore obtain:

$$\begin{aligned}(PX^{\otimes k})_{i_1\ldots i_k} &= \sum_{j_1\ldots j_k}P_{i_1\ldots i_k,j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\\ &= (Px^{\otimes k})_{i_1\ldots i_k}\\ &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\end{aligned}$$

Thus we have an action $G\curvearrowright X_{G,I}^{max}$, and since this action is ergodic by Proposition 8.4, we have an affine homogeneous space, as claimed. □

We can now merge the results that we have, and we obtain:

Theorem 8.6. *Given a closed quantum subgroup $G\subset U_N^+$, and a set $I\subset\{1,\ldots,N\}$, if we consider the following C^*-subalgebra and the following quotient C^*-algebra,*

$$\begin{aligned}C(X_{G,I}^{min}) &= \left\langle\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}\Big|i=1,\ldots,N\right\rangle\subset C(G)\\ C(X_{G,I}^{max}) &= C(S^{N-1}_{\mathbb{C},+})\Big/\left\langle(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\Big|\forall k,\forall i_1,\ldots i_k\right\rangle\end{aligned}$$

then we have maps as follows,

$$G\to X_{G,I}^{min}\subset X_{G,I}^{max}\subset S^{N-1}_{\mathbb{C},+}$$

the space $G\to X_{G,I}^{max}$ is affine homogeneous, and any affine homogeneous space $G\to X$ appears as $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$.

Proof. This follows indeed from Theorem 8.3 and Theorem 8.5 above. □

We will need one more general result from [11], namely an extension of the Weingarten integration formula [23], [63], [143], to the affine homogeneous space setting:

Theorem 8.7. *Assuming that $G\to X$ is an affine homogeneous space, with index set $I\subset\{1,\ldots,N\}$, the Haar integration functional $\int_X=\int_G\alpha$ is given by*

$$\int_X x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\sum_{\pi,\sigma\in D}K_I(\pi)\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)$$

where $\{\xi_\pi|\pi\in D\}$ is a basis of $Fix(u^{\otimes k})$, $W_{kN}=G_{kN}^{-1}$ with

$$G_{kN}(\pi,\sigma)=<\xi_\pi,\xi_\sigma>$$

is the associated Weingarten matrix, and:

$$K_I(\pi)=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}(\xi_\pi)_{j_1\ldots j_k}$$

Proof. By using the Weingarten formula for the quantum group G, we have:

$$\begin{aligned}\int_X x_{i_1}^{e_1}\ldots x_{i_k}^{e_k} &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\ &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)\end{aligned}$$

But this gives the formula in the statement, and we are done. □

Let us go back now to the "minimal vs maximal" discussion, in analogy with the group algebras. Here is a natural example of an intermediate space $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$:

Theorem 8.8. *Given a closed quantum subgroup $G\subset U_N^+$, and a set $I\subset\{1,\ldots,N\}$, if we consider the following quotient algebra*

$$C(X_{G,I}^{med})=C(S^{N-1}_{\mathbb C,+})\Big/\left\langle\sum_{j_1\ldots j_k}\xi_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\xi_{j_1\ldots j_k}\Big|\forall k,\forall\xi\in Fix(u^{\otimes k})\right\rangle$$

we obtain in this way an affine homogeneous space $G\to X_{G,I}$.

Proof. We know from Theorem 8.5 above that $X_{G,I}^{max}\subset S^{N-1}_{\mathbb C,+}$ is constructed by imposing to the standard coordinates the conditions $Px^{\otimes k}=P^I$, where:

$$P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$$

$$P^I_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}$$

According to the Weingarten integration formula for G, we have:

$$\begin{aligned}(Px^{\otimes k})_{i_1\ldots i_k}&=\sum_{j_1\ldots j_k}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\\ P^I_{i_1\ldots i_k}&=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)\end{aligned}$$

Thus $X_{G,I}^{med}\subset X_{G,I}^{max}$, and the other assertions are standard as well. □

We can now put everything together, as follows:

Theorem 8.9. *Given a closed subgroup $G\subset U_N^+$, and a subset $I\subset\{1,\ldots,N\}$, the affine homogeneous spaces over G, with index set I, have the following properties:*

1. *These are exactly the intermediate subspaces $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$ on which G acts affinely, with the action being ergodic.*
2. *For the minimal and maximal spaces $X_{G,I}^{min}$ and $X_{G,I}^{max}$, as well as for the intermediate space $X_{G,I}^{med}$ constructed above, these conditions are satisfied.*
3. *By performing the GNS construction with respect to the Haar integration functional $\int_X=\int_G\alpha$ we obtain the minimal space $X_{G,I}^{min}$.*

We agree to identify all these spaces, via the GNS construction, and denote them $X_{G,I}$.

Proof. This follows indeed by combining the various results and observations formulated above. Once again, for full details on all these facts, we refer to [11]. □

Let us discuss now some basic examples of affine homogeneous spaces, namely those coming from the classical groups, and those coming from the group duals. We will need the following technical result:

Proposition 8.10. *Assuming that a closed subset $X\subset S^{N-1}_{\mathbb C,+}$ is affine homogeneous over a classical group, $G\subset U_N$, then X itself must be classical, $X\subset S^{N-1}_{\mathbb C}$.*

Proof. We use the well-known fact that, since the standard coordinates $u_{ij}\in C(G)$ commute, the corepresentation $u^{\circ\circ\bullet\bullet}=u^{\otimes 2}\otimes\bar{u}^{\otimes 2}$ has the following fixed vector:

$$\xi=\sum_{ij}e_i\otimes e_j\otimes e_i\otimes e_j$$

With $k=\circ\circ\bullet\bullet$ and with this vector ξ, the ergodicity formula reads:

$$\begin{aligned}\sum_{ij}x_ix_jx_i^*x_j^* &= \frac{1}{\sqrt{|I|^4}}\sum_{i,j\in I}1\\ &= 1\end{aligned}$$

By using this formula, along with $\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$, we obtain:

$$\begin{aligned}&\sum_{ij}(x_ix_j-x_jx_i)(x_j^*x_i^*-x_i^*x_j^*)\\ =&\sum_{ij}x_ix_jx_j^*x_i^*-x_ix_jx_i^*x_j^*-x_jx_ix_j^*x_i^*+x_jx_ix_i^*x_j^*\\ =&\ 1-1-1+1\\ =&\ 0\end{aligned}$$

We conclude that for any i,j we have:

$$[x_i,x_j]=0$$

By using now this commutation relation, plus once again the relations defining $S^{N-1}_{\mathbb C,+}$, we have as well:

$$\begin{aligned}&\sum_{ij}(x_ix_j^*-x_j^*x_i)(x_jx_i^*-x_i^*x_j)\\ =&\sum_{ij}x_ix_j^*x_jx_i^*-x_ix_j^*x_i^*x_j-x_j^*x_ix_jx_i^*+x_j^*x_ix_i^*x_j\\ =&\sum_{ij}x_ix_j^*x_jx_i^*-x_ix_i^*x_j^*x_j-x_j^*x_jx_ix_i^*+x_j^*x_ix_i^*x_j\\ =&\ 1-1-1+1\\ =&\ 0\end{aligned}$$

Thus we have $[x_i,x_j^*]=0$ as well, and so $X\subset S^{N-1}_{\mathbb C}$, as claimed. ∎

We can now formulate the result in the classical case, as follows:

Theorem 8.11. *In the classical case, $G\subset U_N$, there is only one affine homogeneous space, for each index set $I=\{1,\ldots,N\}$, namely the quotient space*

$$X=G/(G\cap C_N^I)$$

where $C_N^I\subset U_N$ is the group of unitaries fixing the following vector:

$$\xi_I=\frac{1}{\sqrt{|I|}}(\delta_{i\in I})_i$$

Proof. Consider an affine homogeneous space $G\to X$. We already know from Proposition 8.10 above that X is classical. We will first prove that we have $X=X_{G,I}^{min}$, and then we will prove that $X_{G,I}^{min}$ equals the quotient space in the statement.

(1) We use the well-known fact that the functional $E=(id\otimes\int_G)\Phi$ is the projection onto the fixed point algebra of the action, given by:

$$C(X)^\Phi=\{f\in C(X)|\Phi(f)=f\otimes 1\}$$

Thus our ergodicity condition, namely $E=\int_G\alpha(.)1$, shows that we must have:

$$C(X)^\Phi=\mathbb C1$$

Since in the classical case the condition $\Phi(f)=f\otimes 1$ reads $f(gx)=f(x)$ for any $g\in G$ and $x\in X$, we recover in this way the usual ergodicity condition, stating that whenever a function $f\in C(X)$ is constant on the orbits of the action, it must be constant.

Now observe that for an affine action, the orbits are closed. Thus an affine action which is ergodic must be transitive, and we deduce from this that we have $X=X_{G,I}^{min}$.

(2) We know that the inclusion $C(X)\subset C(G)$ comes via:

$$x_i=\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}$$

Thus, the quotient map $p:G\to X\subset S^{N-1}_\mathbb C$ is given by the following formula:

$$p(g)=\left(\frac{1}{\sqrt{|I|}}\sum_{j\in I}g_{ji}\right)_i$$

In particular, the image of the unit matrix $1\in G$ is the following vector:

$$\begin{aligned}p(1)&=\left(\frac{1}{\sqrt{|I|}}\sum_{j\in I}\delta_{ji}\right)_i\\&=\left(\frac{1}{\sqrt{|I|}}\delta_{i\in I}\right)_i\\&=\xi_I\end{aligned}$$

But this gives the result, and we are done. □

Let us discuss now the group dual case. Given a discrete group $\Gamma =< g_1,\ldots,g_N >$, we can consider the embedding $\widehat{\Gamma}\subset U_N^+$ given by $u_{ij}=\delta_{ij}g_i$. We have then:

Theorem 8.12. *In the group dual case, $G=\widehat{\Gamma}$ with $\Gamma =< g_1,\ldots,g_N >$, we have*

$$X=\widehat{\Gamma}_I$$

$$\Gamma_I =< g_i|i\in I >\subset\Gamma$$

for any affine homogeneous space X, when identifying full and reduced group algebras.

Proof. Assume indeed that we have an affine homogeneous space $G\to X$. In terms of the rescaled coordinates $h_i=\sqrt{|I|}x_i$, our axioms for α,Φ read:

$$\alpha(h_i)=\delta_{i\in I}g_i$$

$$\Phi(h_i)=h_i\otimes g_i$$

As for the ergodicity condition, this translates as follows:

$$\begin{aligned}
&\left(id\otimes\int_G\right)\Phi(h_{i_1}^{e_1}\ldots h_{i_p}^{e_p})=\int_G\alpha(h_{i_1}^{e_p}\ldots h_{i_p}^{e_p})\\
\iff\quad&\left(id\otimes\int_G\right)(h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}\otimes g_{i_1}^{e_1}\ldots g_{i_p}^{e_p})=\int_G\delta_{i_1\in I}\ldots\delta_{i_p\in I}g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}\\
\iff\quad&\delta_{g_{i_1}^{e_1}\ldots g_{i_p}^{e_p},1}h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=\delta_{g_{i_1}^{e_1}\ldots g_{i_p}^{e_p},1}\delta_{i_1\in I}\ldots\delta_{i_p\in I}\\
\iff\quad&\left[g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}=1\implies h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=\delta_{i_1\in I}\ldots\delta_{i_p\in I}\right]
\end{aligned}$$

Now observe that from $g_ig_i^*=g_i^*g_i=1$ we obtain in this way:

$$h_ih_i^*=h_i^*h_i=\delta_{i\in I}$$

Thus the elements h_i vanish for $i\notin I$, and are unitaries for $i\in I$. We conclude that we have $X=\widehat{\Lambda}$, where $\Lambda =< h_i|i\in I >$ is the group generated by these unitaries.

In order to finish the proof, our claim is that for indices $i_x\in I$ we have:

$$g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}=1\iff h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=1$$

Indeed, $\implies$ comes from the ergodicity condition, as processed above, and $\Longleftarrow$ comes from the existence of the morphism α, which is given by $\alpha(h_i)=g_i$, for $i\in I$. □

Let us go back now to the general case, and discuss a number of further axiomatization issues, based on the examples that we have. We will need the following result:

Proposition 8.13. *The closed subspace $C_N^{I+}\subset U_N^+$ defined via*

$$C(C_N^{I+})=C(U_N^+)\Big/\langle u\xi_I=\xi_I\rangle$$

where $\xi_I=\frac{1}{\sqrt{|I|}}(\delta_{i\in I})_i$, is a compact quantum group.

Proof. We must check Woronowicz's axioms, and the proof goes as follows:

(1) Let us set $U_{ij}=\sum_k u_{ik}\otimes u_{kj}$. We have then:

$$\begin{aligned}(U\xi_I)_i&=\frac{1}{\sqrt{|I|}}\sum_{j\in I}U_{ij}\\&=\frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{ik}\otimes u_{kj}\\&=\sum_k u_{ik}\otimes(u\xi_I)_k\end{aligned}$$

Since the vector ξ_I is by definition fixed by u, we obtain:

$$\begin{aligned}(U\xi_I)_i&=\sum_k u_{ik}\otimes(\xi_I)_k\\&=\frac{1}{\sqrt{|I|}}\sum_{k\in I}u_{ik}\otimes 1\\&=(u\xi_I)_i\otimes 1\\&=(\xi_I)_i\otimes 1\end{aligned}$$

Thus we can define indeed a comultiplication map, by $\Delta(u_{ij})=U_{ij}$.

(2) In order to construct the counit map, $\varepsilon(u_{ij})=\delta_{ij}$, we must prove that the identity matrix $1=(\delta_{ij})_{ij}$ satisfies $1\xi_I=\xi_I$. But this is clear.

(3) In order to construct the antipode, $S(u_{ij})=u_{ji}^*$, we must prove that the adjoint matrix $u^*=(u_{ji}^*)_{ij}$ satisfies $u^*\xi_I=\xi_I$. But this is clear from $u\xi_I=\xi_I$. $\square$

Based on the computations that we have so far, we can formulate:

Theorem 8.14. *Given a closed quantum subgroup $G\subset U_N^+$ and a set $I\subset\{1,\ldots,N\}$, we have a quotient map and an inclusion map as follows:*

$$G/(G\cap C_N^{I+})\to X_{G,I}^{min}\subset X_{G,I}^{max}$$

These maps are both isomorphisms in the classical case. In general, they are both proper.

Proof. Consider the quantum group $H=G\cap C_N^{I+}$, which is by definition such that at the level of the corresponding algebras, we have:

$$C(H)=C(G)\Big/\left<u\xi_I=\xi_I\right>$$

In order to construct a quotient map $G/H\to X_{G,I}^{min}$, we must check that the defining relations for $C(G/H)$ hold for the standard generators $x_i\in C(X_{G,I}^{min})$. But if we denote by $\rho : C(G) \to C(H)$ the quotient map, then we have, as desired:

$$\begin{aligned}(id \otimes \rho)\Delta x_i &= (id \otimes \rho)\left(\frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{ik} \otimes u_{kj}\right)\\ &= \sum_k u_{ik} \otimes (\xi_I)_k\\ &= x_i \otimes 1\end{aligned}$$

In the classical case, Theorem 8.11 shows that both the maps in the statement are isomorphisms. For the group duals, however, these maps are not isomorphisms, in general. This follows indeed from Theorem 8.12, and from the general theory in [36]. □

We discuss now a number of further examples. We will need:

Proposition 8.15. *Given a compact matrix quantum group $G = (G, u)$, the pair*

$$G^t = (G, u^t)$$

where $(u^t)_{ij} = u_{ji}$, is a compact matrix quantum group as well.

Proof. The construction of the comultiplication is as follows, where Σ is the flip map:

$$\begin{aligned}&\Delta^t[(u^t)_{ij}] = \sum_k (u^t)_{ik} \otimes (u^t)_{kj}\\ \iff\quad &\Delta^t(u_{ji}) = \sum_k u_{ki} \otimes u_{jk}\\ \iff\quad &\Delta^t = \Sigma\Delta\end{aligned}$$

As for the corresponding counit and antipode, these can be simply taken to be (ε, S), and the axioms are satisfied. □

We will need as well the following result, which is standard as well:

Proposition 8.16. *Given two closed subgroups $G \subset U_N^+$ and $H \subset U_M^+$, with fundamental corepresentations denoted $u = (u_{ij})$ and $v = (v_{ab})$, their product is a closed subgroup*

$$G \times H \subset U_{NM}^+$$

with fundamental corepresentation $w_{ia,jb} = u_{ij} \otimes v_{ab}$.

Proof. Our claim is that the corresponding structural maps are:

$$\Delta(\alpha \otimes \beta) = \Delta(\alpha)_{13}\Delta(\beta)_{24}$$

$$\varepsilon(\alpha \otimes \beta) = \varepsilon(\alpha)\varepsilon(\beta)$$

$$S(\alpha \otimes \beta) = S(\alpha)S(\beta)$$

The verification for the comultiplication is as follows:

$$\begin{aligned}\Delta(w_{ia,jb}) &= \Delta(u_{ij})_{13}\Delta(v_{ab})_{24}\\ &= \sum_{kc} u_{ik}\otimes v_{ac}\otimes u_{kj}\otimes v_{cb}\\ &= \sum_{kc} w_{ia,kc}\otimes w_{kc,jb}\end{aligned}$$

For the counit, we have:

$$\begin{aligned}\varepsilon(w_{ia,jb}) &= \varepsilon(u_{ij})\varepsilon(v_{ab})\\ &= \delta_{ij}\delta_{ab}\\ &= \delta_{ia,jb}\end{aligned}$$

For the antipode, we have:

$$\begin{aligned}S(w_{ia,jb}) &= S(u_{ij})S(v_{ab})\\ &= v_{ba}^*u_{ji}^*\\ &= (u_{ji}v_{ba})^*\\ &= w_{jb,ia}^*\end{aligned}$$

We refer to Wang's paper [140] for more details regarding this construction. □

Let us call a closed quantum subgroup $G\subset U_N^+$ self-transpose when we have an automorphism $T:C(G)\to C(G)$ given by $T(u_{ij})=u_{ji}$. Observe that in the classical case, this amounts in $G\subset U_N$ to be closed under the transposition operation $g\to g^t$.

With these notions in hand, let us go back to the affine homogeneous spaces. As a first result here, any closed subgroup $G\subset U_N^+$ appears as an affine homogeneous space over an appropriate quantum group, as follows:

Theorem 8.17. *Given a closed subgroup $G\subset U_N^+$, we have an identification $X_{\mathcal{G},I}^{min}\simeq G$, given at the level of standard coordinates by $x_{ij}=\frac{1}{\sqrt{N}}u_{ij}$, where:*

(1) *$\mathcal{G}=G^t\times G\subset U_{N^2}^+$, with coordinates $w_{ia,jb}=u_{ji}\otimes u_{ab}$.*
(2) *$I\subset\{1,\ldots,N\}^2$ is the diagonal set, $I=\{(k,k)|k=1,\ldots,N\}$.*

In the self-transpose case we can choose as well $\mathcal{G}=G\times G$, with $w_{ia,jb}=u_{ij}\otimes u_{ab}$.

Proof. As a first observation, our closed subgroup $G\subset U_N^+$ appears as an algebraic submanifold of the free complex sphere on N^2 variables, as follows:

$$G\subset S^{N^2-1}_{\mathbb{C},+}$$

$$x_{ij}=\frac{1}{\sqrt{N}}u_{ij}$$

Let us construct now the affine homogeneous space structure. Our claim is that, with $\mathcal{G} = G^t \times G$ and $I = \{(k,k)\}$ as in the statement, the structural maps are:

$$\alpha = \Delta$$

$$\Phi = (\Sigma \otimes id)\Delta^{(2)}$$

Indeed, in what regards $\alpha = \Delta$, this is given by the following formula:

$$\begin{aligned} \alpha(u_{ij}) &= \sum_k u_{ik} \otimes u_{kj} \\ &= \sum_k w_{kk,ij} \end{aligned}$$

Thus, by dividing by $\sqrt{N}$, we obtain the usual affine homogeneous space formula:

$$\alpha(x_{ij}) = \frac{1}{\sqrt{|I|}} \sum_k w_{kk,ij}$$

Regarding now $\Phi = (\Sigma \otimes id)\Delta^{(2)}$, the formula here is as follows:

$$\begin{aligned} \Phi(u_{ij}) &= (\Sigma \otimes id) \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj} \\ &= \sum_{kl} u_{kl} \otimes u_{ik} \otimes u_{lj} \\ &= \sum_{kl} u_{kl} \otimes w_{kl,ij} \end{aligned}$$

Thus, by dividing by $\sqrt{N}$, we obtain the usual affine homogeneous space formula:

$$\Phi(x_{ij}) = \sum_{kl} x_{kl} \otimes w_{kl,ij}$$

The ergodicity condition being clear as well, this gives the first assertion.

Regarding now the second assertion, assume that we are in the self-transpose case, and so that we have an automorphism $T : C(G) \to C(G)$ given by $T(u_{ij}) = u_{ji}$.

With $w_{ia,jb} = u_{ij} \otimes u_{ab}$, the modified map $\alpha = (T \otimes id)\Delta$ is then given by:

$$\begin{aligned} \alpha(u_{ij}) &= (T \otimes id) \sum_k u_{ik} \otimes u_{kj} \\ &= \sum_k u_{ki} \otimes u_{kj} \\ &= \sum_k w_{kk,ij} \end{aligned}$$

As for the modified map $\Phi=(id\otimes T\otimes id)(\Sigma\otimes id)\Delta^{(2)}$, this is given by:

$$\begin{aligned}\Phi(u_{ij}) &= (id\otimes T\otimes id)\sum_{kl}u_{kl}\otimes u_{ik}\otimes u_{lj}\\ &= \sum_{kl}u_{kl}\otimes u_{ki}\otimes u_{lj}\\ &= \sum_{kl}u_{kl}\otimes w_{kl,ij}\end{aligned}$$

Thus we have the correct affine homogeneous space formulae, and once again the ergodicity condition being clear as well, this gives the result. □

Let us discuss now the generalization of the above result, to the context of the spaces introduced in [36]. We recall from there that we have the following construction:

Definition 8.18. *Given a closed subgroup $G\subset U_N^+$ and an integer $M\leq N$ we set*

$$C(G_{MN})=\left\langle u_{ij}\Big|i\in\{1,\ldots,M\},j\in\{1,\ldots,N\}\right\rangle\subset C(G)$$

and we call row space of G the underlying quotient space $G\to G_{MN}$.

As a basic example here, at $M=N$ we obtain G itself. Also, at $M=1$ we obtain the space whose coordinates are those on the first row of coordinates on G. See [36].

Given $G_N\subset U_N^+$ and an integer $M\leq N$, we can consider the quantum group $G_M=G_N\cap U_M^+$, with the intersection taken inside U_N^+, and with $U_M^+\subset U_N^+$ given by:

$$u=diag(v,1_{N-M})$$

Observe that we have a quotient map $C(G_N)\to C(G_M)$, given by $u_{ij}\to v_{ij}$.

We have the following extension of Theorem 8.17:

Theorem 8.19. *Given a closed subgroup $G_N\subset U_N^+$, we have an identification $X_{\mathcal{G},I}^{min}\simeq G_{MN}$, given at the level of standard coordinates by $x_{ij}=\frac{1}{\sqrt{M}}u_{ij}$, where:*

1. *$\mathcal{G}=G_M^t\times G_N\subset U_{NM}^+$, where $G_M=G_N\cap U_M^+$, with coordinates $w_{ia,jb}=u_{ji}\otimes v_{ab}$.*
2. *$I\subset\{1,\ldots,M\}\times\{1,\ldots,N\}$ is the diagonal set, $I=\{(k,k)|k=1,\ldots,M\}$.*

In the self-transpose case we can choose as well $\mathcal{G}=G_M\times G_N$, with $w_{ia,jb}=u_{ij}\otimes v_{ab}$.

Proof. We will prove that the space $X=G_{MN}$, with coordinates $x_{ij}=\frac{1}{\sqrt{M}}u_{ij}$, coincides with the space $X_{\mathcal{G},I}^{min}$ constructed in the statement, with its standard coordinates.

For this purpose, consider the following composition of morphisms, where in the middle we have the comultiplication, and at left and right we have the canonical maps:

$$C(X)\subset C(G_N)\to C(G_N)\otimes C(G_N)\to C(G_M)\otimes C(G_N)$$

The standard coordinates are then mapped as follows:

$$\begin{aligned} x_{ij} &= \frac{1}{\sqrt{M}} u_{ij} \\ &\to \frac{1}{\sqrt{M}} \sum_k u_{ik} \otimes u_{kj} \\ &\to \frac{1}{\sqrt{M}} \sum_{k \leq M} u_{ik} \otimes v_{kj} \\ &= \frac{1}{\sqrt{M}} \sum_{k \leq M} w_{kk,ij} \end{aligned}$$

Thus we obtain the standard coordinates on the space $X_{\mathcal{G},I}^{min}$, as claimed. Finally, the last assertion is standard as well, by suitably modifying the above morphism. □

Let us discuss now the liberation operation, in the context of the affine homogeneous spaces, and probabilistic aspects. In the easy case, we have the following result:

Proposition 8.20. *When $G \subset U_N^+$ is easy, coming from a category of partitions D, the space $X_{G,I} \subset S^{N-1}_{\mathbb{C},+}$ appears by imposing the relations*

$$\sum_{i_1 \ldots i_k} \delta_\pi(i_1 \ldots i_k) x_{i_1}^{e_1} \ldots x_{i_k}^{e_k} = |I|^{|\pi| - k/2}, \quad \forall k, \forall \pi \in D(k)$$

where $D(k) = D(0,k)$, and where $|.|$ denotes the number of blocks.

Proof. We know by easiness that $Fix(u^{\otimes k})$ is spanned by the vectors $\xi_\pi = T_\pi$, with $\pi \in D(k)$. But these latter vectors are given by:

$$\xi_\pi = \sum_{i_1 \ldots i_k} \delta_\pi(i_1 \ldots i_k) e_{i_1} \otimes \ldots \otimes e_{i_k}$$

We deduce that $X_{G,I} \subset S^{N-1}_{\mathbb{C},+}$ appears by imposing the following relations:

$$\sum_{i_1 \ldots i_k} \delta_\pi(i_1 \ldots i_k) x_{i_1}^{e_1} \ldots x_{i_k}^{e_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \ldots j_k \in I} \delta_\pi(j_1 \ldots j_k), \quad \forall k, \forall \pi \in D(k)$$

Now since the sum on the right equals $|I|^{|\pi|}$, this gives the result. □

More generally now, in view of the examples given above, making the link with [36], it is interesting to work out what happens when G is a product of easy quantum groups, and the index set I appears as $I = \{(c, \ldots, c) | c \in J\}$, for a certain set J.

The result here, in its most general form, is as follows:

Theorem 8.21. *For a product of easy quantum groups, $G = G_{N_1}^{(1)} \times \ldots \times G_{N_s}^{(s)}$, and with $I = \{(c, \ldots, c) | c \in J\}$, the space $X_{G,I} \subset S^{N-1}_{\mathbb{C},+}$ appears by imposing the relations*

$$\sum_{i_1 \ldots i_k} \delta_\pi(i_1 \ldots i_k) x_{i_1}^{e_1} \ldots x_{i_k}^{e_k} = |J|^{|\pi_1 \vee \ldots \vee \pi_s| - k/2}, \quad \forall k, \forall \pi \in D^{(1)}(k) \times \ldots \times D^{(s)}(k)$$

where $D^{(r)} \subset P$ is the category of partitions associated to $G_{N_r}^{(r)} \subset U_{N_r}^+$, and where the partition

$$\pi_1 \vee \ldots \vee \pi_s \in P(k)$$

is the one obtained by superposing $\pi_1, \ldots, \pi_s$.

Proof. Since we are in a direct product situation, $G = G_{N_1}^{(1)} \times \ldots \times G_{N_s}^{(s)}$, the general theory in [140] applies, and shows that a basis for $Fix(u^{\otimes k})$ is provided by the vectors $\rho_\pi = \xi_{\pi_1} \otimes \ldots \otimes \xi_{\pi_s}$ associated to the following partitions:

$$\pi = (\pi_1, \ldots, \pi_s) \in D^{(1)}(k) \times \ldots \times D^{(s)}(k)$$

We conclude that the space $X_{G,I} \subset S^{N-1}_{\mathbb{C},+}$ appears by imposing the following relations to the standard coordinates:

$$\sum_{i_1 \ldots i_k} \delta_\pi(i_1 \ldots i_k) x_{i_1}^{e_1} \ldots x_{i_k}^{e_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \ldots j_k \in I} \delta_\pi(j_1 \ldots j_k), \ \forall k, \forall \pi \in D^{(1)}(k) \times \ldots \times D^{(s)}(k)$$

Since the conditions $j_1, \ldots, j_k \in I$ read $j_1 = (l_1, \ldots, l_1), \ldots, j_k = (l_k, \ldots, l_k)$, for certain elements $l_1, \ldots l_k \in J$, the sums on the right are given by:

$$\begin{aligned}
\sum_{j_1 \ldots j_k \in I} \delta_\pi(j_1 \ldots j_k) &= \sum_{l_1 \ldots l_k \in J} \delta_\pi(l_1, \ldots, l_1, \ldots\ldots, l_k, \ldots, l_k) \\
&= \sum_{l_1 \ldots l_k \in J} \delta_{\pi_1}(l_1 \ldots l_k) \ldots \delta_{\pi_s}(l_1 \ldots l_k) \\
&= \sum_{l_1 \ldots l_k \in J} \delta_{\pi_1 \vee \ldots \vee \pi_s}(l_1 \ldots l_k)
\end{aligned}$$

Now since the sum on the right equals $|J|^{|\pi_1 \vee \ldots \vee \pi_s|}$, this gives the result. □

Finally, let us discuss probabilistic aspects. Following [11], we first have:

Proposition 8.22. *The moments of the variable $\chi_T = \sum_{i \leq T} x_{i \ldots i}$ are given by*

$$\int_X \chi_T^k \simeq \frac{1}{\sqrt{M^k}} \sum_{\pi \in D^{(1)}(k) \cap \ldots \cap D^{(s)}(k)} \left(\frac{TM}{N} \right)^{|\pi|}$$

in the $N_i \to \infty$ limit, $\forall i$, where $M = |I|$, and $N = N_1 \ldots N_s$.

Proof. We have the following formula:

$$\pi(x_{i_1\ldots i_s})=\frac{1}{\sqrt{M}}\sum_{c\in J}u_{i_1c}\otimes\ldots\otimes u_{i_sc}$$

For the variable in the statement, we therefore obtain:

$$\pi(\chi_T)=\frac{1}{\sqrt{M}}\sum_{i\leq T}\sum_{c\in J}u_{ic}\otimes\ldots\otimes u_{ic}$$

Now by raising to the power k and integrating, we obtain:

$$\begin{aligned}\int_X\chi_T^k &= \frac{1}{\sqrt{M^k}}\sum_{i_1\ldots i_k\leq T}\sum_{c_1\ldots c_k\in J}\int_{G^{(1)}}u_{i_1c_1}\ldots u_{i_kc_k}\ldots\ldots\int_{G^{(s)}}u_{i_1c_1}\ldots u_{i_kc_k}\\
&= \frac{1}{\sqrt{M^k}}\sum_{ic}\sum_{\pi\sigma}\delta_{\pi_1}(i)\delta_{\sigma_1}(c)W_{kN_1}^{(1)}(\pi_1,\sigma_1)\ldots\delta_{\pi_s}(i)\delta_{\sigma_s}(c)W_{kN_s}^{(s)}(\pi_s,\sigma_s)\\
&= \frac{1}{\sqrt{M^k}}\sum_{\pi\sigma}T^{|\pi_1\vee\ldots\vee\pi_s|}M^{|\sigma_1\vee\ldots\vee\sigma_s|}W_{kN_1}^{(1)}(\pi_1,\sigma_1)\ldots W_{kN_s}^{(s)}(\pi_s,\sigma_s)\end{aligned}$$

We use now the standard fact that the Weingarten functions are concentrated on the diagonal. Thus in the limit we must have $\pi_i=\sigma_i$ for any i, and we obtain:

$$\begin{aligned}\int_X\chi_T^k &\simeq \frac{1}{\sqrt{M^k}}\sum_{\pi}T^{|\pi_1\vee\ldots\vee\pi_s|}M^{|\pi_1\vee\ldots\vee\pi_s|}N_1^{-|\pi_1|}\ldots N_s^{-|\pi_s|}\\
&\simeq \frac{1}{\sqrt{M^k}}\sum_{\pi\in D^{(1)}\cap\ldots\cap D^{(s)}}T^{|\pi|}M^{|\pi|}(N_1\ldots N_s)^{-|\pi|}\\
&= \frac{1}{\sqrt{M^k}}\sum_{\pi\in D^{(1)}\cap\ldots\cap D^{(s)}}\left(\frac{TM}{N}\right)^{|\pi|}\end{aligned}$$

But this gives the formula in the statement, and we are done. □

As a consequence, we have the following result:

Theorem 8.23. *In the context of a liberation operation for quantum groups, $G^{(i)}\to G^{(i)+}$, the laws of the variables $\sqrt{M}\chi_T$ are in Bercovici-Pata bijection, in the $N_i\to\infty$ limit.*

Proof. Assume indeed that we have easy quantum groups $G^{(1)},\ldots,G^{(s)}$, with free versions $G^{(1)+},\ldots,G^{(s)+}$. At the level of the categories of partitions, we have:

$$\bigcap_i\left(D^{(i)}\cap NC\right)=\left(\bigcap_i D^{(i)}\right)\cap NC$$

Since the intersection of Hom-spaces is the Hom-space for the generated quantum group, we deduce that at the quantum group level, we have:

$$< G^{(1)+},\ldots,G^{(s)+} >=< G^{(1)},\ldots,G^{(s)} >^+$$

Thus the result follows from Proposition 8.22, and from the Bercovici-Pata bijection result for truncated characters for this latter liberation operation [37], [127]. $\square$

9. Half-liberation

We have seen in section 4 that the quadruplets of type (S,T,U,K) can be axiomatized, and that at the level of basic examples we have 4 such quadruplets, corresponding to the usual real and complex geometries $\mathbb{R}^N,\mathbb{C}^N$, and to the free versions of these:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

Our purpose in what follows will be that of extending the above diagram, with the construction of some supplementary examples. There are two methods here:

(1) Look for intermediate geometries $\mathbb{R}^N\subset X\subset\mathbb{R}^N_+$, and their complex analogues.

(2) Look for intermediate geometries $\mathbb{R}^N\subset X\subset\mathbb{C}^N$, and their free analogues.

We will see that, in each case, there is a "standard" solution, and that these solutions can be combined. Thus, we will end up with a total of $3\times 3=9$ solutions, as follows:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{T}\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{C}^N_* \arrow[u] \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

We will see afterwards, in section 10 below, that under certain strong axioms, of combinatorial type, these 9 geometries are conjecturally the only ones.

Let us focus on the first question to be solved, namely finding the intermediate geometries $\mathbb{R}^N\subset X\subset\mathbb{R}^N_+$. Since such a geometry is given by a quadruplet (S,T,U,K), we are led to 4 different intermediate object questions, as follows:

$$S^{N-1}_{\mathbb{R}}\subset S\subset S^{N-1}_{\mathbb{R},+}$$
$$T_N\subset T\subset T_N^+$$
$$O_N\subset U\subset O_N^+$$
$$H_N\subset K\subset H_N^+$$

At the sphere and torus level, there are obviously uncountably many solutions, without supplementary assumptions, and it is hard to get beyond this, with bare hands. Thus, our hopes will basically come from the unitary and reflection quantum groups, where things are more rigid than for spheres and tori. Let us record, however, the following fact regarding the spheres, from [33], which will appear to be relevant, later on:

Theorem 9.1. *The algebraic manifold $S^{(k)}\subset S^{N-1}_{\mathbb R,+}$ obtained by imposing the relations $a_1\ldots a_k=a_k\ldots a_1$ to the standard coordinates of $S^{N-1}_{\mathbb R,+}$ is as follows:*

1. *At $k=1$ we have $S^{(k)}=S^{N-1}_{\mathbb R,+}$.*
2. *At $k=2,4,6,\ldots$ we have $S^{(k)}=S^{N-1}_{\mathbb R}$.*
3. *At $k=3,5,7,\ldots$ we have $S^{(k)}=S^{(3)}$.*

Proof. As a first observation, the commutation relations $ab=ba$ imply the following relations, for any $k\geq 2$:

$$a_1\ldots a_k=a_k\ldots a_1$$

Thus, for any $k\geq 2$, we have an inclusion $S^{(2)}\subset S^{(k)}$. It is also elementary to check that the relations $abc=cba$ imply the following relations, for any $k\geq 3$ odd:

$$a_1\ldots a_k=a_k\ldots a_1$$

Thus, for any $k\geq 3$ odd, we have an inclusion $S^{(3)}\subset S^{(k)}$. Our claim now is that we have $S^{(k+2)}\subset S^{(k)}$, for any $k\geq 2$. In order to prove this, we must show that the relations $a_1\ldots a_{k+2}=a_{k+2}\ldots a_1$ between $x_1,\ldots,x_N$ imply the relations $a_1\ldots a_k=a_k\ldots a_1$ between $x_1,\ldots,x_N$. But this holds indeed, because:

$$\begin{aligned}x_{i_1}\ldots x_{i_{k+2}}=x_{i_{k+2}}\ldots x_{i_1}\quad&\implies\quad x_{i_1}\ldots x_{i_k}x_j^2=x_j^2x_{i_k}\ldots x_{i_1}\\&\implies\quad\sum_j x_{i_1}\ldots x_{i_k}x_j^2=\sum_j x_j^2x_{i_k}\ldots x_{i_1}\\&\implies\quad x_{i_1}\ldots x_{i_k}=x_{i_k}\ldots x_{i_1}\end{aligned}$$

Summing up, we have proved that we have inclusions as follows:

$$S^{(2)}\subset\ldots\ldots\subset S^{(6)}\subset S^{(4)}\subset S^{(2)}$$

$$S^{(3)}\subset\ldots\ldots\subset S^{(7)}\subset S^{(5)}\subset S^{(3)}$$

Thus, we are led to the conclusions in the statement. ∎

Let us focus now on the quantum groups. We will see that there is a lot more rigidity here, which makes things simpler. At the quantum group level, our goal will be that of finding the intermediate objects as follows:

$$O_N\subset U\subset O_N^+$$

$$H_N\subset K\subset H_N^+$$

Quite surprisingly, these two questions are of quite different nature. Indeed, regarding $O_N\subset U\subset O_N^+$, there is a solution here, denoted O_N^*, coming via the relations $abc=cba$, and conjecturally nothing more. Regarding $H_N\subset K\subset H_N^+$, here it is possible to use for instance crossed products, in order to construct uncountably many solutions.

In short, in connection with our intermediate geometry question, we do have in principle our solution, coming via the relations $abc = cba$, and this is compatible with our above $S^{(3)}$ guess for the spheres. In order to get started, let us recall that we have:

Theorem 9.2. *The basic quantum unitary and reflection groups, namely*

$$\begin{tikzcd} & K_N^+ \arrow[rr] & & U_N^+ \\ H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\ & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\ H_N \arrow[rr] \arrow[ur] \arrow[uu] & & O_N \arrow[ur] \arrow[uu] & \end{tikzcd}$$

are all easy, coming from certain categories of partitions.

Proof. This is something that we already discussed, in section 2 above, the corresponding categories of partitions being as follows:

$$\begin{tikzcd} & \mathcal{NC}_{even} \arrow[dl] \arrow[dd] & & \mathcal{NC}_2 \arrow[ll] \arrow[dl] \arrow[dd] \\ NC_{even} \arrow[dd] & & NC_2 \arrow[ll] \arrow[dd] & \\ & \mathcal{P}_{even} \arrow[dl] & & \mathcal{P}_2 \arrow[ll] \arrow[dl] \\ P_{even} & & P_2 \arrow[ll] & \end{tikzcd}$$

Thus, we are led to the conclusion in the statement. □

Getting back now to the half-liberation question, let us start by constructing the solutions. The result here, which is well-known as well, is as follows:

Theorem 9.3. *We have quantum groups as follows, obtained via the "half-commutation" relations $abc = cba$, which fit into the diagram of basic quantum groups:*

$$\begin{tikzcd} K_N^* \arrow[r] & U_N^* \\ H_N^* \arrow[r] \arrow[u] & O_N^* \arrow[u] \end{tikzcd}$$

These quantum groups are all easy, and the corresponding categories of partitions fit into the diagram of categories of partitions for the basic quantum groups.

Proof. This is standard, from [37], [39], the idea being that the half-commutation relations $abc = cba$ come from the operator T_* associated to the half-liberating partition:

$$* \in P(3,3)$$

Thus, the quantum groups in the statement are indeed easy, obtained by adding $*$ to the corresponding categories of noncrossing partitions. We obtain the following categories, with $*$ standing for the fact that, when relabelling clockwise the legs $\circ\bullet\circ\bullet\ldots$, the formula $\#\circ = \#\bullet$ must hold in each block:

$$\begin{tikzcd}
\mathcal{P}^*_{even} \arrow[d] & \mathcal{P}^*_2 \arrow[l] \arrow[d] \\
P^*_{even} & P^*_2 \arrow[l]
\end{tikzcd}$$

Finally, the fact that our new quantum groups and categories fit well into the previous diagrams of quantum groups and categories is clear from this. See [14]. □

The point now is that we have the following result, from [39]:

Theorem 9.4. *There is only one proper intermediate easy quantum group*

$$O_N \subset G \subset O_N^+$$

namely the half-classical orthogonal group O_N^*.

Proof. According to our definition for the easy quantum groups, we must compute here the intermediate categories of pairings, as follows:

$$NC_2 \subset D \subset P_2$$

But this can be done via some standard combinatorics, in three steps, as follows:

(1) Let $\pi \in P_2 - NC_2$, having $s \geq 4$ strings. Our claim is that:

– If $\pi \in P_2 - P_2^*$, there exists a semicircle capping $\pi' \in P_2 - P_2^*$.

– If $\pi \in P_2^* - NC_2$, there exists a semicircle capping $\pi' \in P_2^* - NC_2$.

Indeed, both these assertions can be easily proved, by drawing pictures.

(2) Consider now a partition $\pi \in P_2(k,l) - NC_2(k,l)$. Our claim is that:

– If $\pi \in P_2(k,l) - P_2^*(k,l)$ then $< \pi >= P_2$.

– If $\pi \in P_2^*(k,l) - NC_2(k,l)$ then $< \pi >= P_2^*$.

This can be indeed proved by recurrence on the number of strings, $s = (k+l)/2$, by using (1), which provides us with a descent procedure $s \to s-1$, at any $s \geq 4$.

(3) Finally, assume that we are given an easy quantum group $O_N \subset G \subset O_N^+$, coming from certain sets of pairings $D(k,l) \subset P_2(k,l)$. We have three cases:

– If $D \not\subset P_2^*$, we obtain $G = O_N$.

– If $D \subset P_2, D \not\subset NC_2$, we obtain $G = O_N^*$.

– If $D \subset NC_2$, we obtain $G = O_N^+$.

Thus, we are led to the conclusion in the statement. □

It is actually believed that the above result could still hold, without the easiness assumption. We refer here to [21]. Thus, under a certain natural "easiness" assumption, and perhaps even in general, we can only have an intermediate geometry between classical real and free real, namely half-classical real. In practice now, what we have to do is to construct this geometry, and its complex analogue as well, and check the axioms from section 4. Let us begin by constructing the corresponding quadruplets. We have:

Proposition 9.5. *We have a quadruplet as follows, called half-classical real,*

$$\begin{tikzcd}
S^{N-1}_{\mathbb{R},*} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & T_N^* \arrow[d, dash] \arrow[dl, dash] \\
O_N^* \arrow[r, dash] & H_N^*
\end{tikzcd}$$

and a quadruplet as follows, called half-classical complex,

$$\begin{tikzcd}
S^{N-1}_{\mathbb{C},*} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbb{T}_N^* \arrow[d, dash] \arrow[dl, dash] \\
U_N^* \arrow[r, dash] & K_N^*
\end{tikzcd}$$

obtained by imposing to the standard coordinates the relations $abc = cba$.

Proof. This is more or less an empty statement, with the quantum groups appearing in the above diagrams being those constructed above, and with the corresponding spheres and tori being constructed in a similar way, by imposing the half-commutation relations $abc = cba$ to the standard coordinates, and their adjoints. □

In order to check now our noncommutative geometry axioms, we are in need of a better understanding of the half-liberation operation, via some supplementary results. Let us start with the following simple observation, regarding the real spheres:

Proposition 9.6. *We have a morphism of C^*-algebras as follows,*

$$C(S^{N-1}_{\mathbb{R},*})\to M_2(C(S^{N-1}_{\mathbb{C}}))\quad,\quad x_i\to\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}$$

where z_i are the standard coordinates of $S^{N-1}_{\mathbb{C}}$.

Proof. We have to prove that the matrices X_i on the right satisfy the defining relations for $S^{N-1}_{\mathbb{R},*}$. But these matrices are self-adjoint, and we have:

$$\sum_i X_i^2=\sum_i\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}^2=\sum_i\begin{pmatrix}|z_i|^2&0\\ 0&|z_i|^2\end{pmatrix}=\begin{pmatrix}1&0\\ 0&1\end{pmatrix}$$

As for the half-commutation relations, these follow from the following formula:

$$X_iX_jX_k=\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}\begin{pmatrix}0&z_j\\ \bar{z}_j&0\end{pmatrix}\begin{pmatrix}0&z_k\\ \bar{z}_k&0\end{pmatrix}=\begin{pmatrix}0&z_i\bar{z}_jz_k\\ \bar{z}_iz_j\bar{z}_k&0\end{pmatrix}$$

Indeed, the quantities on the right being symmetric in i,k, this gives the result. □

Regarding the complex spheres, the result here is similar, as follows:

Proposition 9.7. *We have a morphism of C^*-algebras as follows,*

$$C(S^{N-1}_{\mathbb{C},*})\to M_2(C(S^{N-1}_{\mathbb{C}}\times S^{N-1}_{\mathbb{C}}))\quad,\quad x_i\to\begin{pmatrix}0&z_i\\ y_i&0\end{pmatrix}$$

where y_i,z_i are the standard coordinates of $S^{N-1}_{\mathbb{C}}\times S^{N-1}_{\mathbb{C}}$.

Proof. We have to prove that the matrices X_i on the right satisfy the defining relations for $S^{N-1}_{\mathbb{C},*}$. We have the following computation:

$$\sum_i X_iX_i^*=\sum_i\begin{pmatrix}0&z_i\\ y_i&0\end{pmatrix}\begin{pmatrix}0&\bar{y}_i\\ \bar{z}_i&0\end{pmatrix}=\sum_i\begin{pmatrix}|z_i|^2&0\\ 0&|y_i|^2\end{pmatrix}=\begin{pmatrix}1&0\\ 0&1\end{pmatrix}$$

We have as well the following computation:

$$\sum_i X_i^*X_i=\sum_i\begin{pmatrix}0&\bar{y}_i\\ \bar{z}_i&0\end{pmatrix}\begin{pmatrix}0&z_i\\ y_i&0\end{pmatrix}=\sum_i\begin{pmatrix}|y_i|^2&0\\ 0&|z_i|^2\end{pmatrix}=\begin{pmatrix}1&0\\ 0&1\end{pmatrix}$$

As for the half-commutation relations, these follow from the following formula:

$$X_iX_jX_k=\begin{pmatrix}0&z_i\\ y_i&0\end{pmatrix}\begin{pmatrix}0&z_j\\ y_j&0\end{pmatrix}\begin{pmatrix}0&z_k\\ y_k&0\end{pmatrix}=\begin{pmatrix}0&z_iy_jz_k\\ y_iz_jy_k&0\end{pmatrix}$$

Indeed, the quantities on the right being symmetric in i,k, this gives the result. □

Our goal now will be that of proving that the morphisms constructed above are faithful, up to the usual equivalence relation for the quantum algebraic manifolds. For this purpose, we will use some projective geometry arguments, the idea being that of proving that the above morphisms are indeed isomorphisms, at the projective version level, and then lifting these isomorphism results, to the affine setting.

We recall that $P^{N-1}_\mathbb{R}$ is the space of lines in $\mathbb{R}^N$ passing through the origin. We have a quotient map $S^{N-1}_\mathbb{R} \to P^{N-1}_\mathbb{R}$, which produces an embedding $C(P^{N-1}_\mathbb{R}) \subset C(S^{N-1}_\mathbb{R})$, and the image of this embedding is the algebra generated by the variables $p_{ij} = x_i x_j$.

The complex projective space $P^{N-1}_\mathbb{C}$ has a similar description, and we have an embedding $C(P^{N-1}_\mathbb{C}) \subset C(S^{N-1}_\mathbb{C})$, whose image is generated by the variables $p_{ij} = x_i \bar{x}_j$.

The spaces $P^{N-1}_\mathbb{R}, P^{N-1}_\mathbb{C}$ have the following functional analytic description:

Theorem 9.8. *We have presentation results as follows,*

$$\begin{aligned} C(P^{N-1}_\mathbb{C}) &= C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N}\Big| p = p^* = p^2, Tr(p) = 1\right) \\ C(P^{N-1}_\mathbb{R}) &= C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N}\Big| p = \bar{p} = p^* = p^2, Tr(p) = 1\right) \end{aligned}$$

where by C^*_{comm} we mean as usual universal commutative C^*-algebra.

Proof. We use the fact that $P^{N-1}_\mathbb{C}, P^{N-1}_\mathbb{R}$ are respectively the spaces of rank one projections in $M_N(\mathbb{C}), M_N(\mathbb{R})$. With this picture in mind, it is clear that we have arrows $\leftarrow$.

In order to construct now arrows $\to$, consider the universal algebras on the right, A_C, A_R. These algebras being both commutative, by the Gelfand theorem we can write, with X_C, X_R being certain compact spaces:

$$A_C = C(X_C)$$
$$A_R = C(X_R)$$

Now by using the coordinate functions p_{ij}, we conclude that X_C, X_R are certain spaces of rank one projections in $M_N(\mathbb{C}), M_N(\mathbb{R})$. In other words, we have embeddings:

$$X_C \subset P^{N-1}_\mathbb{C}$$
$$X_R \subset P^{N-1}_\mathbb{R}$$

Bsy transposing we obtain arrows $\to$, as desired. $\square$

The above result suggests constructing free projective spaces $P^{N-1}_{\mathbb{R},+}, P^{N-1}_{\mathbb{C},+}$, simply by lifting the commutativity conditions between the variables p_{ij}. However, there is something wrong with this, and more specifically with $P^{N-1}_{\mathbb{R},+}$, coming from the fact that if certain noncommutative coordinates $x_1, \ldots, x_N$ are self-adjoint, then the corresponding projective coordinates $p_{ij} = x_i x_j$ are not necessarily self-adjoint:

$$x_i = x_i^* \not\Longrightarrow x_i x_j = (x_i x_j)^*$$

In short, our attempt to construct free projective spaces $P^{N-1}_{\mathbb{R},+}, P^{N-1}_{\mathbb{C},+}$ as above is not exactly correct, with the space $P^{N-1}_{\mathbb{R},+}$ being rather "irrelevant", and with the space $P^{N-1}_{\mathbb{C},+}$

being probably the good one, but being at the same time "real and complex". Observe that there is some similarity here with the following key result, from section 4 above:

$$PO_N^+=PU_N^+$$

To be more precise, we have good evidence here for the fact that, in the free setting, the projective geometry is at the same time real and complex.

In view of all this, let us formulate the following definition:

Definition 9.9. *Associated to any $N\in\mathbb N$ is the following universal algebra,*

$$C(P^{N-1}_+)=C^*\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=1\right)$$

whose abstract spectrum is called "free projective space".

Observe that we have embeddings of noncommutative spaces, as follows:

$$P^{N-1}_\mathbb R\subset P^{N-1}_\mathbb C\subset P^{N-1}_+$$

Let us compute now the projective versions of the noncommutative spheres that we have, including the half-classical ones. We use the following formalism here:

Definition 9.10. *The projective version of $S\subset S^{N-1}_{\mathbb C,+}$ is the quotient space $S\to PS$ determined by the fact that*

$$C(PS)\subset C(S)$$

is the subalgebra generated by $p_{ij}=x_ix_j^*$, called projective coordinates.

In the classical case, this fits with the usual definition. We will be back with more details in section 15 below, which is dedicated to the study of projective geometry. We have the following result, coming from [5], [32], [33]:

Theorem 9.11. *The projective versions of the basic spheres are as follows,*

$$\begin{matrix}
S^{N-1}_{\mathbb R,+}&\longrightarrow&S^{N-1}_{\mathbb C,+}\\
\uparrow&&\uparrow\\
S^{N-1}_{\mathbb R,*}&\longrightarrow&S^{N-1}_{\mathbb C,*}\\
\uparrow&&\uparrow\\
S^{N-1}_{\mathbb R}&\longrightarrow&S^{N-1}_{\mathbb C}
\end{matrix}\qquad\to\qquad
\begin{matrix}
P^{N-1}_+&\longrightarrow&P^{N-1}_+\\
\uparrow&&\uparrow\\
P^{N-1}_\mathbb C&\longrightarrow&P^{N-1}_\mathbb C\\
\uparrow&&\uparrow\\
P^{N-1}_\mathbb R&\longrightarrow&P^{N-1}_\mathbb C
\end{matrix}$$

modulo, in the free case, a GNS construction with respect to the uniform integration.

Proof. The formulae on the bottom are true by definition. For the formulae on top, we have to prove first that the variables $p_{ij} = x_i x_j^*$ over the free sphere $S^{N-1}_{\mathbb{C},+}$ satisfy the defining relations for $C(P^{N-1}_+)$. We first have:

$$(p^*)_{ij} = p^*_{ji} = (x_j x_i^*)^* = x_i x_j^* = p_{ij}$$

We have as well the following computation:

$$(p^2)_{ij} = \sum_k p_{ik} p_{kj} = \sum_k x_i x_k^* x_k x_j^* = x_i x_j^* = p_{ij}$$

Finally, we have as well the following computation:

$$Tr(p) = \sum_k p_{kk} = \sum_k x_k x_k^* = 1$$

Thus, we have embeddings of algebraic manifolds, as follows:

$$PS^{N-1}_{\mathbb{R},+} \subset PS^{N-1}_{\mathbb{C},+} \subset P^{N-1}_+$$

Regarding now the GNS construction assertion, this follows by reasoning as in the case of the free spheres, the idea being that the uniform integration on these projective spaces comes from the uniform integration over the following quantum group:

$$PO_N^+ = PU_N^+$$

All this is quite technical, and we will not need this result, in what follows. We refer here to [33], and we will back to this in section 15 below. Finally, regarding the middle assertions, concerning the projective versions of the half-classical spheres, it is enough to prove here that we have inclusions as follows:

$$P^{N-1}_{\mathbb{C}} \subset PS^{N-1}_{\mathbb{R},*} \subset PS^{N-1}_{\mathbb{C},*} \subset P^{N-1}_{\mathbb{C}}$$

But this can be done in 3 steps, as follows:

(1) $P^{N-1}_{\mathbb{C}} \subset PS^{N-1}_{\mathbb{R},*}$. In order to prove this, we recall from Proposition 9.6 that we have a morphism as follows, where z_i are the standard coordinates of $S^{N-1}_{\mathbb{C}}$:

$$C(S^{N-1}_{\mathbb{R},*}) \to M_2(C(S^{N-1}_{\mathbb{C}})) \quad , \quad x_i \to \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

Now observe that this model maps the projective coordinates as follows:

$$p_{ij} \to P_{ij} = \begin{pmatrix} z_i \bar{z}_j & 0 \\ 0 & \bar{z}_i z_j \end{pmatrix}$$

Thus, at the level of generated algebras, our model maps:

$$< p_{ij} > \to < P_{ij} > = C(P^{N-1}_{\mathbb{C}})$$

We conclude from this that we have a quotient map as follows:

$$C(PS^{N-1}_{\mathbb{R},*}) \to C(P^{N-1}_{\mathbb{C}})$$

Thus at the level of corresponding spaces, we have, as desired, an inclusion:

$$P^{N-1}_{\mathbb{C}} \subset PS^{N-1}_{\mathbb{R},*}$$

(2) $PS^{N-1}_{\mathbb{R},*} \subset PS^{N-1}_{\mathbb{C},*}$. This is something trivial, coming by functoriality of the operation $S \to PS$, from the inclusion of spheres:

$$S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{C},*}$$

(3) $PS^{N-1}_{\mathbb{C},*} \subset P^{N-1}_{\mathbb{C}}$. This follows from the half-commutation relations, which imply:

$$ab^*cd^* = cb^*ad^* = cd^*ab^*$$

Indeed, this shows that the projective version $PS^{N-1}_{\mathbb{C},*}$ is classical, and so:

$$PS^{N-1}_{\mathbb{C},*} \subset (P^{N-1}_+)_{class} = P^{N-1}_{\mathbb{C}}$$

Thus, we are led to the conclusion in the statement. □

We can go back now to the spheres, and we have the following result:

Theorem 9.12. *We have a morphism of C^*-algebras as follows,*

$$C(S^{N-1}_{\mathbb{R},*}) \subset M_2(C(S^{N-1}_{\mathbb{C}})) \quad , \quad x_i \to \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

where z_i are the standard coordinates of $S^{N-1}_{\mathbb{C}}$.

Proof. We know from Proposition 9.6 that we have a morphism as in the statement, and the injectivity follows from Theorem 9.11, by using a standard grading trick. □

In the case of the complex spheres we have a similar result, as follows:

Theorem 9.13. *We have a morphism of C^*-algebras as follows,*

$$C(S^{N-1}_{\mathbb{C},*}) \to M_2(C(S^{N-1}_{\mathbb{C}} \times S^{N-1}_{\mathbb{C}})) \quad , \quad x_i \to \begin{pmatrix} 0 & z_i \\ y_i & 0 \end{pmatrix}$$

where y_i, z_i are the standard coordinates of $S^{N-1}_{\mathbb{C}} \times S^{N-1}_{\mathbb{C}}$.

Proof. We know from Proposition 9.7 that we have a morphism as in the statement, and the injectivity follows from Theorem 9.11, by using a standard grading trick. □

We will be back later to the above results, which are quite similar to each other, with a number of unifications and generalizations.

Summarizing, we have some interesting affine and projective geometry results regarding the half-classical case, that we will use in what follows.

The point now is that the same arguments apply to the tori, and to the quantum groups. We first have the following result:

Proposition 9.14. *The real and complex half-classical quadruplets*

$$\begin{array}{ccc}
S^{N-1}_{\mathbb{R},*}&-&T_N^*\\
|&\times&|\\
O_N^*&-&H_N^*
\end{array}\qquad\qquad
\begin{array}{ccc}
S^{N-1}_{\mathbb{C},*}&-&\mathbb{T}_N^*\\
|&\times&|\\
U_N^*&-&K_N^*
\end{array}$$

have 2×2 matrix models, constructed by using antidiagonal matrices, as for the spheres.

Proof. This is something that we already know from the spheres. For the other objects, this follows by suitably adapting the proof of Proposition 9.6 and Proposition 9.7. □

Next, we have the following result:

Theorem 9.15. *The real and complex half-classical quadruplets have the same projective version, which is as follows:*

$$\begin{array}{ccc}
P^{N-1}_{\mathbb{C}}&-&P\mathbb{T}_N\\
|&\times&|\\
PU_N&-&PK_N
\end{array}$$

Proof. This is something that we already know from the spheres. For the other objects, this follows from Proposition 9.14, by suitably adapting the proof of Theorem 9.11. □

Finally, we have the following result:

Theorem 9.16. *The 2×2 antidiagonal matrix models for the real and complex half-classical quadruplets, constructed above, are faithful.*

Proof. This is something that we already know from the spheres. For the other objects, this follows by suitably adapting the proof of Theorem 9.12 and Theorem 9.13. □

As already mentioned, all these results are part of a series of more general results, regarding the half-liberation. We will be back to this, in section 12 below.

Let us check now the axioms, for these half-classical quadruplets. We first need some quantum isometry group results:

Theorem 9.17. *The quantum isometry groups of the basic spheres are*

$$\begin{matrix}
S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} & & O_N^+ & \longrightarrow & U_N^+\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{C},*} & \to & O_N^* & \longrightarrow & U_N^*\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} & & O_N & \longrightarrow & U_N
\end{matrix}$$

modulo identifying, as usual, the various C^*-algebraic completions.

Proof. We just have to prove the results in the middle.

Assume $G \curvearrowright S^{N-1}_{\mathbb{C},*}$. From $\Phi(x_a) = \sum_i x_i \otimes u_{ia}$ we obtain, with $p_{ab} = z_a\bar{z}_b$:

$$\Phi(p_{ab}) = \sum_{ij} p_{ij} \otimes u_{ia}u_{jb}^*$$

By multiplying two such arbitrary formulae, we obtain:

$$\begin{aligned}
\Phi(p_{ab}p_{cd}) &= \sum_{ijkl} p_{ij}p_{kl} \otimes u_{ia}u_{jb}^*u_{kc}u_{ld}^*\\
\Phi(p_{ad}p_{cb}) &= \sum_{ijkl} p_{il}p_{kj} \otimes u_{ia}u_{ld}^*u_{kc}u_{jb}^*
\end{aligned}$$

The left terms being equal, and the first terms on the right being equal too, we deduce that, with $[a,b,c] = abc - cba$, we must have the following equality:

$$\sum_{ijkl} p_{ij}p_{kl} \otimes u_{ia}[u_{jb}^*, u_{kc}, u_{ld}^*] = 0$$

Now observe that the products of projective variables $p_{ij}p_{kl} = z_i\bar{z}_jz_k\bar{z}_l$ depend only on the following two cardinalities:

$$|\{i,k\}|, |\{j,l\}| \in \{1,2\}$$

The point now is that this dependence produces the only relations between our variables, we are led in this way to 4 equations, as follows:

(1) $u_{ia}[u_{jb}^*, u_{ka}, u_{lb}^*] = 0$, $\forall a,b$.

(2) $u_{ia}[u_{jb}^*, u_{ka}, u_{ld}^*] + u_{ia}[u_{jd}^*, u_{ka}, u_{lb}^*] = 0$, $\forall a$, $\forall b \neq d$.

(3) $u_{ia}[u_{jb}^*, u_{kc}, u_{lb}^*] + u_{ic}[u_{jb}^*, u_{ka}, u_{lb}^*] = 0$, $\forall a \neq c$, $\forall b$.

(4) $u_{ia}([u_{jb}^*, u_{kc}, u_{ld}^*]+[u_{jd}^*, u_{kc}, u_{lb}^*])+u_{ic}([u_{jb}^*, u_{ka}, u_{ld}^*]+[u_{jd}^*, u_{ka}, u_{lb}^*]) = 0, \forall a \neq c, \forall b \neq d$.

From (1,2) we conclude that (2) holds with no restriction on the indices. By multiplying now this formula to the left by u_{ia}^*, and then summing over i, we obtain:

$$[u_{jb}^*, u_{ka}, u_{ld}^*] + [u_{jd}^*, u_{ka}, u_{lb}^*] = 0$$

By applying now the antipode, then the involution, and finally by suitably relabelling all the indices, we successively obtain from this formula:

$$\begin{aligned}
[u_{dl}, u_{ak}^*, u_{bj}] + [u_{bl}, u_{ak}^*, u_{dj}] = 0 \quad &\implies \quad [u_{dl}^*, u_{ak}, u_{bj}^*] + [u_{bl}^*, u_{ak}, u_{dj}^*] = 0 \\
&\implies \quad [u_{ld}^*, u_{ka}, u_{jb}^*] + [u_{jd}^*, u_{ka}, u_{lb}^*] = 0
\end{aligned}$$

Now by comparing with the original relation, above, we conclude that we have:

$$[u_{jb}^*, u_{ka}, u_{ld}^*] = [u_{jd}^*, u_{ka}, u_{lb}^*] = 0$$

Thus we have reached to the formulae defining U_N^*, and we are done.

Finally, in what regards the universality of the action $O_N^* \curvearrowright S^{N-1}_{\mathbb{R},*}$, this follows from the universality of the following actions:

$$U_N^* \curvearrowright S^{N-1}_{\mathbb{C},*}$$

$$O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$$

Indeed, we have $U_N^* \cap O_N^+ = O_N^*$, and we obtain the result. $\square$

Regarding now the tori, the computation here is as follows:

Theorem 9.18. *The quantum isometry groups of the basic tori are*

$$\begin{tikzcd}
T_N^+ \arrow[r] & \mathbb{T}_N^+ & & H_N^+ \arrow[r] & K_N^+ \\
T_N^* \arrow[u] \arrow[r] & \mathbb{T}_N^* \arrow[u] & \to & H_N^* \arrow[u] \arrow[r] & K_N^* \arrow[u] \\
T_N \arrow[u] \arrow[r] & \mathbb{T}_N \arrow[u] & & \bar{O}_N \arrow[u, dash, dotted] \arrow[r] & \bar{U}_N \arrow[u, dash, dotted]
\end{tikzcd}$$

with all arrows being inclusions, and with no vertical maps at bottom right.

Proof. We just have to prove the results in the middle. In the real case, we must find the conditions on $G \subset O_N^+$ such that $g_a \to \sum_i g_a \otimes u_{ia}$ defines a coaction.

In order for this map to be a coaction, the variables $G_a = \sum_i g_a \otimes u_{ia}$ must satisfy the following relations, which define the groups in the statement:

$$G_a^2 = 1$$

$$G_a G_b G_c = G_c G_b G_a$$

In what regards the squares, we have the following formula:

$$\begin{aligned} G_a^2 &= \sum_{ij} g_i g_j \otimes u_{ia} u_{ja} \\ &= 1 + \sum_{i \neq j} g_i g_j \otimes u_{ia} u_{ja} \end{aligned}$$

As for the products, with the notation $[x, y, z] = xyz - zyx$, we have:

$$[G_a, G_b, G_c] = \sum_{ijk} g_i g_j g_k \otimes [u_{ia}, u_{jb}, u_{kc}]$$

From the first relations, $G_a^2 = 1$, we obtain $G \subset H_N^+$. In order to process now the second relations, $G_a G_b G_c = G_c G_b G_a$, we can split the sum over i, j, k, as follows:

$$\begin{aligned} [G_a, G_b, G_c] &= \sum_{i,j,k \ distinct} g_i g_j g_k \otimes [u_{ia}, u_{jb}, u_{kc}] \\ &+ \sum_{i \neq j} g_i g_j g_i \otimes [u_{ia}, u_{jb}, u_{ic}] \\ &+ \sum_{i \neq j} g_i \otimes [u_{ia}, u_{jb}, u_{jc}] \\ &+ \sum_{i \neq k} g_k \otimes [u_{ia}, u_{ib}, u_{kc}] \\ &+ \sum_{i} g_i \otimes [u_{ia}, u_{ib}, u_{ic}] \end{aligned}$$

Our claim is that the last three sums vanish. Indeed, observe that we have:

$$[u_{ia}, u_{ib}, u_{ic}] = \delta_{abc} u_{ia} - \delta_{abc} u_{ia} = 0$$

Thus the last sum vanishes. Regarding now the fourth sum, we have:

$$\begin{aligned} \sum_{i \neq k} [u_{ia}, u_{ib}, u_{kc}] &= \sum_{i \neq k} u_{ia} u_{ib} u_{kc} - u_{kc} u_{ib} u_{ia} \\ &= \sum_{i \neq k} \delta_{ab} u_{ia}^2 u_{kc} - \delta_{ab} u_{kc} u_{ia}^2 \\ &= \delta_{ab} \sum_{i \neq k} [u_{ia}^2, u_{kc}] \\ &= \delta_{ab} \left[\sum_{i \neq k} u_{ia}^2, u_{kc} \right] \\ &= \delta_{ab} [1 - u_{ka}^2, u_{kc}] \\ &= 0 \end{aligned}$$

The proof for the third sum is similar. Thus, we are left with the first two sums. By using $g_ig_jg_k = g_kg_jg_i$ for the first sum, the formula becomes:

$$\begin{aligned}[G_a, G_b, G_c] &= \sum_{i<k, j\neq i,k} g_ig_jg_k \otimes ([u_{ia}, u_{jb}, u_{kc}] + [u_{ka}, u_{jb}, u_{ic}]) \\ &+ \sum_{i\neq j} g_ig_jg_i \otimes [u_{ia}, u_{jb}, u_{ic}]\end{aligned}$$

In order to have a coaction, the above coefficients must vanish. Now observe that, when setting $i = k$ in the coefficients of the first sum, we obtain twice the coefficients of the second sum. Thus, our vanishing conditions can be formulated as follows:

$$[u_{ia}, u_{jb}, u_{kc}] + [u_{ka}, u_{jb}, u_{ic}] = 0, \forall j \neq i, k$$

Now observe that at $a = b$ or $b = c$ this condition reads $0 + 0 = 0$. Thus, we can formulate our vanishing conditions in a more symmetric way, as follows:

$$[u_{ia}, u_{jb}, u_{kc}] + [u_{ka}, u_{jb}, u_{ic}] = 0, \forall j \neq i, k, \forall b \neq a, c$$

We use now the trick from [44]. We apply the antipode to this formula, and then we relabel the indices $i \leftrightarrow c, j \leftrightarrow b, k \leftrightarrow a$. We succesively obtain in this way:

$$[u_{ck}, u_{bj}, u_{ai}] + [u_{ci}, u_{bj}, u_{ak}] = 0, \forall j \neq i, k, \forall b \neq a, c$$

$$[u_{ia}, u_{jb}, u_{kc}] + [u_{ic}, u_{jb}, u_{ka}] = 0, \forall b \neq a, c, \forall j \neq i, k$$

Since we have $[x, y, z] = -[z, y, x]$, by comparing the last formula with the original one, we conclude that our vanishing relations reduce to a single formula, as follows:

$$[u_{ia}, u_{jb}, u_{kc}] = 0, \forall j \neq i, k, \forall b \neq a, c$$

Our first claim is that this formula implies $G \subset H_N^{[\infty]}$, where $H_N^{[\infty]} \subset O_N^+$ is defined via the relations $xyz = 0$, for any $x \neq z$ on the same row or column of u. In order to prove this, we will just need the $c = a$ particular case of this formula, which reads:

$$u_{ia}u_{jb}u_{ka} = u_{ka}u_{jb}u_{ia}, \forall j \neq i, k, \forall a \neq b$$

It is enough to check that the assumptions $j \neq i, k$ and $a \neq b$ can be dropped. But this is what happens indeed, because at $j = i$ we have:

$$\begin{aligned}[u_{ia}, u_{ib}, u_{ka}] &= u_{ia}u_{ib}u_{ka} - u_{ka}u_{ib}u_{ia} \\ &= \delta_{ab}(u_{ia}^2u_{ka} - u_{ka}u_{ia}^2) \\ &= 0\end{aligned}$$

Also, at $j = k$ we have:

$$\begin{aligned}[u_{ia}, u_{kb}, u_{ka}] &= u_{ia}u_{kb}u_{ka} - u_{ka}u_{kb}u_{ia} \\ &= \delta_{ab}(u_{ia}u_{ka}^2 - u_{ka}^2u_{ia}) \\ &= 0\end{aligned}$$

Finally, at $a=b$ we have:

$$\begin{aligned}[u_{ia},u_{ja},u_{ka}] &= u_{ia}u_{ja}u_{ka}-u_{ka}u_{ja}u_{ia}\\ &= \delta_{ijk}(u_{ia}^3-u_{ia}^3)\\ &= 0\end{aligned}$$

Our second claim now is that, due to $G\subset H_N^{[\infty]}$, we can drop the assumptions $j\neq i,k$ and $b\neq a,c$ in the original relations $[u_{ia},u_{jb},u_{kc}]=0$. Indeed, at $j=i$ we have:

$$\begin{aligned}[u_{ia},u_{ib},u_{kc}] &= u_{ia}u_{ib}u_{kc}-u_{kc}u_{ib}u_{ia}\\ &= \delta_{ab}(u_{ia}^2u_{kc}-u_{kc}u_{ia}^2)\\ &= 0\end{aligned}$$

The proof at $j=k$ and at $b=a$, $b=c$ being similar, this finishes the proof of our claim. We conclude that the half-commutation relations $[u_{ia},u_{jb},u_{kc}]=0$ hold without any assumption on the indices, and so we obtain $G\subset H_N^*$, as claimed.

As for the proof in the complex case, this is similar. See [8]. □

By intersecting now with K_N^+, as required by our (S,T,U,K) axioms, we obtain:

Theorem 9.19. *The quantum reflection groups of the basic tori are*

$$\begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ & & H_N^+ & \longrightarrow & K_N^+\\ \uparrow & & \uparrow & & \uparrow & & \uparrow\\ T_N^* & \longrightarrow & \mathbb{T}_N^* & \to & H_N^* & \longrightarrow & K_N^*\\ \uparrow & & \uparrow & & \uparrow & & \uparrow\\ T_N & \longrightarrow & \mathbb{T}_N & & H_N & \longrightarrow & K_N \end{matrix}$$

with all the arrows being inclusions.

Proof. We already know that the results on the left and on the right hold indeed. As for the results in the middle, these follow from Theorem 9.18 above. □

We can now formulate our extension result, as follows:

Theorem 9.20. *We have basic noncommutative geometries, as follows,*

$$\begin{array}{ccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow \\
\mathbb{R}^N_* & \longrightarrow & \mathbb{C}^N_* \\
\uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

with each $\mathbb{K}^N_\times$ symbol standing for the corresponding (S,T,U,K) quadruplet.

Proof. We have to check the axioms from section 4, for the half-classical geometries.

The algebraic axioms are all clear, and the quantum isometry axioms follow from the above computations. Next in line, we have to prove the following formulae:

$$O_N^* =< O_N, T_N^* >$$

$$U_N^* =< U_N, \mathbb{T}_N^* >$$

By using standard generation results, it is enough to prove the first formula. Moreover, once again by standard generation results, it is enough to check that:

$$H_N^* =< H_N, T_N^* >$$

The inclusion $\supset$ being clear, we are left with proving the inclusion $\subset$. But this follows from the formula $H_N^* = T_N^* \rtimes S_N$, established in [120], as follows:

$$\begin{aligned}
H_N^* &= T_N^* \rtimes S_N \\
&= < S_N, T_N^* > \\
&\subset < H_N, T_N^* >
\end{aligned}$$

Alternatively, these formulae can be established by using the technology in [48], or by doing some combinatorial computations, using categories and easiness.

Finally, the axiom $S = S_U$ can be proved as in the classical and free cases, by using the Weingarten formula, and the following ergodicity property:

$$\left(id \otimes \int_U\right)\Phi(x) = \int_S x$$

Our claim, which will finish the proof, is that this holds as well in the half-classical case. Indeed, in the real case, where $x_i = x_i^*$, it is enough to check the above equality on an arbitrary product of coordinates, $x_{i_1}\ldots x_{i_k}$. The left term is as follows:

$$\begin{aligned}
&\left(id\otimes\int_{O_N^*}\right)\Phi(x_{i_1}\ldots x_{i_k})\\
&= \sum_{j_1\ldots j_k}x_{j_1}\ldots x_{j_k}\int_{O_N^*}u_{j_1i_1}\ldots u_{j_ki_k}\\
&= \sum_{j_1\ldots j_k}\sum_{\pi,\sigma\in P_2^*(k)}\delta_\pi(j)\delta_\sigma(i)W_{kN}(\pi,\sigma)x_{j_1}\ldots x_{j_k}\\
&= \sum_{\pi,\sigma\in P_2^*(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\sum_{j_1\ldots j_k}\delta_\pi(j)x_{j_1}\ldots x_{j_k}
\end{aligned}$$

Let us look now at the last sum on the right. We have to sum there quantities of type $x_{j_1}\ldots x_{j_k}$, over all choices of multi-indices $j=(j_1,\ldots,j_k)$ which fit into our given pairing $\pi\in P_2^*(k)$. But by using the relations $x_ix_jx_k=x_kx_jx_i$, and then $\sum_i x_i^2=1$ in order to simplify, we conclude that the sum of these quantities is 1. Thus, we obtain:

$$\left(id\otimes\int_{O_N^*}\right)\Phi(x_{i_1}\ldots x_{i_k})=\sum_{\pi,\sigma\in P_2^*(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)$$

On the other hand, another application of the Weingarten formula gives:

$$\begin{aligned}
\int_{S^{N-1}_{\mathbb{R},*}}x_{i_1}\ldots x_{i_k} &= \int_{O_N^*}u_{1i_1}\ldots u_{1i_k}\\
&= \sum_{\pi,\sigma\in P_2^*(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\
&= \sum_{\pi,\sigma\in P_2^*(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)
\end{aligned}$$

Thus, we are done. In the complex case the proof is similar, by adding exponents. For further details, we refer to [32] for the real case, and to [5] for the complex case. □

Summarizing, we have done so far half of our extension program.

10. Hybrid geometries

In order to finish the extension program outlined in the beginning of the previous section, we must discuss now the second question, concerning the "hybrid" case. To be more precise, we have seen so far that have basic noncommutative geometries as follows, with each $\mathbb{K}^N_\times$ symbol standing for the corresponding (S,T,U,K) quadruplet:

$$\begin{array}{ccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow \\
\mathbb{R}^N_* & \longrightarrow & \mathbb{C}^N_* \\
\uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

We will see in this section that there are some privileged intermediate geometries between the real and the complex ones, as follows:

$$\begin{array}{ccccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{T}\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N_* & \longrightarrow & \mathbb{T}\mathbb{R}^N_* & \longrightarrow & \mathbb{C}^N_* \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{T}\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

We will see as well that, that under strong combinatorial axioms, of "easiness" and "uniformity" type, these 9 geometries are the only ones.

In order to get started, an intermediate geometry $\mathbb{R}^N \subset X \subset \mathbb{C}^N$ is given by a quadruplet (S,T,U,K), whose components are subject to the following conditions:

$$S^{N-1}_\mathbb{R} \subset S \subset S^{N-1}_\mathbb{C}$$

$$T_N \subset T \subset \mathbb{T}_N$$

$$O_N \subset U \subset U_N$$

$$H_N \subset K \subset K_N$$

Our plan will be that of investigating first these intermediate object questions. Then, we will discuss the verification of the geometric axioms, for the solutions that we found. And then, afterwards, we will discuss the half-classical and the free cases as well.

In what regards the $S^{N-1}_\mathbb{R}\subset S\subset S^{N-1}_\mathbb{C}$ problem, there are obviously infinitely many solutions. However, we have a "privileged" solution, constructed as follows:

Theorem 10.1. *We have an intermediate sphere as follows,*

$$S^{N-1}_\mathbb{R}\subset\mathbb{T}S^{N-1}_\mathbb{R}\subset S^{N-1}_\mathbb{C}$$

which appears as the affine lift of $P^{N-1}_\mathbb{R}$, inside the complex sphere $S^{N-1}_\mathbb{C}$.

Proof. The projective version of the intermediate sphere $\mathbb{T}S^{N-1}_\mathbb{R}$ is given by:

$$P\mathbb{T}S^{N-1}_\mathbb{R}=PS^{N-1}_\mathbb{R}=P^{N-1}_\mathbb{R}$$

Conversely, assume that $S\subset S^{N-1}_\mathbb{C}$ satisfies $PS\subset P^{N-1}_\mathbb{R}$. For $x\in S$ the projective coordinates $p_{ij}=x_i\bar{x}_j$ must be real, $x_i\bar{x}_j=\bar{x}_ix_j$, Thus, we must have:

$$\frac{x_1}{\bar{x}_1}=\frac{x_2}{\bar{x}_2}=\ldots=\frac{x_N}{\bar{x}_N}$$

Now if we denote by $\lambda\in\mathbb{T}$ this common number, we succesively have:

$$\begin{aligned}\frac{x_i}{\bar{x}_i}=\lambda\quad&\iff\quad x_i=\lambda\bar{x}_i\\&\iff\quad x_i^2=\lambda|x_i|^2\\&\iff\quad x_i=\pm\sqrt{\lambda}|x_i|\end{aligned}$$

Thus we obtain $x\in\sqrt{\lambda}S^{N-1}_\mathbb{R}$, and this gives the result. $\square$

In the case of the tori, we have a similar result, as follows:

Theorem 10.2. *We have an intermediate torus as follows, which appears as the affine lift of the Clifford torus $PT_N=T_{N-1}$, inside the complex torus $\mathbb{T}_N$:*

$$T_N\subset\mathbb{T}T_N\subset\mathbb{T}_N$$

More generally, we have intermediate tori as follows, with $r\in\mathbb{N}\cup\{\infty\}$,

$$T_N\subset\mathbb{Z}_rT_N\subset\mathbb{T}_N$$

all whose projective versions equal the Clifford torus $PT_N=T_{N-1}$.

Proof. The first assertion, regarding $\mathbb{T}T_N$, follows exactly as for the spheres, as in proof of Theorem 10.1. The second assertion is clear as well, because we have:

$$P\mathbb{Z}_rT_N=PT_N=T_{N-1}$$

Thus, we are led to the conclusion in the statement. $\square$

In connection with the above statement, an interesting question is that of classifying the intermediate tori, which in our case are usual compact groups, as follows:

$$T_N\subset T\subset\mathbb{T}_N$$

At the group dual level, we must classify the following intermediate quotients:

$$\mathbb{Z}^N \to \Gamma \to \mathbb{Z}_2^N$$

There are many examples of such groups, and this even when imposing strong supplementary conditions, such as having an action of the symmetric group S_N on the generators. We will not go further in this direction, our main idea being anyway that of basing our study mostly on quantum group theory, and on the related notion of easiness.

At the unitary group level now, the situation is of course much more rigid, and becomes quite interesting. We have the following result from [21], to start with:

Theorem 10.3. *The following inclusion of compact groups is maximal,*

$$\mathbb{T}O_N \subset U_N$$

in the sense that there is no intermediate compact group in between.

Proof. In order to prove this result, consider as well the group $\mathbb{T}SO_N$.

Observe that we have $\mathbb{T}SO_N = \mathbb{T}O_N$ if N is odd. If N is even the group $\mathbb{T}O_N$ has two connected components, with $\mathbb{T}SO_N$ being the component containing the identity.

Let us denote by $\mathfrak{so}_N, \mathfrak{u}_N$ the Lie algebras of SO_N, U_N. It is well-known that $\mathfrak{u}_N$ consists of the matrices $M \in M_N(\mathbb{C})$ satisfying $M^* = -M$, and that:

$$\mathfrak{so}_N = \mathfrak{u}_N \cap M_N(\mathbb{R})$$

Also, it is easy to see that the Lie algebra of $\mathbb{T}SO_N$ is $\mathfrak{so}_N \oplus i\mathbb{R}$.

Step 1. Our first claim is that if $N \geq 2$, the adjoint representation of SO_N on the space of real symmetric matrices of trace zero is irreducible.

Let indeed $X \in M_N(\mathbb{R})$ be symmetric with trace zero. We must prove that the following space consists of all the real symmetric matrices of trace zero:

$$V = span\left\{UXU^t \Big| U \in SO_N\right\}$$

We first prove that V contains all the diagonal matrices of trace zero. Since we may diagonalize X by conjugating with an element of SO_N, our space V contains a nonzero diagonal matrix of trace zero. Consider such a matrix:

$$D = diag(d_1, d_2, \ldots, d_N)$$

We can conjugate this matrix by the following matrix:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} \in SO_N$$

We conclude that our space V contains as well the following matrix:

$$D' = diag(d_2, d_1, d_3, \ldots, d_N)$$

More generally, we see that for any $1 \leq i, j \leq N$ the diagonal matrix obtained from D by interchanging d_i and d_j lies in V. Now since S_N is generated by transpositions, it follows that V contains any diagonal matrix obtained by permuting the entries of D. But it is well-known that this representation of S_N on the diagonal matrices of trace zero is irreducible, and hence V contains all such diagonal matrices, as claimed.

In order to conclude now, assume that Y is an arbitrary real symmetric matrix of trace zero. We can find then an element $U \in SO_N$ such that UYU^t is a diagonal matrix of trace zero. But we then have $UYU^t \in V$, and hence also $Y \in V$, as desired.

Step 2. Our claim is that the inclusion $\mathbb{T}SO_N \subset U_N$ is maximal in the category of connected compact groups.

Let indeed G be a connected compact group satisfying $\mathbb{T}SO_N \subset G \subset U_N$. Then G is a Lie group. Let $\mathfrak{g}$ denote its Lie algebra, which satisfies:

$$\mathfrak{so}_N \oplus i\mathbb{R} \subset \mathfrak{g} \subset \mathfrak{u}_N$$

Let ad_G be the action of G on $\mathfrak{g}$ obtained by differentiating the adjoint action of G on itself. This action turns $\mathfrak{g}$ into a G-module. Since $SO_N \subset G$, $\mathfrak{g}$ is also a SO_N-module.

Now if $G \neq \mathbb{T}SO_N$, then since G is connected we must have $\mathfrak{so}_N \oplus i\mathbb{R} \neq \mathfrak{g}$. It follows from the real vector space structure of the Lie algebras $\mathfrak{u}_N$ and $\mathfrak{so}_N$ that there exists a nonzero symmetric real matrix of trace zero X such that:

$$iX \in \mathfrak{g}$$

We know that the space of symmetric real matrices of trace zero is an irreducible representation of SO_N under the adjoint action. Thus $\mathfrak{g}$ must contain all such X, and hence $\mathfrak{g} = \mathfrak{u}_N$. But since U_N is connected, it follows that $G = U_N$.

Step 3. Our claim is that the commutant of SO_N in $M_N(\mathbb{C})$ is as follows:

(1) $SO_2' = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \Big| \alpha, \beta \in \mathbb{C} \right\}$.
(2) If $N \geq 3$, $SO_N' = \{\alpha I_N | \alpha \in \mathbb{C}\}$.

Indeed, at $N = 2$ this is a direct computation.

At $N \geq 3$, an element in $X \in SO_N'$ commutes with any diagonal matrix having exactly $N-2$ entries equal to 1 and two entries equal to -1. Hence X is a diagonal matrix.

Now since X commutes with any even permutation matrix and $N \geq 3$, it commutes in particular with the permutation matrix associated with the cycle (i, j, k) for any $1 < i < j < k$, and hence all the entries of X are the same.

We conclude that X is a scalar matrix, as claimed.

Step 4. Our claim is that the set of matrices with nonzero trace is dense in SO_N.

At $N=2$ this is clear, since the set of elements in SO_2 having a given trace is finite. So assume $N>2$, and let:

$$T\in SO_N\simeq SO(\mathbb{R}^N)$$
$$Tr(T)=0$$

Let $E\subset\mathbb{R}^N$ be a 2-dimensional subspace preserved by T, such that:

$$T_{|E}\in SO(E)$$

Let $\varepsilon>0$ and let $S_\varepsilon\in SO(E)$ with $||T_{|E}-S_\varepsilon||<\varepsilon$, and with $Tr(T_{|E})\neq Tr(S_\varepsilon)$, in the $N=2$ case. Now define $T_\varepsilon\in SO(\mathbb{R}^N)=SO_N$ by:

$$T_{\varepsilon|E}=S_\varepsilon\quad,\quad T_{\varepsilon|E^\perp}=T_{|E^\perp}$$

It is clear that we have the following estimate:

$$||T-T_\varepsilon||\leq||T_{|E}-S_\varepsilon||<\varepsilon$$

Also, we have the following estimate:

$$Tr(T_\varepsilon)=Tr(S_\varepsilon)+Tr(T_{|E^\perp})\neq 0$$

Thus, we have proved our claim.

Step 5. Our claim is that $\mathbb{T}O_N$ is the normalizer of $\mathbb{T}SO_N$ in U_N, i.e. is the subgroup of U_N consisting of the unitaries U for which, for all $X\in\mathbb{T}SO_N$:

$$U^{-1}XU\in\mathbb{T}SO_N$$

It is clear that the group $\mathbb{T}O_N$ normalizes $\mathbb{T}SO_N$, so in order to prove the result, we must show that if $U\in U_N$ normalizes $\mathbb{T}SO_N$ then $U\in\mathbb{T}O_N$.

First note that U normalizes SO_N. Indeed if $X\in SO_N$ then:

$$U^{-1}XU\in\mathbb{T}SO_N$$

Thus $U^{-1}XU=\lambda Y$ for some $\lambda\in\mathbb{T}$ and $Y\in SO_N$.

If $Tr(X)\neq 0$, we have $\lambda\in\mathbb{R}$ and hence:

$$\lambda Y=U^{-1}XU\in SO_N$$

The set of matrices having nonzero trace being dense in SO_N, we conclude that $U^{-1}XU\in SO_N$ for all $X\in SO_N$. Thus, we have:

$$\begin{aligned}X\in SO_N\quad&\implies\quad(UXU^{-1})^t(UXU^{-1})=I_N\\&\implies\quad X^tU^tUX=U^tU\\&\implies\quad U^tU\in SO_N'\end{aligned}$$

It follows that at $N\geq 3$ we have $U^tU=\alpha I_N$, with $\alpha\in\mathbb{T}$, since U is unitary. Hence we have $U=\alpha^{1/2}(\alpha^{-1/2}U)$ with:

$$\alpha^{-1/2}U\in O_N\quad,\quad U\in\mathbb{T}O_N$$

If $N=2$, $(U^tU)^t=U^tU$ gives again that $U^tU=\alpha I_2$, and we conclude as in the previous case.

Step 6. Our claim is that the inclusion $\mathbb{T}O_N\subset U_N$ is maximal in the category of compact groups.

Suppose indeed that $\mathbb{T}O_N\subset G\subset U_N$ is a compact group such that $G\neq U_N$. It is a well-known fact that the connected component of the identity in G is a normal subgroup, denoted G_0. Since we have $\mathbb{T}SO_N\subset G_0\subset U_N$, we must have:

$$G_0=\mathbb{T}SO_N$$

But since G_0 is normal in G, the group G normalizes $\mathbb{T}SO_N$, and hence $G\subset\mathbb{T}O_N$, which finishes the proof. □

Following [21], we have as well the following result:

Theorem 10.4. *The following inclusion of compact groups is maximal,*

$$PO_N\subset PU_N$$

in the sense that there is no intermediate compact group in between.

Proof. This follows from Theorem 10.3. Indeed, if $PO_N\subset G\subset PU_N$ is a proper intermediate subgroup, then its preimage under the quotient map $U_N\to PU_N$ would be a proper intermediate subgroup of $\mathbb{T}O_N\subset U_N$, which is a contradiction. □

Finally, still following [21], we have as well the following result:

Theorem 10.5. *The following inclusion of compact quantum groups is maximal,*

$$O_N\subset O_N^*$$

in the sense that there is no intermediate compact quantum group in between.

Proof. The idea is that this follows from the result regarding $PO_N\subset PU_N$, by taking affine lifts, and using algebraic techniques. Consider indeed a sequence of surjective Hopf $*$-algebra maps as follows, whose composition is the canonical surjection:

$$C(O_N^*)\xrightarrow{f}A\xrightarrow{g}C(O_N)$$

This produces a diagram of Hopf algebra maps with pre-exact rows, as follows:

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & C(PO_N^*) \arrow[r] \arrow[d, "f_|"] & C(O_N^*) \arrow[r] \arrow[d, "f"] & C(\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & \mathbb{C} \\
\mathbb{C} \arrow[r] & PA \arrow[r] \arrow[d, "g_|"] & A \arrow[r] \arrow[d, "g"] & C(\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & \mathbb{C} \\
\mathbb{C} \arrow[r] & PC(O_N) \arrow[r] & C(O_N) \arrow[r] & C(\mathbb{Z}_2) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

Consider now the following composition, with the isomorphism on the left being something well-known, coming from [48], as explained in section 9 above:

$$C(PU_N)\simeq C(PO_N^*)\xrightarrow{f_|}PA\xrightarrow{g_|}PC(O_N)\simeq C(PO_N)$$

This induces, at the group level, the embedding $PO_N\subset PU_N$. Thus $f_|$ or $g_|$ is an isomorphism. If $f_|$ is an isomorphism we get a commutative diagram of Hopf algebra morphisms with pre-exact rows, as follows:

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & C(PO_N^*) \arrow[r] \arrow[d, equal] & C(O_N^*) \arrow[r] \arrow[d, "f"] & C(\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & \mathbb{C} \\
\mathbb{C} \arrow[r] & C(PO_N^*) \arrow[r] & A \arrow[r] & C(\mathbb{Z}_2) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

Then f is an isomorphism. Similarly if $g_|$ is an isomorphism, then g is an isomorphism. For further details on all this, we refer to [21]. □

In connection now with our question, which is that of classifying the intermediate groups $O_N\subset G\subset U_N$, the above results lead to a dichotomy, coming from:

$$PG\in\{PO_N,PU_N\}$$

In the lack of a classification result here, which is surely known, but that we were unable to find in the literature, here are some basic examples of such intermediate groups:

Proposition 10.6. *We have compact groups $O_N\subset G\subset U_N$ as follows:*

(1) *The following groups, depending on a parameter $r\in\mathbb{N}\cup\{\infty\}$,*

$$\mathbb{Z}_rO_N=\left\{wU\Big|w\in\mathbb{Z}_r,U\in O_N\right\}$$

whose projective versions equal PO_N, and the biggest of which is the group $\mathbb{T}O_N$, which appears as affine lift of PO_N.

(2) *The following groups, depending on a parameter $d\in 2\mathbb{N}\cup\{\infty\}$,*

$$U_N^d=\left\{U\in U_N\Big|\det U\in\mathbb{Z}_d\right\}$$

interpolating between U_N^2 and $U_N^\infty=U_N$, whose projective versions equal PU_N.

Proof. All the assertions are elementary, the idea being as follows:

(1) We have indeed compact groups $\mathbb{Z}_rO_N$ with $r\in\mathbb{N}\cup\{\infty\}$ as in the statement, whose projective versions are given by:

$$P\mathbb{Z}_rO_N=PO_N$$

At $r=\infty$ we obtain the group $\mathbb{T}O_N$, and the fact that this group appears as the affine lift of PO_N follows exactly as in the sphere case, by using the computation from the proof of Theorem 10.1.

(2) As a first observation, the following formula, with $d\in\mathbb N\cup\{\infty\}$, defines indeed a closed subgroup $U_N^d\subset U_N$:

$$U_N^d=\left\{U\in U_N\Big|\det U\in\mathbb Z_d\right\}$$

In the case where d is even, this subgroup contains the orthogonal group O_N. As for the last assertion, namely $PU_N^d=PU_N$, this follows either be suitably rescaling the unitary matrices, or by applying the result in Theorem 10.3. □

The above results suggest that the solutions of $O_N\subset G\subset U_N$ should come from O_N,U_N, by succesively applying the following constructions:

$$G\to\mathbb Z_rG\quad,\quad G\to G\cap U_N^d$$

These operations do not exactly commute, but normally we should be led in this way to a 2-parameter series, unifying the two 1-parameter series from (1,2) above. However, some other groups like $\mathbb Z_NSO_N$ work too, so all this is probably a bit more complicated. As already mentioned, all this looks like quite standard group and Lie algebra theory, but we unable to find a good reference here. So, in the lack of something better, the above results will be our final saying on the subject, along with the reference to [21].

In what follows we will be mostly interested in the group $\mathbb TO_N$, which fits with the spheres and tori that we already have. This group, and the whole series $\mathbb Z_rO_N$ with $r\in\mathbb N\cup\{\infty\}$ that it is part of, is easy, the precise result being as follows:

Theorem 10.7. *We have the following results:*

1. *$\mathbb TO_N$ is easy, the corresponding category $\bar P_2\subset P_2$ consisting of the pairings having the property that when flatenning, we have the global formula $\#\circ=\#\bullet$.*
2. *$\mathbb Z_rO_N$ is easy, the corresponding category $P_2^r\subset P_2$ consisting of the pairings having the property that when flatenning, we have the global formula $\#\circ=\#\bullet(r)$.*

Proof. These results are standard and well-known, the proof being as follows:

(1) If we denote the standard corepresentation by $u=zv$, with $z\in\mathbb T$ and with $v=\bar v$, then in order to have $Hom(u^{\otimes k},u^{\otimes l})\neq\emptyset$, the z variabes must cancel, and in the case where they cancel, we obtain the same Hom-space as for O_N.

Now since the cancelling property for the z variables corresponds precisely to the fact that k,l must have the same numbers of $\circ$ symbols minus $\bullet$ symbols, the associated Tannakian category must come from the category of pairings $\bar P_2\subset P_2$, as claimed.

(2) This is something that we already know at $r=1,\infty$, where the group in question is $O_N,\mathbb TO_N$. The proof in general is similar, by writing $u=zv$ as above. □

Quite remarkably, the above result has the following converse:

Theorem 10.8. *The proper intermediate easy compact groups*

$$O_N \subset G \subset U_N$$

are precisely the groups $\mathbb{Z}_r O_N$, with $r \in \{2, 3, \ldots, \infty\}$.

Proof. According to our conventions for the easy quantum groups, which apply of course to the classical case, we must compute the following intermediate categories:

$$\mathcal{P}_2 \subset D \subset P_2$$

So, assume that we have such a category, $D \neq \mathcal{P}_2$, and pick an element $\pi \in D - \mathcal{P}_2$, assumed to be flat. We can modify π, by performing the following operations:

(1) First, we can compose with the basic crossing, in order to assume that π is a partition of type $\cap \ldots\ldots \cap$, consisting of consecutive semicircles. Our assumption $\pi \notin \mathcal{P}_2$ means that at least one semicircle is colored black, or white.

(2) Second, we can use the basic mixed-colored semicircles, and cap with them all the mixed-colored semicircles. Thus, we can assume that π is a nonzero partition of type $\cap \ldots\ldots \cap$, consisting of consecutive black or white semicircles.

(3) Third, we can rotate, as to assume that π is a partition consisting of an upper row of white semicircles, $\cup \ldots\ldots \cup$, and a lower row of white semicircles, $\cap \ldots\ldots \cap$. Our assumption $\pi \notin \mathcal{P}_2$ means that this latter partition is nonzero.

For $a, b \in \mathbb{N}$ consider the partition consisting of an upper row of a white semicircles, and a lower row of b white semicircles, and set:

$$\mathcal{C} = \left\{ \pi_{ab} \Big| a, b \in \mathbb{N} \right\} \cap D$$

According to the above, we have $\pi \in <\mathcal{C}>$. The point now is that we have:

(1) There exists $r \in \mathbb{N} \cup \{\infty\}$ such that $\mathcal{C}$ equals the following set:

$$\mathcal{C}_r = \left\{ \pi_{ab} \Big| a = b(r) \right\}$$

This is indeed standard, by using the categorical axioms.

(2) We have the following formula, with P_2^r being as above:

$$<\mathcal{C}_r> = P_2^r$$

This is standard as well, by doing some diagrammatic work.

With these results in hand, the conclusion now follows. Indeed, with $r \in \mathbb{N} \cup \{\infty\}$ being as above, we know from the beginning of the proof that any $\pi \in D$ satisfies:

$$\pi \in <\mathcal{C}> = <\mathcal{C}_r> = P_2^r$$

We conclude from this that we have an inclusion as follows:

$$D \subset P_2^r$$

Conversely, we have as well the following inclusion:

$$P_2^r =< \mathcal{C}_r >=< \mathcal{C} >\subset< D >= D$$

Thus we have $D = P_2^r$, and this finishes the proof. See [127]. □

As a conclusion, $\mathbb{T}O_N$ is definitely the "privileged" unitary group that we were looking for, with the remark that its arithmetic versions $\mathbb{Z}_rO_N$ are interesting as well.

Finally, let us discuss the reflection group case. Here the problem is that of classifying the intermediate compact groups $H_N \subset G \subset K_N$, and this looks of course well-known. In practice, however, the situation is considerably more complicated than in the continuous group case, with the expected 2-parameter series there being replaced by an expected 3-parameter series. So, instead of getting into this quite technical subject, let us just formulate a basic result, explaining what the 3 parameters are:

Proposition 10.9. *We have compact groups $H_N \subset G \subset K_N$ as follows:*
1. *The groups $\mathbb{Z}_rH_N$, with $r \in \mathbb{N} \cup \{\infty\}$.*
2. *The groups $H_N^s = \mathbb{Z}_s \wr S_N$, with $s \in 2\mathbb{N}$.*
3. *The groups $H_N^{sd} = H_N^s \cap U_N^d$, with $d|s$ and $s \in 2\mathbb{N}$.*

Proof. The constructions in the statement produce indeed closed subgroups $G \subset K_N$, for all the possible values of the parameters.

Regarding now the condition $H_N \subset G$, the situation is as follows:

(1) Here this condition is automatic.

(2) Here this condition follows from $s \in 2\mathbb{N}$.

(3) Here this condition follows from $d|s$ and $s \in 2\mathbb{N}$. □

The same discussion as in the continuous case applies, the idea being that the constructions $G \to \mathbb{Z}_rG$ and $G \to G \cap H_N^{sd}$ can be combined, and that all this leads in principle to a 3-parameter series. All this is, however, quite technical, and we do not really know if it is so. We will actually not need all this, so we will just stop our study here, and recommend here [121] and subsequent papers.

As in the continuous case, a solution to these classification problems comes from the notion of easiness. We have indeed the following result, coming from [15], [127]:

Theorem 10.10. *The following groups are easy:*
1. *$\mathbb{Z}_rH_N$, the corresponding category $P_{even}^r \subset P_{even}$ consisting of the partitions having the property that when flatenning, we have the global formula $\#\circ = \#\bullet\,(r)$.*
2. *$H_N^s = \mathbb{Z}_s \wr S_N$, the corresponding category $P_{even}^{(s)} \subset P_{even}$ consisting of the partitions having the property that we have the formula $\#\circ = \#\bullet\,(s)$, in each block.*

In addition, the easy solutions of $H_N \subset G \subset K_N$ appear by combining these examples.

Proof. All this is well-known, the idea being as follows:

(1) The computation here is similar to the one in the proof of Theorem 10.7, by writing the fundamental representation $u = zv$ as there.

(2) This is something very standard and fundamental, known since the paper [15], and which follows from a long, routine computation, perfomed there.

As for the last assertion, things here are quite technical, and for the precise statement and proof of the classification result, we refer here to paper [127]. $\square$

Summarizing, the situation here is more complicated than in the continuous group case. However, in what regards the "standard" solution, this is definitely $\mathbb{T}H_N$.

With all this preliminary work done, let us turn now to our main question, namely constructing new geometries. We have here the following result:

Theorem 10.11. *We have correspondences as follows,*

$$\begin{tikzcd}
\mathbb{T}S^{N-1}_\mathbb{R} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \mathbb{T}T_N \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
\mathbb{T}O_N \arrow[r, leftrightarrow] & \mathbb{T}H_N
\end{tikzcd}$$

which produce a new geometry.

Proof. We have indeed a quadruplet (S, T, U, K) as in the statement, produced by the various constructions above. Regarding now the verification of the axioms:

(1) We have the following computation:

$$\begin{aligned}
P(\mathbb{T}S^{N-1}_\mathbb{R} \cap \mathbb{T}_N^+) &= P(\mathbb{T}S^{N-1}_\mathbb{R} \cap \mathbb{T}_N) \\
&\subset P\mathbb{T}S^{N-1}_\mathbb{R} \cap P\mathbb{T}_N \\
&= P^{N-1}_\mathbb{R} \cap \mathbb{T}_{N-1} \\
&= T_{N-1}
\end{aligned}$$

By lifting, we obtain from this that we have:

$$\mathbb{T}S^{N-1}_\mathbb{R} \cap \mathbb{T}_N^+ \subset \mathbb{T}T_N$$

The inclusion "$\supset$" being clear as well, we are done with checking the first axiom.

(2) The second axiom states that we must have:

$$\mathbb{T}H_N \cap \mathbb{T}_N^+ = \mathbb{T}T_N$$

The verification of this second axiom is similar.

(3) The third axiom states that we must have:

$$\mathbb{T}O_N \cap K_N^+ = \mathbb{T}H_N$$

But can be checked either directly, or by proceeding as above, by taking projective versions, and then lifting.

(4) The quantum isometry group axiom states that we must have:

$$G^+(\mathbb{T}S^{N-1}_{\mathbb{R}}) = \mathbb{T}O_N$$

The verification of this axiom is routine, and all this is explained for instance in [9].

(5) The quantum reflection group axiom states that we must have:

$$G^+(\mathbb{T}T_N) \cap K_N^+ = \mathbb{T}H_N$$

But this can be checked in a similar way, by adapting the computation from the classical real case.

(6) Regarding now the hard liberation axiom, this is clear, because we have:

$$\begin{aligned}
< O_N, \mathbb{T}T_N > &= < O_N, \mathbb{T}, T_N > \\
&= < O_N, \mathbb{T} > \\
&= \mathbb{T}O_N
\end{aligned}$$

(7) Finally, we have as well the last axiom, namely:

$$S_{\mathbb{T}O_N} = \mathbb{T}S^{N-1}_{\mathbb{R}}$$

But this completes the proof. □

Let us discuss now the half-classical and free extensions of Theorem 10.11, and of some of the results preceding it. In order to have no redundant discussion and diagrams, we will talk directly about the $\times 9$ extension of the theory that we have so far. We first need to complete our collection of spheres S, tori T, unitary groups U, and reflection groups K. In what regards the spheres, the result is as follows:

Proposition 10.12. *We have noncommutative spheres as follows,*

$$\begin{tikzcd}
S^{N-1}_{\mathbb{R},+} \arrow[r] & \mathbb{T}S^{N-1}_{\mathbb{R},+} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\
S^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[u] & \mathbb{T}S^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C},*} \arrow[u] \\
S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & \mathbb{T}S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C}} \arrow[u]
\end{tikzcd}$$

with the middle vertical objects coming via the relations $ab^* = a^*b$.

Proof. We can indeed construct new spheres via the relations $ab^* = a^*b$, and these fit into previous 6-diagram of spheres as indicated. As for the fact that in the classical case we obtain the previously constructed sphere $\mathbb{T}S^{N-1}_\mathbb{R}$, this follows from Theorem 10.1 and its proof, because the relations used there are precisely those of type $a\bar{b} = \bar{a}b$. □

There are many things that can be done with the above spheres. As a basic result here, let us record the following fact, regarding the corresponding projective spaces:

Theorem 10.13. *The projective spaces associated to the basic spheres are*

$$\begin{array}{ccccc}
P^{N-1}_+ & \longrightarrow & P^{N-1}_+ & \longrightarrow & P^{N-1}_+ \\
\uparrow & & \uparrow & & \uparrow \\
P^{N-1}_\mathbb{C} & \longrightarrow & P^{N-1}_\mathbb{C} & \longrightarrow & P^{N-1}_\mathbb{C} \\
\uparrow & & \uparrow & & \uparrow \\
P^{N-1}_\mathbb{R} & \longrightarrow & P^{N-1}_\mathbb{R} & \longrightarrow & P^{N-1}_\mathbb{C}
\end{array}$$

via the standard identifications for noncommutative algebraic manifolds.

Proof. This is something that we already know for the 6 previous spheres. As for the 3 new spheres, this follows from the defining relations $ab^* = a^*b$, which tell us that the coordinates of the corresponding projective spaces must be self-adjoint. □

At the torus level now, the construction is similar, as follows:

Proposition 10.14. *We have noncommutative tori as follows,*

$$\begin{array}{ccccc}
T_N^+ & \longrightarrow & \mathbb{T}T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\
\uparrow & & \uparrow & & \uparrow \\
T_N^* & \longrightarrow & \mathbb{T}T_N^* & \longrightarrow & \mathbb{T}_N^* \\
\uparrow & & \uparrow & & \uparrow \\
T_N & \longrightarrow & \mathbb{T}T_N & \longrightarrow & \mathbb{T}_N
\end{array}$$

with the middle vertical objects coming via the relations $ab^* = a^*b$.

Proof. This is clear from Proposition 10.12, by intersecting everything with $\mathbb{T}_N^+$. □

In what regards the unitary quantum groups, the result is as follows:

Theorem 10.15. *We have quantum groups as follows, which are all easy,*

$$\begin{tikzcd}
O_N^+ \arrow[r] & \mathbb{T}O_N^+ \arrow[r] & U_N^+ \\
O_N^* \arrow[r] \arrow[u] & \mathbb{T}O_N^* \arrow[r] \arrow[u] & U_N^* \arrow[u] \\
O_N \arrow[r] \arrow[u] & \mathbb{T}O_N \arrow[r] \arrow[u] & U_N \arrow[u]
\end{tikzcd}$$

with the middle vertical objects coming via the relations $ab^* = a^*b$.

Proof. This is standard, indeed, the categories of partitions being as follows:

$$\begin{tikzcd}
NC_2 \arrow[d] & \bar{NC}_2 \arrow[l] \arrow[d] & \mathcal{NC}_2 \arrow[l] \arrow[d] \\
P_2^* \arrow[d] & \bar{P}_2^* \arrow[l] \arrow[d] & \mathcal{P}_2^* \arrow[l] \arrow[d] \\
P_2 & \bar{P}_2 \arrow[l] & \mathcal{P}_2 \arrow[l]
\end{tikzcd}$$

Observe that our diagrams are both intersection diagrams. □

Regarding the quantum reflection groups, we have here:

Theorem 10.16. *We have quantum groups as follows, which are all easy,*

$$\begin{tikzcd}
H_N^+ \arrow[r] & \mathbb{T}H_N^+ \arrow[r] & K_N^+ \\
H_N^* \arrow[r] \arrow[u] & \mathbb{T}H_N^* \arrow[r] \arrow[u] & K_N^* \arrow[u] \\
H_N \arrow[r] \arrow[u] & \mathbb{T}H_N \arrow[r] \arrow[u] & K_N \arrow[u]
\end{tikzcd}$$

with the middle vertical objects coming via the relations $ab^* = a^*b$.

Proof. This is standard, indeed, the categories of partitions being as follows:

$$\begin{tikzcd}
NC_{even} & \bar{NC}_{even} \arrow[l] & \mathcal{NC}_{even} \arrow[l] \\
P^*_{even} \arrow[u, leftarrow] & \bar{P}^*_{even} \arrow[l] \arrow[u, leftarrow] & \mathcal{P}^*_{even} \arrow[l] \arrow[u, leftarrow] \\
P_{even} \arrow[u, leftarrow] & \bar{P}_{even} \arrow[l] \arrow[u, leftarrow] & \mathcal{P}_{even} \arrow[l] \arrow[u, leftarrow]
\end{tikzcd}$$

Observe that our diagrams are both intersection diagrams. $\square$

Let us point out that we have some interesting questions, regarding the classification of the intermediate compact quantum groups for the following 4 inclusions:

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur, Rightarrow] & & O_N^+ \arrow[ur, Rightarrow] & \\
 & K_N^* \arrow[rr] \arrow[uu] & & U_N^* \arrow[uu] \\
H_N^* \arrow[rr] \arrow[uu] \arrow[ur, Rightarrow] & & O_N^* \arrow[uu] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

In what regards the half-classical questions, these can be in principle fully investigated by using the technology in [48], but we do not know what the final answer is. As for the free questions, these are more delicate, but in the easy case, they are solved by [127].

Getting back now to the verification of the axioms, we first have:

Theorem 10.17. *The quantum isometries of the basic spheres, namely*

$$\begin{tikzcd}
S^{N-1}_{\mathbb{R},+} \arrow[r] & \mathbb{T}S^{N-1}_{\mathbb{R},+} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\
S^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[u] & \mathbb{T}S^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C},*} \arrow[u] \\
S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & \mathbb{T}S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C}} \arrow[u]
\end{tikzcd}$$

are the basic unitary quantum groups.

Proof. This is routine, by lifting the results that we already have. ☐

Regarding now the tori, we first have here:

Proposition 10.18. *The quantum isometries of the basic tori are*

$$\begin{array}{ccccc}
H_N^+ & \longrightarrow & \mathbb{T}H_N^+ & \longrightarrow & K_N^+\\
\uparrow & & \uparrow & & \uparrow\\
H_N^* & \longrightarrow & \mathbb{T}H_N^* & \longrightarrow & K_N^*\\
\uparrow & & \uparrow & & \uparrow\\
\bar{O}_N & \longrightarrow & \mathbb{T}\bar{O}_N & \longrightarrow & \bar{U}_N
\end{array}$$

with the bars denoting as usual Schur-Weyl twists.

Proof. The result follows by lifting the results that we already have. ☐

By looking now at quantum reflections, we obtain:

Theorem 10.19. *The quantum reflections of the tori,*

$$\begin{array}{ccccc}
T_N^+ & \longrightarrow & \mathbb{T}T_N^+ & \longrightarrow & \mathbb{T}_N^+\\
\uparrow & & \uparrow & & \uparrow\\
T_N^* & \longrightarrow & \mathbb{T}T_N^* & \longrightarrow & \mathbb{T}_N^*\\
\uparrow & & \uparrow & & \uparrow\\
T_N & \longrightarrow & \mathbb{T}T_N & \longrightarrow & \mathbb{T}_N
\end{array}$$

are the basic quantum reflection groups.

Proof. This is indeed routine, by intersecting. ☐

Finally, we have hard liberation results, as follows:

Theorem 10.20. *We have hard liberation formulae of type*

$$U =< O_N, T >$$

for all the basic unitary quantum groups.

Proof. We only need to check this for the "hybrid" examples, constructed in this section. But for these hybrid examples, $U = \mathbb{T}O_N^\times$, the results follow from:

$$\begin{aligned}\mathbb{T}O_N^\times &= <\mathbb{T}, O_N^\times> \\ &= <\mathbb{T}, <O_N, T_N^\times>> \\ &= <O_N, <\mathbb{T}, T_N^\times>> \\ &= <O_N, \mathbb{T}T_N^\times>\end{aligned}$$

Thus, we have indeed complete hard liberation results, as claimed. □

We can now formulate our main result, as follows:

Theorem 10.21. *We have 9 noncommutative geometries, as follows,*

$$\begin{tikzcd} \mathbb{R}^N_+ \arrow[r] & \mathbb{T}\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\ \mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{C}^N_* \arrow[u] \\ \mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u] \end{tikzcd}$$

with each of the $\mathbb{K}^\times$ symbols standing for the corresponding quadruplet.

Proof. This follows indeed by putting everything together, a bit as in the proof of Theorem 10.11, the idea being that the intersection axioms are clear, the quantum isometry axioms follow from the above computations, and the remaining axioms are elementary. □

Getting now into classification results, let us recall from section 4 that a geometry coming from a quadruplet (S, T, U, K) is easy when U, K are easy, and when the easy generation formula $U = \{O_N, K\}$ is satisfied. Combinatorially, this gives:

Proposition 10.22. *An easy geometry is uniquely determined by a pair (D, E) of categories of partitions, which must be as follows,*

$$\mathcal{NC}_2 \subset D \subset P_2$$

$$\mathcal{NC}_{even} \subset E \subset P_{even}$$

and which are subject to the following intersection and generation conditions,

$$D = E \cap P_2$$

$$E = <D, \mathcal{NC}_{even}>$$

and to the usual axioms for the associated quadruplet (S, T, U, K), where U, K are respectively the easy quantum groups associated to the categories D, E.

Proof. This statement simply comes from the following conditions:

$$U = \{O_N, K\}$$
$$K = U \cap K_N^+$$

Indeed, U, K must be easy, coming from certain categories of partitions D, E. It is clear that D, E must appear as intermediate categories, as in the statement, and the fact that the intersection and generation conditions must be satisfied follows from:

$$U = \{O_N, K\} \iff D = E \cap P_2$$
$$K = U \cap K_N^+ \iff E =< D, \mathcal{NC}_{even} >$$

Thus, we are led to the conclusion in the statement. □

Here is now a classification result, based on the above:

Theorem 10.23. *There are exactly 4 geometries which are easy, uniform and pure, with purity meaning that the geometry must be real, classical, complex or free, namely:*

$$\begin{tikzcd} \mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\ \mathbb{R}^N \arrow[u] \arrow[r] & \mathbb{C}^N \arrow[u] \end{tikzcd}$$

When lifting the uniformity and purity conditions, and replacing them with a "slicing" axiom, we have 9 such geometries, namely those in Theorem 10.21.

Proof. All this is quite technical, the idea being as follows:

(1) Assume first that we have an easy geometry which is pure, in the sense that it lies on one of the 4 edges of the square in the statement. We know from Proposition 10.22 that its unitary group U must come from a category of pairings D satisfying:

$$D =< D, \mathcal{NC}_{even} > \cap P_2$$

But this equation can be solved by using the results in [107], [108], [120], [127], and by using the uniformity axiom, which excludes the half-liberations and the hybrids, we are led to the conclusion that the only solutions are the 4 vertices of the square.

(2) Regarding the second assertion, this can be obtained by using the same technology, by using the "slicing" axiom from [14], which amounts in saying that U, or the geometry itself, can be reconstructed from its projections on the edges of the square. See [14]. □

11. Twisted geometry

We have seen so far that the abstract noncommutative geometries, taken in a "spherical" sense, with coordinates bounded by $||x_i|| \leq 1$, can be axiomatized with the help of quadruplets (S, T, U, K). There are 9 main such geometries, as follows:

$$\begin{array}{ccccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{TR}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N_* & \longrightarrow & \mathbb{TR}^N_* & \longrightarrow & \mathbb{C}^N_* \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{TR}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

As a first related question, we would like to investigate the $q = -1$ twists of these geometries. In order to get started, the best is to deform first the simplest objects that we have, namely the quantum spheres. This can be done as follows:

Theorem 11.1. *We have quantum spheres as follows, obtained via the twisted commutation relations $ab = \pm ba$, and twisted half-commutation relations $abc = \pm cba$,*

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb{R},+} & \longrightarrow & \mathbb{T}S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\
\uparrow & & \uparrow & & \uparrow \\
\bar{S}^{N-1}_{\mathbb{R},*} & \longrightarrow & \mathbb{T}\bar{S}^{N-1}_{\mathbb{R},*} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{C},*} \\
\uparrow & & \uparrow & & \uparrow \\
\bar{S}^{N-1}_{\mathbb{R}} & \longrightarrow & \mathbb{T}\bar{S}^{N-1}_{\mathbb{R}} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{C}}
\end{array}$$

with the precise signs being as follows:

(1) *The signs on the bottom correspond to the anticommutation of distinct coordinates, and their adjoints. That is, with $z_i = x_i, x_i^*$ and $\varepsilon_{ij} = 1 - \delta_{ij}$, the formula is:*

$$z_i z_j = (-1)^{\varepsilon_{ij}} z_j z_i$$

(2) *The signs in the middle come from functoriality, as for the spheres in the middle to contain those on the bottom. That is, the formula is:*

$$z_i z_j z_k = (-1)^{\varepsilon_{ij} + \varepsilon_{jk} + \varepsilon_{ik}} z_k z_j z_i$$

Proof. As a first observation, we are using here bars in order to denote the $q = -1$ twists, and this in view of the discussion above, leading us to $q = \pm 1$, and with the $q = -1$ theory that we want to develop being different from the usual one.

(1) Here there is nothing to prove, because we can define the spheres on the bottom by the following formulae, with $z_i=x_i,x_i^*$ and $\varepsilon_{ij}=1-\delta_{ij}$ being as above:

$$C(\bar{S}^{N-1}_\mathbb{R})=C(S^{N-1}_{\mathbb{R},+})\Big/\Big\langle x_ix_j=(-1)^{\varepsilon_{ij}}x_jx_i\Big\rangle$$

$$C(\bar{S}^{N-1}_\mathbb{C})=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle z_iz_j=(-1)^{\varepsilon_{ij}}z_jz_i\Big\rangle$$

(2) Here our claim is that, if we want to construct half-classical twisted spheres, via relations of type $abc=\pm cba$ between the coordinates x_i and their adjoints x_i^*, as for these spheres to contain the twisted spheres constructed in (1), the only possible choice for these relations is as follows, with $z_i=x_i,x_i^*$ and $\varepsilon_{ij}=1-\delta_{ij}$ being as above:

$$z_iz_jz_k=(-1)^{\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}}z_kz_jz_i$$

But this is something clear, coming from the following computation, inside of the quotient algebras corresponding to the twisted spheres constructed in (1) above:

$$\begin{aligned}z_iz_jz_k&=(-1)^{\varepsilon_{ij}}z_jz_iz_k\\&=(-1)^{\varepsilon_{ij}+\varepsilon_{ik}}z_jz_kz_i\\&=(-1)^{\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}}z_kz_jz_i\end{aligned}$$

Thus, we are led to the conclusion in the statement, the spheres being given by:

$$C(\bar{S}^{N-1}_{\mathbb{R},*})=C(S^{N-1}_{\mathbb{R},+})\Big/\Big\langle x_ix_jx_k=(-1)^{\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}}x_kx_jx_i\Big\rangle$$

$$C(\bar{S}^{N-1}_{\mathbb{C},*})=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle z_iz_jz_k=(-1)^{\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}}z_kz_jz_i\Big\rangle$$

Thus, we have constructed our spheres, and embeddings, as desired. □

With the above result in hand, let us go ahead now, and twist the whole quadruplets (S,T,U,K) that we have. Things are quite tricky here, and let us start with the unitary quantum groups U. We would like these quantum groups to act on the corresponding spheres, $U\curvearrowright S$. Thus, we would like to have morphisms of algebras, as follows:

$$\Phi(x_i)=\sum_jx_j\otimes u_{ji}$$

Now with $z_i=x_i,x_i^*$ being as before, and with $v_{ij}=u_{ij},u_{ij}^*$ constructed accordingly, the above formula and its adjoint tell us that we must have:

$$\Phi(z_i)=\sum_jz_j\otimes v_{ji}$$

Thus the variables $Z_i=\sum_jz_j\otimes v_{ji}$ on the right must satisfy the twisted commutation or half-commutation relations in Theorem 11.1, and this will lead us to the correct twisted commutation or half-commutation relations to be satisfied by the variables v_{ij}.

In practice now, let us first discuss the twisting of O_N,U_N. Following [20] in the orthogonal case, and then [5] in the unitary case, the result here is as follows:

Theorem 11.2. *We have twisted orthogonal and unitary groups, as follows,*

$$\begin{array}{ccc}O_N^+&\longrightarrow&U_N^+\\ \uparrow&&\uparrow\\ \bar{O}_N&\longrightarrow&\bar{U}_N\end{array}$$

defined via the following relations, with the convention $\alpha=a,a^*$ and $\beta=b,b^*$:

$$\alpha\beta=\begin{cases}-\beta\alpha&\text{for } a,b\in\{u_{ij}\}\text{ distinct, on the same row or column of } u\\ \beta\alpha&\text{otherwise}\end{cases}$$

These quantum groups act on the corresponding twisted real and complex spheres.

Proof. Let us first discuss the construction of the quantum group $\bar{O}_N$. We must prove that the algebra $C(\bar{O}_N)$ obtained from $C(O_N^+)$ via the relations in the statement has a comultiplication Δ, a counit ε, and an antipode S. Regarding Δ, let us set:

$$U_{ij}=\sum_k u_{ik}\otimes u_{kj}$$

For $j\neq k$ we have the following computation:

$$\begin{aligned}U_{ij}U_{ik}&=\sum_{s\neq t}u_{is}u_{it}\otimes u_{sj}u_{tk}+\sum_s u_{is}u_{is}\otimes u_{sj}u_{sk}\\ &=\sum_{s\neq t}-u_{it}u_{is}\otimes u_{tk}u_{sj}+\sum_s u_{is}u_{is}\otimes(-u_{sk}u_{sj})\\ &=-U_{ik}U_{ij}\end{aligned}$$

Also, for $i\neq k,j\neq l$ we have the following computation:

$$\begin{aligned}U_{ij}U_{kl}&=\sum_{s\neq t}u_{is}u_{kt}\otimes u_{sj}u_{tl}+\sum_s u_{is}u_{ks}\otimes u_{sj}u_{sl}\\ &=\sum_{s\neq t}u_{kt}u_{is}\otimes u_{tl}u_{sj}+\sum_s(-u_{ks}u_{is})\otimes(-u_{sl}u_{sj})\\ &=U_{kl}U_{ij}\end{aligned}$$

Thus, we can define a comultiplication map for $C(\bar{O}_N)$, by setting:

$$\Delta(u_{ij})=U_{ij}$$

Regarding now the counit ε and the antipode S, things are clear here, by using the same method, and with no computations needed, the formulae to be satisfied being trivially satisfied. We conclude that $\bar{O}_N$ is a compact quantum group, and the proof for $\bar{U}_N$ is similar, by adding $*$ exponents everywhere in the above computations.

Finally, the last assertion is clear too, by doing some elementary computations, of the same type as above, and with the remark that the converse holds too, in the sense that if we want a quantum group $U\subset U_N^+$ to be defined by relations of type $ab=\pm ba$, and to have an action $U\curvearrowright S$ on the corresponding twisted sphere, we are led to the relations in the statement. We refer to [5] for further details on all this. $\square$

In order to discuss now the half-classical case, given three coordinates $a,b,c\in\{u_{ij}\}$, let us set $span(a,b,c)=(r,c)$, where $r,c\in\{1,2,3\}$ are the number of rows and columns spanned by a,b,c. In other words, if we write $a=u_{ij},b=u_{kl},c=u_{pq}$ then $r=\#\{i,k,p\}$ and $l=\#\{j,l,q\}$. With this convention, we have the following result:

Theorem 11.3. *We have intermediate quantum groups as follows,*

$$\begin{tikzcd}
O_N^+ \arrow[r] & \mathbb{T}O_N^+ \arrow[r] & U_N^+ \\
\bar{O}_N^* \arrow[r] \arrow[u] & \mathbb{T}\bar{O}_N^* \arrow[r] \arrow[u] & \bar{U}_N^* \arrow[u] \\
\bar{O}_N \arrow[r] \arrow[u] & \mathbb{T}\bar{O}_N \arrow[r] \arrow[u] & \bar{U}_N \arrow[u]
\end{tikzcd}$$

defined via the following relations, with $\alpha=a,a^*$, $\beta=b,b^*$ and $\gamma=c,c^*$,

$$\alpha\beta\gamma=\begin{cases}-\gamma\beta\alpha & \text{for } a,b,c\in\{u_{ij}\} \text{ with } span(a,b,c)=(\leq 2,3) \text{ or } (3,\leq 2)\\ \gamma\beta\alpha & \text{otherwise}\end{cases}$$

which act on the corresponding twisted half-classical real and complex spheres.

Proof. We use the same method as in the proof of Theorem 11.2, but with the combinatorics being now more complicated. Observe first that the rules for the various commutation and anticommutation signs in the statement can be summarized as follows:

$r\backslash c$	1	2	3
1	$+$	$+$	$-$
2	$+$	$+$	$-$
3	$-$	$-$	$+$

Let us first prove the result for $\bar{O}_N^*$. We must construct here morphisms Δ,ε,S, and the proof, similar to the proof of Theorem 11.2, goes as follows:

(1) We first construct Δ. For this purpose, we must prove that $U_{ij} = \sum_k u_{ik} \otimes u_{kj}$ satisfy the relations in the statement. We have the following computation:

$$\begin{aligned}
U_{ia}U_{jb}U_{kc} &= \sum_{xyz} u_{ix}u_{jy}u_{kz} \otimes u_{xa}u_{yb}u_{zc} \\
&= \sum_{xyz} \pm u_{kz}u_{jy}u_{ix} \otimes \pm u_{zc}u_{yb}u_{xa} \\
&= \pm U_{kc}U_{jb}U_{ia}
\end{aligned}$$

We must show that, when examining the precise two $\pm$ signs in the middle formula, their product produces the correct $\pm$ sign at the end. But the point is that both these signs depend only on $s = span(x, y, z)$, and for $s = 1, 2, 3$ respectively, we have:

– For a $(3, 1)$ span we obtain $+-$, $+-$, $-+$, so a product $-$ as needed.

– For a $(2, 1)$ span we obtain $++$, $++$, $--$, so a product $+$ as needed.

– For a $(3, 3)$ span we obtain $--$, $--$, $++$, so a product $+$ as needed.

– For a $(3, 2)$ span we obtain $+-$, $+-$, $-+$, so a product $-$ as needed.

– For a $(2, 2)$ span we obtain $++$, $++$, $--$, so a product $+$ as needed.

Together with the fact that our problem is invariant under $(r, c) \to (c, r)$, and with the fact that for a $(1, 1)$ span there is nothing to prove, this finishes the proof for Δ.

(2) The construction of the counit, via the formula $\varepsilon(u_{ij}) = \delta_{ij}$, requires the Kronecker symbols δ_{ij} to commute/anticommute according to the above table. Equivalently, we must prove that the situation $\delta_{ij}\delta_{kl}\delta_{pq} = 1$ can appear only in a case where the above table indicates "+". But this is clear, because $\delta_{ij}\delta_{kl}\delta_{pq} = 1$ implies $r = c$.

(3) Finally, the construction of the antipode, via the formula $S(u_{ij}) = u_{ji}$, is clear too, because this requires the choice of our $\pm$ signs to be invariant under transposition, and this is true, the above table being symmetric.

We conclude that $\bar{O}_N^*$ is indeed a compact quantum group, and the proof for $\bar{U}_N^*$ is similar, by adding $*$ exponents everywhere in the above computations.

Finally, the last assertion is clear too, by doing some elementary computations, of the same type as above, and with the remark that the converse holds too, in the sense that if we want a quantum group $U \subset U_N^+$ to be defined by relations of type $abc = \pm cba$, and to have an action $U \curvearrowright S$ on the corresponding half-classical twisted sphere, we are led to the relations in the statement. We refer to [5] for further details on all this. □

The above results can be summarized as follows:

Theorem 11.4. *We have quantum groups as follows, obtained via the twisted commutation relations $ab = \pm ba$, and twisted half-commutation relations $abc = \pm cba$,*

$$\begin{tikzcd}
O_N^+ \arrow[r] & \mathbb{T}O_N^+ \arrow[r] & U_N^+ \\
\bar{O}_N^* \arrow[r] \arrow[u] & \mathbb{T}\bar{O}_N^* \arrow[r] \arrow[u] & \bar{U}_N^* \arrow[u] \\
\bar{O}_N \arrow[r] \arrow[u] & \mathbb{T}\bar{O}_N \arrow[r] \arrow[u] & \bar{U}_N \arrow[u]
\end{tikzcd}$$

with the various signs coming as follows:

1. *The signs for $\bar{O}_N$ correspond to anticommutation of distinct entries on rows and columns, and commutation otherwise, with this coming from $\bar{O}_N \curvearrowright \bar{S}^{N-1}_\mathbb{R}$.*
2. *The signs for $\bar{O}_N^*, \bar{U}_N, \bar{U}_N^*$ come as well from the signs for $\bar{S}^{N-1}_\mathbb{R}$, either via the requirement $\bar{O}_N \subset U$, or via the requirement $U \curvearrowright S$.*

Proof. This is a summary of Theorem 11.2 and Theorem 11.3, and their proofs. □

Moving ahead now, and back to our geometric program, we have twisted the spheres and unitary groups S, U, and we are left with twisting the tori and reflection groups T, K. But these are "discrete" objects, which can only be rigid, so let us formulate:

Definition 11.5. *The twists of the basic quantum tori and reflection groups,*

$$\begin{tikzcd}
T_N^+ \arrow[r] & \mathbb{T}T_N^+ \arrow[r] & \mathbb{T}_N^+ \\
T_N^* \arrow[r] \arrow[u] & \mathbb{T}T_N^* \arrow[r] \arrow[u] & \mathbb{T}_N^* \arrow[u] \\
T_N \arrow[r] \arrow[u] & \mathbb{T}T_N \arrow[r] \arrow[u] & \mathbb{T}_N \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
H_N^+ \arrow[r] & \mathbb{T}H_N^+ \arrow[r] & K_N^+ \\
H_N^* \arrow[r] \arrow[u] & \mathbb{T}H_N^* \arrow[r] \arrow[u] & K_N^* \arrow[u] \\
H_N \arrow[r] \arrow[u] & \mathbb{T}H_N \arrow[r] \arrow[u] & K_N \arrow[u]
\end{tikzcd}$$

are by definition these tori and reflection groups themselves.

With this definition in hand, we are done with our twisting program for the triples (S, T, U, K), and we have now candidates $\bar{\mathbb{R}}^N$, $\bar{\mathbb{C}}^N$ and $\bar{\mathbb{R}}^N_*$, $\bar{\mathbb{C}}^N_*$ for new noncommutative geometries, to be checked from our axiomatic viewpoint, and then to be developed.

In order to discuss these questions, we must first review the above construction of the twists of S, T, U, K, which was something quite ad-hoc, and replace all that has being said above by something more conceptual. Let us start with:

Proposition 11.6. *The intermediate easy quantum groups*
$$H_N \subset G \subset U_N^+$$
come via Tannakian duality from the intermediate categories of partitions
$$P_{even} \supset D \supset \mathcal{NC}_2$$
with $P_{even}(k,l) \subset P(k,l)$ being the category of partitions whose blocks have even size.

Proof. This is something coming from the general easiness theory for quantum groups, discussed in section 2 above. Indeed, as explained there, the easy quantum groups appear as certain intermediate compact quantum groups, as follows:
$$S_N \subset G \subset U_N^+$$

To be more precise, such a quantum group is easy when the corresponding Tannakian category comes from an intermediate category of partitions, as follows:
$$P \supset D \supset \mathcal{NC}_2$$

Now since this correspondence makes correspond $H_N \leftrightarrow P_{even}$, once again as explained in section 2 above, we are led to the conclusion in the statement. $\square$

Summarizing, we must do some combinatorics, for the partitions having even blocks. Given a partition $\tau \in P(k,l)$, let us call "switch" the operation which consists in switching two neighbors, belonging to different blocks, in the upper row, or in the lower row. Also, we use the standard embedding $S_k \subset P_2(k,k)$, via the pairings having only up-to-down strings. With these conventions, we have the following result:

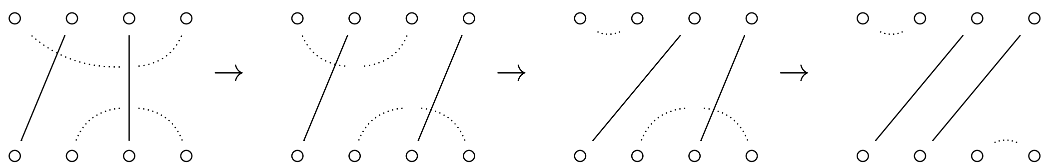
Theorem 11.7. *There is a signature map $\varepsilon : P_{even} \to \{-1,1\}$, given by*
$$\varepsilon(\tau) = (-1)^c$$
where c is the number of switches needed to make τ noncrossing. In addition:

(1) *For $\tau \in S_k$, this is the usual signature.*
(2) *For $\tau \in P_2$ we have $(-1)^c$, where c is the number of crossings.*
(3) *For $\tau \leq \pi \in NC_{even}$, the signature is 1.*

Proof. In order to show that ε is well-defined, we must prove that the number c in the statement is well-defined modulo 2. It is enough to perform the verification for the noncrossing partitions. More precisely, given $\tau, \tau' \in NC_{even}$ having the same block structure, we must prove that the number of switches c required for the passage $\tau \to \tau'$ is even.

In order to do so, observe that any partition $\tau \in P(k,l)$ can be put in "standard form", by ordering its blocks according to the appearence of the first leg in each block, counting clockwise from top left, and then by performing the switches as for block 1 to be at left, then for block 2 to be at left, and so on. Here the required switches are also uniquely determined, by the order coming from counting clockwise from top left.

Here is an example of such an algorithmic switching operation, with block 1 being first put at left, by using two switches, then with block 2 left unchanged, and then with block 3 being put at left as well, but at right of blocks 1 and 2, with one switch:



The point now is that, under the assumption $\tau \in NC_{even}(k,l)$, each of the moves required for putting a leg at left, and hence for putting a whole block at left, requires an even number of switches. Thus, putting τ is standard form requires an even number of switches. Now given $\tau,\tau' \in NC_{even}$ having the same block structure, the standard form coincides, so the number of switches c required for the passage $\tau \to \tau'$ is indeed even.

Regarding now the remaining assertions, these are all elementary:

(1) For $\tau \in S_k$ the standard form is $\tau' = id$, and the passage $\tau \to id$ comes by composing with a number of transpositions, which gives the signature.

(2) For a general $\tau \in P_2$, the standard form is of type $\tau' = |\ldots|^{\cup\ldots\cup}_{\cap\ldots\cap}$, and the passage $\tau \to \tau'$ requires c mod 2 switches, where c is the number of crossings.

(3) Assuming that $\tau \in P_{even}$ comes from $\pi \in NC_{even}$ by merging a certain number of blocks, we can prove that the signature is 1 by proceeding by recurrence. $\square$

We define the kernel of a multi-index $\binom{i}{j}$ to be the partition obtained by joining the equal indices. Also, we write $\pi \leq \sigma$ if each block of π is contained in a block of σ. With these conventions, and the above result in hand, we can now formulate:

Definition 11.8. *Associated to any partition $\pi \in P_{even}(k,l)$ is the linear map*

$$\bar{T}_\pi : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$$

given by the following formula, with $e_1,\ldots,e_N$ being the standard basis of $\mathbb{C}^N$,

$$\bar{T}_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k}) = \sum_{j_1\ldots j_l} \bar{\delta}_\pi\begin{pmatrix} i_1 & \ldots & i_k\\ j_1 & \ldots & j_l\end{pmatrix} e_{j_1}\otimes\ldots\otimes e_{j_l}$$

and where $\bar{\delta}_\pi \in \{-1,0,1\}$ is $\bar{\delta}_\pi = \varepsilon(\tau)$ if $\tau \geq \pi$, and $\bar{\delta}_\pi = 0$ otherwise, with:

$$\tau = \ker\begin{pmatrix} i\\ j\end{pmatrix}$$

In other words, what we are doing here is to add signatures to the usual formula of T_π. Indeed, observe that the usual formula for T_π can be written as folllows:

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j:\ker\binom{i}{j}\geq\pi}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

Now by inserting signs, coming from the signature map $\varepsilon:P_{even}\to\{\pm 1\}$, we are led to the following formula, which coincides with the one given above:

$$\bar{T}_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{\tau\geq\pi}\varepsilon(\tau)\sum_{j:\ker\binom{i}{j}=\tau}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

We will be back later to this analogy, with more details on what can be done with it. For the moment, we must first prove a key categorical result, as follows:

Proposition 11.9. *The assignement $\pi\to\bar{T}_\pi$ is categorical, in the sense that*

$$\bar{T}_\pi\otimes\bar{T}_\sigma=\bar{T}_{[\pi\sigma]}\quad,\quad\bar{T}_\pi\bar{T}_\sigma=N^{c(\pi,\sigma)}\bar{T}_{[^\sigma_\pi]}\quad,\quad\bar{T}_\pi^*=\bar{T}_{\pi^*}$$

where $c(\pi,\sigma)$ are certain positive integers.

Proof. We have to go back to the proof from the untwisted case, from section 2 above, and insert signs. We have to check three conditions, as follows:

1. Concatenation. In the untwisted case, this was based on the following formula:

$$\delta_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1\ldots k_r\\ l_1\ldots l_s\end{pmatrix}=\delta_{[\pi\sigma]}\begin{pmatrix}i_1\ldots i_p&k_1\ldots k_r\\ j_1\ldots j_q&l_1\ldots l_s\end{pmatrix}$$

In the twisted case, it is enough to check the following formula:

$$\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\right)\varepsilon\left(\ker\begin{pmatrix}k_1\ldots k_r\\ l_1\ldots l_s\end{pmatrix}\right)=\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p&k_1\ldots k_r\\ j_1\ldots j_q&l_1\ldots l_s\end{pmatrix}\right)$$

Let us denote by τ,ν the partitions on the left, so that the partition on the right is of the form $\rho\leq[\tau\nu]$. Now by switching to the noncrossing form, $\tau\to\tau'$ and $\nu\to\nu'$, the partition on the right transforms into $\rho\to\rho'\leq[\tau'\nu']$. Now since the partition $[\tau'\nu']$ is noncrossing, we can use Theorem 11.7 (3), and we obtain the result.

2. Composition. In the untwisted case, this was based on the following formula:

$$\sum_{j_1\ldots j_q}\delta_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}j_1\ldots j_q\\ k_1\ldots k_r\end{pmatrix}=N^{c(\pi,\sigma)}\delta_{[^\pi_\sigma]}\begin{pmatrix}i_1\ldots i_p\\ k_1\ldots k_r\end{pmatrix}$$

In order to prove now the result in the twisted case, it is enough to check that the signs match. More precisely, we must establish the following formula:

$$\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\right)\varepsilon\left(\ker\begin{pmatrix}j_1\ldots j_q\\ k_1\ldots k_r\end{pmatrix}\right)=\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ k_1\ldots k_r\end{pmatrix}\right)$$

Let τ,ν be the partitions on the left, so that the partition on the right is of the form $\rho\leq[^\tau_\nu]$. Our claim is that we can jointly switch τ,ν to the noncrossing form. Indeed, we can first switch as for $\ker(j_1\ldots j_q)$ to become noncrossing, and then switch the upper legs of τ, and the lower legs of ν, as for both these partitions to become noncrossing. Now observe that when switching in this way to the noncrossing form, $\tau\to\tau'$ and $\nu\to\nu'$, the partition on the right transforms into $\rho\to\rho'\leq[^{\tau'}_{\nu'}]$. Now since the partition $[^{\tau'}_{\nu'}]$ is noncrossing, we can apply Theorem 11.7 (3), and we obtain the result.

3. Involution. Here we must prove the following formula:

$$\bar{\delta}_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}=\bar{\delta}_{\pi^*}\begin{pmatrix}j_1\ldots j_q\\ i_1\ldots i_p\end{pmatrix}$$

But this is clear from the definition of $\bar{\delta}_\pi$, and we are done. $\square$

As a conclusion, our twisted construction $\pi\to\bar{T}_\pi$ has all the needed properties for producing quantum groups, via Tannakian duality, and we can now formulate:

Theorem 11.10. *Given a category of partitions $D\subset P_{even}$, the construction*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\bar{T}_\pi\Big|\pi\in D(k,l)\right)$$

produces via Tannakian duality a quantum group $\bar{G}_N\subset U_N^+$, for any $N\in\mathbb{N}$.

Proof. This follows indeed from the Tannakian results from section 2 above, exactly as in the easy case, by using this time Proposition 11.9 as technical ingredient.

To be more precise, Proposition 11.9 shows that the linear spaces on the right form a Tannakian category, and so the results in section 2 apply, and give the result. $\square$

We can unify the easy quantum groups, or at least the examples coming from categories $D\subset P_{even}$, with the quantum groups constructed above, as follows:

Definition 11.11. *A closed subgroup $G\subset U_N^+$ is called q-easy, or quizzy, with deformation parameter $q=\pm1$, when its tensor category appears as follows,*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\dot{T}_\pi\Big|\pi\in D(k,l)\right)$$

for a certain category of partitions $D\subset P_{even}$, where, for $q=-1,1$:

$$\dot{T}=\bar{T},T$$

The Schur-Weyl twist of G is the quizzy quantum group $\bar{G}\subset U_N^+$ obtained via $q\to-q$.

We will see later on that the easy quantum group associated to P_{even} itself is the hyperochahedral group H_N, and so that our assumption $D\subset P_{even}$, replacing $D\subset P$, simply corresponds to $H_N\subset G$, replacing the usual condition $S_N\subset G$.

For the moment, our most pressing task is that of checking that, when applying the Schur-Weyl twisting to the basic unitary quantum groups, we obtain the ad-hoc twists that we previously constructed. This is indeed the case:

Theorem 11.12. *The twisted unitary quantum groups introduced before,*

$$\begin{tikzcd}
O_N^+ \arrow[r] & \mathbb{T}O_N^+ \arrow[r] & U_N^+ \\
\bar{O}_N^* \arrow[r] \arrow[u] & \mathbb{T}\bar{O}_N^* \arrow[r] \arrow[u] & \bar{U}_N^* \arrow[u] \\
\bar{O}_N \arrow[r] \arrow[u] & \mathbb{T}\bar{O}_N \arrow[r] \arrow[u] & \bar{U}_N \arrow[u]
\end{tikzcd}$$

appear as Schur-Weyl twists of the basic unitary quantum groups.

Proof. This is something routine, in several steps, as follows:

(1) The basic crossing, $\ker\binom{ij}{ji}$ with $i\neq j$, comes from the transposition $\tau\in S_2$, so its signature is -1. As for its degenerated version $\ker\binom{ii}{ii}$, this is noncrossing, so here the signature is 1. We conclude that the linear map associated to the basic crossing is:

$$\bar{T}_{\slash\!\!\!\backslash}(e_i\otimes e_j)=\begin{cases}-e_j\otimes e_i & \text{for } i\neq j\\ e_j\otimes e_i & \text{otherwise}\end{cases}$$

For the half-classical crossing, namely $\ker\binom{ijk}{kji}$ with i,j,k distinct, the signature is once again -1, and by examining the signatures of the various degenerations of this half-classical crossing, we are led to the following formula:

$$\bar{T}_{\slash\!\!\!|\!\!\!\backslash}(e_i\otimes e_j\otimes e_k)=\begin{cases}-e_k\otimes e_j\otimes e_i & \text{for } i,j,k \text{ distinct}\\ e_k\otimes e_j\otimes e_i & \text{otherwise}\end{cases}$$

(2) Our claim now if that for an orthogonal quantum group G, the following holds, with the quantum group $\bar{O}_N$ being the one in Theorem 11.2:

$$\bar{T}_{\slash\!\!\!\backslash}\in End(u^{\otimes 2})\iff G\subset\bar{O}_N$$

Indeed, by using the formula of $\bar{T}_{\slash\!\!\!\backslash}$ found in (1) above, we obtain:

$$\begin{aligned}(\bar{T}_{\slash\!\!\!\backslash}\otimes 1)u^{\otimes 2}(e_i\otimes e_j\otimes 1)\ &=\ \sum_k e_k\otimes e_k\otimes u_{ki}u_{kj}\\ &-\ \sum_{k\neq l} e_l\otimes e_k\otimes u_{ki}u_{lj}\end{aligned}$$

On the other hand, we have as well the following formula:

$$u^{\otimes 2}(\bar{T}_{/\!\!\!\backslash}\otimes 1)(e_i\otimes e_j\otimes 1)\ =\ \begin{cases}\sum_{kl}e_l\otimes e_k\otimes u_{li}u_{ki} & \text{if } i=j\\ -\sum_{kl}e_l\otimes e_k\otimes u_{lj}u_{ki} & \text{if } i\neq j\end{cases}$$

For $i=j$ the conditions are $u_{ki}^2=u_{ki}^2$ for any k, and $u_{ki}u_{li}=-u_{li}u_{ki}$ for any $k\neq l$. For $i\neq j$ the conditions are $u_{ki}u_{kj}=-u_{kj}u_{ki}$ for any k, and $u_{ki}u_{lj}=u_{lj}u_{ki}$ for any $k\neq l$. Thus we have exactly the relations between the coordinates of $\bar{O}_N$, and we are done.

(3) Our claim now if that for an orthogonal quantum group G, the following holds, with the quantum group $\bar{O}_N^*$ being the one in Theorem 11.3:

$$\bar{T}_{/\!\!\!|\!\!\!\backslash}\in End(u^{\otimes 3})\iff G\subset\bar{O}_N^*$$

Indeed, by using the formula of $\bar{T}_{/\!\!\!|\!\!\!\backslash}$ found in (1) above, we obtain:

$$\begin{aligned}(\bar{T}_{/\!\!\!|\!\!\!\backslash}\otimes 1)u^{\otimes 2}(e_i\otimes e_j\otimes e_k\otimes 1)\ &=\ \sum_{abc\ not\ distinct}e_c\otimes e_b\otimes e_a\otimes u_{ai}u_{bj}u_{ck}\\ &\quad -\sum_{a,b,c\ distinct}e_c\otimes e_b\otimes e_a\otimes u_{ai}u_{bj}u_{ck}\end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned}&u^{\otimes 2}(\bar{T}_{/\!\!\!|\!\!\!\backslash}\otimes 1)(e_i\otimes e_j\otimes e_k\otimes 1)\\ &=\begin{cases}\sum_{abc}e_c\otimes e_b\otimes e_a\otimes u_{ck}u_{bj}u_{ai} & \text{for } i,j,k \text{ not distinct}\\ -\sum_{abc}e_c\otimes e_b\otimes e_a\otimes u_{ck}u_{bj}u_{ai} & \text{for } i,j,k \text{ distinct}\end{cases}\end{aligned}$$

For i,j,k not distinct the conditions are $u_{ai}u_{bj}u_{ck}=u_{ck}u_{bj}u_{ai}$ for a,b,c not distinct, and $u_{ai}u_{bj}u_{ck}=-u_{ck}u_{bj}u_{ai}$ for a,b,c distinct. For i,j,k distinct the conditions are $u_{ai}u_{bj}u_{ck}=-u_{ck}u_{bj}u_{ai}$ for a,b,c not distinct, and $u_{ai}u_{bj}u_{ck}=u_{ck}u_{bj}u_{ai}$ for a,b,c distinct. Thus we have exactly the relations between the coordinates of $\bar{O}_N^*$, and we are done.

(4) Now with the above results in hand, we obtain that the Schur-Weyl twists of O_N,O_N^* are indeed the quantum groups $\bar{O}_N,\bar{O}_N^*$ from Theorem 11.2 and Theorem 11.3.

(4) The proof in the unitary case is similar, by adding signs in the above computations (2,3), the conclusion being that the Schur-Weyl twists of U_N,U_N^* are indeed $\bar{U}_N,\bar{U}_N^*$. □

Let us clarify now the relation between the maps $T_\pi,\bar{T}_\pi$. We recall that the Möbius function of any lattice, and in particular of P_{even}, is given by:

$$\mu(\sigma,\pi)=\begin{cases}1 & \text{if } \sigma=\pi\\ -\sum_{\sigma\leq\tau<\pi}\mu(\sigma,\tau) & \text{if } \sigma<\pi\\ 0 & \text{if } \sigma\not\leq\pi\end{cases}$$

With this notation, we have the following result:

Proposition 11.13. *For any partition $\pi\in P_{even}$ we have the formula*

$$\bar{T}_\pi=\sum_{\tau\leq\pi}\alpha_\tau T_\tau$$

where $\alpha_\sigma=\sum_{\sigma\leq\tau\leq\pi}\varepsilon(\tau)\mu(\sigma,\tau)$, with μ being the Möbius function of P_{even}.

Proof. The linear combinations $T=\sum_{\tau\leq\pi}\alpha_\tau T_\tau$ acts on tensors as follows:

$$\begin{aligned}
T(e_{i_1}\otimes\ldots\otimes e_{i_k}) &= \sum_{\tau\leq\pi}\alpha_\tau T_\tau(e_{i_1}\otimes\ldots\otimes e_{i_k})\\
&= \sum_{\tau\leq\pi}\alpha_\tau\sum_{\sigma\leq\tau}\sum_{j:\ker\binom{i}{j}=\sigma} e_{j_1}\otimes\ldots\otimes e_{j_l}\\
&= \sum_{\sigma\leq\pi}\left(\sum_{\sigma\leq\tau\leq\pi}\alpha_\tau\right)\sum_{j:\ker\binom{i}{j}=\sigma} e_{j_1}\otimes\ldots\otimes e_{j_l}
\end{aligned}$$

Thus, in order to have $\bar{T}_\pi=\sum_{\tau\leq\pi}\alpha_\tau T_\tau$, we must have $\varepsilon(\sigma)=\sum_{\sigma\leq\tau\leq\pi}\alpha_\tau$, for any $\sigma\leq\pi$. But this problem can be solved by using the Möbius inversion formula, and we obtain the numbers $\alpha_\sigma=\sum_{\sigma\leq\tau\leq\pi}\varepsilon(\tau)\mu(\sigma,\tau)$ in the statement. □

With the above results in hand, let us go back now to the question of twisting the quantum reflection groups. It is convenient to include in our discussion two more quantum groups, coming from [120] and denoted $H_N^{[\infty]},K_N^{[\infty]}$, constructed as follows:

Proposition 11.14. *We have intermediate liberations $H_N^{[\infty]},K_N^{[\infty]}$ as follows, constructed by using the relations $\alpha\beta\gamma=0$ for any $a\neq c$ on the same row or column of u:*

$$\begin{array}{ccccccc}
K_N & \longrightarrow & K_N^* & \longrightarrow & K_N^{[\infty]} & \longrightarrow & K_N^+\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^{[\infty]} & \longrightarrow & H_N^+
\end{array}$$

These quantum groups are both easy, with the corresponding categories of partitions, denoted $P_{even}^{[\infty]}\subset P_{even}$ and $\mathcal{P}_{even}^{[\infty]}\subset\mathcal{P}_{even}$, being generated by $\eta=\ker\binom{iij}{jii}$.

Proof. This is routine, by using the fact that the relations $\alpha\beta\gamma=0$ in the statement are equivalent to the condition $\eta\in End(u^{\otimes k})$, with $|k|=3$. For details here, and for more on these two quantum groups, which are very interesting objects, and that we have actually already met in section 4 above, we refer to the paper of Raum-Weber [120]. □

In order to discuss now the Schur-Weyl twisting of the various quantum reflection groups that we have, we will need the following technical result:

Proposition 11.15. *We have the following equalities,*

$$\begin{aligned}
P_{even}^* &= \left\{\pi\in P_{even}\Big|\varepsilon(\tau)=1,\forall\tau\leq\pi,|\tau|=2\right\}\\
P_{even}^{[\infty]} &= \left\{\pi\in P_{even}\Big|\sigma\in P_{even}^*,\forall\sigma\subset\pi\right\}\\
P_{even}^{[\infty]} &= \left\{\pi\in P_{even}\Big|\varepsilon(\tau)=1,\forall\tau\leq\pi\right\}
\end{aligned}$$

where $\varepsilon:P_{even}\to\{\pm1\}$ is the signature of even permutations.

Proof. This is routine combinatorics, from [9], [120], the idea being as follows:

(1) Given $\pi\in P_{even}$, we have $\tau\leq\pi,|\tau|=2$ precisely when $\tau=\pi^\beta$ is the partition obtained from π by merging all the legs of a certain subpartition $\beta\subset\pi$, and by merging as well all the other blocks.

Now observe that π^β does not depend on π, but only on β, and that the number of switches required for making π^β noncrossing is $c=N_\bullet-N_\circ$ modulo 2, where $N_\bullet/N_\circ$ is the number of black/white legs of β, when labelling the legs of π counterclockwise $\circ\bullet\circ\bullet\ldots$

Thus $\varepsilon(\pi^\beta)=1$ holds precisely when $\beta\in\pi$ has the same number of black and white legs, and this gives the result.

(2) This simply follows from the equality $P_{even}^{[\infty]}=<\eta>$ coming from Proposition 11.14, by computing $<\eta>$, and for the complete proof here we refer to [120].

(3) We use the fact, also from [120], that the relations $g_ig_ig_j=g_jg_ig_i$ are trivially satisfied for real reflections. Thus, we have:

$$P_{even}^{[\infty]}(k,l)=\left\{\ker\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}\Big|g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l}\text{ inside }\mathbb Z_2^{*N}\right\}$$

In other words, the partitions in $P_{even}^{[\infty]}$ are those describing the relations between free variables, subject to the conditions $g_i^2=1$.

We conclude that $P_{even}^{[\infty]}$ appears from NC_{even} by "inflating blocks", in the sense that each $\pi\in P_{even}^{[\infty]}$ can be transformed into a partition $\pi'\in NC_{even}$ by deleting pairs of consecutive legs, belonging to the same block.

Now since this inflation operation leaves invariant modulo 2 the number $c\in\mathbb N$ of switches in the definition of the signature, it leaves invariant the signature $\varepsilon=(-1)^c$ itself, and we obtain in this way the inclusion "$\subset$" in the statement.

Conversely, given $\pi\in P_{even}$ satisfying $\varepsilon(\tau)=1$, $\forall\tau\leq\pi$, our claim is that:

$$\rho\leq\sigma\subset\pi,|\rho|=2\implies\varepsilon(\rho)=1$$

Indeed, let us denote by α,β the two blocks of ρ, and by γ the remaining blocks of π, merged altogether. We know that the partitions $\tau_1=(\alpha\wedge\gamma,\beta)$, $\tau_2=(\beta\wedge\gamma,\alpha)$, $\tau_3=(\alpha,\beta,\gamma)$ are all even. On the other hand, putting these partitions in noncrossing form requires respectively $s+t, s'+t, s+s'+t$ switches, where t is the number of switches needed for putting $\rho=(\alpha,\beta)$ in noncrossing form. Thus t is even, and we are done.

With the above claim in hand, we conclude, by using the second equality in the statement, that we have $\sigma\in P^*_{even}$. Thus we have $\pi\in P^{[\infty]}_{even}$, which ends the proof of "$\supset$". $\square$

With the above result in hand, we can now prove:

Theorem 11.16. *The basic quantum reflection groups, namely*

$$\begin{array}{ccccc}
H_N^+ & \longrightarrow & \mathbb{T}H_N^+ & \longrightarrow & K_N^+\\
\uparrow & & \uparrow & & \uparrow\\
H_N^* & \longrightarrow & \mathbb{T}H_N^* & \longrightarrow & K_N^*\\
\uparrow & & \uparrow & & \uparrow\\
H_N & \longrightarrow & \mathbb{T}H_N & \longrightarrow & K_N
\end{array}$$

equal their own Schur-Weyl twists.

Proof. This result, established in [8], basically comes from the results that we have:

(1) In the real case, the verifications are as follows:

– H_N^+. We know from Theorem 11.7 above that for $\pi\in NC_{even}$ we have $\bar{T}_\pi=T_\pi$, and since we are in the situation $D\subset NC_{even}$, the definitions of $G,\bar{G}$ coincide.

– $H_N^{[\infty]}$. Here we can use the same argument as in (1), based this time on the description of $P^{[\infty]}_{even}$ involving the signatures found in Proposition 11.15.

– H_N^*. We have $H_N^*=H_N^{[\infty]}\cap O_N^*$, so $\bar{H}_N^*\subset H_N^{[\infty]}$ is the subgroup obtained via the defining relations for $\bar{O}_N^*$. But all the $abc=-cba$ relations defining $\bar{H}_N^*$ are automatic, of type $0=0$, and it follows that $\bar{H}_N^*\subset H_N^{[\infty]}$ is the subgroup obtained via the relations $abc=cba$, for any $a,b,c\in\{u_{ij}\}$. Thus we have $\bar{H}_N^*=H_N^{[\infty]}\cap O_N^*=H_N^*$, as claimed.

– H_N. We have $H_N=H_N^*\cap O_N$, and by functoriality, $\bar{H}_N=\bar{H}_N^*\cap\bar{O}_N=H_N^*\cap\bar{O}_N$. But this latter intersection is easily seen to be equal to H_N, as claimed.

(2) In the complex case the proof is similar, and we refer here to [9]. $\square$

In relation now with the tori, we have the following result:

Theorem 11.17. *The diagonal tori of the twisted quantum groups are*

$$\begin{array}{ccccc}T_N^+ & \to & \mathbb{T}T_N^+ & \to & \mathbb{T}_N^+\\ \uparrow & & \uparrow & & \uparrow\\ T_N^* & \to & \mathbb{T}T_N^* & \to & \mathbb{T}_N^*\\ \uparrow & & \uparrow & & \uparrow\\ T_N & \to & \mathbb{T}T_N & \to & \mathbb{T}_N\end{array}$$

exactly as in the untwisted case.

Proof. This is clear for the quantum reflection groups, which are not twistable, and for the quantum unitary groups this is elementary as well, coming from definitions. □

Before getting into the spheres, let us discuss integration questions. The result here, valid for any Schur-Weyl twist in our sense, is as follows:

Theorem 11.18. *We have the Weingarten type formula*

$$\int_{\dot{G}} u_{i_1j_1}^{e_1}\ldots u_{i_kj_k}^{e_k}=\sum_{\pi,\sigma\in P_\times(\alpha)}\dot{\delta}_\pi(i_1\ldots i_k)\dot{\delta}_\sigma(j_1\ldots j_k)W_{kN}(\pi,\sigma)$$

where $W_{kN}=G_{kN}^{-1}$, with $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$, for $\pi,\sigma\in D(k)$.

Proof. This follows exactly as in the untwisted case, the idea being that the signs will cancel. Let us recall indeed from Definition 11.8 and the comments afterwards that the twisted vectors $\bar{\xi}_\pi$ associated to the partitions $\pi\in P_{even}(k)$ are as follows:

$$\bar{\xi}_\pi=\sum_{\tau\geq\pi}\varepsilon(\tau)\sum_{i:\ker(i)=\tau}e_{i_1}\otimes\ldots\otimes e_{i_k}$$

Thus, the Gram matrix of these vectors is given by:

$$\begin{aligned}<\xi_\pi,\xi_\sigma> &= \sum_{\tau\geq\pi\vee\sigma}\varepsilon(\tau)^2\left|\left\{(i_1,\ldots,i_k)\Big|\ker i=\tau\right\}\right|\\ &= \sum_{\tau\geq\pi\vee\sigma}\left|\left\{(i_1,\ldots,i_k)\Big|\ker i=\tau\right\}\right|\\ &= N^{|\pi\vee\sigma|}\end{aligned}$$

Thus the Gram matrix is the same as in the untwisted case, and so the Weingarten matrix is the same as well as in the untwisted case, and this gives the result. □

In relation now with the spheres, we have the following result:

Theorem 11.19. *The twisted spheres have the following properties:*

(1) *They have affine actions of the twisted unitary quantum groups.*
(2) *They have unique invariant Haar functionals, which are ergodic.*
(3) *Their Haar functionals are given by Weingarten type formulae.*
(4) *They appear, via the GNS construction, as first row spaces.*

Proof. The proofs here are similar to those from the untwisted case, via a long and routine computation, by adding signs where needed, and with the main technical ingredient, namely the Weingarten formula, being available from Theorem 11.18 above. ☐

As a conclusion now, we have shown that the various quadruplets (S,T,U,K) constructed in sections 1-10 above have twisted counterparts $(\bar{S},T,\bar{U},K)$. The question that we would like to solve now is that of finding correspondences, as follows:

$$\begin{tikzcd}
\bar{S} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
\bar{U} \arrow[r, leftrightarrow] & K
\end{tikzcd}$$

In order to discuss this, let us get back to the axiomatics from section 4. We have seen there that the 12 correspondences come in fact from 7 correspondences, as follows:

$$\begin{tikzcd}
S \arrow[r] \arrow[d, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl] \\
U \arrow[r] & K
\end{tikzcd}$$

In the twisted case, 6 of these correspondences hold as well, but the remaining one, namely $S \to T$, definitely does not hold as stated, and must be modified. Let us begin our discussion with the quantum isometry group results. We have here:

Theorem 11.20. *We have the quantum isometry group formula*

$$\bar{U} = G^+(\bar{S})$$

in all the 9 *main twisted cases.*

Proof. The proofs here are similar to those from the untwisted case, via a long and routine computation, by adding signs where needed, which amounts in replacing the usual commutators $[a,b] = ab - ba$ by twisted commutators, given by:

$$[[a,b]] = ab + ba$$

There is one subtle point, however, coming from the fact that the linear independence of various products of coordinates of length 1,2,3, which was something clear in the untwisted case, is now a non-trivial question. But this can be solved via a technical application of the Weingarten formula, from Theorem 11.18. For details here, we refer to [5]. □

Regarding now the $K = G^+(T) \cap K_N^+$ axiom, this is something that we already know. However, regarding the correspondence $S \to T$, things here fail in the twisted case. Our "fix" for this, or at least the best fix that we could find, is as follows:

Theorem 11.21. *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, define its toral isometry group as being the biggest subgroup of $\mathbb{T}_N^+$ acting affinely on X:*

$$\mathcal{G}^+(X) = G^+(X) \cap \mathbb{T}_N^+$$

With this convention, for the 9 basic spheres S, and for their twists as well, the toral isometry group equals the torus T.

Proof. We recall from section 3 that the affine quantum isometry group $G^+(X) \subset U_N^+$ of a noncommutative manifold $X \subset S^{N-1}_{\mathbb{C},+}$ coming from certain polynomial relations P is constructed according to the following procedure:

$$P(x_i) = 0 \implies P\left(\sum_j x_j \otimes u_{ji}\right) = 0$$

Similarly, the toral isometry group $\mathcal{G}^+(X) \subset \mathbb{T}_N^+$ is constructed as follows:

$$P(x_i) = 0 \implies P(x_i \otimes u_i) = 0$$

In the monomial case one can prove that the following formula holds:

$$G^+(\bar{S}) = \overline{G^+(S)}$$

By intersecting with $\mathbb{T}_N^+$, we obtain from this that we have:

$$\mathcal{G}^+(\bar{S}) = \mathcal{G}^+(S)$$

The result can be of course be proved as well directly. For $\bar{S}^{N-1}_{\mathbb{R}}$ we have:

$$\Phi(x_i x_j) = x_i x_j \otimes u_i u_j$$

$$\Phi(x_j x_i) = x_j x_i \otimes u_j u_i$$

Thus we obtain $u_i u_j = -u_j u_i$ for $i \neq j$, and so the quantum group is T_N.

The proof in the complex, half-liberated and hybrid cases is similar. □

Regarding the hard liberation axiom, this seems to hold indeed in all the cases under consideration, but this is non-trivial, and not known yet. As a conclusion, we conjecturally have an extension of our (S, T, U, K) formalism, with the $S \to T$ axiom needing a modification as above, which covers the twisted objects $(\bar{S}, T, \bar{U}, K)$ as well.

12. Matrix models

We have seen in section 9 above that a useful technique for the study of the half-classical manifolds, $X\subset S^{N-1}_{\mathbb{C},*}$, is that of modelling the standard coordinates $x_1,\ldots,x_N\in C(X)$ by certain explicit variables, namely some suitable antidiagonal 2×2 matrices $T_1,\ldots,T_N\in M_2(C(Y))$, over a certain classical manifold Y, associated to X.

In this section we discuss modelling questions for the general manifolds $X\subset S^{N-1}_{\mathbb{C},+}$. Let us first recall the GNS representation theorem, in a detailed form:

Theorem 12.1. *Any C^*-algebra A appears as closed $*$-algebra of operators on a Hilbert space, $A\subset B(H)$, in the following way:*

1. *In the commutative case, where $A=C(X)$, we can set $H=L^2(X)$, with respect to some probability measure on X, and use the embedding $g\to(g\to fg)$.*
2. *In general, we can set $H=L^2(A)$, with respect to some faithful positive trace $tr:A\to\mathbb{C}$, and then use a similar embedding, $a\to(b\to ab)$.*

Proof. This is something that we already know, from section 1 above, coming from basic measure theory and functional analysis, the idea being as follows:

(1) This is something elementary, modulo the fact that any compact space X has a probability measure, which follows from basic measure theory.

(2) Here the subtle point is the construction of the trace $tr:A\to\mathbb{C}$, which can be done via abstract functional analysis methods. □

In the case of the algebras $A=C(X)$ with $X\subset S^{N-1}_{\mathbb{C},+}$ that we are interested in, the above result tells us that we can always find operators $T_i\in B(H)$ which model the standard coordinates $x_i\in C(X)$. To be more precise, we have:

Proposition 12.2. *Given an algebraic manifold $X\subset S^{N-1}_{\mathbb{C},+}$, coming via*

$$C(X)=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle f_\alpha(x_1,\ldots,x_N)=0\Big\rangle$$

we have a morphism of C^*-algebras as follows, whenever the operators $T_i\in B(H)$ satisfy the relations $\sum_i T_iT_i^*=\sum_i T_i^*T_i=1$ and $f_\alpha(T_1,\ldots,T_N)=0$,

$$\pi:C(X)\to B(H)\quad,\quad x_i\to T_i$$

and we can always find a Hilbert space H and operators (T_i) such that π is faithful.

Proof. Here the first assertion is more of an empty statement, explaining the definition of the algebra $C(X)$, via generators and relations, and the second assertion is something non-trivial, coming as a consequence of the GNS theorem. □

In practice, all this is a bit too general, and not very useful. However, and here comes our point, by replacing the operator algebra models $C(X) \to B(H)$ by suitable models of type $C(X) \to B$, with B being a C^*-algebra, not necessarily equal to a full operator algebra over a Hilbert space, we are led to some interesting and useful theory.

In order to discuss this, we need a good family of target algebras B, that can we can say that we understand very well. And here, we can use:

Definition 12.3. *A random matrix C^*-algebra is an algebra of type*

$$B = M_K(C(T))$$

with T being a compact space, and $K \in \mathbb{N}$ being an integer.

The terminology here comes from the fact that, in practice, the space T usually comes with a probability measure on it, which makes the elements of B "random matrices". Observe that we can write our random matrix algebra as follows:

$$B = M_K(\mathbb{C}) \otimes C(T)$$

Thus, the random matrix algebras appear by definition as tensor products of the simplest types of C^*-algebras that we know, namely the full matrix algebras, $M_K(\mathbb{C})$ with $K \in \mathbb{N}$, and the commutative algebras, $C(T)$, with T being a compact space.

Getting back now to our modelling questions for manifolds, we can formulate:

Definition 12.4. *A matrix model for a noncommutative algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$ is a morphism of C^*-algebras of the following type,*

$$\pi : C(X) \to M_K(C(T))$$

with T being a compact space, and $K \in \mathbb{N}$ being an integer.

As a first observation, when X happens to be classical, we can take $K = 1$ and $T = X$, and we have a faithful model for our manifold, namely:

$$id : C(X) \to M_1(C(X))$$

In general, we will be looking of course for faithful models for our manifolds, or at least for models having some suitable, weaker faithfulness properties. For this purpose we cannot use of course $K = 1$, and the smallest value $K \in \mathbb{N}$ doing the job, if any, will correspond somehow to the "degree of noncommutativity" of our manifold.

Before getting into all this, we would like to clarify a few more abstract issues. As mentioned above, the C^*-algebras of type $B = M_K(C(T))$ are called "random matrix C^*-algebras". The reason for this is the fact that most of the interesting compact spaces T come by definition with a natural probability measure of them. Thus, B is a subalgebra of the algebra $B'' = M_K(L^\infty(T))$, usually known as a "random matrix algebra".

This perspective is quite interesting for us, because most of our examples of manifolds $X \subset X^{N-1}_{\mathbb{C},+}$ appear as homogeneous spaces, and so are measured spaces too. Thus, we can further ask for our models $C(X) \to M_K(C(T))$ to extend into models of type $L^\infty(X) \to M_K(L^\infty(T))$, which can help in connection with integration problems.

In short, time now to talk about L^∞-functions, in the noncommutative setting. In order to discuss all this, we will need some basic von Neumann algebra theory, coming as a complement to the C^*-algebra theory from section 1 above. Let us start with:

Proposition 12.5. *For an operator algebra $A \subset B(H)$, the following are equivalent:*

1. *A is closed under the weak operator topology, making each of the linear maps $T \to < Tx, y >$ continuous.*
2. *A is closed under the strong operator topology, making each of the linear maps $T \to Tx$ continuous.*

In the case where these conditions are satisfied, A is closed under the norm topology.

Proof. There are several statements here, the proof being as follows:

(1) It is clear that the norm topology is stronger than the strong operator topology, which is in turn stronger than the weak operator topology. At the level of the subsets $S \subset B(H)$ which are closed things get reversed, in the sense that weakly closed implies strongly closed, which in turn implies norm closed. Thus, we are left with proving that for any algebra $A \subset B(H)$, strongly closed implies weakly closed.

(2) But this latter fact is something standard, which can be proved via an amplification trick. Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \ldots \oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \to B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

Assume now that we are given an operator $T \in \bar{A}$, with the bar denoting the weak closure. We have then, by using the Hahn-Banach theorem, for any $x \in K$:

$$\begin{aligned} T \in \bar{A} &\implies \pi(T) \in \overline{\pi(A)} \\ &\implies \pi(T)x \in \overline{\pi(A)x} \\ &\implies \pi(T)x \in \overline{\pi(A)x}^{||.||} \end{aligned}$$

Now observe that the last formula tells us that for any $x = (x_1, \ldots, x_n)$, and any $\varepsilon > 0$, we can find $S \in A$ such that the following holds, for any i:

$$||Sx_i - Tx_i|| < \varepsilon$$

Thus T belongs to the strong operator closure of A, as desired. □

In the above statement the terminology, while standard, is a bit confusing, because the norm topology is stronger than the strong operator topology. As a solution to this, we agree to call the norm topology "strong", and the weak and strong operator topologies "weak", whenever these two topologies coincide. With this convention, the algebras from Proposition 12.5 are those which are weakly closed, and we can formulate:

Definition 12.6. *A von Neumann algebra is a $*$-algebra of operators*

$$A \subset B(H)$$

which is closed under the weak topology.

As basic examples, we have the algebra $B(H)$ itself, then the singly generated von Neumann algebras, $A =< T >$, with $T \in B(H)$, and then the multiply generated von Neumann algebras, namely $A =< T_i >$, with $T_i \in B(H)$. There are many other examples, and general methods for constructing examples, and we will discuss this later.

At the level of the general results, we first have the bicommutant theorem of von Neumann, which provides a useful alternative to Definition 12.6, as follows:

Theorem 12.7. *For a $*$-algebra $A \subset B(H)$, the following are equivalent:*

1. *A is weakly closed, so it is a von Neumann algebra.*
2. *A equals its algebraic bicommutant A'', taken inside $B(H)$.*

Proof. Since the commutants are automatically weakly closed, it is enough to show that weakly closed implies $A = A''$. For this purpose, we will prove something a bit more general, stating that given a $*$-algebra of operators $A \subset B(H)$, the following holds, with A'' being the bicommutant inside $B(H)$, and with $\bar{A}$ being the weak closure:

$$A'' = \bar{A}$$

We prove this equality by double inclusion, as follows:

"$\supset$" Since any operator commutes with the operators that it commutes with, we have a trivial inclusion $S \subset S''$, valid for any set $S \subset B(H)$. In particular, we have:

$$A \subset A''$$

Our claim now is that the algebra A'' is closed, with respect to the strong operator topology. Indeed, assuming that we have $T_i \to T$ in this topology, we have:

$$\begin{aligned} T_i \in A'' &\implies ST_i = T_iS, \ \forall S \in A' \\ &\implies ST = TS, \ \forall S \in A' \\ &\implies T \in A \end{aligned}$$

Thus our claim is proved, and together with Proposition 12.5, which allows us to pass from the strong to the weak operator topology, this gives the desired inclusion:

$$\bar{A} \subset A''$$

"$\subset$" Here we must prove that we have the following implication, valid for any $T \in B(H)$, with the bar denoting as usual the weak operator closure:

$$T \in A'' \implies T \in \bar{A}$$

For this purpose, we use the same amplification trick as in the proof of Proposition 12.5 above. Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \ldots \oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \to B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T && \\ & \ddots & \\ && T \end{pmatrix}$$

The idea will be that of doing the computations in this representation. First, in this representation, the image of our algebra $A \subset B(H)$ is given by:

$$\pi(A) = \left\{ \begin{pmatrix} T && \\ & \ddots & \\ && T \end{pmatrix} \Bigg| T \in A \right\}$$

We can compute the commutant of this image, exactly as in the usual scalar matrix case, and we obtain the following formula:

$$\pi(A)' = \left\{ \begin{pmatrix} S_{11} & \ldots & S_{1n} \\ \vdots && \vdots \\ S_{n1} & \ldots & S_{nn} \end{pmatrix} \Bigg| S_{ij} \in A' \right\}$$

We conclude from this that, given an operator $T \in A''$ as above, we have:

$$\begin{pmatrix} T && \\ & \ddots & \\ && T \end{pmatrix} \in \pi(A)''$$

In other words, the conclusion of all this is that we have:

$$T \in A'' \implies \pi(T) \in \pi(A)''$$

Now given a vector $x \in K$, consider the orthogonal projection $P \in B(K)$ on the norm closure of the vector space $\pi(A)x \subset K$. Since the subspace $\pi(A)x \subset K$ is invariant under the action of $\pi(A)$, so is its norm closure inside K, and we obtain from this:

$$P \in \pi(A)'$$

By combining this with what we found above, we conclude that we have:

$$T \in A'' \implies \pi(T)P = P\pi(T)$$

Now since this holds for any $x \in K$, we conclude that any $T \in A''$ belongs to the strong operator closure of A. By using now Proposition 12.5, which allows us to pass from the strong to the weak operator closure, we conclude that we have $A'' \subset \bar{A}$, as desired. □

As an interesting consequence of Theorem 12.7, we have the following result:

Proposition 12.8. *Given a von Neumann algebra $A \subset B(H)$, its center*

$$Z(A) = A \cap A'$$

regarded as an algebra $Z(A) \subset B(H)$, is a von Neumann algebra too.

Proof. This follows from the fact that the commutants are weakly closed, that we know from the above, which shows that $A' \subset B(H)$ is a von Neumann algebra. Thus, the intersection $Z(A) = A \cap A'$ must be a von Neumann algebra too, as claimed. □

In order to develop now some general theory, let us start by investigating the finite dimensional case. Here the ambient operator algebra is $B(H) = M_N(\mathbb{C})$, and any subspace $A \subset B(H)$ is automatically closed, for all 3 topologies from Proposition 12.5 above.

Thus, we are left with the question of investigating the $*$-algebras of usual matrices $A \subset M_N(\mathbb{C})$. But this is a purely algebraic question, whose answer is as follows:

Theorem 12.9. *The $*$-algebras $A \subset M_N(\mathbb{C})$ are exactly the algebras of the form*

$$A = M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C})$$

depending on parameters $k \in \mathbb{N}$ and $r_1, \ldots, r_k \in \mathbb{N}$ satisfying

$$r_1 + \ldots + r_k = N$$

embedded into $M_N(\mathbb{C})$ via the obvious block embedding, twisted by a unitary $U \in U_N$.

Proof. We have two assertions to be proved, the idea being as follows:

(1) Given numbers $r_1, \ldots, r_k \in \mathbb{N}$ satisfying $r_1 + \ldots + r_k = N$, we have an obvious embedding of $*$-algebras, via matrix blocks, as follows:

$$M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

In addition, we can twist this embedding by a unitary $U \in U_N$, as follows:

$$M \to UMU^*$$

Thus, we have proved one of the implications.

(2) In the other sense now, consider an arbitrary $*$-algebra of the $N \times N$ matrices:

$$A \subset M_N(\mathbb{C})$$

Let us first look at the center of this algebra, which given by:

$$Z(A) = A \cap A'$$

It is elementary to prove that this center, as an algebra, is of the following form:

$$Z(A) \simeq \mathbb{C}^k$$

Consider now the standard basis $e_1, \ldots, e_k \in \mathbb{C}^k$, and let $p_1, \ldots, p_k \in Z(A)$ be the images of these vectors via the above identification. In other words, these elements $p_1, \ldots, p_k \in A$ are central minimal projections, summing up to 1:

$$p_1 + \ldots + p_k = 1$$

The idea is then that this partition of the unity will eventually lead to the block decomposition of A, as in the statement. We prove this in 4 steps, as follows:

Step 1. We first construct the matrix blocks, our claim here being that each of the following linear subspaces of A are non-unital $*$-subalgebras of A:

$$A_i = p_i A p_i$$

But this is clear, with the fact that each A_i is closed under the various non-unital $*$-subalgebra operations coming from the projection equations $p_i^2 = p_i = p_i^*$.

Step 2. We prove now that the above algebras $A_i \subset A$ are in a direct sum position, in the sense that we have a non-unital $*$-algebra sum decomposition, as follows:

$$A = A_1 \oplus \ldots \oplus A_k$$

As with any direct sum question, we have two things to be proved here. First, by using the formula $p_1 + \ldots + p_k = 1$ and the projection equations $p_i^2 = p_i = p_i^*$, we conclude that we have the needed generation property, namely:

$$A_1 + \ldots + A_k = A$$

As for the fact that the sum is indeed direct, this follows as well from the formula $p_1 + \ldots + p_k = 1$, and from the projection equations $p_i^2 = p_i = p_i^*$.

Step 3. Our claim now, which will finish the proof, is that each of the $*$-subalgebras $A_i = p_i A p_i$ constructed above is a full matrix algebra. To be more precise here, with $r_i = rank(p_i)$, our claim is that we have isomorphisms, as follows:

$$A_i \simeq M_{r_i}(\mathbb{C})$$

In order to prove this claim, recall that the projections $p_i \in A$ were chosen central and minimal. Thus, the center of each of the algebras A_i reduces to the scalars:

$$Z(A_i) = \mathbb{C}$$

But this shows, either via a direct computation, or via the bicommutant theorem, that the each of the algebras A_i is a full matrix algebra, as claimed.

Step 4. We can now obtain the result, by putting together what we have. Indeed, by using the results from Step 2 and Step 3, we obtain an isomorphism as follows:

$$\begin{aligned} A &= A_1 \oplus \ldots \oplus A_k \\ &\simeq M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C}) \end{aligned}$$

Moreover, a careful look at the isomorphisms established in Step 3 shows that at the global level, of the algebra A itself, the above isomorphism comes by twisting the standard multimatrix embedding $M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C}) \subset M_N(\mathbb{C})$, discussed in the beginning of the proof, (1) above, by a certain unitary $U \in U_N$. Thus, we obtain the result. □

As an application of Theorem 12.9, clarifying the relation with linear algebra, or operator theory in finite dimensions, we have the following result:

Proposition 12.10. *Given an operator $T \in B(H)$ in finite dimensions, $H = \mathbb{C}^N$, the von Neumann algebra $A =< T >$ that it generates inside $B(H) = M_N(\mathbb{C})$ is*

$$A = M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C})$$

with the sizes of the blocks $r_1, \ldots, r_k \in \mathbb{N}$ coming from the spectral theory of the associated matrix $M \in M_N(\mathbb{C})$. In the normal case $TT^* = T^*T$, this decomposition comes from

$$T = UDU^*$$

with $D \in M_N(\mathbb{C})$ diagonal, and with $U \in U_N$ unitary.

Proof. This is standard, by using the basic linear algebra theory and spectral theory for the usual matrices $M \in M_N(\mathbb{C})$. □

Let us get now to infinite dimensions, with Proposition 12.10 as our main source of inspiration. We have here the following result:

Theorem 12.11. *Given an operator $T \in B(H)$ which is normal,*

$$TT^* = T^*T$$

the von Neumann algebra $A =< T >$ that it generates inside $B(H)$ is

$$< T >= L^\infty(\sigma(T))$$

with $\sigma(T)$ being its spectrum, formed of numbers $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not invertible.

Proof. This is standard as well, by using the spectral theory for the normal operators $T \in B(H)$, coming from section 1 above. □

More generally, along the same lines, we have the following result, dealing this time with commuting families of normal operators:

Theorem 12.12. *Given operators $T_i \in B(H)$ which are normal, and which commute, the von Neumann algebra $A =< T_i >$ that these operators generates inside $B(H)$ is*

$$< T_i >= L^\infty(X)$$

with X being a certain measured space, associated to the family $\{T_i\}$.

Proof. This is once again routine, by using the spectral theory for the families of commuting normal operators $T_i \in B(H)$, coming from section 1 above. □

As an interesting abstract consequence of this, we have:

Theorem 12.13. *The commutative von Neumann algebras are the algebras of type*

$$A = L^\infty(X)$$

with X being a measured space.

Proof. We have two assertions to be proved, the idea being as follows:

(1) In one sense, we must prove that given a measured space X, we can realize the commutative algebra $A = L^\infty(X)$ as a von Neumann algebra, on a certain Hilbert space H. But this is something that we already know, coming from the multiplicity operators $T_f(g) = fg$ discussed in section 1 above, the representation being as follows:

$$L^\infty(X) \subset B(L^2(X))$$

(2) In the other sense, given a commutative von Neumann algebra $A \subset B(H)$, we must construct a certain measured space X, and an identification $A = L^\infty(X)$. But this follows from Theorem 12.12, because we can write our von Neumann algebra as follows:

$$A =< T_i >$$

To be more precise, A being commutative, any element $T \in A$ is normal. Thus, we can pick a basis $\{T_i\} \subset A$, and then we have $A =< T_i >$ as above, with $T_i \in B(H)$ being commuting normal operators. Thus Theorem 12.12 applies, and gives the result. □

The above result is not the end of the story with the commutative von Neumann algebras, because we still have to understand how a given commutative algebra $A = L^\infty(X)$ can be represented as an operator algebra, $A \subset B(H)$, over the various Hilbert spaces H. The answer here is that the commutative von Neumann algebras appear as $L^\infty(X) \subset B(L^2(X))$, up to a certain multiplicity, but we will not need this here.

Moving ahead now, we can combine Proposition 12.8 with Theorem 12.13, and by building along the lines of Theorem 12.9, but this time in infinite dimensions, we are led to the following statement, due to Murray-von Neumann and Connes:

Theorem 12.14. *Given a von Neumann algebra $A\subset B(H)$, if we write its center as*

$$Z(A)=L^\infty(X)$$

then we have a decomposition as follows, with the fibers A_x having trivial center:

$$A=\int_XA_x\,dx$$

Moreover, the factors, $Z(A)=\mathbb C$, can be basically classified in terms of the II_1 *factors, which are those satisfying* $\dim A=\infty$*, and having a faithful trace $tr:A\to\mathbb C$.*

Proof. This is something that we know to hold in finite dimensions, as a consequence of Theorem 12.9 above. In general, this is something heavy, the idea being as follows:

(1) This is von Neumann's reduction theory main result, whose statement is already quite hard to understand, and whose proof uses advanced functional analysis.

(2) This is heavy, due to Murray-von Neumann and Connes, the idea being that the other factors can be basically obtained via crossed product constructions. □

All the above was of course very brief. We recommend here the original papers of Murray-von Neumann and Connes, [111], [112], [138], [139], and then [64], [65].

We can now extend our noncommutative space setting, as follows:

Theorem 12.15. *Consider the category of "noncommutative measure spaces", having as objects the pairs (A,tr) consisting of a von Neumann algebra with a faithful trace, and with the arrows reversed, which amounts in writing $A=L^\infty(X)$ and $tr=\int_X$.*

1. *The category of usual measured spaces embeds into this category, and we obtain in this way the objects whose associated von Neumann algebra is commutative.*
2. *Each C^*-algebra given with a trace produces as well a noncommutative measure space, by performing the GNS construction, and taking the weak closure.*
3. *In what regards the finitely generated group duals, or more generally the compact matrix quantum groups, the corresponding identification is injective.*
4. *Even more generally, for noncommutative algebraic manifolds having an integratiuon functional, like the spheres, the identification is injective.*

Proof. This is clear indeed from the basic properties of the GNS construction, from Theorem 12.1, and from the general theory from Theorem 12.14. □

Before getting into matrix modelling questions, we would like to formulate the following result, that we announced long ago, in section 1 above, but had not discussed yet:

Theorem 12.16. *In the context of the noncommutative geometries coming from quadruplets (S,T,U,K), we have von Neumann algebras, with traces, as follows,*

$$\begin{tikzcd}
L^\infty(S) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & L^\infty(T) \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
L^\infty(U) \arrow[r, leftrightarrow] & L^\infty(K)
\end{tikzcd}$$

with $L^\infty(S) \subset L^\infty(U)$ being obtained by taking the first row algebra.

Proof. This follows indeed from the results that we already have, by using the general formalism from Theorem 12.15. $\square$

In relation now with the modelling questions, we can now go ahead with our program, and discuss von Neumann algebraic extensions. We have the following result:

Theorem 12.17. *Given a matrix model $\pi : C(X) \to M_K(C(T))$, with both X, T being assumed to have integration functionals, the following are equivalent:*

(1) *π is stationary, in the sense that $\int_X = (tr \otimes \int_T)\pi$.*
(2) *π produces an inclusion $\pi' : C_{red}(X) \subset M_K(X(T))$.*
(3) *π produces an inclusion $\pi'' : L^\infty(X) \subset M_K(L^\infty(T))$.*

Moreover, in the quantum group case, these conditions imply that π is faithful.

Proof. This is standard functional analysis. Consider indeed the following diagram, with all solid arrows being the canonical maps between the algebras concerned:

$$\begin{tikzcd}
M_K(C(T)) \arrow[rr] & & M_K(L^\infty(T)) \\
C(X) \arrow[r] \arrow[u, "\pi"] & C_{red}(X) \arrow[r] \arrow[ul, dotted, "\pi'"'] & L^\infty(X) \arrow[u, dotted, "\pi''"']
\end{tikzcd}$$

With this picture in hand, the implications $(1) \iff (2) \iff (3)$ are all clear, coming from the basic properties of the GNS construction, and of the von Neumann algebras.

As for the last assertion, this is something more subtle, coming from the fact that if $L^\infty(G)$ is of type I, as required by (3), then G must be coamenable. See [115]. $\square$

The above result raises a number of interesting questions, notably in what regards the extension of the last assertion, to the case of more general homogeneous spaces.

Before going further, we would like to record as well the following key result regarding the matrix models, valid so far in the quantum group case only:

Theorem 12.18. *Consider a matrix model $\pi : C(G) \to M_K(C(T))$ for a closed subgroup $G \subset U_N^+$, with T being assumed to be a compact probability space.*

(1) *There exists a smallest subgroup $G' \subset G$, producing a factorization of type:*
$$\pi : C(G) \to C(G') \to M_K(C(T))$$
The algebra $C(G')$ is called Hopf image of π.

(2) *When π is inner faithful, in the sense that $G = G'$, we have the formula*
$$\int_G = \lim_{k\to\infty} \sum_{r=1}^k \varphi^{*r}$$
where $\varphi = (tr \otimes \int_T)\pi$, and $\phi * \psi = (\phi \otimes \psi)\Delta$.

Proof. All this is well-known, but quite specialized, the idea being as follows:

(1) This follows by dividing the algebra $C(G)$ by a suitable ideal, namely the Hopf ideal generated by the kernel of the matrix model map $\pi : C(G) \to M_K(C(T))$.

(2) This follows by suitably adapting Woronowicz's proof for the existence and formula of the Haar integration functional from [149], to the matrix model situation. □

The above result is quite important, for a number of reasons. Indeed, as a main application of it, while the existence of a faithful matrix model $\pi : C(G) \subset M_K(C(T))$ forces the C^*-algebra $C(G)$ to be of type I, and so G to be coamenable, as already mentioned in the proof of Theorem 12.17 above, there is no known restriction coming from the existence of an inner faithful model $\pi : C(G) \to M_K(C(T))$. See [17], [60].

In the general manifold setting, talking about such things is in general not possible, unless our manifold X has some extra special structure, as for instance being an homogeneous space, in the spirit of the spaces discussed in sections 6-8 above.

Let us go back now to our basic notion of a matrix model, from Definition 12.4 above, and develop some more general theory, in that setting. We first have:

Proposition 12.19. *A 1×1 model for a manifold $X \subset S^{N-1}_{\mathbb{C},+}$ must come from a map*
$$p : T \to X_{class} \subset X$$
and π is faithful precisely when $X = X_{class}$, and when p is surjective.

Proof. According to our conventions, a 1×1 model for a manifold $X \subset S^{N-1}_{\mathbb{C},+}$ is simply a morphism of C^*-algebras as follows:
$$\pi : C(X) \to C(T)$$

Now since the algebra $C(T)$ is commutative, this morphism must factorize through the abelianization of $C(X)$, as follows:
$$\pi : C(X) \to C(X_{class}) \to C(T)$$

Thus, our morphism π must come by transposition from a map p, as claimed. $\square$

In order to generalize the above trivial fact, we use the following definition:

Definition 12.20. *Let $X \subset S^{N-1}_{\mathbb{C},+}$. We define a closed subspace $X^{(K)} \subset X$ by*

$$C(X^{(K)}) = C(X)/J_K$$

where J_K is the common null space of matrix representations of $C(X)$, of size $L \leq K$,

$$J_K = \bigcap_{L \leq K} \bigcap_{\pi : C(X) \to M_L(\mathbb{C})} \ker(\pi)$$

and we call $X^{(K)}$ the "part of X which is realizable with $K \times K$ models".

As a basic example here, the first such space, at $K = 1$, is the classical version:

$$X^{(1)} = X_{class}$$

Observe that we have embeddings of quantum spaces, as follows:

$$X^{(1)} \subset X^{(2)} \subset X^{(3)} \ldots \ldots \subset X$$

As a first result now on these spaces, we have the following well-known fact:

Theorem 12.21. *The increasing union of compact quantum spaces*

$$X^{(\infty)} = \bigcup_{K \geq 1} X^{(K)}$$

equals X precisely when the algebra $C(X)$ is residually finite dimensional.

Proof. This is something well-known, coming from the general theory from [139]. We refer to [58] for a discussion on this topic, in the context of the quantum groups. $\square$

Getting back now to the case $K < \infty$, we first have, following [18]:

Proposition 12.22. *Consider an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$.*

1. *Given a closed subspace $Y \subset X \subset S^{N-1}_{\mathbb{C},+}$, we have $Y \subset X^{(K)}$ precisely when any irreducible representation of $C(Y)$ has dimension $\leq K$.*
2. *In particular, we have $X^{(K)} = X$ precisely when any irreducible representation of $C(X)$ has dimension $\leq K$.*

Proof. This follows by using the general C^*-algebra theory, as follows:

(1) If any irreducible representation of $C(Y)$ has dimension $\leq K$, then we have $Y \subset X^{(K)}$, because the irreducible representations of a C^*-algebra separate its points.

Conversely, assuming $Y \subset X^{(K)}$, it is enough to show that any irreducible representation of the algebra $C(X^{(K)})$ has dimension $\leq K$. But this is once again well-known.

(2) This follows indeed from (1). $\square$

The connection with the previous considerations comes from:

Theorem 12.23. *If $X\subset S^{N-1}_{\mathbb C,+}$ has a faithful matrix model*
$$C(X)\to M_K(C(T))$$
then we have $X=X^{(K)}$.

Proof. This follows from the above and from standard representation theory of the C^*-algebras. For full details on all this, we refer to [18]. $\square$

We now discuss the universal $K\times K$-matrix model, constructed as follows:

Theorem 12.24. *Given $X\subset S^{N-1}_{\mathbb C,+}$ algebraic, the category of its $K\times K$ matrix models, with $K\geq 1$ being fixed, has a universal object as follows:*
$$\pi_K:C(X)\to M_K(C(T_K))$$
That is, given a matrix model
$$\rho:C(X)\to M_K(C(T))$$
we have a diagram of type
$$\begin{tikzcd}
C(X) \arrow[rr, "\pi"] \arrow[dr, "\rho"'] & & M_K(C(T_K)) \arrow[dl] \\
& M_K(C(T)) &
\end{tikzcd}$$
where the map on the right is unique and arises from a continuous map $T\to T_K$.

Proof. Consider the universal commutative C^*-algebra generated by elements $x_{ij}(a)$, with $1\leq i,j\leq K$, $a\in\mathcal O(X)$, subject to the relations ($a,b\in\mathcal O(X)$, $\lambda\in\mathbb C$, $1\leq i,j\leq K$):
$$x_{ij}(a+\lambda b)=x_{ij}(a)+\lambda x_{ij}(b)$$
$$x_{ij}(ab)=\sum_k x_{ik}(a)x_{kj}(b)$$
$$x_{ij}(1)=\delta_{ij}$$
$$x_{ij}(a)^*=x_{ji}(a^*)$$
This is indeed well-defined because of the following relations:
$$\sum_l\sum_k x_{ik}(z_l^*)x_{ki}(z_l)=1$$
Let T_K be the spectrum of this C^*-algebra. Since X is algebraic, we have:
$$\pi:C(X)\to M_K(C(T_K))$$
$$\pi(z_k)=(x_{ij}(z_k))$$
By construction of T_K and π, we have the universal matrix model. See [18]. $\square$

Getting now to the case of the algebraic manifolds, we first have here:

Proposition 12.25. *Let $X\subset S^{N-1}_{\mathbb C,+}$ with X algebraic and $X_{class}\neq\emptyset$, and let*

$$\pi:C(X)\to M_K(C(T_K))$$

be the universal matrix model. Then we have

$$C(X^{(K)})=C(X)/Ker(\pi)$$

and hence $X=X^{(K)}$ if and only if X has a faithful $K\times K$-matrix model.

Proof. We have to show that $Ker(\pi)=J_K$, the latter ideal being the intersection of the kernels of all matrix representations $C(X)\to M_L(\mathbb C)$, for any $L\leq K$. For $a\notin Ker(\pi)$, we see that $a\notin J_K$ by evaluating at an appropriate element of T_K.

Conversely, assume that we are given $a\in Ker(\pi)$. Let $\rho:C(X)\to M_L(\mathbb C)$ be a representation with $L\leq K$, and let $\varepsilon:C(X)\to\mathbb C$ be a representation. We can extend ρ to a representation $\rho':C(X)\to M_K(\mathbb C)$ by letting, for any $b\in C(X)$:

$$\rho'(b)=\begin{pmatrix}\rho(b)&0\\0&\varepsilon(b)I_{K-L}\end{pmatrix}$$

The universal property of the universal matrix model yields that $\rho'(a)=0$, since $\pi(a)=0$. Thus $\rho(a)=0$. We therefore have $a\in J_K$, and $Ker(\pi)\subset J_K$, and the first statement is proved. The last statement follows from the first one. See [18]. ☐

Next, we have the following result, also from [18]:

Proposition 12.26. *Let $X\subset S^{N-1}_{\mathbb C,+}$ be algebraic, and satisfying:*

$$X_{class}\neq\emptyset$$

Then $X^{(K)}$ is algebraic as well.

Proof. We keep the notations above, and consider the following map:

$$\pi_0:\mathcal O(X)\to M_K(C(T_K))$$

$$z_l\to(x_{ij}(z_l))$$

This induces a $*$-algebra map, as follows:

$$\tilde\pi_0:C^*(\mathcal O(X)/Ker(\pi_0))\to M_K(C(T_K))$$

We need to show that $\tilde\pi_0$ is injective. For this purpose, observe that the universal model factorizes as follows, where p is canonical surjection:

$$\pi:C(X)\xrightarrow{p}C^*(\mathcal O(X)/Ker(\pi_0))\xrightarrow{\tilde\pi_0}M_K(C(T_K))$$

We therefore obtain $Ker(\pi)=Ker(p)$, and we conclude that:

$$\begin{aligned}C(X^{(K)})&=C(X)/Ker(p)\\&=C^*(\mathcal O(X)/Ker(\pi_0))\end{aligned}$$

Thus $X^{(K)}$ is indeed algebraic. Since $\mathcal{O}(X)/Ker(\pi_0)$ is isomorphic to a $*$-subalgebra of $M_K(C(T_K))$, it satisfies the standard Amitsur-Levitski polynomial identity:

$$S_{2K}(x_1,\ldots,x_{2K})=0$$

By density, so does $C^*(\mathcal{O}(X)/Ker(\pi_0))$.

Thus any irreducible representation of $C^*(\mathcal{O}(X)/Ker(\pi_0))$ has dimension $\leq K$. Now if $a\in C^*(\mathcal{O}(X)/Ker(\pi_0))$ is a nonzero element, we can, by the same reasoning as in the previous proof, find a representation as follows, such that $\rho(a)\neq 0$:

$$\rho:C^*(\mathcal{O}(X)/Ker(\pi_0))\to M_K(\mathbb{C})$$

Indeed, given algebra map $\varepsilon:C(X)\to\mathbb{C}$ induces an algebra map:

$$C(T_K)\to\mathbb{C}$$

$$x_{ij}(a)\to\delta_{ij}\varepsilon(a)$$

This map enables us to extend representations similarly as before.

By construction the universal model space yields an algebra map as follows:

$$M_K(C(T_K))\to M_K(\mathbb{C})$$

The composition of this map with $\tilde{\pi}_0 p=\pi$ is ρp, so $\tilde{\pi}_0(a)\neq 0$, and $\tilde{\pi}_0$ is injective. □

Summarizing, we have proved the following result:

Theorem 12.27. *Let $X\subset S^{N-1}_{\mathbb{C},+}$ be algebraic, satisfying:*

$$X_{class}\neq\emptyset$$

Then we have an increasing sequence of algebraic submanifolds

$$X_{class}=X^{(1)}\subset X^{(2)}\subset X^{(3)}\subset\ldots\ldots\subset X$$

where $X^{(K)}$ is given by the fact that

$$C(X^{(K)})\subset M_K(C(T_K))$$

is obtained by factorizing the universal matrix model.

Proof. This follows indeed from the above results. See [18]. □

As an illustration, let us discuss the half-liberation operation, which is connected to $X^{(2)}$. We restrict the attention to the real case. Let us start with:

Definition 12.28. *The half-classical version of a manifold X is given by:*

$$C(X^*)=C(X)\Big/\Big\langle abc=cba\Big|\forall a,b,c\in\{x_i\}\Big\rangle$$

We say that X is half-classical when $X=X^*$.

In order to understand the structure of X^*, we use an old matrix model method, which goes back to [48], and then to [47]. This is based on the following observation:

Proposition 12.29. *For any $z\in\mathbb C^N$, the matrices*

$$X_i=\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}$$

are self-adjoint, and half-commute.

Proof. This is something elementary, that we know from section 9 above. □

In order to connect the algebra of the classical coordinates z_i to that of the noncommutative coordinates X_i, we will need an abstract definition, as follows:

Definition 12.30. *Given a noncommutative polynomial $f\in\mathbb R<x_1,\ldots,x_N>$, we define a usual polynomial $f^\circ\in\mathbb R[z_1,\ldots,z_N,\bar{z}_1,\ldots,\bar{z}_N]$ by setting*

$$f=x_{i_1}x_{i_2}x_{i_3}x_{i_4}\ldots\implies f^\circ=z_{i_1}\bar{z}_{i_2}z_{i_3}\bar{z}_{i_4}\ldots$$

in the monomial case, and then by extending this correspondence, by linearity.

As a basic example here, the polynomial defining the free real sphere $S^{N-1}_{\mathbb R,+}$ produces in this way the polynomial defining the complex sphere $S^{N-1}_{\mathbb C}$:

$$f=x_1^2+\ldots+x_N^2\implies f^\circ=|z_1|^2+\ldots+|z_N|^2$$

Given a polynomial $f\in\mathbb R<x_1,\ldots,x_N>$, we can decompose it into its even and odd parts, $f=g+h$, by putting into g/h the monomials of even/odd length. Observe that with $z=(z_1,\ldots,z_N)$, these odd and even parts are given by:

$$g(z)=\frac{f(z)+f(-z)}{2}\quad,\quad h(z)=\frac{f(z)-f(-z)}{2}$$

With these conventions, we have the following result:

Proposition 12.31. *Given a manifold X, coming from a family of polynomials*

$$\{f_\alpha\}\subset\mathbb R<x_1,\ldots,x_N>$$

we have a morphism of unital C^*-algebras as follows,

$$\pi:C(X)\to M_2(\mathbb C)\quad,\quad \pi(x_i)=\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}$$

precisely when $z=(z_1,\ldots,z_N)\in\mathbb C^N$ belongs to the real algebraic manifold

$$Y=\left\{z\in\mathbb C^N\Big|g_\alpha^\circ(z_1,\ldots,z_N)=h_\alpha^\circ(z_1,\ldots,z_N)=0,\forall\alpha\right\}$$

where $f_\alpha=g_\alpha+h_\alpha$ is the even/odd decomposition of f_α.

Proof. Let X_i be the matrices in the statement. In order for $x_i\to X_i$ to define a morphism of algebras, these matrices must satisfy the equations defining X. Thus, the model space Z in the statement consists of those points $z=(z_1,\ldots,z_N)\in\mathbb C^N$ satisfying:

$$f_\alpha(X_1,\ldots,X_N)=0\quad,\quad\forall\alpha$$

Now observe that the matrices X_i in the statement multiply as follows:

$$X_{i_1}X_{j_1}\ldots X_{i_k}X_{j_k}=\begin{pmatrix}z_{i_1}\bar{z}_{j_1}\ldots z_{i_k}\bar{z}_{j_k}&0\\0&\bar{z}_{i_1}z_{j_1}\ldots\bar{z}_{i_k}z_{j_k}\end{pmatrix}$$

$$X_{i_1}X_{j_1}\ldots X_{i_k}X_{j_k}X_{i_{k+1}}=\begin{pmatrix}0&z_{i_1}\bar{z}_{j_1}\ldots z_{i_k}\bar{z}_{j_k}z_{i_{k+1}}\\ \bar{z}_{i_1}z_{j_1}\ldots\bar{z}_{i_k}z_{j_k}\bar{z}_{i_{k+1}}&0\end{pmatrix}$$

We therefore obtain, in terms of the even/odd decomposition $f_\alpha=g_\alpha+h_\alpha$:

$$f_\alpha(X_1,\ldots,X_N)=\begin{pmatrix}g_\alpha^\circ(z_1,\ldots,z_N)&h_\alpha^\circ(z_1,\ldots,z_N)\\ \\ \overline{h_\alpha^\circ(z_1,\ldots,z_N)}&\overline{g_\alpha^\circ(z_1,\ldots,z_N)}\end{pmatrix}$$

Thus, we obtain the equations for Y from the statement. $\square$

Following now [47], we have the following result:

Theorem 12.32. *Given a half-classical manifold X which is symmetric, in the sense that all its defining polynomials f_α are even, its universal 2×2 antidiagonal model,*

$$\pi:C(X)\to M_2(C(Y))$$

where Y is the manifold constructed in Proposition 12.31, is faithful. In addition, the construction $X\to Y$ is such that X exists precisely when Y is compact.

Proof. We can proceed as in [47]. Indeed, the universal model π in the statement induces, at the level of projective versions, a certain representation:

$$C(PX)\to M_2(C(PY))$$

By using the multiplication formulae from the proof of Proposition 12.31, the image of this representation consists of diagonal matrices, and the upper left components of these matrices are the standard coordinates of PY. Thus, we have an isomorphism:

$$PX\simeq PY$$

We can conclude as in [47], by using a grading trick. See [47]. $\square$

13. Free coordinates

We discuss here and in the next 3 sections a number of more specialized questions, of algebraic, geometric, analytic and probabilistic nature. We will be interested in the main 9 examples of noncommutative geometries in our sense, which are as follows:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{T}\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{C}^N_* \arrow[u] \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

Our purpose will be that of going beyond the basic level, where we are now, with a number of results regarding the coordinates $x_1, \ldots, x_N$ of such spaces:

(1) A first question, which is algebraic, is that of understanding the precise relations satisfied by these coordinates. We will see that this is related to the question of unifying the twisted and untwisted geometries, via intersection.

(2) A second question, which is analytic, is that of understanding the fixed N behavior of these coordinates. This can be done via deformation methods. We will see as well that there is an unexpected link here with quantum permutations.

Let us begin by discussing algebraic aspects. This is something quite fundamental. Indeed, in the classical case, the algebraic manifolds X can be identified with the corresponding ideals of vanishing polynomials J, and the correspondence $X \leftrightarrow J$ is the foundation for all the known algebraic geometric theory, ancient or more modern.

In the free setting, things are in a quite primitive status, and a suitable theory of "noncommutative algebra", useful in connection with our present considerations, is so far missing. Computing J for the free spheres, and perhaps for some other spheres as well, is a problem which is difficult enough for us, and that we will investigate here.

As a starting point, we know that the above 9 geometries are easy, and looking in detail at this easiness property will be our first task. Let us first recall that we have:

Definition 13.1. *A geometry (S,T,U,K) is called easy when U,K are easy, and*

$$U = \{O_N, K\}$$

with the operation on the right being the easy generation operation.

In other words, the easiness condition asks of course for U,K to be easy, and asks as well for the following condition to be satisfied:

$$< O_N,K >= \{O_N,K\}$$

Here the operation on the right is the easy generation one, discussed in section 2 above, given at the level of the associated categories of partitions by:

$$D_{\{G,H\}}=D_G\cap D_H$$

The easy geometries in the above sense can be investigated by using:

Proposition 13.2. *An easy geometry is uniquely determined by a pair (D,E) of categories of partitions, which must be as follows,*

$$\mathcal{NC}_2\subset D\subset P_2$$

$$\mathcal{NC}_{even}\subset E\subset P_{even}$$

and which are subject to the following intersection and generation conditions,

$$D=E\cap P_2$$

$$E=< D,\mathcal{NC}_{even} >$$

and to the usual axioms for the associated quadruplet (S,T,U,K), where U,K are respectively the easy quantum groups associated to the categories D,E.

Proof. This statement simply comes from the following conditions:

$$U=\{O_N,K\}$$

$$K=U\cap K_N^+$$

To be more precise, let us look at Definition 13.1. The main condition there tells us that U,K must be easy, coming from certain categories D,E.

It is clear that D,E must appear as intermediate categories, as in the statement, and the fact that the intersection and generation conditions must be satisfied follows from:

$$U=\{O_N,K\}\iff D=E\cap P_2$$

$$K=U\cap K_N^+\iff E=< D,\mathcal{NC}_{even} >$$

Thus, we are led to the conclusion in the statement. □

Generally speaking, the idea now is that everything can be reformulated in terms of (D,E), which must satisfy the conditions in Proposition 13.2.

Instead of discussing the full reformulation, let us work out at least the construction of the quadruplet (S,T,U,K). In what regards the quantum groups, these come from the categories of partitions via Tannakian duality, as follows:

Theorem 13.3. *In the context of an easy geometry (S,T,U,K), we have:*

$$C(U)=C(U_N^+)\Big/\left\langle T_\pi\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall k,l,\forall\pi\in D(k,l)\right\rangle$$

Also, we have the following formula:

$$C(K)=C(K_N^+)\Big/\left\langle T_\pi\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall k,l,\forall\pi\in D(k,l)\right\rangle$$

In fact, these formulae simply follow from the fact that U is easy.

Proof. This is clear indeed by applying Tannakian duality, in its "soft" form, to the unitary quantum group U, and to the quantum reflection group K, with the remark that, in what regards K, this appears indeed as a quantum subgroup of K_N^+.

To be more precise, the Tannakian duality from [149], in its soft form from [106], which was discussed in section 2 above, states that for a closed subgroup $G\subset U_N$, with fundamental corepresentation denoted $v=(v_{ij})$, we have:

$$C(G)=C(U_N^+)\Big/\left\langle T\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall k,l,\forall T\in Hom(v^{\otimes k},v^{\otimes l}))\right\rangle$$

But in the easy case, and in particular for the quantum groups U,K that we are interested in, this gives the formulae in the statement. □

Regarding now the associated torus T, which is not exactly covered by the easy quantum group formalism, the result here is a bit different, as follows:

Theorem 13.4. *In the context of an easy geometry (S,T,U,K), we have:*

$$\Gamma=F_N\Big/\left\langle g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l}\Big|\forall i,j,k,l,\exists\pi\in D(k,l),\delta_\pi\begin{pmatrix}i\\ j\end{pmatrix}\neq0\right\rangle$$

In fact, this formula simply follows from the fact that U is easy.

Proof. Let us denote by $g_i=u_{ii}$ the standard coordinates on the associated torus T, and consider the diagonal matrix formed by these coordinates:

$$g=\begin{pmatrix}g_1&&\\&\ddots&\\&&g_N\end{pmatrix}$$

We have the following computation:

$$\begin{aligned}C(T)&=\left[C(U_N^+)\Big/\left\langle T_\pi\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall\pi\in D\right\rangle\right]\Big/\left\langle u_{ij}=0\Big|\forall i\neq j\right\rangle\\&=\left[C(U_N^+)\Big/\left\langle u_{ij}=0\Big|\forall i\neq j\right\rangle\right]\Big/\left\langle T_\pi\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall\pi\in D\right\rangle\\&=C^*(F_N)\Big/\left\langle T_\pi\in Hom(g^{\otimes k},g^{\otimes l})\Big|\forall\pi\in D\right\rangle\end{aligned}$$

Now observe that, with $g=diag(g_1,\ldots,g_N)$ as before, we have:

$$T_\pi g^{\otimes k}(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}\cdot g_{i_1}\ldots g_{i_k}$$

On the other hand, we have as well:

$$g^{\otimes l}T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}\cdot g_{j_1}\ldots g_{j_l}$$

Thus, the commutation relation $T_\pi\in Hom(g^{\otimes k},g^{\otimes l})$ reads:

$$\begin{aligned}&\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}\cdot g_{i_1}\ldots g_{i_k}\\ =&\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}\cdot g_{j_1}\ldots g_{j_l}\end{aligned}$$

Thus we obtain the formula in the statement, and the last assertion is clear. □

Finally, regarding the sphere S, which is not a quantum group, but rather an homogeneous space, here the result is a bit more complicated, as follows:

Theorem 13.5. *In the context of an easy geometry (S,T,U,K), we have*

$$C(S)=C(S^{N-1}_{\mathbb C,+})\Big/\left\langle x_{i_1}\ldots x_{i_k}=x_{j_1}\ldots x_{j_k}\Big|\forall i,j,k,l,\exists\pi\in D(k)\cap I_k,\delta_\pi\begin{pmatrix}i\\ j\end{pmatrix}\neq0\right\rangle$$

where the set on the right, $I_k\subset P_2(k,k)$, is the set of colored permutations.

Proof. This follows indeed from Theorem 13.3 above, by applying the construction $U\to S$, which amounts in taking the first row space. □

Let us discuss now an alternative take on these questions, following [33], based on the notion of "monomiality", which applies to the spheres, which are not easy.

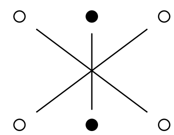
Looking back at the definition of the spheres that we have, and at the precise relations between the coordinates, we are led into the following notion:

Definition 13.6. *A monomial sphere is a subset $S\subset S^{N-1}_{\mathbb C,+}$ obtained via relations of type*

$$x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=x_{i_{\sigma(1)}}^{f_1}\ldots x_{i_{\sigma(k)}}^{f_k}\quad,\quad\forall(i_1,\ldots,i_k)\in\{1,\ldots,N\}^k$$

with $\sigma\in S_k$ being certain permutations, and with $e_r,f_r\in\{1,*\}$ being certain exponents.

This definition is quite broad, and we have for instance the sphere $S^{N-1}_{\mathbb{C},\times}$ coming from the relations $ab^*c = cb^*a$, corresponding to the following diagram:



This latter sphere is actually a quite interesting object, coming from the considerations in [42], [43]. However, while being monomial, this sphere does not exactly fit with our noncommutative geometry considerations here.

To be more precise, according to the work in [9], [19], this sphere is part of a triple $(S^{N-1}_{\mathbb{C},\times}, \mathbb{T}^\times_N, U^\times_N)$, satisfying a simplified set of noncommutative geometry axioms. However, according to the work in [107], [108], the quantum group $U^\times_N$ has no reflection group counterpart $K^\times_N$. Thus, this sphere does not exactly fit with our axiomatics here.

In view of these difficulties, we will now restrict the attention to the real case. Let us first recall, from the various classification results established above, that we have:

Theorem 13.7. *There are exactly* 3 *real easy geometries, whose associated spheres, tori and quantum unitary and reflection groups are as follows,*

$$S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$$

$$T_N \subset T_N^* \subset T_N^+$$

$$O_N \subset O_N^* \subset O_N^+$$

$$H_N \subset H_N^* \subset H_N^+$$

and whose associated categories of pairings and partitions D, E are as follows:

$$P_2 \supset P_2^* \supset NC_2$$

$$P_{even} \supset P^*_{even} \supset NC_{even}$$

Proof. This is something that we already know, coming from the fact that $G = O_N^*$ is the unique intermediate easy quantum group $O_N \subset G \subset O_N^+$. □

Let us focus now on the spheres, and try to understand their "easiness" property. That is, our objects of interest in what follows will be the 3 real spheres, namely:

$$S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$$

In order to talk about monomiality, it is convenient to introduce the following group:

$$S_\infty = \bigcup_{k\geq 0} S_k$$

To be more precise, this group appears by definition as an inductive limit, with the inclusions $S_k \subset S_{k+1}$ that we use being given by:

$$\sigma \in S_k \implies \sigma(k+1) = k+1$$

In terms of S_∞, the definition of the monomial spheres reformulates as follows:

Proposition 13.8. *The monomial spheres are the subsets $S \subset S^{N-1}_{\mathbb{R},+}$ obtained via relations*

$$x_{i_1} \ldots x_{i_k} = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}, \ \forall (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$$

associated to certain elements $\sigma \in S_\infty$, where $k \in \mathbb{N}$ is such that $\sigma \in S_k$.

Proof. We must prove that the relations $x_{i_1} \ldots x_{i_k} = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}$ are left unchanged when replacing $k \to k+1$. But this follows from $\sum_i x_i^2 = 1$, because:

$$\begin{aligned}
& x_{i_1} \ldots x_{i_k} x_{i_{k+1}} = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}} x_{i_{k+1}} \\
\implies \quad & x_{i_1} \ldots x_{i_k} x_{i_{k+1}}^2 = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}} x_{i_{k+1}}^2 \\
\implies \quad & \sum_{i_{k+1}} x_{i_1} \ldots x_{i_k} x_{i_{k+1}}^2 = \sum_{i_{k+1}} x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}} x_{i_{k+1}}^2 \\
\implies \quad & x_{i_1} \ldots x_{i_k} = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}
\end{aligned}$$

Thus we can indeed "simplify at right", and this gives the result. □

Following [33], our goal in what follows will be that of proving that the 3 main spheres are the only monomial ones, in our sense.

In order to prove this result, we will use group theory methods. We call a subgroup $G \subset S_\infty$ filtered when it is stable under concatenation, in the sense that when writing $G = (G_k)$ with $G_k \subset S_k$, we have the following formula:

$$\sigma \in G_k, \pi \in G_l \implies \sigma\pi \in G_{k+l}$$

With this convention, we have the following result:

Theorem 13.9. *The monomial spheres are the subsets $S_G \subset S^{N-1}_{\mathbb{R},+}$ given by*

$$C(S_G) = C(S^{N-1}_{\mathbb{R},+}) \Big/ \Big\langle x_{i_1} \ldots x_{i_k} = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}, \forall (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k, \forall \sigma \in G_k \Big\rangle$$

where $G = (G_k)$ is a filtered subgroup of $S_\infty = (S_k)$.

Proof. We know from Proposition 13.8 that the construction in the statement produces a monomial sphere. Conversely, given a monomial sphere $S \subset S^{N-1}_{\mathbb{R},+}$, let us set:

$$G_k = \left\{ \sigma \in S_k \Big| x_{i_1} \ldots x_{i_k} = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}, \forall (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k \right\}$$

With $G = (G_k)$ we have $S = S_G$. Thus, it remains to prove that G is a filtered group.

Since the relations $x_{i_1}\ldots x_{i_k}=x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$ can be composed and reversed, each G_k follows to be stable under composition and inversion, and is therefore a group.

Also, since the relations $x_{i_1}\ldots x_{i_k}=x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$ can be concatenated as well, our group $G=(G_k)$ is stable under concatenation, and we are done. $\square$

At the level of examples, the groups $\{1\}\subset S_\infty$ produce the following spheres:

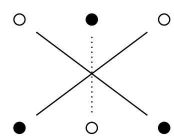
$$S^{N-1}_{\mathbb{R},+}\supset S^{N-1}_{\mathbb{R}}$$

In order to discuss now the half-liberated case, we will need:

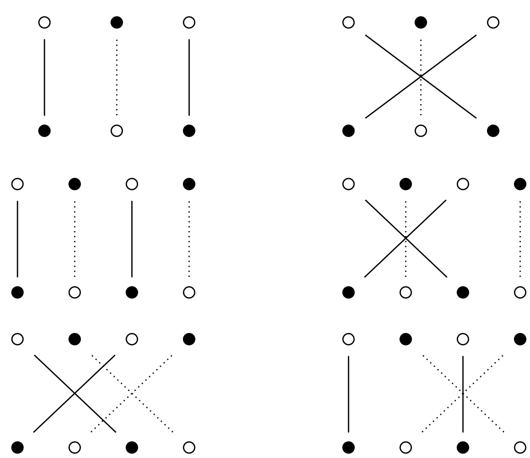
Proposition 13.10. *Let $S_\infty^*\subset S_\infty$ be the set of permutations having the property that when labelling cyclically the legs $\bullet\circ\bullet\circ\ldots$, each string joins a black leg to a white leg.*

1. *S_∞^* is a filtered subgroup of S_∞, generated by the half-liberated crossing.*
2. *We have $S_{2k}^*\simeq S_k\times S_k$, and $S_{2k+1}^*\simeq S_k\times S_{k+1}$, for any $k\in\mathbb{N}$.*

Proof. The fact that S_∞^* is indeed a subgroup of S_∞, which is filtered, is clear. Observe now that the half-liberated crossing has the "black-to-white" joining property:



Thus this crossing belongs to S_3^*, and it is routine to check, by double inclusion, that the filtered subgroup of S_∞ generated by it is the whole S_∞^*. Regarding now the last assertion, observe first that S_3^*,S_4^* consist of the following permutations:



Thus we have $S_3^*=S_1\times S_2$ and $S_4^*=S_2\times S_2$, with the first component coming from dotted permutations, and with the second component coming from the solid line permutations. The same argument works in general, and gives the last assertion. $\square$

Now back to the main 3 real spheres, the result is as follows:

Proposition 13.11. *The basic monomial real spheres, namely*

$$S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$$

come respectively from the filtered groups

$$S_\infty \supset S_\infty^* \supset \{1\}$$

via the above correspondence.

Proof. This is clear by definition in the classical and in the free cases. In the half-liberated case, the result follows from Proposition 13.10 (1) above. □

Now back to the general case, consider a monomial sphere $S_G \subset S^{N-1}_{\mathbb{R},+}$, with the filtered group $G \subset S_\infty$ taken to be maximal, as in the proof of Theorem 13.9. We have:

Proposition 13.12. *The filtered group $G \subset S_\infty$ associated to a monomial sphere $S \subset S^{N-1}_{\mathbb{R},+}$ is stable under the following operations, on the corresponding diagrams:*

1. *Removing outer strings.*
2. *Removing neighboring strings.*

Proof. Both these results follow by using the quadratic condition:

(1) Regarding the outer strings, by summing over a, we have:

$$\begin{aligned} Xa = Ya &\implies Xa^2 = Ya^2 \\ &\implies X = Y \end{aligned}$$

We have as well the following computation:

$$\begin{aligned} aX = aY &\implies a^2X = a^2Y \\ &\implies X = Y \end{aligned}$$

(2) Regarding the neighboring strings, once again by summing over a, we have:

$$\begin{aligned} XabY = ZabT &\implies Xa^2Y = Za^2T \\ &\implies XY = ZT \end{aligned}$$

We have as well the following computation:

$$\begin{aligned} XabY = ZbaT &\implies Xa^2Y = Za^2T \\ &\implies XY = ZT \end{aligned}$$

Thus $G = (G_k)$ has both the properties in the statement. □

We are now in position of stating and proving a main result, as follows:

Theorem 13.13. *There is only one intermediate monomial sphere*

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{R},+}$$

namely the half-classical real sphere $S^{N-1}_{\mathbb{R},*}$.

Proof. We will prove that the only filtered groups $G \subset S_\infty$ satisfying the conditions in Proposition 13.12 are those correspoding to our 3 spheres, namely:

$$\{1\} \subset S_\infty^* \subset S_\infty$$

In order to do so, consider such a filtered group:

$$G \subset S_\infty$$

We assume this group to be non-trivial, $G \neq \{1\}$.

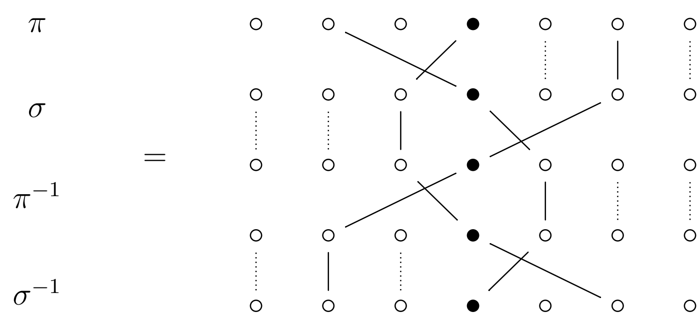
Step 1. Our first claim is that G contains a 3-cycle. Assume indeed that two permutations $\pi, \sigma \in S_\infty$ have support overlapping on exactly one point, say:

$$supp(\pi) \cap supp(\sigma) = \{i\}$$

The point is then that the commutator $\sigma^{-1}\pi^{-1}\sigma\pi$ is a 3-cycle, namely:

$$(i, \sigma^{-1}(i), \pi^{-1}(i))$$

Indeed the computation of the commutator goes as follows:

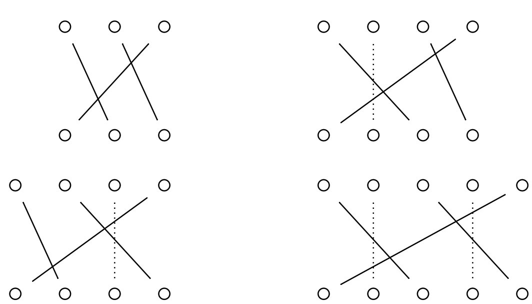


Now let us pick a non-trivial element $\tau \in G$. By removing outer strings at right and at left we obtain permutations $\tau' \in G_k, \tau'' \in G_s$ having a non-trivial action on their right/left leg, and the trick applies, with:

$$\pi = \tau' \otimes id_{s-1}$$
$$\sigma = id_{k-1} \otimes \tau''$$

Thus, G contains a 3-cycle, as claimed.

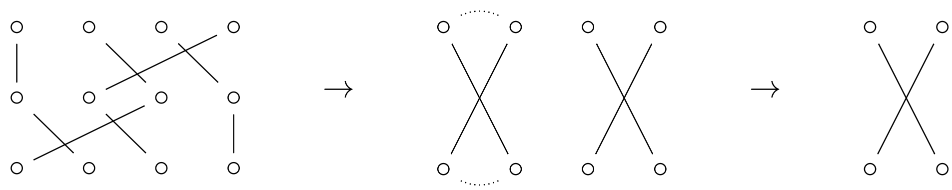
Step 2. Our second claim is G must contain one of the following permutations:



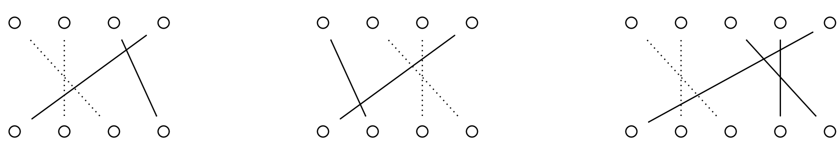
Indeed, consider the 3-cycle that we just constructed. By removing all outer strings, and then all pairs of adjacent vertical strings, we are left with these permutations.

<u>Step 3</u>. Our claim now is that we must have $S_\infty^* \subset G$. Indeed, let us pick one of the permutations that we just constructed, and apply to it our various diagrammatic rules.

From the first permutation we can obtain the basic crossing, as follows:



Also, by removing a suitable ╳ shaped configuration, which is represented by dotted lines in the diagrams below, we can obtain the basic crossing from the second and third permutation, and the half-liberated crossing from the fourth permutation:



Thus, in all cases we have a basic or half-liberated crossing, and so, as desired:

$$S_\infty^* \subset G$$

<u>Step 4</u>. Our last claim, which will finish the proof, is that there is no proper intermediate subgroup as follows:

$$S_\infty^* \subset G \subset S_\infty$$

In order to prove this, observe that $S_\infty^* \subset S_\infty$ is the subgroup of parity-preserving permutations, in the sense that "i even $\implies$ $\sigma(i)$ even".

Now let us pick an element $\sigma \in S_k - S_k^*$, with $k \in \mathbb{N}$. We must prove that the group $G =< S_\infty^*, \sigma >$ equals the whole S_∞. In order to do so, we use the fact that σ is not parity preserving. Thus, we can find i even such that $\sigma(i)$ is odd. In addition, up to passing to $\sigma|$, we can assume that $\sigma(k) = k$, and then, up to passing one more time to $\sigma|$, we can further assume that k is even. Since both i, k are even we have:

$$(i, k) \in S_k^*$$

We conclude that the following element belongs to G:

$$\sigma(i, k)\sigma^{-1} = (\sigma(i), k)$$

But, since $\sigma(i)$ is odd, by deleting an appropriate number of vertical strings, $(\sigma(i), k)$ reduces to the basic crossing $(1, 2)$. Thus $G = S_\infty$, and we are done. $\square$

Our purpose now will be that of going beyond this, with a number of results regarding the coordinates $x_1, \ldots, x_N$ of our real spheres.

To be more precise, a first question that we would like to solve, which is of purely algebraic nature, is that of understanding the precise relations satisfied by these coordinates $x_1, \ldots, x_N$ over our real spheres. We will see that this is related to the question of unifying the twisted and untwisted geometries, via intersection.

Let us begin by recalling the construction of the twisted real spheres, which was discussed in section 11 above. This is something very simple, as follows:

Definition 13.14. *The subspheres $\bar{S}^{N-1}_{\mathbb{R}}, \bar{S}^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$ are constructed by imposing the following conditions on the standard coordinates $x_1, \ldots, x_N$:*

1. $\bar{S}^{N-1}_{\mathbb{R}}$: *$x_i x_j = -x_j x_i$, for any $i \neq j$.*
2. $\bar{S}^{N-1}_{\mathbb{R},*}$: *$x_i x_j x_k = -x_k x_j x_i$ for any i,j,k distinct, $x_i x_j x_k = x_k x_j x_i$ otherwise.*

Here the fact that we have indeed $\bar{S}^{N-1}_{\mathbb{R}} \subset \bar{S}^{N-1}_{\mathbb{R},*}$ comes from the following computations, for $a,b,c \in \{x_i\}$ distinct, where $x_1, \ldots, x_N$ are the standard coordinates on $\bar{S}^{N-1}_{\mathbb{R}}$:

$$abc = -bac = bca = -cba$$

$$aab = -aba = baa$$

We refer to section 11 for more details regarding the above spheres.

Summarizing, we have a total of 5 real spheres, or rather a total of $3+3=6$ real spheres, with the convention that the free real sphere equals its twist:

$$S^{N-1}_{\mathbb{R},+} = \bar{S}^{N-1}_{\mathbb{R},+}$$

The point now is that we can intersect these $3+3=6$ spheres, and we end up with a total of $3 \times 3 = 9$ real spheres, in a generalized sense, as follows:

Definition 13.15. *Associated to any integer $N \in \mathbb{N}$ are the generalized spheres*

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+} \\
\uparrow & & \uparrow & & \uparrow \\
S^{N-1}_{\mathbb{R}} \cap \bar{S}^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},*} \cap \bar{S}^{N-1}_{\mathbb{R},*} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R},*} \\
\uparrow & & \uparrow & & \uparrow \\
S^{N-1}_{\mathbb{R}} \cap \bar{S}^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*} \cap \bar{S}^{N-1}_{\mathbb{R}} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R}}
\end{array}$$

obtained by intersecting the 3 twisted real spheres and the 3 untwisted real spheres.

In order to compute the various intersections appearing above, which in general cannot be thought of as being smooth, let us introduce the following objects:

Definition 13.16. *The polygonal spheres are real algebraic manifolds, defined as*

$$S^{N-1,d-1}_{\mathbb{R}}=\left\{x\in S^{N-1}_{\mathbb{R}}\Big|x_{i_0}\ldots x_{i_d}=0,\forall i_0,\ldots,i_d\text{ distinct}\right\}$$

depending on integers $1\leq d\leq N$*.*

These spheres are not smooth in general, but recall that we are currently doing algebraic geometry, rather than differential geometry. To be more precise, the point is that the problem that we want to solve, namely understanding the precise relations satisfied by the coordinates $x_1,\ldots,x_N$ for the real spheres, naturally leads into polygonal spheres.

More generally now, we have the following construction of "generalized polygonal spheres", which applies to the half-classical and twisted cases too:

$$C(\dot{S}^{N-1,d-1}_{\mathbb{R},\times})=C\big(\dot{S}^{N-1}_{\mathbb{R},\times}\big)\Big/\Big\langle x_{i_0}\ldots x_{i_d}=0,\forall i_0,\ldots,i_d\text{ distinct}\Big\rangle$$

Here the fact that in the classical case we obtain the polygonal spheres from Definition 13.16 comes from a straightforward application of the Gelfand theorem.

With these conventions, we have the following result, dealing with all the spheres that we have so far in real case, namely twisted, untwisted and intersections:

Theorem 13.17. *The diagram obtained by intersecting the twisted and untwisted real spheres, from Definition 13.15 above, is given by*

$$\begin{tikzcd}
S^{N-1}_{\mathbb{R}} \arrow[r] & S^{N-1}_{\mathbb{R},*} \arrow[r] & S^{N-1}_{\mathbb{R},+} \\
S^{N-1,1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{N-1,1}_{\mathbb{R},*} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{R},*} \arrow[u] \\
S^{N-1,0}_{\mathbb{R}} \arrow[r] \arrow[u] & \bar{S}^{N-1,1}_{\mathbb{R}} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{R}} \arrow[u]
\end{tikzcd}$$

and so all these spheres are generalized polygonal spheres.

Proof. We must prove that the 4-diagram obtained by intersecting the 5 main spheres coincides with the 4-diagram appearing at bottom left in the statement:

$$\begin{matrix} S^{N-1}_{\mathbb{R}}\cap\bar{S}^{N-1}_{\mathbb{R},*} & \rightarrow & S^{N-1}_{\mathbb{R},*}\cap\bar{S}^{N-1}_{\mathbb{R},*} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}}\cap\bar{S}^{N-1}_{\mathbb{R}} & \rightarrow & S^{N-1}_{\mathbb{R},*}\cap\bar{S}^{N-1}_{\mathbb{R}} \end{matrix} \quad = \quad \begin{matrix} S^{N-1,1}_{\mathbb{R}} & \rightarrow & S^{N-1,1}_{\mathbb{R},*} \\ \uparrow & & \uparrow \\ S^{N-1,0}_{\mathbb{R}} & \rightarrow & \bar{S}^{N-1,1}_{\mathbb{R}} \end{matrix}$$

But this is clear, because combining the commutation and anticommutation relations leads to the vanishing relations defining the spheres of type $\dot{S}^{N-1,d-1}_{\mathbb{R},\times}$. More precisely:

(1) $S^{N-1}_{\mathbb{R}}\cap\bar{S}^{N-1}_{\mathbb{R}}$ consists of the points $x\in S^{N-1}_{\mathbb{R}}$ such that, for any $i\neq j$:

$$x_ix_j=-x_jx_i$$

Now since we have as well $x_ix_j=x_jx_i$, for any i,j, this relation reads $x_ix_j=0$ for $i\neq j$, which means that we have $x\in S^{N-1,0}_{\mathbb{R}}$, as desired.

(2) $S^{N-1}_{\mathbb{R}}\cap\bar{S}^{N-1}_{\mathbb{R},*}$ consists of the points $x\in S^{N-1}_{\mathbb{R}}$ such that, for i,j,k distinct:

$$x_ix_jx_k=-x_kx_jx_i$$

Once again by commutativity, this relation is equivalent to $x\in S^{N-1,1}_{\mathbb{R}}$, as desired.

(3) $S^{N-1}_{\mathbb{R},*}\cap\bar{S}^{N-1}_{\mathbb{R}}$ is obtained from $\bar{S}^{N-1}_{\mathbb{R}}$ by imposing to the standard coordinates the half-commutation relations $abc=cba$. On the other hand, we know from $\bar{S}^{N-1}_{\mathbb{R}}\subset\bar{S}^{N-1}_{\mathbb{R},*}$ that the standard coordinates on $\bar{S}^{N-1}_{\mathbb{R}}$ satisfy $abc=-cba$ for a,b,c distinct, and $abc=cba$ otherwise. Thus, the relations brought by intersecting with $S^{N-1}_{\mathbb{R},*}$ reduce to the relations $abc=0$ for a,b,c distinct, and so we are led to the sphere $\bar{S}^{N-1,1}_{\mathbb{R}}$.

(4) $S^{N-1}_{\mathbb{R},*}\cap\bar{S}^{N-1}_{\mathbb{R},*}$ is obtained from $\bar{S}^{N-1}_{\mathbb{R},*}$ by imposing the relations $abc=-cba$ for a,b,c distinct, and $abc=cba$ otherwise. Since we know that $abc=cba$ for any a,b,c, the extra relations reduce to $abc=0$ for a,b,c distinct, and so we are led to $S^{N-1,1}_{\mathbb{R},*}$. ☐

Summarizing, whether we want it or not, when talking about intersections between twisted and untwisted geometries, we are led into polygonal spheres, and into non-smooth objects in general. This will be of course not an issue, in what follows.

In view of this, and also in connection with general axiomatization questions, let us find now a suitable axiomatic framework for the 9 spheres in Theorem 13.17.

We denote by $P(k,l)$ the set of partitons between an upper row of k points, and a lower row of l points, we set $P=\bigcup_{kl}P(k,l)$, and we denote by $P_{even}\subset P$ the subset of partitions having all the blocks of even size. Observe that $P_{even}(k,l)=\emptyset$ for $k+l$ odd.

We use the fact that there is a signature map $\varepsilon : P_{even} \to \{-1,1\}$, extending the usual signature of permutations, $\varepsilon : S_\infty \to \{-1,1\}$. This map is obtained by setting $\varepsilon(\pi) = (-1)^c$, where $c \in \mathbb{N}$ is the number of switches between neighbors required for making π noncrossing, and which can be shown to be well-defined modulo 2.

We have the following definition, once again from [6]:

Definition 13.18. *Given variables $x_1,\ldots,x_N$, any permutation $\sigma \in S_k$ produces two collections of relations between these variables, as follows:*

(1) *Untwisted relations, namely, for any $i_1,\ldots,i_k$:*

$$x_{i_1}\ldots x_{i_k} = x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$$

(2) *Twisted relations, namely, for any $i_1,\ldots,i_k$:*

$$x_{i_1}\ldots x_{i_k} = \varepsilon\left(\ker\begin{pmatrix}i_1&\ldots&i_k\\ i_{\sigma(1)}&\ldots&i_{\sigma(k)}\end{pmatrix}\right)x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$$

The untwisted relations are denoted $\mathcal{R}_\sigma$, and the twisted ones are denoted $\bar{\mathcal{R}}_\sigma$.

Observe that the untwisted relations $\mathcal{R}_\sigma$ are trivially satisfied for the standard coordinates on $S^{N-1}_{\mathbb{R}}$, for any permutation $\sigma \in S_k$. A twisted analogue of this fact holds, in the sense that the standard coordinates on $\bar{S}^{N-1}_{\mathbb{R}}$ satisfy the relations $\bar{\mathcal{R}}_\sigma$, for any $\sigma \in S_k$. Indeed, by using the anticommutation relations between the distinct coordinates of these latter spheres, we must have a formula of the following type:

$$x_{i_1}\ldots x_{i_k} = \pm x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$$

But the sign $\pm$ obtained in this way is precisely the one given above, namely:

$$\pm = \varepsilon\left(\ker\begin{pmatrix}i_1&\ldots&i_k\\ i_{\sigma(1)}&\ldots&i_{\sigma(k)}\end{pmatrix}\right)$$

We have now all the needed ingredients for axiomatizing the various spheres appearing so far, namely the twisted and untwisted ones, and their intersections:

Definition 13.19. *We have 3 types of quantum spheres $S \subset S^{N-1}_{\mathbb{R},+}$, as follows:*

(1) *Monomial, namely $\dot{S}^{N-1}_{\mathbb{R},E}$, with $E \subset S_\infty$, obtained via the following relations:*

$$\left\{\dot{\mathcal{R}}_\sigma\Big|\sigma \in E\right\}$$

(2) *Mixed monomial, which appear as intersections as follows, with $E,F \subset S_\infty$:*

$$S^{N-1}_{\mathbb{R},E,F} = S^{N-1}_{\mathbb{R},E} \cap \bar{S}^{N-1}_{\mathbb{R},F}$$

(3) *Polygonal, which are again intersections, with $E,F \subset S_\infty$, and $d \in \{1,\ldots,N\}$:*

$$S^{N-1,d-1}_{\mathbb{R},E,F} = S^{N-1}_{\mathbb{R},E,F} \cap S^{N-1,d-1}_{\mathbb{R},+}$$

Here the subsphere $S^{N-1,d-1}_{\mathbb{R},+}\subset S^{N-1}_{\mathbb{R},+}$ appearing in (3) is constructed as in Definition 13.16 above, by imposing the following relations, with $i_0,\ldots,i_d$ distinct:

$$x_{i_0}\ldots x_{i_d}=0$$

With the above notions, we cover all spheres appearing so far. More precisely, the 5 basic spheres in are monomial, the 9 spheres in Theorem 13.17 are mixed monomial, and the polygonal sphere formalism covers all the examples constructed so far.

Observe that the set of mixed monomial spheres is closed under intersections. The same holds for the set of polygonal spheres, because we have the following formula:

$$S^{N-1,d-1}_{\mathbb{R},E,F}\cap S^{N-1,d'-1}_{\mathbb{R},E',F'}=S^{N-1,min(d,d')-1}_{\mathbb{R},E\cup E',F\cup F'}$$

Let us try now to understand the structure of the various types of spheres, by using the real sphere technology developed before.

We call a group of permutations $G\subset S_\infty$ filtered if, with $G_k=G\cap S_k$, we have $G_k\times G_l\subset G_{k+l}$, for any k,l.

We use the following simple fact, coming from [33]:

Proposition 13.20. *The various spheres can be parametrized by groups, as follows:*

(1) *Monomial case: $\dot{S}^{N-1}_{\mathbb{R},G}$, with $G\subset S_\infty$ filtered group.*
(2) *Mixed monomial case: $S^{N-1}_{\mathbb{R},G,H}$, with $G,H\subset S_\infty$ filtered groups.*
(3) *Polygonal case: $S^{N-1,d-1}_{\mathbb{R},G,H}$, with $G,H\subset S_\infty$ filtered groups, and $d\in\{1,\ldots,N\}$.*

Proof. This basically follows from the theory developed before, as follows:

(1) As explained before, in order to prove this assertion, for a monomial sphere $S=\dot{S}_{\mathbb{R},E}$, we can take $G\subset S_\infty$ to be the set of permutations $\sigma\in S_\infty$ having the property that the relations $\dot{\mathcal{R}}_\sigma$ hold for the standard coordinates of S. We have then $E\subset G$, we have as well $S=\dot{S}^{N-1}_{\mathbb{R},G}$, and the fact that G is a filtered group is clear as well.

(2) This follows from (1), by taking intersections.

(3) Once again this follows from (1), by taking intersections. □

The idea in what follows will be that of writing the 9 main polygonal spheres as in Proposition 13.20 (2), as to reach to a "standard parametrization" for our spheres.

We recall from the beginning of this section that the permutations $\sigma\in S_\infty$ having the property that when labelling clockwise their legs $\circ\bullet\circ\bullet\ldots$, and string joins a white leg to a black leg, form a filtered group, denoted $S^*_\infty\subset S_\infty$.

This group comes from the general half-liberation considerations from section 9 above, and its algebraic structure is very simple, as follows:

$$S_{2n}^* \simeq S_n \times S_n$$

$$S_{2n+1}^* \simeq S_n \times S_{n+1}$$

We call a mixed monomial sphere parametrization $S = S^{N-1}_{\mathbb{R},G,H}$ standard when both filtered groups $G, H \subset S_\infty$ are chosen to be maximal. In this case, Proposition 13.20 and its proof tell us that G, H encode all the monomial relations which hold in S.

With these conventions, we have the following result from [6], [8], extending some previous findings from above, regarding the untwisted spheres:

Theorem 13.21. *The standard parametrization of the 9 main spheres is*

$$\begin{array}{ccccccc}
S_\infty & & S_\infty^* & & \{1\} & & G/H\\
\vdots & & \vdots & & \vdots & & \\
S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+} & \cdots\cdots & \{1\}\\
\uparrow & & \uparrow & & \uparrow & & \\
S^{N-1,1}_{\mathbb{R}} & \longrightarrow & S^{N-1,1}_{\mathbb{R},*} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R},*} & \cdots\cdots & S_\infty^*\\
\uparrow & & \uparrow & & \uparrow & & \\
S^{N-1,0}_{\mathbb{R}} & \longrightarrow & \bar{S}^{N-1,1}_{\mathbb{R}} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R}} & \cdots\cdots & S_\infty
\end{array}$$

so these spheres come from the $3 \times 3 = 9$ pairs of groups among $\{1\} \subset S_\infty^* \subset S_\infty$.

Proof. The fact that we have parametrizations as above is known to hold for the 5 untwisted and twisted spheres. For the remaining 4 spheres the result follows by intersecting, by using the following formula, valid for any $E, F \subset S_\infty$:

$$S^{N-1}_{\mathbb{R},E,F} \cap S^{N-1}_{\mathbb{R},E',F'} = S^{N-1}_{\mathbb{R},E\cup E',F\cup F'}$$

In order to prove now that the parametrizations are standard, we must compute the following two filtered groups, and show that we get the groups in the statement:

$$G = \left\{\sigma \in S_\infty \Big| \text{the relations } \mathcal{R}_\sigma \text{ hold over } S\right\}$$

$$H = \left\{\sigma \in S_\infty \Big| \text{the relations } \bar{\mathcal{R}}_\sigma \text{ hold over } S\right\}$$

As a first observation, by using the various inclusions between spheres, we just have to compute G for the spheres on the bottom, and H for the spheres on the left:

$$X = S^{N-1,0}_{\mathbb{R}}, \bar{S}^{N-1,1}_{\mathbb{R}}, \bar{S}^{N-1}_{\mathbb{R}} \implies G = S_\infty, S_\infty^*, \{1\}$$

$$X = S^{N-1,0}_{\mathbb{R}}, S^{N-1,1}_{\mathbb{R}}, S^{N-1}_{\mathbb{R}} \implies H = S_\infty, S_\infty^*, \{1\}$$

The results for $S^{N-1,0}_{\mathbb{R}}$ being clear, we are left with computing the remaining 4 groups, for the spheres $S^{N-1}_{\mathbb{R}}, \bar{S}^{N-1}_{\mathbb{R}}, S^{N-1,1}_{\mathbb{R}}, \bar{S}^{N-1,1}_{\mathbb{R}}$. The proof here goes as follows:

(1) $S^{N-1}_{\mathbb{R}}$. According to the definition of $H = (H_k)$, we have:

$$\begin{aligned} H_k &= \left\{\sigma \in S_k \Big| x_{i_1} \ldots x_{i_k} = \varepsilon\left(\ker\left(\begin{smallmatrix} i_1 & \ldots & i_k \\ i_{\sigma(1)} & \ldots & i_{\sigma(k)}\end{smallmatrix}\right)\right) x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}, \forall i_1, \ldots, i_k\right\} \\ &= \left\{\sigma \in S_k \Big| \varepsilon\left(\ker\left(\begin{smallmatrix} i_1 & \ldots & i_k \\ i_{\sigma(1)} & \ldots & i_{\sigma(k)}\end{smallmatrix}\right)\right) = 1, \forall i_1, \ldots, i_k\right\} \\ &= \left\{\sigma \in S_k \Big| \varepsilon(\tau) = 1, \forall \tau \leq \sigma\right\} \end{aligned}$$

Now observe that for any $\sigma \in S_k, \sigma \neq 1_k$, we can always find a partition $\tau \leq \sigma$ satisfying $\varepsilon(\tau) = -1$. We deduce that we have $H_k = \{1_k\}$, and so $H = \{1\}$, as desired.

(2) $\bar{S}^{N-1}_{\mathbb{R}}$. The proof of $G = \{1\}$ here is similar to the proof of $H = \{1\}$ in (1) above, by using the same combinatorial ingredient at the end.

(3) $S^{N-1,1}_{\mathbb{R}}$. By definition of $H = (H_k)$, a permutation $\sigma \in S_k$ belongs to H_k when the following condition is satisfied, for any choice of the indices $i_1, \ldots, i_k$:

$$x_{i_1} \ldots x_{i_k} = \varepsilon\left(\ker\begin{pmatrix} i_1 & \ldots & i_k \\ i_{\sigma(1)} & \ldots & i_{\sigma(k)}\end{pmatrix}\right) x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}$$

We have three cases here, as follows:

– When $|\ker i| = 1$ this formula reads $x_r^k = x_r^k$, which is true.

– When $|\ker i| \geq 3$ this formula is automatically satisfied as well, because by using the relations $ab = ba$, and $abc = 0$ for a, b, c distinct, which both hold over $S^{N-1,1}_{\mathbb{R}}$, this formula reduces to $0 = 0$.

– Thus, we are left with studying the case $|\ker i| = 2$. Here the quantities on the left $x_{i_1} \ldots x_{i_k}$ will not vanish, so the sign on the right must be 1, and we therefore have:

$$H_k = \left\{\sigma \in S_k \Big| \varepsilon(\tau) = 1, \forall \tau \leq \sigma, |\tau| = 2\right\}$$

Now by coloring the legs of σ clockwise $\circ \bullet \circ \bullet \ldots$, the above condition is satisfied when each string of σ joins a white leg to a black leg. Thus $H_k = S_k^*$, as desired.

(4) $\bar{S}^{N-1,1}_{\mathbb{R}}$. The proof of $G = S_\infty^*$ here is similar to the proof of $H = S_\infty^*$ in (3) above, by using the same combinatorial ingredient at the end. □

We can now formulate a classification result, as follows:

Theorem 13.22. *The following hold:*

1. $S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$ *are the only untwisted monomial spheres.*
2. $\bar{S}^{N-1}_{\mathbb{R}} \subset \bar{S}^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$ *are the only twisted monomial spheres.*
3. *The* 9 *spheres in Theorem 13.21 are the only polygonal ones.*

Proof. By using standard parametrizations, the above 3 statements are equivalent. Now since (1) was proved before, all the results hold true. □

14. Polygonal spheres

We have seen in the previous section that the study of the algebraic relations between the coordinates $x_1,\ldots,x_N$ of the real spheres $S^{N-1}_\mathbb{R}\subset S^{N-1}_{\mathbb{R},*}\subset S^{N-1}_{\mathbb{R}_+}$ naturally leads to the twisted versions of these spheres, $\bar{S}^{N-1}_\mathbb{R}\subset \bar{S}^{N-1}_{\mathbb{R},*}\subset S^{N-1}_{\mathbb{R}_+}$, and more specifically to the intersections between the twisted and untwisted spheres, which are as follows:

$$\begin{array}{ccccc}
S^{N-1}_\mathbb{R} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+}\\
\uparrow & & \uparrow & & \uparrow\\
S^{N-1}_\mathbb{R}\cap \bar{S}^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},*}\cap \bar{S}^{N-1}_{\mathbb{R},*} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R},*}\\
\uparrow & & \uparrow & & \uparrow\\
S^{N-1}_\mathbb{R}\cap \bar{S}^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*}\cap \bar{S}^{N-1}_{\mathbb{R}} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R}}
\end{array}$$

We have seen as well that these intersections all appear as "polygonal spheres", which are certain real algebraic manifolds, according to the following result:

Theorem 14.1. *The 5 main spheres, and the intersections between them, are*

$$\begin{array}{ccccc}
S^{N-1}_\mathbb{R} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+}\\
\uparrow & & \uparrow & & \uparrow\\
S^{N-1,1}_\mathbb{R} & \longrightarrow & S^{N-1,1}_{\mathbb{R},*} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R},*}\\
\uparrow & & \uparrow & & \uparrow\\
S^{N-1,0}_\mathbb{R} & \longrightarrow & \bar{S}^{N-1,1}_{\mathbb{R}} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R}}
\end{array}$$

where $\dot{S}^{N-1,d-1}_{\mathbb{R},\times}\subset \dot{S}^{N-1}_{\mathbb{R},\times}$ is obtained by assuming $x_{i_0}\ldots x_{i_d}=0$, for $i_0,\ldots,i_d$ distinct.

Proof. This is something that we know from section 13, the idea being that commutation and anticommutation produces vanishing relations. □

We refer to section 13 for more on these spheres, their algebraic axiomatization and main properties, and the "standard parametrization" result there.

In this section we discuss the extension of the axiomatics that we have, in order to cover both the twisted and untwisted cases, and the intersections.

For this purpose, we are in need of some new quantum isometry group computations. In order to deal with the polygonal spheres, we will need the following standard result:

Proposition 14.2. *Assume that $X\subset S^{N-1}_\mathbb{R}$ is invariant under $x_i\to -x_i$, for any i.*

(1) *If the coordinates $x_1,\ldots,x_N$ are linearly independent inside $C(X)$, then the group $G(X)=G^+(X)\cap O_N$ consists of the usual isometries of X.*
(2) *In addition, in the case where the products of coordinates $\{x_ix_j|i\leq j\}$ are linearly independent inside $C(X)$, we have $G^+(X)=G(X)$.*

Proof. This follows from [44], [91], the idea being as follows:

(1) The assertion here is well-known, $G(X)=G^+(X)\cap O_N$ being by definition the biggest subgroup $G\subset O_N$ acting affinely on X. We refer to [91] for details, and for a number of noncommutative extensions of this fact, with $G(X)$ replaced by $G^+(X)$.

(2) Here we must prove that, whenever we have a coaction $\Phi:C(X)\to C(G)\otimes C(X)$, given by $\Phi(x_i)=\sum_ju_{ij}\otimes x_j$, the variables u_{ij} commute. But this follows by using a strandard trick, from [44], that we will briefly recall now. We can write:

$$\Phi([x_i,x_j])=\sum_{k\leq l}([u_{ik},u_{jl}]-[u_{jk},u_{il}])\otimes\left(1-\frac{\delta_{kl}}{2}\right)x_kx_l$$

Now since the variables $\{x_kx_l|k\leq l\}$ are linearly independent, we obtain from this:

$$[u_{ik},u_{jl}]=[u_{jk},u_{il}]$$

Moreover, if we apply now the antipode we further obtain:

$$[u_{lj},u_{ki}]=[u_{li},u_{kj}]$$

By relabelling, this gives the following formula:

$$[u_{ik},u_{jl}]=[u_{il},u_{jk}]$$

We therefore conclude that we have a commutation relation, as follows:

$$[u_{ik},u_{jl}]=0$$

Thus, we are led to the conclusion in the statement. See [44]. □

With the above notion in hand, let us investigate the polygonal spheres. We recall that the quantum isometry groups of the 5 main spheres are as follows:

$$\begin{tikzcd}
S^{N-1}_\mathbb{R} \arrow[r] \arrow[d, squiggly, no head] & S^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[d, squiggly, no head] & S^{N-1}_{\mathbb{R},+} \arrow[d, squiggly, no head] & \bar{S}^{N-1}_{\mathbb{R},*} \arrow[l] \arrow[d, squiggly, no head] & \bar{S}^{N-1}_\mathbb{R} \arrow[l] \arrow[d, squiggly, no head] \\
O_N \arrow[r] & O_N^* \arrow[r] & O_N^+ & \bar{O}_N^* \arrow[l] & \bar{O}_N \arrow[l]
\end{tikzcd}$$

In the polygonal case now, we begin with the computations of the quantum isometry groups in the classical case. We have here the following result, from [6]:

Theorem 14.3. *The quantum isometry group of $S^{N-1,d-1}_\mathbb{R}$ is as follows:*

(1) *At $d=1$ we obtain the free hyperoctahedral group H_N^+.*
(2) *At $d=2,\ldots,N-1$ we obtain the hyperoctahedral group H_N.*
(3) *At $d=N$ we obtain the orthogonal group O_N.*

Proof. Observe first that the sphere $S^{N-1,d-1}_\mathbb{R}$ appears by definition as a union on $\binom{N}{d}$ copies of the sphere $S^{d-1}_\mathbb{R}$, one for each choice of d coordinate axes, among the coordinate axes of $\mathbb{R}^N$. We can write this decomposition as follows, with $I_N=\{1,\ldots,N\}$:

$$S^{N-1,d-1}_\mathbb{R}=\bigcup_{I\subset I_N,|I|=d}(S^{d-1}_\mathbb{R})^I$$

(1) At $d=1$ our sphere is $S^{N-1,0}_\mathbb{R}=\mathbb{Z}_2^{\oplus N}$, formed by the endpoints of the N copies of $[-1,1]$ on the coordinate axes of $\mathbb{R}^N$. Thus by [20] the quantum isometry group is H_N^+.

(2) Our first claim is that at $d\geq 2$, the elements $\{x_ix_j|i\leq j\}$ are linearly independent. Since $S^{N-1,1}_\mathbb{R}\subset S^{N-1,d}_\mathbb{R}$, we can restrict attention to the case $d=2$. Here the above decomposition is as follows, where $\mathbb{T}^{\{i,j\}}$ denote the various copies of $\mathbb{T}$:

$$S^{N-1,d-1}_\mathbb{R}=\bigcup_{i<j}\mathbb{T}^{\{i,j\}}$$

Now since $\{x^2,y^2,xy\}$ are linearly independent over $\mathbb{T}\subset\mathbb{R}^2$, we deduce from this that $\{x_ix_j|i\leq j\}$ are linearly independent over $S^{N-1,d-1}_\mathbb{R}$, and we are done. Thus, our claim is proved, and so Proposition 14.2 (2) above applies, and gives $G^+(X)=G(X)$.

We are therefore left with proving $G(X)=H_N$, for any $d\in\{2,\ldots,N-1\}$.

Let us first discuss the case $d=2$. Here any affine isometric action $U\curvearrowright S^{N-1,1}_\mathbb{R}$ must permute the $\binom{N}{2}$ circles $\mathbb{T}^I$, so we can write $U(\mathbb{T}^I)=\mathbb{T}^{I'}$, for a certain permutation of the indices $I\to I'$. Now since U is bijective, we deduce that for any I,J we have:

$$U\left(\mathbb{T}^I\cap\mathbb{T}^J\right)=\mathbb{T}^{I'}\cap\mathbb{T}^{J'}$$

Since for $|I\cap J|=0,1,2$ we have $\mathbb{T}^I\cap\mathbb{T}^J\simeq\emptyset,\{-1,1\},\mathbb{T}$, by taking the union over I,J with $|I\cap J|=1$, we deduce that $U(\mathbb{Z}_2^{\oplus N})=\mathbb{Z}_2^{\oplus N}$. Thus $U\in H_N$, and we are done.

In the general case now, $d\in\{2,\ldots,N-1\}$, we can proceed similarly, by recurrence. Indeed, for any subsets $I,J\subset I_N$ with $|I|=|J|=d$ we have:

$$(S^{d-1}_\mathbb{R})^I\cap(S^{d-1}_\mathbb{R})^J=(S^{|I\cap J|-1}_\mathbb{R})^{I\cap J}$$

By using $d\leq N-1$, we deduce that we have the following formula:

$$S^{N-1,d-2}_\mathbb R=\bigcup_{|I|=|J|=d,|I\cap J|=d-1}(S^{|I\cap J|-1}_\mathbb R)^{I\cap J}$$

On the other hand, by using the same argument as in the $d=2$ case, we deduce that the space on the right is invariant, under any affine isometric action on $S^{N-1,d-1}_\mathbb R$. Thus by recurrence we obtain $G(S^{N-1,d-1}_\mathbb R)=G(S^{N-1,d-2}_\mathbb R)=H_N$, and we are done.

(3) At $d=N$ the result is known since [32], with the proof coming from the equality $G^+(X)=G(X)$, deduced from Proposition 14.2 (2), as explained above. □

The study in the twisted case is considerably more difficult than in the classical case, and we have complete results only at $d=1,2,N$, as follows:

Theorem 14.4. *The quantum isometry group of $\bar{S}^{N-1,d-1}_\mathbb R$ is as follows:*

1. *At $d=1$ we obtain the free hyperoctahedral group H_N^+.*
2. *At $d=2$ we obtain the hyperoctahedral group H_N.*
3. *At $d=N$ we obtain the twisted orthogonal group $\bar{O}_N$.*

Proof. The idea is to adapt the proof of Theorem 14.3 above:

(1) At $d=1$ we have $\bar{S}^{N-1,0}_\mathbb R=S^{N-1,0}_\mathbb R=\mathbb Z_2^{\oplus N}$, and by Proposition 14.2 (1) above, coming from [20], the corresponding quantum isometry group is indeed H_N^+.

(2) As a first ingredient, we will need the twisted analogue of the trick from [44], explained in the proof of Proposition 14.2 (2) above. This twisted trick was already worked out in [7], for the sphere $\bar{S}^{N-1}_\mathbb R$ itself, and the situation is similar for any closed subset $X\subset\bar{S}^{N-1}_\mathbb R$, having the property that the variables $\{x_ix_j|i\leq j\}$ are linearly indepedent. More presisely, our claim is that if $G\subset O_N^+$ acts on X, then we must have $G\subset\bar{O}_N$.

Indeed, given a coaction $\Phi(x_i)=\sum_ju_{ij}\otimes x_j$, we can write:

$$\Phi(x_ix_j)=\sum_ku_{ik}u_{jk}\otimes x_k^2+\sum_{k<l}(u_{ik}u_{jl}-u_{il}u_{jk})\otimes x_kx_l$$

We deduce that with $[[a,b]]=ab+ba$ we have the following formula:

$$\Phi([[x_i,x_j]])=\sum_k[[u_{ik},u_{jk}]]\otimes x_k^2+\sum_{k<l}([u_{ik},u_{jl}]-[u_{il},u_{jk}])\otimes x_kx_l$$

Now assuming $i\neq j$, we have $[[x_i,x_j]]=0$, and we therefore obtain, for any k:

$$[[u_{ik},u_{jk}]]=0$$

We also have, for any $k<l$, the following formula:

$$[u_{ik},u_{jl}]=[u_{il},u_{jk}]$$

By applying the antipode and then by relabelling, the latter relation gives:

$$[u_{ik},u_{jl}]=0$$

Thus we have reached to the defining relations for the quantum group $\bar{O}_N$, from section 11 above, and so we have $G\subset\bar{O}_N$, as claimed.

Our second claim is that the above trick applies to any $\bar{S}^{N-1,d-1}_\mathbb{R}$ with $d\geq 2$. Indeed, by using the maps $\pi_{ij}:C(\bar{S}^{N-1,d-1}_\mathbb{R})\to C(\bar{S}^1_\mathbb{R})$ obtained by setting $x_k=0$ for $k\neq i,j$, we conclude that the variables $\{x_ix_j|i\leq j\}$ are indeed linearly independent over $\bar{S}^{N-1,d-1}_\mathbb{R}$.

Summarizing, we have proved so far that if a compact quantum group $G\subset O_N^+$ acts on a polygonal sphere $\bar{S}^{N-1,d-1}_\mathbb{R}$ with $d\geq 2$, then we must have $G\subset\bar{O}_N$. We must now adapt the second part of the proof of Proposition 14.2, and since this is quite unobvious at $d\geq 3$, we will restrict now attention to the case $d=2$, as in the statement.

So, consider a compact quantum group $G\subset\bar{O}_N$. In order to have a coaction map $\Phi:C(\bar{S}^{N-1,1}_\mathbb{R})\to C(G)\otimes C(\bar{S}^{N-1,1}_\mathbb{R})$, given as usual by $\Phi(x_i)=\sum_j u_{ij}\otimes x_j$, the elements $X_i=\sum_j u_{ij}\otimes x_j$ must satisfy the relations $X_iX_jX_k=0$, for any i,j,k distinct.

So, let us compute $X_iX_jX_k$ for i,j,k distinct. We have:

$$\begin{aligned}
X_iX_jX_k &= \sum_{abc}u_{ia}u_{jb}u_{kc}\otimes x_ax_bx_c\\
&= \sum_{a,b,c\ not\ distinct}u_{ia}u_{jb}u_{kc}\otimes x_ax_bx_c\\
&= \sum_{a\neq b}u_{ia}u_{ja}u_{kb}\otimes x_a^2x_b+\sum_{a\neq b}u_{ia}u_{jb}u_{ka}\otimes x_ax_bx_a\\
&+ \sum_{a\neq b}u_{ib}u_{ja}u_{ka}\otimes x_bx_a^2+\sum_a u_{ia}u_{ja}u_{ka}\otimes x_a^3
\end{aligned}$$

By using $x_ax_bx_a=-x_a^2x_b$ and $x_bx_a^2=x_a^2x_b$, we deduce that we have:

$$\begin{aligned}
X_iX_jX_k &= \sum_{a\neq b}(u_{ia}u_{ja}u_{kb}-u_{ia}u_{jb}u_{ka}+u_{ib}u_{ja}u_{ka})\otimes x_a^2x_b\\
&+ \sum_a u_{ia}u_{ja}u_{ka}\otimes x_a^3\\
&= \sum_{ab}(u_{ia}u_{ja}u_{kb}-u_{ia}u_{jb}u_{ka}+u_{ib}u_{ja}u_{ka})\otimes x_a^2x_b
\end{aligned}$$

By using now the defining relations for $\bar{O}_N$, which apply to the variables u_{ij}, this formula can be written in a cyclic way, as follows:

$$X_iX_jX_k=\sum_{ab}(u_{ia}u_{ja}u_{kb}+u_{ja}u_{ka}u_{ib}+u_{ka}u_{ia}u_{jb})\otimes x_a^2x_b$$

We use now the fact that the variables on the right $x_a^2x_b$ are linearly independent. We conclude that, in order for our quantum group $G\subset\bar{O}_N$ to act on $\bar{S}^{N-1,1}_\mathbb{R}$, its coordinates must satisfy the following relations, for any i,j,k distinct:

$$u_{ia}u_{ja}u_{kb}+u_{ja}u_{ka}u_{ib}+u_{ka}u_{ia}u_{jb}=0$$

By multiplying to the right by u_{kb} and then by summing over b, we deduce from this that we have $u_{ia}u_{ja}=0$, for any i,j. Now since the quotient of $C(\bar{O}_N)$ by these latter relations is $C(H_N)$, we conclude that we have $G^+(\bar{S}^{N-1,1}_\mathbb{R})=H_N$, as claimed.

(3) At $d=N$ the result is already known, and its proof follows in fact from the "twisted trick" explained in the proof of (2) above, applied to $\bar{S}^{N-1}_\mathbb{R}$. $\square$

In general now, the idea will be that the quantum isometry groups of the intersections of the spheres will basically appear as intersections of the quantum isometry groups.

To start with, we must compute the intersections between the quantum orthogonal groups and their twists. The result here, which is similar to the one for the corresponding spheres, established in section 13 above, is as follows:

Proposition 14.5. *The 5 orthogonal groups and their twists, and the intersections between them, are as follows, at any $N\geq3$:*

$$\begin{array}{ccccc}
O_N & \to & O_N^* & \to & O_N^+\\
\uparrow && \uparrow && \uparrow\\
H_N & \to & H_N^* & \to & \bar{O}_N^*\\
\uparrow && \uparrow && \uparrow\\
H_N & \to & H_N & \to & \bar{O}_N
\end{array}$$

At $N=2$ the same holds, with the lower left square being replaced by:

$$\begin{array}{ccc}
O_2 & \to & O_2^+\\
\uparrow && \uparrow\\
H_2 & \to & \bar{O}_2
\end{array}$$

Proof. We have to study 4 quantum group intersections, as follows:

(1) $O_N\cap\bar{O}_N$. Here an element $U\in O_N$ belongs to the intersection when its entries satisfy $ab=0$ for any $a\neq b$ on the same row or column of U. But this means that our matrix $U\in O_N$ must be monomial, and so we get $U\in H_N$, as claimed.

(2) $O_N \cap \bar{O}_N^*$. At $N=2$ the defining relations for $\bar{O}_N^*$ dissapear, and so we have the following computation, which leads to the conclusion in the statement:

$$O_2 \cap \bar{O}_2^* = O_2 \cap O_2^+ = O_2$$

At $N \geq 3$ now, the inclusion $H_N \subset O_N \cap \bar{O}_N^*$ is clear. In order to prove the converse inclusion, pick $U \in O_N$ in the intersection, and assume that U is not monomial. By permuting the entries we can further assume $U_{11} \neq 0, U_{12} \neq 0$, and from $U_{11}U_{12}U_{i3} = 0$ for any i we deduce that the third column of U is filled with 0 entries, a contradiction. Thus we must have $U \in H_N$, as claimed.

(3) $O_N^* \cap \bar{O}_N$. At $N=2$ we have the following computation, as claimed:

$$O_2^* \cap \bar{O}_2 = O_2^+ \cap \bar{O}_2 = \bar{O}_2$$

At $N \geq 3$ now, the best is to use the result in (4) below. Indeed, knowing that we have $O_N^* \cap \bar{O}_N^* = H_N^*$, our intersection is then:

$$G = H_N^* \cap \bar{O}_N$$

Now since the standard coordinates on H_N^* are known to satisf $ab = 0$ for $a \neq b$ on the same row or column of u, the commutation/anticommutation relations defining $\bar{O}_N$ reduce to plain commutation relations. Thus G follows to be classical, $G \subset O_N$, and by using (1) above we obtain the following formula, as claimed:

$$\begin{aligned} G &= H_N^* \cap \bar{O}_N \cap O_N \\ &= H_N^* \cap H_N \\ &= H_N \end{aligned}$$

(4) $O_N^* \cap \bar{O}_N^*$. The result here is non-trivial, and we must use the half-liberation technology from [48]. The quantum group $H_N^\times = O_N^* \cap \bar{O}_N^*$ is indeed half-classical in the sense of [48], and since we have $H_N^* \subset H_N^\times$, this quantum group is not classical. Thus the main result in [48] applies, and shows that $H_N^\times \subset O_N^*$ must come, via the crossed product construction there, from an intermediate compact group, as follows:

$$\mathbb{T} \subset G \subset U_N$$

Now observe that the standard coordinates on $H_N^\times$ are by definition subject to the conditions $abc = 0$ when $(r,s) = (\leq 2, 3), (3, \leq 2)$, with the notations and conventions from section 11 above. It follows that the standard coordinates on G are subject to the conditions $\alpha\beta\gamma = 0$ when $(r,s) = (\leq 2, 3), (3, \leq 2)$, where $r, s = span(a,b,c)$, and $\alpha = a, a^*, \beta = b, b^*, \gamma = c, c^*$. Thus we have an inclusion as follows:

$$G \subset \bar{U}_N^*$$

We deduce that we have an inclusion as follows, with $K_N^\circ = U_N \cap \bar{U}_N^*$:

$$G \subset K_N^\circ$$

But this intersection can be computed exactly as in the real case, in the proof of (2) above, and we obtain $K_2^\circ = U_2$, and $K_N^\circ = \mathbb{T} \wr S_N$ at $N \geq 3$.

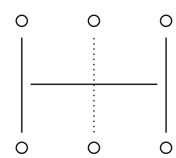
But the half-liberated quantum groups obtained from U_2 and $\mathbb{T} \wr S_N$ via the half-liberation construction in [48] are well-known, these being $O_2^* = O_2^+$ and H_N^*. Thus by functoriality we have $H_2^\times \subset O_2^+$ and $H_N^\times \subset H_N^*$ at $N \geq 3$, and since the reverse inclusions are clear, we obtain $H_2^\times = O_2^+$ and $H_N^\times = H_N^*$ at $N \geq 3$, as claimed. □

Let us go back now to the sphere left, namely $S^{N-1,1}_{\mathbb{R},*}$. We will need:

Proposition 14.6. *Let $H_N^{[\infty]} \subset O_N^+$ be the compact quantum group obtained via the relations $abc = 0$, whenever $a \neq c$ are on the same row or column of u.*

1. *We have inclusions $H_N^* \subset H_N^{[\infty]} \subset H_N^+$.*
2. *We have $ab_1 \ldots b_r c = 0$, whenever $a \neq c$ are on the same row or column of u.*
3. *We have $ab^2 = b^2 a$, for any two entries a, b of u.*

Proof. We briefly recall the proof in [120], for future use in what follows. Our first claim is that $H_N^{[\infty]}$ comes, as an easy quantum group, from the following diagram:

$\pi =$ 

Indeed, this diagram acts via the following linear map:

$$T_\pi(e_{ijk}) = \delta_{ik} e_{ijk}$$

We therefore have the following formulae:

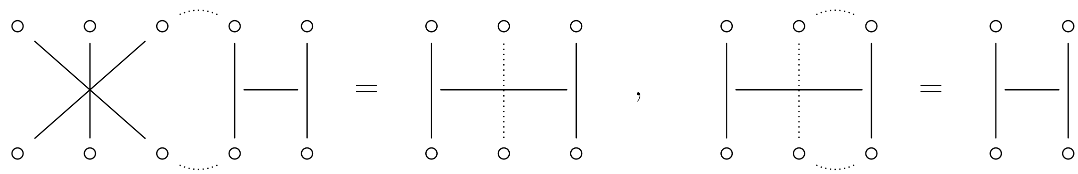
$$\begin{aligned} T_\pi u^{\otimes 3} e_{abc} &= T_\pi \sum_{ijk} e_{ijk} \otimes u_{ia} u_{jb} u_{kc} = \sum_{ijk} e_{ijk} \otimes \delta_{ik} u_{ia} u_{jb} u_{kc} \\ u^{\otimes 3} T_\pi e_{abc} &= u^{\otimes 3} \delta_{ac} e_{abc} = \sum_{ijk} e_{ijk} \otimes \delta_{ac} u_{ia} u_{jb} u_{kc} \end{aligned}$$

Thus the condition $T_\pi \in End(u^{\otimes 3})$ is equivalent to the following relations:

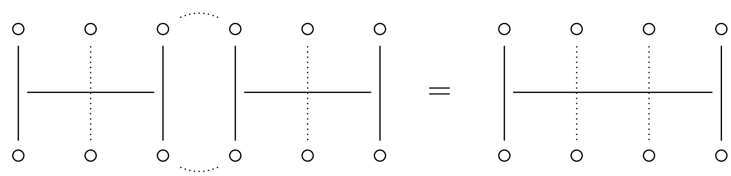
$$(\delta_{ik} - \delta_{ac}) u_{ia} u_{jb} u_{kc} = 0$$

The non-trivial cases are $i = k, a \neq c$ and $i \neq k, a = c$, and these produce the relations $u_{ia} u_{jb} u_{ic} = 0$ for any $a \neq c$, and $u_{ia} u_{jb} u_{ka} = 0$, for any $i \neq k$. Thus, we have reached to the standard relations for the quantum group $H_N^{[\infty]}$.

(1) The fact that we have inclusions $H_N^* \subset H_N^{[\infty]} \subset H_N^+$ comes from:

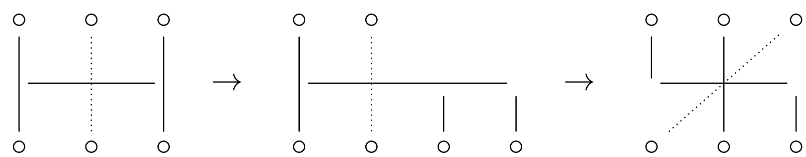


(2) At $r = 2$, the relations $ab_1b_2c = 0$ come indeed from the following diagram:



In the general case $r \geq 2$ the proof is similar, see [28] for details.

(3) We use here an idea from [120], [120]. By rotating π, we obtain:



Let us denote by σ the partition on the right. Since $T_\sigma(e_{ijk}) = \delta_{ij}e_{kji}$, we obtain:

$$\begin{aligned} T_\sigma u^{\otimes 3} e_{abc} &= T_\sigma \sum_{ijk} e_{ijk} \otimes u_{ia}u_{jb}u_{kc} = \sum_{ijk} e_{kji} \otimes \delta_{ij} u_{ia}u_{jb}u_{kc} \\ u^{\otimes 3} T_\sigma e_{abc} &= u^{\otimes 3}\delta_{ab}e_{cba} = \sum_{ijk} e_{kji} \otimes \delta_{ab} u_{kc}u_{jb}u_{ia} \end{aligned}$$

Thus $T_\sigma \in End(u^{\otimes 3})$ is equivalent to the following relations:

$$\delta_{ij}u_{ia}u_{jb}u_{kc} = \delta_{ab}u_{kc}u_{jb}u_{ia}$$

Now by setting $j = i, b = a$ we obtain the commutation relation $u_{ia}^2 u_{kc} = u_{kc}u_{ia}^2$ in the statement, which finishes the proof. $\square$

The relation of $H_N^{[\infty]}$ with the polygonal spheres comes from the following fact:

Proposition 14.7. *Let $X \subset S^{N-1}_{\mathbb{R},+}$ be closed, let $d \geq 2$, and set $X^{d-1} = X \cap S^{N-1,d-1}_{\mathbb{R},+}$. Then for a quantum group $G \subset H_N^{[\infty]}$ the following are equivalent:*

(1) *$x_i \to \sum_j u_{ij} \otimes x_j$ defines a coaction $\Phi : C(X^{d-1}) \to C(G) \otimes C(X^{d-1})$.*

(2) *$x_i \to \sum_j u_{ij} \otimes x_j$ defines a morphism $\widetilde{\Phi} : C(X) \to C(G) \otimes C(X^{d-1})$.*

In particular, $G^+(X) \cap H_N^{[\infty]}$ acts on X^{d-1}, for any $d \geq 2$.

Proof. The idea here is to use the relations in Proposition 14.6 (2) above:

(1) $\implies$ (2) This is clear, by composing Φ with the following projection map:

$$\pi : C(X) \to C(X^{d-1})$$

(2) $\implies$ (1) In order for a coaction $C(X^{d-1}) \to C(G) \otimes C(X^{d-1})$ to exist, the variables $X_i = \sum_j u_{ij} \otimes x_j$ must satisfy the relations defining X, which hold indeed by (2), and must satisfy as well the relations $X_{i_0} \dots X_{i_d} = 0$ for $i_0, \dots, i_d$ distinct, which define $S^{N-1,d-1}_{\mathbb{R},+}$.

The point now is that, under the assumption $G \subset H_N^{[\infty]}$, these latter relations are automatic. Indeed, by using Proposition 14.6 (2), for $i_0, \dots, i_d$ distinct we obtain:

$$\begin{aligned}
X_{i_0} \dots X_{i_d} &= \sum_{j_0 \dots j_d} u_{i_0 j_0} \dots u_{i_d j_d} \otimes x_{j_0} \dots x_{j_d} \\
&= \sum_{j_0 \dots j_d \ distinct} u_{i_0 j_0} \dots u_{i_d j_d} \otimes 0 \quad + \sum_{j_0 \dots j_d \ not\ distinct} 0 \otimes x_{j_0} \dots x_{j_d} \\
&= 0 + 0 = 0
\end{aligned}$$

Thus the coaction in (1) exists precisely when (2) is satisfied, and we are done.

Finally, the last assertion is clear from (2) $\implies$ (1), because the universal coaction of $G = G^+(X)$ gives rise to a map $\widetilde{\Phi} : C(X) \to C(G) \otimes C(X^{d-1})$ as in (2). □

As an illustration, we have the following result:

Theorem 14.8. *The compact quantum groups*

$$H_N, H_N, H_N^*, H_N^*, H_N^{[\infty]}$$

act respectively on the spheres

$$S^{N-1,d-1}_{\mathbb{R}}, \bar{S}^{N-1,d-1}_{\mathbb{R}}, S^{N-1,d-1}_{\mathbb{R},*}, \bar{S}^{N-1,d-1}_{\mathbb{R},*}, S^{N-1,d-1}_{\mathbb{R},+}$$

at any $d \geq 2$.

Proof. We use Proposition 14.7. We know that the quantum isometry groups at $d = N$ are respectively equal to the following quantum groups:

$$O_N, \bar{O}_N, O_N^*, \bar{O}_N^*, O_N^+$$

Our claim is that, by intersecting these quantum groups with $H_N^{[\infty]}$, we obtain the quantum groups in the statement. Indeed:

(1) $O_N \cap H_N^{[\infty]} = H_N$ is clear from definitions.

(2) $\bar{O}_N \cap H_N^{[\infty]} = H_N$ follows from $\bar{O}_N \cap H_N^+ \subset O_N$, which in turn follows from the computation (3) in the proof of Proposition 14.5, with H_N^* replaced by H_N^+.

(3) $O_N^* \cap H_N^{[\infty]} = H_N^*$ follows from $O_N^* \cap H_N^+ = H_N^*$.

(4) $\bar{O}_N^* \cap H_N^{[\infty]} \supset H_N^*$ is clear, and the reverse inclusion can be proved by a direct computation, similar to the computation (3) in the proof of Proposition 14.5.

(5) $O_N^+ \cap H_N^{[\infty]} = H_N^{[\infty]}$ is clear from definitions. □

Observe that the above result is "sharp", in the sense that the actions there are the universal ones, in the classical case at any $d \in \{2, \ldots, N-1\}$, as well as in the twisted case at $d = 2$. Indeed, this follows from the various results established above.

Let us discuss now the computation for $S^{N-1,1}_{\mathbb{R},*}$. We know that the quantum group H_N^* acts on $S^{N-1,1}_{\mathbb{R},*}$. This action, however, is not universal, because we have:

Proposition 14.9. *$\widehat{\mathbb{Z}_2^{*N}}$ acts on $S^{N-1,1}_{\mathbb{R},*}$.*

Proof. The standard coordinates on $S^{N-1,1}_{\mathbb{R},*}$ are subject to the following relations:

$$x_ix_jx_k = \begin{cases} 0 & \text{for } i,j,k \text{ distinct} \\ x_kx_jx_i & \text{otherwise} \end{cases}$$

Thus, in order to have a coaction map $\Phi : C(S^{N-1,1}_{\mathbb{R},*}) \to C(G) \otimes C(S^{N-1,1}_{\mathbb{R},*})$, given by $\Phi(x_i) = \sum_j u_{ij} \otimes x_j$, the variables $X_i = \sum_j u_{ij} \otimes x_j$ must satisfy the above relations.

For the group dual $G = \widehat{\mathbb{Z}_2^{*N}}$ we have by definition $u_{ij} = \delta_{ij} g_i$, where $g_1, \ldots, g_N$ are the standard generators of $\mathbb{Z}_2^{*N}$, and we therefore have:

$$X_iX_jX_k = g_ig_jg_k \otimes x_ix_jx_k$$

$$X_kX_jX_i = g_kg_jg_i \otimes x_kx_jx_i$$

Thus the formula $X_iX_kX_k = 0$ for i,j,k distinct is clear, and the formula $X_iX_jX_k = X_kX_jX_i$ for i,j,k not distinct requires $g_ig_jg_k = g_kg_jg_i$ for i,j,k not distinct, which is clear as well. Indeed, at $i = j$ this latter relation reduces to $g_k = g_k$, at $i = k$ this relation is trivial, $g_ig_jg_i = g_ig_jg_i$, and at $j = k$ this relation reduces to $g_i = g_i$. □

More generally, we have the following result:

Proposition 14.10. *$H_N^{[\infty]}$ acts on $S^{N-1,1}_{\mathbb{R},*}$.*

Proof. We proceed as in the proof of Theorem 14.4 above. By expanding the formula of $X_iX_jX_k$ and by using the relations for the sphere $S^{N-1,1}_{\mathbb{R},*}$, we have:

$$\begin{aligned}
X_iX_jX_k &= \sum_{abc} u_{ia}u_{jb}u_{kc}\otimes x_ax_bx_c\\
&= \sum_{a,b,c\ not\ distinct} u_{ia}u_{jb}u_{kc}\otimes x_ax_bx_c\\
&= \sum_{a\neq b}(u_{ia}u_{ja}u_{kb}+u_{ib}u_{ja}u_{ka})\otimes x_a^2x_b\\
&+ \sum_{a\neq b} u_{ia}u_{jb}u_{ka}\otimes x_ax_bx_a+\sum_a u_{ia}u_{ja}u_{ka}\otimes x_a^3
\end{aligned}$$

Now by assuming $G=H_N^{[\infty]}$, and by using the various formulae in Proposition 14.6 above, we obtain, for any i,j,k distinct:

$$X_iX_jX_k=\sum_{a\neq b}(0\cdot u_{kb}+u_{ib}\cdot 0)\otimes x_a^2x_b+\sum_{a\neq b}0\otimes x_ax_bx_a+\sum_a(0\cdot u_{ka})\otimes x_a^3=0$$

It remains to prove that we have $X_iX_jX_k=X_kX_jX_i$, for i,j,k not distinct. By replacing $i\leftrightarrow k$ in the above formula of $X_iX_jX_k$, we obtain:

$$\begin{aligned}
X_kX_jX_i &= \sum_{a\neq b}(u_{ka}u_{ja}u_{ib}+u_{kb}u_{ja}u_{ia})\otimes x_a^2x_b\\
&+ \sum_{a\neq b} u_{ka}u_{jb}u_{ia}\otimes x_ax_bx_a+\sum_a u_{ka}u_{ja}u_{ia}\otimes x_a^3
\end{aligned}$$

Let us compare this formula with the above formula of $X_iX_jX_k$. The last sum being 0 in both cases, we must prove that for any i,j,k not distinct and any $a\neq b$ we have:

$$u_{ia}u_{ja}u_{kb}+u_{ib}u_{ja}u_{ka}=u_{ka}u_{ja}u_{ib}+u_{kb}u_{ja}u_{ia}$$

$$u_{ia}u_{jb}u_{ka}=u_{ka}u_{jb}u_{ia}$$

By symmetry the three cases $i=j,i=k,j=k$ reduce to two cases, $i=j$ and $i=k$. The case $i=k$ being clear, we are left with the case $i=j$, where we must prove:

$$u_{ia}u_{ia}u_{kb}+u_{ib}u_{ia}u_{ka}=u_{ka}u_{ia}u_{ib}+u_{kb}u_{ia}u_{ia}$$

$$u_{ia}u_{ib}u_{ka}=u_{ka}u_{ib}u_{ia}$$

By using $a\neq b$, the first equality reads $u_{ia}^2u_{kb}+0\cdot u_{ka}=u_{ka}\cdot 0+u_{kb}u_{ia}^2$, and since by Proposition 14.6 (3) we have $u_{ia}^2u_{kb}=u_{kb}u_{ia}^2$, we are done. As for the second equality, this reads $0\cdot u_{ka}=u_{ka}\cdot 0$, which is true as well, and this ends the proof. □

We will prove now that the action in Proposition 14.10 is universal. In order to do so, we need to convert the formulae of type $X_iX_jX_k=0$ and $X_iX_jX_k=X_kX_jX_i$ into relations between the quantum group coordinates u_{ij}, and this requires a good knowledge of the linear relations between the variables $x_a^2x_b,x_ax_bx_a,x_a^3$ over the sphere $S^{N-1,1}_{\mathbb R,*}$.

So, we must first study these variables. The answer here is given by:

Proposition 14.11. *The variables*

$$\left\{x_a^2x_b,x_ax_bx_a,x_a^3\Big|a\neq b\right\}$$

are linearly independent over the sphere $S^{N-1,1}_{\mathbb R,*}$.

Proof. We use a trick from [48]. Consider the 1-dimensional polygonal version of the complex sphere $S^{N-1}_{\mathbb C}$, which is by definition given by:

$$S^{N-1,1}_{\mathbb C}=\left\{z\in S^{N-1}_{\mathbb C}\Big|z_iz_jz_k=0,\forall i,j,k\text{ distinct}\right\}$$

We have then a 2×2 matrix model for the coordinates of $S^{N-1,1}_{\mathbb R,*}$, as follows:

$$x_i\to\gamma_i=\begin{pmatrix}0&z_i\\ \bar z_i&0\end{pmatrix}$$

Indeed, the matrices γ_i on the right are all self-adjoint, their squares sum up to 1, they half-commute, and they satisfy $\gamma_i\gamma_j\gamma_k=0$ for i,j,k distinct. Thus we have indeed a morphism $C(S^{N-1,1}_{\mathbb R,*})\to M_2(C(S^{N-1,1}_{\mathbb C}))$ mapping $x_i\to\gamma_i$, as claimed.

We can use this model in order to prove the linear independence. Indeed, the variables $x_a^2x_b,x_ax_bx_a,x_a^3$ that we are interested in are mapped to the following variables:

$$\gamma_a^2\gamma_b=\begin{pmatrix}0&|z_a|^2z_b\\ |z_a|^2\bar z_b&0\end{pmatrix}$$

$$\gamma_a\gamma_b\gamma_a=\begin{pmatrix}0&z_a^2\bar z_b\\ \bar z_a^2z_b&0\end{pmatrix}$$

$$\gamma_a^3=\begin{pmatrix}0&|z_a|^2z_a\\ |z_a|^2\bar z_a&0\end{pmatrix}$$

Now observe that the following variables are linearly independent over $S^1_{\mathbb C}$:

$$|z_1|^2z_2,|z_2|^2z_1,z_1^2\bar z_2,z_2^2\bar z_1,|z_1|^2z_1,|z_2|^2z_2$$

Thus the upper right entries of the above matrices are linearly independent over $S^{N-1,1}_{\mathbb C}$. Thus the matrices themselves are linearly independent, and this proves our result. □

With the above result in hand, we can now reformulate the coaction problem into a purely quantum group-theoretical problem, as follows:

Proposition 14.12. *A quantum group $G\subset O_N^+$ acts on $S^{N-1,1}_{\mathbb R,*}$ precisely when its standard coordinates u_{ij} satisfy the following relations:*

1. $u_{ia}u_{ja}u_{kb}+u_{ib}u_{ja}u_{ka}=0$ *for any i,j,k distinct.*
2. $u_{ia}u_{jb}u_{ka}=0$ *for any i,j,k distinct.*
3. $u_{ia}^2u_{kb}=u_{kb}u_{ia}^2$.
4. $u_{ka}u_{ia}u_{ib}=u_{ib}u_{ia}u_{ka}$.
5. $u_{ia}u_{ib}u_{ka}=u_{kb}u_{ib}u_{ia}$.

Proof. We use notations from the beginning of the proof of Proposition 14.10, along with the following formula, also established there:

$$\begin{aligned}X_iX_jX_k&=\sum_{a\neq b}(u_{ia}u_{ja}u_{kb}+u_{ib}u_{ja}u_{ka})\otimes x_a^2x_b\\&+\sum_{a\neq b}u_{ia}u_{jb}u_{ka}\otimes x_ax_bx_a+\sum_a u_{ia}u_{ja}u_{ka}\otimes x_a^3\end{aligned}$$

In order to have an action as in the statement, these quantities must satisfy $X_iX_kX_k=0$ for i,j,k disctinct, and $X_iX_kX_k=X_kX_jX_i$ for i,j,k not distinct. Now by using Proposition 14.11, we conclude that the relations to be satisfied are as follows:

(A) For i,j,k distinct, the following must hold:

$$u_{ia}u_{ja}u_{kb}+u_{ib}u_{ja}u_{ka}=0,\forall a\neq b$$

$$u_{ia}u_{jb}u_{ka}=0,\forall a\neq b$$

$$u_{ia}u_{ja}u_{ka}=0,\forall a$$

(B) For i,j,k not distinct, the following must hold:

$$u_{ia}u_{ja}u_{kb}+u_{ib}u_{ja}u_{ka}=u_{ka}u_{ja}u_{ib}+u_{kb}u_{ja}u_{ia},\forall a\neq b$$

$$u_{ia}u_{jb}u_{ka}=u_{ka}u_{jb}u_{ia},\forall a\neq b$$

$$u_{ia}u_{ja}u_{ka}=u_{ka}u_{ja}u_{ia},\forall a$$

In order to simplify this set of relations, the first observation is that the last relations in both (A) and (B) can be merged with the other ones, and we are led to:

(A') For i,j,k distinct, the following must hold:

$$u_{ia}u_{ja}u_{kb}+u_{ib}u_{ja}u_{ka}=0,\forall a,b$$

$$u_{ia}u_{jb}u_{ka}=0,\forall a,b$$

(B') For i,j,k not distinct, the following must hold:

$$u_{ia}u_{ja}u_{kb}+u_{ib}u_{ja}u_{ka}=u_{ka}u_{ja}u_{ib}+u_{kb}u_{ja}u_{ia},\forall a,b$$

$$u_{ia}u_{jb}u_{ka}=u_{ka}u_{jb}u_{ia},\forall a,b$$

Observe that the relations (A') are exactly the relations (1,2) in the statement.

Let us further process the relations (B'). In the case $i=k$ the relations are automatic, and in the cases $j=i, j=k$ the relations that we obtain coincide, via $i\leftrightarrow k$. Thus (B') reduces to the set of relations obtained by setting $j=i$, which are as follows:

$$u_{ia}u_{ia}u_{kb}+u_{ib}u_{ia}u_{ka}=u_{ka}u_{ia}u_{ib}+u_{kb}u_{ia}u_{ia}$$

$$u_{ia}u_{ib}u_{ka}=u_{ka}u_{ib}u_{ia}$$

Observe that the second relation is the relation (5) in the statement. Regarding now the first relation, with the notation $[x,y,z]=xyz-zyx$, this is as follows:

$$[u_{ia},u_{ia},u_{kb}]=[u_{ka},u_{ia},u_{ib}]$$

By applying the antipode, we obtain $[u_{bk},u_{ai},u_{ai}]=[u_{bi},u_{ai},u_{ak}]$, and then relabelling $a\leftrightarrow i$ and $b\leftrightarrow k$, this relation becomes $[u_{kb},u_{ia},u_{ia}]=[u_{ka},u_{ia},u_{ib}]$. Now since we have $[x,y,z]=-[z,y,x]$, by comparing this latter relation with the original one, a simplification occurs, and the resulting relations are as follows:

$$[u_{ia},u_{ia},u_{kb}]=[u_{ka},u_{ia},u_{ib}]=0$$

But these are exactly the relations (3,4) in the statement, and we are done. $\square$

Now by solving the quantum group problem raised by Proposition 14.12, we obtain:

Proposition 14.13. *We have $G^+(S^{N-1,1}_{\mathbb{R},*})=H_N^{[\infty]}$.*

Proof. The inclusion $\supset$ is clear from Proposition 14.10. For the converse, we already have the result at $N=2$, so assume $N\geq 3$. We will use many times the conditions (1-5) in Proposition 14.12. By using (2), for $i\neq j$ we have:

$$\begin{aligned}
u_{ia}u_{jb}u_{ka}=0,\forall k\neq i,j\quad&\implies\quad u_{ia}u_{jb}u_{ka}^2=0,\forall k\neq i,j\\
&\implies\quad u_{ia}u_{jb}\left(\sum_{k\neq i,j}u_{ka}^2\right)=0,\forall i\neq j\\
&\implies\quad u_{ia}u_{jb}(1-u_{ia}^2-u_{ja}^2)=0,\forall i\neq j
\end{aligned}$$

Now by using (3), we can move the variable u_{jb} to the right. By further multiply by u_{jb} to the right, and then summing over b, we obtain:

$$\begin{aligned}
&u_{ia}u_{jb}(1-u_{ia}^2-u_{ja}^2)=0,\forall i\neq j\\
\implies\quad&u_{ia}(1-u_{ia}^2-u_{ja}^2)u_{jb}=0,\forall i\neq j\\
\implies\quad&u_{ia}(1-u_{ia}^2-u_{ja}^2)u_{jb}^2=0,\forall i\neq j\\
\implies\quad&u_{ia}(1-u_{ia}^2-u_{ja}^2)=0,\forall i\neq j
\end{aligned}$$

We can proceed now as follows, by summing over $j\neq i$:

$$\begin{aligned}
&u_{ia}(1-u_{ia}^2-u_{ja}^2)=0,\forall i\neq j\\
\implies\ &u_{ia}u_{ja}^2=u_{ia}-u_{ia}^3,\forall i\neq j\\
\implies\ &u_{ia}(1-u_{ia}^2)=(N-1)(u_{ia}-u_{ia}^3)\\
\implies\ &u_{ia}=u_{ia}^3
\end{aligned}$$

Thus the standard coordinates are partial isometries, and so $G\subset H_N^+$. On the other hand, we know from the proof of Proposition 14.6 (3) that the quantum subgroup $G\subset H_N^+$ obtained via the relations $[a,b^2]=0$ is $H_N^{[\infty]}$, and this finishes the proof. □

We have now complete results for the 9 main spheres, as follows:

Theorem 14.14. *The quantum isometry groups of the 9 polygonal spheres are*

$$\begin{tikzcd}
O_N \arrow[r] & O_N^* \arrow[r] & O_N^+\\
H_N \arrow[r] \arrow[u] & H_N^{[\infty]} \arrow[r] \arrow[u] & \bar{O}_N^* \arrow[u]\\
H_N^+ \arrow[r] \arrow[u] & H_N \arrow[r] \arrow[u] & \bar{O}_N \arrow[u]
\end{tikzcd}$$

where $H_N^+,H_N^{[\infty]}$ *and* $\bar{O}_N,O_N^*,\bar{O}_N^*,O_N^*$ *are quantum versions of* H_N,O_N.

Proof. This follows indeed by putting together the above results. □

Let us discuss now a straightforward complex extension of the above results. Our starting point will be the following definition:

Definition 14.15. *The complex polygonal spheres, denoted*

$$S^{N-1,d-1}_{\mathbb{C}},\bar{S}^{N-1,d-1}_{\mathbb{C}},\bar{S}^{N-1,d-1}_{\mathbb{C},*},S^{N-1,d-1}_{\mathbb{C},+}$$

are constructed from $S^{N-1}_{\mathbb{C},+}$ *in the same way as their real versions, namely*

$$S^{N-1,d-1}_{\mathbb{R}},\bar{S}^{N-1,d-1}_{\mathbb{R}},\bar{S}^{N-1,d-1}_{\mathbb{R},*},S^{N-1,d-1}_{\mathbb{R},+}$$

are constructed from $S^{N-1}_{\mathbb{R},+}$, *namely by assuming that the corresponding vanishing relations hold between the variables* $x_i=z_i,z_i^*$.

As in the real case, we will restrict now the attention to the 5 main spheres, coming from [9], and to their intersections. We have 9 such spheres here, as follows:

$$\begin{tikzcd}
S^{N-1}_{\mathbb{C}} \arrow[r] & S^{N-1}_{\mathbb{C},*} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\
S^{N-1}_{\mathbb{C}} \cap \bar{S}^{N-1}_{\mathbb{C},*} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C},*} \cap \bar{S}^{N-1}_{\mathbb{C},*} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{C},*} \arrow[u] \\
S^{N-1}_{\mathbb{C}} \cap \bar{S}^{N-1}_{\mathbb{C}} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C},*} \cap \bar{S}^{N-1}_{\mathbb{C}} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{C}} \arrow[u]
\end{tikzcd}$$

The intersections can be computed as in the real case, and we have:

Proposition 14.16. *The* 5 *main spheres, and the intersections between them, are*

$$\begin{tikzcd}
S^{N-1}_{\mathbb{C}} \arrow[r] & S^{N-1}_{\mathbb{C},*} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\
S^{N-1,1}_{\mathbb{C}} \arrow[r] \arrow[u] & S^{N-1,1}_{\mathbb{C},*} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{C},*} \arrow[u] \\
S^{N-1,0}_{\mathbb{C}} \arrow[r] \arrow[u] & \bar{S}^{N-1,1}_{\mathbb{C}} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{C}} \arrow[u]
\end{tikzcd}$$

with all the maps being inclusions.

Proof. This is similar to the proof from the real case, by replacing in all the computations there the variables x_i by the variables $x_i = z_i, z_i^*$. □

Next, we have the following result:

Theorem 14.17. *The quantum isometry groups of the* 9 *main complex spheres are*

$$\begin{tikzcd}
U_N \arrow[r] & U_N^{**} \arrow[r] & U_N^+ \\
K_N \arrow[r] \arrow[u] & K_N^{[\infty]} \arrow[r] \arrow[u] & \bar{U}_N^{**} \arrow[u] \\
K_N^+ \arrow[r] \arrow[u] & K_N \arrow[r] \arrow[u] & \bar{U}_N \arrow[u]
\end{tikzcd}$$

where K_N *and its versions are the complex analogues of* H_N *and its versions.*

Proof. The idea is that the proof here is quite similar to the proof in the real case, by replacing H_N, O_N with their complex analogues K_N, U_N. □

As a conclusion, we have many technical results available, but there are still many questions left, regarding the extension of our (S, T, U, K) formalism, as to cover the intersections between the twisted and untwisted geometries.

15. Projective geometry

We discuss here analogues of the various structure results and axiomatization and classification questions developed above, in the projective geometry setting. This section will be quite elementary, with full details given, even for results that we already know, to be repeated here, with the aim of making this presentation as independent as possible from the previous sections, as a beginning of something new.

The point is that things become considerably simpler in the projective geometry setting. Consider indeed the diagram of 9 main geometries, that we found above:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{TR}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{TR}^N_* \arrow[r] \arrow[u] & \mathbb{C}^N_* \arrow[u] \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{TR}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

As explained in sections 9-10, when looking at the projective versions of the corresponding spheres, the diagram drastically simplies, and becomes as follows:

$$\begin{tikzcd}
P^{N-1}_+ \arrow[r] & P^{N-1}_+ \arrow[r] & P^{N-1}_+ \\
P^{N-1}_\mathbb{C} \arrow[r] \arrow[u] & P^{N-1}_\mathbb{C} \arrow[r] \arrow[u] & P^{N-1}_\mathbb{C} \arrow[u] \\
P^{N-1}_\mathbb{R} \arrow[r] \arrow[u] & P^{N-1}_\mathbb{R} \arrow[r] \arrow[u] & P^{N-1}_\mathbb{C} \arrow[u]
\end{tikzcd}$$

Thus, we are led to the conclusion that, under certain combinatorial axioms, there should be only 3 projective geometries, namely the real, complex and free one:

$$P^{N-1}_\mathbb{R} \subset P^{N-1}_\mathbb{C} \subset P^{N-1}_+$$

We will discuss this in what follows, with analogues and improvements of the affine results. Also, we would like to study the corresponding quadruplets (P, PT, PU, PK), and to axiomatize the projective geometries, with correspondences as follows:

$$\begin{tikzcd}
P \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & PT \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
PU \arrow[r, leftrightarrow] & PK
\end{tikzcd}$$

Summarizing, there is a lot of work to be done, on one hand in reformulating and improving the results from the affine case, and on the other hand, in starting to develop the projective theory independently from the affine theory.

Let us begin with a short summary of the various projective geometry results that we have so far. We will give full details here, with the aim of making the present section as independent as possible from the previous sections, as a beginning of something new.

Our starting point is the following functional analytic description of $P^{N-1}_\mathbb{R},P^{N-1}_\mathbb{C}$:

Proposition 15.1. *We have presentation results as follows,*

$$C(P^{N-1}_\mathbb{R}) = C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=\bar{p}=p^t=p^2,Tr(p)=1\right)$$
$$C(P^{N-1}_\mathbb{C}) = C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=1\right)$$

for the algebras of continuous functions on the real and complex projective spaces.

Proof. We use the fact that $P^{N-1}_\mathbb{C},P^{N-1}_\mathbb{R}$ are respectively the spaces of rank one projections in $M_N(\mathbb{C}),M_N(\mathbb{R})$. With this picture in mind, it is clear that we have arrows $\leftarrow$.

In order to construct now arrows $\to$, consider the universal algebras on the right, A_C,A_R. These algebras being both commutative, by the Gelfand theorem we can write, with X_C,X_R being certain compact spaces:

$$A_C=C(X_C)$$
$$A_R=C(X_R)$$

Now by using the coordinate functions p_{ij}, we conclude that X_C,X_R are certain spaces of rank one projections in $M_N(\mathbb{C}),M_N(\mathbb{R})$. In other words, we have embeddings:

$$X_C\subset P^{N-1}_\mathbb{C}$$
$$X_R\subset P^{N-1}_\mathbb{R}$$

Bsy transposing we obtain arrows $\to$, as desired. □

The above result suggests the following definition:

Definition 15.2. *Associated to any $N\in\mathbb N$ is the following universal algebra,*

$$C(P^{N-1}_+)=C^*\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=1\right)$$

whose abstract spectrum is called "free projective space".

Observe that we have embeddings of compact quantum spaces, as follows:

$$P^{N-1}_\mathbb{R}\subset P^{N-1}_\mathbb{C}\subset P^{N-1}_+$$

Also, the complex projective space $P^{N-1}_\mathbb{C}$ is the classical version of P^{N-1}_+.

Let us discuss now the relation with the spheres. Given a closed subset $X\subset S^{N-1}_{\mathbb R,+}$, its projective version is by definition the quotient space $X\to PX$ determined by the fact that $C(PX)\subset C(X)$ is the subalgebra generated by the following variables:

$$p_{ij}=x_ix_j$$

In the classical case, we recover in this way the usual projective version.

In order to discuss now the relation with the quantum spheres, it is convenient to neglect the material from section 10, regarding the "hybrid" case, the projective versions of the spheres there bringing nothing new, for obvious reasons.

On the other hand, it is also convenient to neglect the material regarding the complex quantum spheres, because, as explained in section 9, the projective versions of these spheres bring nothing new, due to the various results worked out there.

Thus, we are left with the 3 real spheres, and we have the following result:

Proposition 15.3. *The projective versions of the 3 real spheres are as follows,*

$$\begin{tikzcd}
S^{N-1}_\mathbb{R} \arrow[r] \arrow[d] & S^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[d] & S^{N-1}_{\mathbb{R},+} \arrow[d] \\
P^{N-1}_\mathbb{R} \arrow[r] & P^{N-1}_\mathbb{C} \arrow[r] & P^{N-1}_+
\end{tikzcd}$$

modulo the standard equivalence relation for the quantum algebraic manifolds.

Proof. The assertion at left is true by definition. For the assertion at right, we have to prove that the variables $p_{ij}=z_iz_j$ over the free sphere $S^{N-1}_{\mathbb R,+}$ satisfy the defining relations for $C(P^{N-1}_+)$, from Definition 15.2. We first have the following computation:

$$\begin{aligned}(p^*)_{ij} &= p_{ji}^*\\ &= (z_jz_i)^*\\ &= z_iz_j\\ &= p_{ij}\end{aligned}$$

We have as well the following computation:

$$\begin{aligned}(p^2)_{ij} &= \sum_k p_{ik}p_{kj}\\ &= \sum_k z_iz_k^2z_j\\ &= z_iz_j\\ &= p_{ij}\end{aligned}$$

Finally, we have as well the following computation:

$$\begin{aligned}Tr(p) &= \sum_k p_{kk}\\ &= \sum_k z_k^2\\ &= 1\end{aligned}$$

Regarding now the middle assertion, stating that we have $PS^{N-1}_{\mathbb{R},*} = P^{N-1}_{\mathbb{C}}$, the inclusion "$\subset$" follows from the relations $abc = cba$, which imply:

$$abcd = cbad = cbda$$

In the other sense now, the point is that we have a matrix model, as follows:

$$\pi : C(S^{N-1}_{\mathbb{R},*}) \to M_2(C(S^{N-1}_{\mathbb{C}}))$$

$$x_i \to \begin{pmatrix}0 & z_i\\ \bar{z}_i & 0\end{pmatrix}$$

But this gives the missing inclusion "$\supset$", and we are done. See [32]. □

In addition to the above result, let us mention that, as already discussed above, passing to the complex case brings nothing new. This is because the projective version of the free complex sphere is equal to the free projective space constructed above:

$$PS^{N-1}_{\mathbb{C},+} = P^{N-1}_+$$

For details on all this, we refer to section 9 above.

In what regards the tori, we have the following result:

Proposition 15.4. *The projective versions of the* 3 *real tori are as follows,*

$$\begin{array}{ccccc}
T_N & \longrightarrow & T_N^* & \longrightarrow & T_N^+\\
\downarrow & & \downarrow & & \downarrow\\
PT_N & \longrightarrow & P\mathbb{T}_N & \longrightarrow & PT_N^+
\end{array}$$

modulo the standard equivalence relation for the quantum algebraic manifolds.

Proof. This follows by using the same arguments as for the spheres. □

In what regards the unitary groups, that we will call in what follows orthogonal groups, because we are now in the real case, we have here the following result:

Proposition 15.5. *The projective versions of the* 3 *orthogonal groups are as follows,*

$$\begin{array}{ccccc}
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+\\
\downarrow & & \downarrow & & \downarrow\\
PO_N & \longrightarrow & PU_N & \longrightarrow & PO_N^+
\end{array}$$

modulo the standard equivalence relation for the compact quantum groups.

Proof. This follows by using the same arguments as for the spheres. □

Finally, in what regards the reflection groups, that we will call hyperoctahedral groups, because we are now in the real case, we have here the following result:

Proposition 15.6. *The projective versions of the* 3 *hyperoctahedral groups are as follows,*

$$\begin{array}{ccccc}
H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^+\\
\downarrow & & \downarrow & & \downarrow\\
PH_N & \longrightarrow & PK_N & \longrightarrow & PH_N^+
\end{array}$$

modulo the standard equivalence relation for the compact quantum groups.

Proof. This follows by using the same arguments as for the spheres. □

In addition to the above results, let us mention that, as it was the case for the spheres, passing to the complex case brings nothing new. This is because we have isomorphisms $P\mathbb{T}_N^+ = PT_N^+$ and $PU_N^+ = PO_N^+$ and $PK_N^+ = PH_N^+$, as explained in section 9 above.

Getting back now to our general program, we are done with the construction work, for the various projective geometry basic objects. Our next task will be that of working out axiomatization and classification results, first in analogy with the affine results, and then independently of what we already have, with a number of new results, of true projective nature. Let us begin with a summary of the constructions discussed above. As a conclusion to what we did, we have 3 projective quadruplets, as follows:

Theorem 15.7. *We have "basic" projective quadruplets (P, PT, PU, PK) as follows,*

(1) *A classical real quadruplet, as follows,*

$$\begin{matrix} P^{N-1}_{\mathbb{R}} & - & PT_N \\ | & \times & | \\ PO_N & - & PH_N \end{matrix}$$

(2) *A classical complex quadruplet, as follows,*

$$\begin{matrix} P^{N-1}_{\mathbb{C}} & - & P\mathbb{T}_N \\ | & \times & | \\ PU_N & - & PK_N \end{matrix}$$

(3) *A free quadruplet, as follows,*

$$\begin{matrix} P^{N-1}_{+} & - & PT^+_N \\ | & \times & | \\ PO^+_N & - & PH^+_N \end{matrix}$$

which appear as projective versions of the main 3 real quadruplets.

Proof. This follows from the results that already have. To be more precise:

(1) Consider the classical affine real quadruplet, which is as follows:

$$\begin{tikzcd} S^{N-1}_{\mathbb{R}} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & T_N \arrow[d, dash] \arrow[dl, dash] \\ O_N \arrow[r, dash] & H_N \end{tikzcd}$$

The projective version of this quadruplet is then the quadruplet in (1).

(2) Consider the half-classical affine real quadruplet, which is as follows:

$$\begin{tikzcd} S^{N-1}_{\mathbb{R},*} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & T_N^* \arrow[d, dash] \arrow[dl, dash] \\ O_N^* \arrow[r, dash] & H_N^* \end{tikzcd}$$

The projective version of this quadruplet is then the quadruplet in (2).

(3) Consider the free affine real quadruplet, which is as follows:

$$\begin{tikzcd} S^{N-1}_{\mathbb{R},+} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & T_N^+ \arrow[d, dash] \arrow[dl, dash] \\ O_N^+ \arrow[r, dash] & H_N^+ \end{tikzcd}$$

The projective version of this quadruplet is then the quadruplet in (3). □

Getting back now to our general projective geometry program, we would like to have axiomatization and classification results for such quadruplets.

In order to do this, following [33], we can axiomatize our various projective spaces, as follows:

Definition 15.8. *A monomial projective space is a closed subset $P \subset P^{N-1}_+$ obtained via relations of type*

$$p_{i_1 i_2} \ldots p_{i_{k-1} i_k} = p_{i_{\sigma(1)} i_{\sigma(2)}} \ldots p_{i_{\sigma(k-1)} i_{\sigma(k)}}, \ \forall (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$$

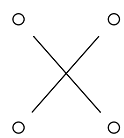
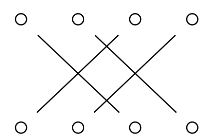
with σ ranging over a certain subset of $\bigcup_{k \in 2\mathbb{N}} S_k$, which is stable under $\sigma \to |\sigma|$.

Observe the similarity with the corresponding notion for the spheres, from section 13. The only subtlety in the projective case is the stability under the operation $\sigma \to |\sigma|$, which in practice means that if the above relation associated to σ holds, then the following relation, associated to $|\sigma|$, must hold as well:

$$p_{i_0i_1}\ldots p_{i_ki_{k+1}} = p_{i_0i_{\sigma(1)}}p_{i_{\sigma(2)}i_{\sigma(3)}}\ldots p_{i_{\sigma(k-2)}i_{\sigma(k-1)}}p_{i_{\sigma(k)}i_{k+1}}$$

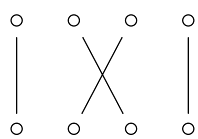
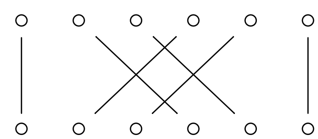
As an illustration, the basic projective spaces are all monomial:

Proposition 15.9. *The* 3 *projective spaces are all monomial, with the permutations*

producing respectively the spaces $P^{N-1}_\mathbb{R}, P^{N-1}_\mathbb{C}$.

Proof. We must divide the algebra $C(P^{N-1}_+)$ by the relations associated to the diagrams in the statement, as well as those associated to their shifted versions, given by:

(1) The basic crossing, and its shifted version, produce the following relations:

$$p_{ab} = p_{ba}$$

$$p_{ab}p_{cd} = p_{ac}p_{bd}$$

Now by using these relations several times, we obtain:

$$\begin{aligned} p_{ab}p_{cd} &= p_{ac}p_{bd} \\ &= p_{ca}p_{db} \\ &= p_{cd}p_{ab} \end{aligned}$$

Thus, the space produced by the basic crossing is classical, $P \subset P^{N-1}_\mathbb{C}$. By using one more time the relations $p_{ab} = p_{ba}$ we conclude that we have $P = P^{N-1}_\mathbb{R}$, as claimed.

(2) The fattened crossing, and its shifted version, produce the following relations:

$$p_{ab}p_{cd} = p_{cd}p_{ab}$$

$$p_{ab}p_{cd}p_{ef} = p_{ad}p_{eb}p_{cf}$$

The first relations tell us that the projective space must be classical, $P \subset P^{N-1}_\mathbb{C}$. Now observe that with $p_{ij} = z_i\bar{z}_j$, the second relations read:

$$z_a\bar{z}_bz_c\bar{z}_dz_e\bar{z}_f = z_a\bar{z}_dz_e\bar{z}_bz_c\bar{z}_f$$

Since these relations are automatic, we have $P = P^{N-1}_\mathbb{C}$, and we are done. $\square$

Following [33], we can now formulate our classification result, as follows:

Theorem 15.10. *The basic projective spaces, namely*

$$P^{N-1}_{\mathbb{R}} \subset P^{N-1}_{\mathbb{C}} \subset P^{N-1}_{+}$$

are the only monomial ones.

Proof. We follow the proof from the affine case, from section 13 above.

Let $\mathcal{R}_\sigma$ be the collection of relations associated to a permutation $\sigma \in S_k$ with $k \in 2\mathbb{N}$, as in Definition 15.8. We fix a monomial projective space $P \subset P^{N-1}_{+}$, and we associate to it subsets $G_k \subset S_k$, as follows:

$$G_k = \begin{cases} \{\sigma \in S_k | \mathcal{R}_\sigma \text{ hold over } P\} & (k \text{ even}) \\ \{\sigma \in S_k | \mathcal{R}_{|\sigma} \text{ hold over } P\} & (k \text{ odd}) \end{cases}$$

As in the affine case, we obtain in this way a filtered group $G = (G_k)$, which is stable under removing outer strings, and under removing neighboring strings.

Thus the computations in section 13 apply, and show that we have only 3 possible situations, corresponding to the 3 projective spaces in Proposition 15.9 above. □

Let us discuss now similar results for the projective quantum groups. Given a closed subgroup $G \subset O_N^+$, its projective version $G \to PG$ is by definition given by the fact that $C(PG) \subset C(G)$ is the subalgebra generated by the following variables:

$$w_{ij,ab} = u_{ia}u_{jb}$$

In the classical case we recover in this way the usual projective version:

$$PG = G/(G \cap \mathbb{Z}_2^N)$$

Let us discuss now the analogues of the classification results in section 13, for the quantum groups introduced above. First, we have the following key result, from [20]:

Theorem 15.11. *We have the following results:*

(1) *The group inclusion $\mathbb{T}O_N \subset U_N$ is maximal.*
(2) *The group inclusion $PO_N \subset PU_N$ is maximal.*
(3) *The quantum group inclusion $O_N \subset O_N^*$ is maximal.*

Proof. The idea here is as follows:

(1) This can be obtained by using standard Lie group methods.

(2) This follows from (1), by taking projective versions.

(3) This follows from (2), via standard algebraic lifting results.

For details on all this, we refer to [20]. □

Our claim now is that, under suitable assumptions, O_N^* is the only intermediate object $O_N\subset G\subset O_N^+$, and PU_N is the only intermediate object $PO_N\subset G\subset PO_N^+$.

In order to formulate a precise statement here, we recall the following notion, from [37]:

Definition 15.12. *An intermediate compact quantum group*

$$O_N\subset G\subset O_N^+$$

is called easy when the corresponding Tannakian category

$$span(NC_2(k,l))\subset Hom(u^{\otimes k},u^{\otimes l})\subset span(P_2(k,l))$$

comes via the following formula, using the standard $\pi\to T_\pi$ construction,

$$Hom(u^{\otimes k},u^{\otimes l})=span(D(k,l))$$

from a certain collection of sets of pairings $D=(D(k,l))$.

As explained in [37], by "saturating" the sets $D(k,l)$, we can assume that the collection $D=(D(k,l))$ is a category of pairings, in the sense that it is stable under vertical and horizontal concatenation, upside-down turning, and contains the semicircle. See [37].

In the projective case now, we have the following related definition:

Definition 15.13. *A projective category of pairings is a collection of subsets*

$$NC_2(2k,2l)\subset E(k,l)\subset P_2(2k,2l)$$

stable under the usual categorical operations, and satisfying:

$$\sigma\in E\implies|\sigma|\in E$$

As basic examples here, we have the following projective categories of pairings, where P_2^* is the category of matching pairings:

$$NC_2\subset P_2^*\subset P_2$$

This follows indeed from definitions. Now with the above notion in hand, we can formulate the following projective analogue of Definition 15.12:

Definition 15.14. *An intermediate compact quantum group*

$$PO_N\subset H\subset PO_N^+$$

is called projectively easy when its Tannakian category

$$span(NC_2(2k,2l))\subset Hom(v^{\otimes k},v^{\otimes l})\subset span(P_2(2k,2l))$$

comes via via the following formula, using the standard $\pi\to T_\pi$ construction,

$$Hom(v^{\otimes k},v^{\otimes l})=span(E(k,l))$$

for a certain projective category $E=(E(k,l))$.

Observe that, given any easy quantum group $O_N \subset G \subset O_N^+$, its projective version $PO_N \subset PG \subset PO_N^+$ is projectively easy in our sense.

In particular the quantum groups $PO_N \subset PU_N \subset PO_N^+$ are all projectively easy, coming from $NC_2 \subset P_2^* \subset P_2$.

We have in fact the following general result, from [33]:

Theorem 15.15. *We have a bijective correspondence between the affine and projective categories of partitions, given by*

$$G \to PG$$

at the quantum group level.

Proof. The construction of correspondence $D \to E$ is clear, simply by setting:

$$E(k,l) = D(2k,2l)$$

Indeed, due to the axioms for the categories of partitions, from [37], the conditions in Definition 15.13 are satisfied.

Conversely, given $E = (E(k,l))$ as in Definition 15.13, we can set:

$$D(k,l) = \begin{cases} E(k,l) & (k,l \text{ even}) \\ \{\sigma : |\sigma \in E(k+1,l+1)\} & (k,l \text{ odd}) \end{cases}$$

Our claim is that $D = (D(k,l))$ is a category of partitions. Indeed:

(1) The composition action is clear. Indeed, when looking at the numbers of legs involved, in the even case this is clear, and in the odd case, this follows from:

$$\begin{aligned} & |\sigma, |\sigma' \in E \\ \implies & |^\sigma_\tau \in E \\ \implies & {}^\sigma_\tau \in D \end{aligned}$$

(2) For the tensor product axiom, we have 4 cases to be investigated, depending on the parity of the number of legs of σ, τ, as follows:

– The even/even case is clear.

– The odd/even case follows from the following computation:

$$\begin{aligned} |\sigma, \tau \in E \quad & \implies \quad |\sigma\tau \in E \\ & \implies \quad \sigma\tau \in D \end{aligned}$$

– Regarding now the even/odd case, this can be solved as follows:

$$\begin{aligned}\sigma,|\tau\in E&\implies|\sigma|,|\tau\in E\\&\implies|\sigma||\tau\in E\\&\implies|\sigma\tau\in E\\&\implies\sigma\tau\in D\end{aligned}$$

– As for the remaining odd/odd case, here the computation is as follows:

$$\begin{aligned}|\sigma,|\tau\in E&\implies||\sigma|,|\tau\in E\\&\implies||\sigma||\tau\in E\\&\implies\sigma\tau\in E\\&\implies\sigma\tau\in D\end{aligned}$$

(3) Finally, the conjugation axiom is clear from definitions.

It is clear that both compositions $D\to E\to D$ and $E\to D\to E$ are the identities, as claimed. As for the quantum group assertion, this is clear as well. □

We refer to [33] for further details, and comments on the above correspondence.

Now back to uniqueness issues, we have here the following result, also from [33]:

Theorem 15.16. *We have the following results:*

(1) *O_N^* is the only intermediate easy quantum group, as follows:*

$$O_N\subset G\subset O_N^+$$

(2) *PU_N is the only intermediate projectively easy quantum group, as follows:*

$$PO_N\subset G\subset PO_N^+$$

Proof. The idea here is as follows:

(1) The assertion regarding $O_N\subset O_N^*\subset O_N^+$ is from [39], and this is something that we already know, explained in section 11 above.

(2) The assertion regarding $PO_N\subset PU_N\subset PO_N^+$ follows from the classification result in (1), and from the duality in Theorem 15.15. □

Summarizing, we have analogues of the various affine classification results, with the remark that everything becomes simpler in the projective setting.

Let us discuss now the relation between the projective spaces and the projective orthogonal groups, with quantum isometry group computations.

We use the following action formalism, in the projective setting, which is quite similar to the affine action formalism introduced in section 3 above:

Definition 15.17. *Consider a closed subgroup $G\subset O_N^+$, and a closed subset $X\subset S^{N-1}_{\mathbb R,+}$.*

(1) *We write $G\curvearrowright X$ when the formula*
$$\Phi(z_i)=\sum_a u_{ia}\otimes z_a$$
defines a morphism of C^*-algebras, as follows:
$$\Phi:C(X)\to C(G)\otimes C(X)$$
(2) *We write $PG\curvearrowright PX$ when the formula*
$$\Phi(z_iz_j)=\sum_a u_{ia}u_{jb}\otimes z_az_b$$
defines a morphism of C^*-algebras, as follows:
$$\Phi:C(PX)\to C(PG)\otimes C(PX)$$

Observe that the above morphisms Φ, if they exist, are automatically coaction maps. Observe also that an affine action $G\curvearrowright X$ produces a projective action $PG\curvearrowright PX$.

Finally, let us mention that given an algebraic subset $X\subset S^{N-1}_{\mathbb R,+}$, it is routine to prove that there exist universal quantum groups $G\subset O_N^+$ acting as (1), and as in (2).

We have the following result, with respect to the above notions:

Theorem 15.18. *The quantum isometry groups of the basic real spheres and projective spaces, namely*
$$\begin{array}{ccccc}
S^{N-1}_{\mathbb R} & \longrightarrow & S^{N-1}_{\mathbb R,*} & \longrightarrow & S^{N-1}_{\mathbb R,+}\\
\downarrow & & \downarrow & & \downarrow\\
P^{N-1}_{\mathbb R} & \longrightarrow & P^{N-1}_{\mathbb C} & \longrightarrow & P^{N-1}_{+}
\end{array}$$
are the following affine and projective quantum groups,
$$\begin{array}{ccccc}
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+\\
\downarrow & & \downarrow & & \downarrow\\
PO_N & \longrightarrow & PU_N & \longrightarrow & PO_N^+
\end{array}$$
with respect to the affine and projective action notions introduced above.

Proof. The fact that the 3 quantum groups on top act affinely on the corresponding 3 spheres is known since [32], and is elementary. By restriction, the 3 quantum groups on the bottom follow to act on the corresponding 3 projective spaces.

We must prove now that all these actions are universal. At right there is nothing to prove, so we are left with studying the actions on $S^{N-1}_{\mathbb{R}},S^{N-1}_{\mathbb{R},*}$ and on $P^{N-1}_{\mathbb{R}},P^{N-1}_{\mathbb{C}}$.

$\underline{P^{N-1}_{\mathbb{R}}}$. Consider the following projective coordinates:

$$w_{ia,jb}=u_{ia}u_{jb}$$

$$p_{ij}=z_iz_j$$

In terms of these projective coordinates, the coaction map is given by:

$$\Phi(p_{ij})=\sum_{ab}w_{ia,jb}\otimes p_{ab}$$

Thus, we have the following formulae:

$$\begin{aligned}\Phi(p_{ij})&=\sum_{a<b}(w_{ij,ab}+w_{ij,ba})\otimes p_{ab}+\sum_a w_{ij,aa}\otimes p_{aa}\\ \Phi(p_{ji})&=\sum_{a<b}(w_{ji,ab}+w_{ji,ba})\otimes p_{ab}+\sum_a w_{ji,aa}\otimes p_{aa}\end{aligned}$$

By comparing these two formulae, and then by using the linear independence of the variables $p_{ab}=z_az_b$ for $a\leq b$, we conclude that we must have:

$$w_{ij,ab}+w_{ij,ba}=w_{ji,ab}+w_{ji,ba}$$

Let us apply now the antipode to this formula. For this purpose, observe that:

$$\begin{aligned}S(w_{ij,ab})&=S(u_{ia}u_{jb})\\&=S(u_{jb})S(u_{ia})\\&=u_{bj}u_{ai}\\&=w_{ba,ji}\end{aligned}$$

Thus by applying the antipode we obtain:

$$w_{ba,ji}+w_{ab,ji}=w_{ba,ij}+w_{ab,ij}$$

By relabelling, we obtain the following formula:

$$w_{ji,ba}+w_{ij,ba}=w_{ji,ab}+w_{ij,ab}$$

Now by comparing with the original relation, we obtain:

$$w_{ij,ab}=w_{ji,ba}$$

But, with $w_{ij,ab}=u_{ia}u_{jb}$, this formula reads:

$$u_{ia}u_{jb}=u_{jb}u_{ia}$$

Thus our quantum group $G\subset O_N^+$ must be classical:

$$G\subset O_N$$

It follows that we have $PG\subset PO_N$, as claimed.

$\underline{P^{N-1}_\mathbb{C}}$. Consider a coaction map, written as follows, with $p_{ab}=z_a\bar{z}_b$:

$$\Phi(p_{ij})=\sum_{ab}u_{ia}u_{jb}\otimes p_{ab}$$

The idea here will be that of using the following formula:

$$p_{ab}p_{cd}=p_{ad}p_{cb}$$

We have the following formulae:

$$\begin{aligned}\Phi(p_{ij}p_{kl})&=\sum_{abcd}u_{ia}u_{jb}u_{kc}u_{ld}\otimes p_{ab}p_{cd}\\ \Phi(p_{il}p_{kj})&=\sum_{abcd}u_{ia}u_{ld}u_{kc}u_{jb}\otimes p_{ad}p_{cb}\end{aligned}$$

The terms at left being equal, and the last terms at right being equal too, we deduce that, with $[a,b,c]=abc-cba$, we must have the following formula:

$$\sum_{abcd}u_{ia}[u_{jb},u_{kc},u_{ld}]\otimes p_{ab}p_{cd}=0$$

Now since the quantities $p_{ab}p_{cd}=z_a\bar{z}_bz_c\bar{z}_d$ at right depend only on the numbers $|\{a,c\}|,|\{b,d\}|\in\{1,2\}$, and this dependence produces the only possible linear relations between the variables $p_{ab}p_{cd}$, we are led to $2\times 2=4$ equations, as follows:

(1) $u_{ia}[u_{jb},u_{ka},u_{lb}]=0$, $\forall a,b$.

(2) $u_{ia}[u_{jb},u_{ka},u_{ld}]+u_{ia}[u_{jd},u_{ka},u_{lb}]=0$, $\forall a$, $\forall b\neq d$.

(3) $u_{ia}[u_{jb},u_{kc},u_{lb}]+u_{ic}[u_{jb},u_{ka},u_{lb}]=0$, $\forall a\neq c$, $\forall b$.

(4) $u_{ia}[u_{jb},u_{kc},u_{ld}]+u_{ia}[u_{jd},u_{kc},u_{lb}]+u_{ic}[u_{jb},u_{ka},u_{ld}]+u_{ic}[u_{jd},u_{ka},u_{lb}]=0$, $\forall a\neq c$, $\forall b\neq d$.

We will need in fact only the first two formulae. Since (1) corresponds to (2) at $b=d$, we conclude that (1,2) are equivalent to (2), with no restriction on the indices. By multiplying now this formula to the left by u_{ia}, and then summing over i, we obtain:

$$[u_{jb},u_{ka},u_{ld}]+[u_{jd},u_{ka},u_{lb}]=0$$

We use now the antipode/relabel trick from [44]. By applying the antipode we obtain:

$$[u_{dl},u_{ak},u_{bj}]+[u_{bl},u_{ak},u_{dj}]=0$$

By relabelling we obtain the following formula:

$$[u_{ld},u_{ka},u_{jb}]+[u_{jd},u_{ka},u_{lb}]=0$$

Now by comparing with the original relation, we obtain:

$$[u_{jb}, u_{ka}, u_{ld}] = [u_{jd}, u_{ka}, u_{lb}] = 0$$

Thus our quantum group is half-classical:

$$G \subset O_N^*$$

It follows that we have $PG \subset PU_N$, and we are done. □

The above results can be probably improved. As an example, let us say that a closed subgroup $G \subset U_N^+$ acts projectively on PX when we have a coaction map as follows:

$$\Phi(z_i z_j) = \sum_{ab} u_{ia} u_{jb}^* \otimes z_a z_b$$

The above proof can be adapted, by putting $*$ signs where needed, and Theorem 15.18 still holds, under this general formalism. However, establishing general universality results, involving arbitrary subgroups $H \subset PO_N^+$, looks like a quite non-trivial question.

Let us discuss now the axiomatization question for the projective quadruplets of type (P, PT, PU, PK), in the spirit of the axiomatization from section 4 above.

We recall that we first have a classical real quadruplet, as follows:

$$\begin{tikzcd}
P^{N-1}_{\mathbb{R}} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & PT_N \arrow[d, dash] \arrow[dl, dash] \\
PO_N \arrow[r, dash] & PH_N
\end{tikzcd}$$

We have then a classical complex quadruplet, which can be thought of as well as being a real half-classical quadruplet, which is as follows:

$$\begin{tikzcd}
P^{N-1}_{\mathbb{C}} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & P\mathbb{T}_N \arrow[d, dash] \arrow[dl, dash] \\
PU_N \arrow[r, dash] & PK_N
\end{tikzcd}$$

Finally, we have a free quadruplet, which can be thought of as being the same time real and complex, which is as follows:

$$\begin{array}{ccc} P^{N-1}_+ & - & PT_N^+ \\ | & \times & | \\ PO_N^+ & - & PH_N^+ \end{array}$$

In analogy with what happens in the affine case, the problem is that of axiomatizing these geometries, with correspondences as follows:

$$\begin{tikzcd} P \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & PT \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ PU \arrow[r, leftrightarrow] & PK \end{tikzcd}$$

Modulo this problem, which is for the moment open, things are potentially quite nice, because we seem to have only 3 geometries, namely real, complex and free.

Generally speaking, we are led here into several questions:

(1) We first need functoriality results for the operations $<,>$ and $\cap$, in relation with taking the projective version, and taking affine lifts, as to deduce most of our 7 axioms, in their obvious projective formulation, from the affine ones.

(2) Then, we need quantum isometry results in the projective setting, for the projective spaces themselves, and for the projective tori, either established ad-hoc, or by using the affine results. For the projective spaces, this was done above.

(3) We need as well some further functoriality results, in order to axiomatize the intermediate objects that we are dealing, the problem here being whether we want to use projective objects, or projective versions, perhaps saturated, of affine objects.

In short, we need functoriality results a bit everywhere, in connection with the various questions to be solved. Modulo this, things are quite clear, with the final result being the fact that we have indeed only 3 projective geometries, in analogy with the fact that we have only 3 geometries. Technically, the proof should be using the fact that, in the easy setting, $PO_N\subset PU_N\subset PO_N^+$ are the only possible unitary groups.

Let us also mention that, in the noncommutative setting, there are several ways of defining the projective versions, with the one used here being the "simplest". As explained in [9], [19], it is possible to construct a left projective version, a right projective version, and a mixed projective version, with all these operations being interesting. Thus, the results and problems presented above are just the "tip of the iceberg", with the general projective space and version problematics being much wider then this.

Yet another question concerns the study of the projective spaces associated to the twisted spheres, from section 13 above, and to the intersections studied in section 14.

Finally, at a more concrete level, the question of developing these projective geometries, and notably the free one, remains open, and extremely interesting. There is of course a lot of material which can be "imported" from the affine setting, but at the genuine projective geometry level nothing much is known, passed a handful of quantum group results.

16. Hyperspherical laws

We discuss in this final section a number of more advanced results, mixing algebra, geometry, analysis and probability, twisted and untwisted objects, affine and projective manifolds, and many more. At the core of all this will be a subtle twisting result, relating the free projective orthogonal group PO_N^+ and the quantum permutation group $S_{N^2}^+$. We believe that all this material should be relevant to certain questions in quantum physics, but nothing much is known here for the moment. We will comment on this at the end.

We follow the papers [16], [22], [25], [26], [27], [32], [34], where these results were found. As a starting point, we have the very natural question, first investigated in [32], of computing the laws of individual coordinates of the main 3 real spheres, namely:

$$S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$$

We already know from section 5 above the $N \to \infty$ behavior of these laws, called "hyperspherical". To be more precise, for $S^{N-1}_{\mathbb{R}}$ we obtain the normal law, and for $S^{N-1}_{\mathbb{R},+}$ we obtain the semicircle law. As for the sphere $S^{N-1}_{\mathbb{R},*}$, this has the same projective version as $S^{N-1}_{\mathbb{C}}$, where the corresponding law becomes complex Gaussian with $N \to \infty$, as explained in section 5, and so we obtain a symmetrized Rayleigh variable. See [32].

The problem that we want to investigate here, and that will bring us into a lot of interesting mathematics, is that of computing these hyperspherical laws at fixed values of $N \in \mathbb{N}$. Let us begin with a full discussion in the classical case. At $N = 2$ the sphere is the unit circle $\mathbb{T}$, and with $z = e^{it}$ the coordinates are $\cos t, \sin t$. The integrals of the arbitrary products of such coordinates can be computed as follows:

Theorem 16.1. *We have the following formula,*

$$\int_0^{\pi/2} \cos^p t \sin^q t \, dt = \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)} \frac{p!!q!!}{(p+q+1)!!}$$

where $\varepsilon(p) = 1$ if p is even, and $\varepsilon(p) = 0$ if p is odd, and where

$$m!! = (m-1)(m-3)(m-5)\ldots$$

with the product ending at 2 if m is odd, and ending at 1 if m is even.

Proof. This is standard calculus, with particular cases of this formula being very familiar to everyone loving and teaching calculus, as we all should. Let us set:

$$I_p = \int_0^{\pi/2} \cos^p t \, dt$$

We compute I_p by partial integration. We have the following formula:

$$\begin{aligned}
&(\cos^p t\sin t)'\\
=\ & p\cos^{p-1}t(-\sin t)\sin t+\cos^p t\cos t\\
=\ & p\cos^{p+1}t-p\cos^{p-1}t+\cos^{p+1}t\\
=\ & (p+1)\cos^{p+1}t-p\cos^{p-1}t
\end{aligned}$$

By integrating between 0 and $\pi/2$, we obtain the following formula:

$$(p+1)I_{p+1}=pI_{p-1}$$

Thus we can compute I_p by recurrence, and we obtain:

$$\begin{aligned}
I_p &= \frac{p-1}{p}\,I_{p-2}\\
&= \frac{p-1}{p}\cdot\frac{p-3}{p-2}\,I_{p-4}\\
&= \frac{p-1}{p}\cdot\frac{p-3}{p-2}\cdot\frac{p-5}{p-4}\,I_{p-6}\\
&\ \ \vdots\\
&= \frac{p!!}{(p+1)!!}\,I_{1-\varepsilon(p)}
\end{aligned}$$

Together with $I_0=\frac{\pi}{2}$ and $I_1=1$, which are both clear, we obtain:

$$I_p=\left(\frac{\pi}{2}\right)^{\varepsilon(p)}\frac{p!!}{(p+1)!!}$$

Summarizing, we have proved the following formula, with one equality coming from the above computation, and with the other equality coming from this, via $t=\frac{\pi}{2}-s$:

$$\int_0^{\pi/2}\cos^p t\,dt=\int_0^{\pi/2}\sin^p t\,dt=\left(\frac{\pi}{2}\right)^{\varepsilon(p)}\frac{p!!}{(p+1)!!}$$

In relation with the formula in the statement, we are therefore done with the case $p=0$ or $q=0$. Let us investigate now the general case. We must compute:

$$I_{pq}=\int_0^{\pi/2}\cos^p t\sin^q t\,dt$$

In order to do the partial integration, observe that we have:

$$\begin{aligned}
&(\cos^p t\sin^q t)'\\
=\ & p\cos^{p-1}t(-\sin t)\sin^q t\\
+\ & \cos^p t\cdot q\sin^{q-1}t\cos t\\
=\ & -p\cos^{p-1}t\sin^{q+1}t+q\cos^{p+1}t\sin^{q-1}t
\end{aligned}$$

By integrating between 0 and $\pi/2$, we obtain, for $p, q > 0$:

$$pI_{p-1,q+1} = qI_{p+1,q-1}$$

Thus, we can compute I_{pq} by recurrence. When q is even we have:

$$\begin{aligned}
I_{pq} &= \frac{q-1}{p+1} I_{p+2,q-2} \\
&= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} I_{p+4,q-4} \\
&= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} \cdot \frac{q-5}{p+5} I_{p+6,q-6} \\
&= \vdots \\
&= \frac{p!!q!!}{(p+q)!!} I_{p+q}
\end{aligned}$$

But the last term was already computed above, and we obtain the result:

$$\begin{aligned}
I_{pq} &= \frac{p!!q!!}{(p+q)!!} I_{p+q} \\
&= \frac{p!!q!!}{(p+q)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(p+q)} \frac{(p+q)!!}{(p+q+1)!!} \\
&= \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)} \frac{p!!q!!}{(p+q+1)!!}
\end{aligned}$$

Observe that this gives the result for p even as well, by symmetry. Indeed, we have $I_{pq} = I_{qp}$, by using the following change of variables:

$$t = \frac{\pi}{2} - s$$

In the remaining case now, where both p, q are odd, we can use once again the formula $pI_{p-1,q+1} = qI_{p+1,q-1}$ established above, and the recurrence goes as follows:

$$\begin{aligned}
I_{pq} &= \frac{q-1}{p+1} I_{p+2,q-2} \\
&= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} I_{p+4,q-4} \\
&= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} \cdot \frac{q-5}{p+5} I_{p+6,q-6} \\
&= \vdots \\
&= \frac{p!!q!!}{(p+q-1)!!} I_{p+q-1,1}
\end{aligned}$$

In order to compute the last term, observe that we have:

$$\begin{aligned}
I_{p1} &= \int_0^{\pi/2} \cos^p t \sin t \, dt \\
&= -\frac{1}{p+1} \int_0^{\pi/2} (\cos^{p+1} t)' \, dt \\
&= \frac{1}{p+1}
\end{aligned}$$

Thus, we can finish our computation in the case p, q odd, as follows:

$$\begin{aligned}
I_{pq} &= \frac{p!!q!!}{(p+q-1)!!} I_{p+q-1,1} \\
&= \frac{p!!q!!}{(p+q-1)!!} \cdot \frac{1}{p+q} \\
&= \frac{p!!q!!}{(p+q+1)!!}
\end{aligned}$$

Thus, we obtain the formula in the statement, the exponent of $\pi/2$ appearing there being $\varepsilon(p)\varepsilon(q) = 0 \cdot 0 = 0$ in the present case, and this finishes the proof. □

In order to discuss the higher spheres, we will use spherical coordinates:

Theorem 16.2. *We have spherical coordinates in N dimensions,*

$$\begin{cases}
x_1 &= r \cos t_1 \\
x_2 &= r \sin t_1 \cos t_2 \\
\vdots & \\
x_{N-1} &= r \sin t_1 \sin t_2 \ldots \sin t_{N-2} \cos t_{N-1} \\
x_N &= r \sin t_1 \sin t_2 \ldots \sin t_{N-2} \sin t_{N-1}
\end{cases}$$

the corresponding Jacobian being given by the following formula:

$$J(r,t) = r^{N-1} \sin^{N-2} t_1 \sin^{N-3} t_2 \, \ldots \, \sin^2 t_{N-3} \sin t_{N-2}$$

Proof. The fact that we have coordinates is clear. Regarding the Jacobian, the proof is similar to the one from 2 or 3 dimensions, by developing the determinant over the last column, and then by proceeding by recurrence. Indeed, by developing, we have:

$$\begin{aligned}
J_N &= r \sin t_1 \ldots \sin t_{N-2} \sin t_{N-1} \times \sin t_{N-1} J_{N-1} \\
&+ r \sin t_1 \ldots \sin t_{N-2} \cos t_{N-1} \times \cos t_{N-1} J_{N-1} \\
&= r \sin t_1 \ldots \sin t_{N-2} (\sin^2 t_{N-1} + \cos^2 t_{N-1}) J_{N-1} \\
&= r \sin t_1 \ldots \sin t_{N-2} J_{N-1}
\end{aligned}$$

Thus, we obtain the formula in the statement, by recurrence. □

With the above results in hand, we can now compute arbitrary polynomial integrals, over the spheres of arbitrary dimension, the result being is as follows:

Theorem 16.3. *The spherical integral of $x_{i_1}\ldots x_{i_k}$ vanishes, unless each $a\in\{1,\ldots,N\}$ appears an even number of times in the sequence $i_1,\ldots,i_k$. We have*

$$\int_{S^{N-1}_\mathbb{R}}x_{i_1}\ldots x_{i_k}\,dx=\frac{(N-1)!!l_1!!\ldots l_N!!}{(N+\Sigma l_i-1)!!}$$

with l_a being this number of occurrences.

Proof. First, the result holds indeed at $N=2$, due to the following formula proved above, where $\varepsilon(p)=1$ when $p\in\mathbb N$ is even, and $\varepsilon(p)=0$ when p is odd:

$$\int_0^{\pi/2}\cos^pt\sin^qt\,dt=\left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)}\frac{p!!q!!}{(p+q+1)!!}$$

In general, we can restrict attention to the case $l_a\in 2\mathbb N$, since the other integrals vanish. The integral in the statement can be written in spherical coordinates, as follows:

$$I=\frac{2^N}{V}\int_0^{\pi/2}\ldots\int_0^{\pi/2}x_1^{l_1}\ldots x_N^{l_N}J\,dt_1\ldots dt_{N-1}$$

In this formula, indeed:

– V is the volume of the sphere.

– J is the Jacobian.

– The 2^N factor comes from the restriction to the $1/2^N$ part of the sphere where all the coordinates are positive.

The normalization constant in front of the integral is:

$$\begin{aligned}\frac{2^N}{V}&=\frac{2^N}{N\pi^{N/2}}\cdot\Gamma\left(\frac{N}{2}+1\right)\\&=\left(\frac{2}{\pi}\right)^{[N/2]}(N-1)!!\end{aligned}$$

As for the unnormalized integral, this is given by:

$$\begin{aligned}I'=\int_0^{\pi/2}\ldots\int_0^{\pi/2}\quad&(\cos t_1)^{l_1}(\sin t_1\cos t_2)^{l_2}\\&\vdots\\&(\sin t_1\sin t_2\ldots\sin t_{N-2}\cos t_{N-1})^{l_{N-1}}\\&(\sin t_1\sin t_2\ldots\sin t_{N-2}\sin t_{N-1})^{l_N}\\&\sin^{N-2}t_1\sin^{N-3}t_2\ldots\sin^2t_{N-3}\sin t_{N-2}\\&dt_1\ldots dt_{N-1}\end{aligned}$$

By rearranging the terms, we obtain:

$$\begin{aligned}
I' \;=\; & \int_0^{\pi/2} \cos^{l_1} t_1 \sin^{l_2+\ldots+l_N+N-2} t_1 \, dt_1 \\
& \int_0^{\pi/2} \cos^{l_2} t_2 \sin^{l_3+\ldots+l_N+N-3} t_2 \, dt_2 \\
& \vdots \\
& \int_0^{\pi/2} \cos^{l_{N-2}} t_{N-2} \sin^{l_{N-1}+l_N+1} t_{N-2} \, dt_{N-2} \\
& \int_0^{\pi/2} \cos^{l_{N-1}} t_{N-1} \sin^{l_N} t_{N-1} \, dt_{N-1}
\end{aligned}$$

Now by using the above-mentioned formula at $N=2$, this gives:

$$\begin{aligned}
I' \;=\; & \frac{l_1!!(l_2+\ldots+l_N+N-2)!!}{(l_1+\ldots+l_N+N-1)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(N-2)} \\
& \frac{l_2!!(l_3+\ldots+l_N+N-3)!!}{(l_2+\ldots+l_N+N-2)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(N-3)} \\
& \vdots \\
& \frac{l_{N-2}!!(l_{N-1}+l_N+1)!!}{(l_{N-2}+l_{N-1}+l_N+2)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(1)} \\
& \frac{l_{N-1}!!l_N!!}{(l_{N-1}+l_N+1)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(0)}
\end{aligned}$$

Now observe that the various double factorials multiply up to quantity in the statement, modulo a $(N-1)!!$ factor, and that the $\frac{\pi}{2}$ factors multiply up to:

$$F = \left(\frac{\pi}{2}\right)^{[N/2]}$$

Thus by multiplying with the normalization constant, we obtain the result. □

In connection now with our probabilistic questions, we have:

Theorem 16.4. *The even moments of the hyperspherical variables are*

$$\int_{S^{N-1}_\mathbb{R}} x_i^k dx = \frac{(N-1)!!k!!}{(N+k-1)!!}$$

and the variables $y_i = x_i/\sqrt{N}$ become normal and independent with $N \to \infty$.

Proof. The moment formula in the statement follows from Theorem 16.3. Now observe that with $N \to \infty$ we have the following estimate:

$$\begin{aligned}\int_{S^{N-1}_{\mathbb{R}}} x_i^k dx &= \frac{(N-1)!!}{(N+k-1)!!} \times k!! \\ &\simeq N^{k/2} \times k!! \\ &= N^{k/2} M_k(g_1)\end{aligned}$$

Thus, we have $x_i/\sqrt{N} \sim g_1$, as claimed. Finally, the independence assertion follows as well from the formula in Theorem 16.3, via standard probability theory. □

In the case of the half-classical sphere, we have the following integration result:

Theorem 16.5. *The half-classical integral of $x_{i_1} \ldots x_{i_k}$ vanishes, unless each index a appears the same number of times at odd and even positions in $i_1, \ldots, i_k$. We have*

$$\int_{S^{N-1}_{\mathbb{R},*}} x_{i_1} \ldots x_{i_k}\, dx = 4^{\sum l_i} \frac{(2N-1)! l_1! \ldots l_n!}{(2N + \sum l_i - 1)!}$$

where l_a denotes this number of common occurrences.

Proof. As before, we can assume that k is even, $k = 2l$. The corresponding integral can be viewed as an integral over $S^{N-1}_{\mathbb{C}}$, as follows:

$$I = \int_{S^{N-1}_{\mathbb{C}}} z_{i_1} \bar{z}_{i_2} \ldots z_{i_{2l-1}} \bar{z}_{i_{2l}}\, dz$$

Now by using transformations of type $p \to \lambda p$ with $|\lambda| = 1$, we see that I vanishes, unless each z_a appears as many times as $\bar{z}_a$ does, and this gives the first assertion.

Assume now that we are in the non-vanishing case. Then the l_a copies of z_a and the l_a copies of $\bar{z}_a$ produce by multiplication a factor $|z_a|^{2l_a}$, so we have:

$$I = \int_{S^{N-1}_{\mathbb{C}}} |z_1|^{2l_1} \ldots |z_N|^{2l_N}\, dz$$

Now by using the standard identification $S^{N-1}_{\mathbb{C}} \simeq S^{2N-1}_{\mathbb{R}}$, we obtain:

$$\begin{aligned} I &= \int_{S^{2N-1}_{\mathbb{R}}} (x_1^2 + y_1^2)^{l_1} \ldots (x_N^2 + y_N^2)^{l_N}\, d(x,y) \\ &= \sum_{r_1 \ldots r_N} \binom{l_1}{r_1} \ldots \binom{l_N}{r_N} \int_{S^{2N-1}_{\mathbb{R}}} x_1^{2l_1 - 2r_1} y_1^{2r_1} \ldots x_N^{2l_N - 2r_N} y_N^{2r_N}\, d(x,y)\end{aligned}$$

By using the formula in Theorem 16.3, we obtain:

$$\begin{aligned} I &= \sum_{r_1\ldots r_N}\binom{l_1}{r_1}\ldots\binom{l_N}{r_N}\frac{(2N-1)!!(2r_1)!!\ldots(2r_N)!!(2l_1-2r_1)!!\ldots(2l_N-2r_N)!!}{(2N+2\sum l_i-1)!!}\\ &= \sum_{r_1\ldots r_N}\binom{l_1}{r_1}\ldots\binom{l_N}{r_N}\frac{(2N-1)!(2r_1)!\ldots(2r_N)!(2l_1-2r_1)!\ldots(2l_N-2r_N)!}{(2N+\sum l_i-1)!r_1!\ldots r_N!(l_1-r_1)!\ldots(l_N-r_N)!}\end{aligned}$$

We can rewrite the sum on the right in the following way:

$$\begin{aligned} I &= \sum_{r_1\ldots r_N}\frac{l_1!\ldots l_N!(2N-1)!(2r_1)!\ldots(2r_N)!(2l_1-2r_1)!\ldots(2l_N-2r_N)!}{(2N+\sum l_i-1)!(r_1!\ldots r_N!(l_1-r_1)!\ldots(l_N-r_N)!)^2}\\ &= \sum_{r_1}\binom{2r_1}{r_1}\binom{2l_1-2r_1}{l_1-r_1}\ldots\sum_{r_N}\binom{2r_N}{r_N}\binom{2l_N-2r_N}{l_N-r_N}\frac{(2N-1)!l_1!\ldots l_N!}{(2N+\sum l_i-1)!}\end{aligned}$$

The sums on the right being $4^{l_1},\ldots,4^{l_N}$, this gives the formula in the statement. □

As before, we can deduce from this a probabilistic result, as follows:

Theorem 16.6. *The even moments of the half-classical hyperspherical variables are*

$$\int_{S^{N-1}_{\mathbb{R},*}}x_i^k\,dx=4^k\frac{(2N-1)!k!}{(2N+k-1)!}$$

and the variables $y_i=x_i/(4N)$ become symmetrized Rayleigh with $N\to\infty$.

Proof. The moment formula in the statement follows from Theorem 16.5. Now observe that with $N\to\infty$ we have the following estimate:

$$\begin{aligned}\int_{S^{N-1}_{\mathbb{R},*}}x_i^k\,dx &= 4^k\times\frac{(N-1)!}{(N+k-1)!}\times k!\\ &\simeq 4^k\times N^k\times k!\\ &= (4N)^kM_k(|c|)\end{aligned}$$

Here c is a standard complex Gaussian variable, and this gives the result. □

As a comment here, it is possible to prove, based once again on the general integration formula from Theorem 16.5 above, that the rescaled variables $y_i=x_i/(4N)$ become "half-independent" with $N\to\infty$. For a discussion of the notion of half-independence, and various related topics, we refer to the series of papers [28], [29], [30].

In the case of the free sphere now, the computations are substantially more complicated. Let us start with the following result, that we basically know from section 5 above:

Theorem 16.7. *For the free sphere $S^{N-1}_{\mathbb{R},+}$, the rescaled coordinates*

$$y_i = \sqrt{N} x_i$$

become semicircular and free, in the $N \to \infty$ limit.

Proof. As explained in section 5 above, the Weingarten formula for the free sphere, together with the standard fact that the Gram matrix, and hence the Weingarten matrix too, is asymptotically diagonal, gives the following estimate:

$$\int_{S^{N-1}_{\mathbb{R},+}} x_{i_1} \ldots x_{i_k} \, dx \simeq N^{-k/2} \sum_{\sigma \in NC_2(k)} \delta_\sigma(i_1, \ldots, i_k)$$

With this formula in hand, we can compute the asymptotic moments of each coordinate x_i. Indeed, by setting $i_1 = \ldots = i_k = i$, all Kronecker symbols are 1, and we obtain:

$$\int_{S^{N-1}_{\mathbb{R},+}} x_i^k \, dx \simeq N^{-k/2} |NC_2(k)|$$

Thus the rescaled coordinates $y_i = \sqrt{N} x_i$ become semicircular in the $N \to \infty$ limit, as claimed. As for the asymptotic freeness result, this follows as well from the above general joint moment estimate, via standard free probability theory. See [23], [32]. □

The problem now, which is highly non-trivial, is that of computing the moments of the coordinates of the free sphere at fixed values of $N \in \mathbb{N}$. The answer here, from [26], based on advanced quantum group and calculus techniques, is as follows:

Theorem 16.8. *The moments of the free hyperspherical law are given by*

$$\int_{S^{N-1}_{\mathbb{R},+}} x_1^{2l} \, dx = \frac{1}{(N+1)^l} \cdot \frac{q+1}{q-1} \cdot \frac{1}{l+1} \sum_{r=-l-1}^{l+1} (-1)^r \binom{2l+2}{l+r+1} \frac{r}{1+q^r}$$

where $q \in [-1, 0)$ is such that $q + q^{-1} = -N$.

Proof. The idea is that $x_1 \in C(S^{N-1}_{\mathbb{R},+})$ has the same law as $u_{11} \in C(O_N^+)$, which has the same law as a certain variable $w \in C(SU_2^q)$, which can be in turn modelled by an explicit operator on $l^2(\mathbb{N})$, whose law can be computed by using advanced calculus.

Let us first explain the relation between O_N^+ and SU_2^q. To any matrix $F \in GL_N(\mathbb{R})$ satisfying $F^2 = 1$ we associate the following universal algebra:

$$C(O_F^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N} \middle| u = F\bar{u}F = \text{unitary} \right)$$

Observe that $O_{I_N}^+ = O_N^+$. In general, the above algebra satisfies Woronowicz's generalized axioms in [148], which do not include the strong antipode axiom $S^2 = id$.

At $N=2$, up to a trivial equivalence relation on the matrices F, and on the quantum groups O_F^+, we can assume that F is as follows, with $q\in[-1,0)$:

$$F=\begin{pmatrix}0&\sqrt{-q}\\ 1/\sqrt{-q}&0\end{pmatrix}$$

Our claim is that for this matrix we have:

$$O_F^+=SU_2^q$$

Indeed, the relations $u=F\bar{u}F$ tell us that u must be of the following special form:

$$u=\begin{pmatrix}\alpha&-q\gamma^*\\ \gamma&\alpha^*\end{pmatrix}$$

Thus $C(O_F^+)$ is the universal algebra generated by two elements α,γ, with the relations making the above matrix u unitary. But these unitarity conditions are:

$$\alpha\gamma=q\gamma\alpha$$
$$\alpha\gamma^*=q\gamma^*\alpha$$
$$\gamma\gamma^*=\gamma^*\gamma$$
$$\alpha^*\alpha+\gamma^*\gamma=1$$
$$\alpha\alpha^*+q^2\gamma\gamma^*=1$$

We recognize here the relations in [148] defining the algebra $C(SU_2^q)$, and it follows that we have an isomorphism of Hopf C^*-algebras:

$$C(O_F^+)\simeq C(SU_2^q)$$

Now back to the general case, let us try to understand the integration over O_F^+. Given $\pi\in NC_2(2k)$ and $i=(i_1,\ldots,i_{2k})$, we set:

$$\delta_\pi^F(i)=\prod_{s\in\pi}F_{i_{s_l}i_{s_r}}$$

Here the product is over all strings $s=\{s_l\curvearrowright s_r\}$ of π. Our claim is that the following family of vectors, with $\pi\in NC_2(2k)$, spans the space of fixed vectors of $u^{\otimes 2k}$:

$$\xi_\pi=\sum_i\delta_\pi^F(i)e_{i_1}\otimes\ldots\otimes e_{i_{2k}}$$

Indeed, having $\xi_\cap$ fixed by $u^{\otimes 2}$ is equivalent to assuming that $u=F\bar{u}F$ is unitary.

By using now the above vectors, we obtain the following Weingarten formula:

$$\int_{O_F^+}u_{i_1j_1}\ldots u_{i_{2k}j_{2k}}=\sum_{\pi\sigma}\delta_\pi^F(i)\delta_\sigma^F(j)W_{kN}(\pi,\sigma)$$

With these preliminaries in hand, let us start the computation. Let $N\in\mathbb{N}$, and consider the number $q\in[-1,0)$ satisfying:

$$q+q^{-1}=-N$$

Our claim is that we have:

$$\int_{O_N^+}\varphi(\sqrt{N+2}\,u_{ij})=\int_{SU_2^q}\varphi(\alpha+\alpha^*+\gamma-q\gamma^*)$$

Indeed, the moments of the variable on the left are given by:

$$\int_{O_N^+}u_{ij}^{2k}=\sum_{\pi\sigma}W_{kN}(\pi,\sigma)$$

On the other hand, the moments of the variable on the right, which in terms of the fundamental corepresentation $v=(v_{ij})$ is given by $w=\sum_{ij}v_{ij}$, are given by:

$$\int_{SU_2^q}w^{2k}=\sum_{ij}\sum_{\pi\sigma}\delta_\pi^F(i)\delta_\sigma^F(j)W_{kN}(\pi,\sigma)$$

We deduce that $w/\sqrt{N+2}$ has the same moments as u_{ij}, which proves our claim.

In order to do now the computation over SU_2^q, we can use a matrix model due to Woronowicz [148], where the standard generators α,γ are mapped as follows:

$$\begin{aligned}\pi_u(\alpha)e_k&=\sqrt{1-q^{2k}}e_{k-1}\\ \pi_u(\gamma)e_k&=uq^ke_k\end{aligned}$$

Here $u\in\mathbb T$ is a parameter, and (e_k) is the standard basis of $l^2(\mathbb N)$. The point with this representation is that it allows the computation of the Haar functional. Indeed, if D is the diagonal operator given by $D(e_k)=q^{2k}e_k$, then the formula is as follows:

$$\int_{SU_2^q}x=(1-q^2)\int_{\mathbb T}tr(D\pi_u(x))\frac{du}{2\pi iu}$$

With the above model in hand, the law of the variable that we are interested in is of the following form:

$$\int_{SU_2^q}\varphi(\alpha+\alpha^*+\gamma-q\gamma^*)=(1-q^2)\int_{\mathbb T}tr(D\varphi(M))\frac{du}{2\pi iu}$$

To be more precise, this formula holds indeed, with:

$$M(e_k)=e_{k+1}+q^k(u-qu^{-1})e_k+(1-q^{2k})e_{k-1}$$

The point now is that the integral on the right can be computed, by using advanced calculus methods, and this gives the result. We refer here to [26]. □

The computation of the joint free hyperspherical laws remains an open problem. Open as well is the question of finding a more conceptual proof for the above formula.

Following now [22], let us discuss an interesting relation of all this with the quantum permutations, and with the free hypergeometric laws. The idea will be that of working out some abstract algebraic results, regarding twists of quantum automorphism groups, which will particularize into results relating quantum rotations and permutations, having no classical counterpart (!) both at the algebraic and the probabilistic level.

In order to explain this material, from [22], which is quite technical, requiring good algebraic knowledge, let us begin with some generalities. We first have:

Definition 16.9. *A finite quantum space X is the abstract dual of a finite dimensional C^*-algebra B, according to the following formula:*

$$C(X) = B$$

The number of elements of such a space is $|X| = \dim B$. By decomposing the algebra B, we have a formula of the following type:

$$C(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

With $n_1 = \ldots = n_k = 1$ we obtain in this way the space $X = \{1, \ldots, k\}$. Also, when $k = 1$ the equation is $C(X) = M_n(\mathbb{C})$, and the solution will be denoted $X = M_n$.

Following [2], we endow each finite quantum space X with its counting measure, corresponding as the algebraic level to the integration functional obtained by applying the regular representation, and then the normalized matrix trace:

$$tr : C(X) \to B(l^2(X)) \to \mathbb{C}$$

As basic examples, for both $X = \{1, \ldots, k\}$ and $X = M_n$ we obtain the usual trace. In general, we can write the algebra $C(X)$ as follows:

$$C(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

In terms of this writing, the weights of tr are as follows:

$$c_i = \frac{n_i^2}{\sum_i n_i^2}$$

With these conventions, we have the following result, from [2], [141]:

Theorem 16.10. *Given a finite quantum space X, there is a universal compact quantum group S_X^+ acting on X, leaving the counting measure invariant. We have*

$$C(S_X^+) = C(U_N^+) \Big/ \Big\langle \mu \in Hom(u^{\otimes 2}, u), \eta \in Fix(u) \Big\rangle$$

where $N = |X|$ and where μ, η are the multiplication and unit maps of $C(X)$. Also:

1. *For $X = \{1, \ldots, N\}$ we have $S_X^+ = S_N^+$.*
2. *For $X = M_n$ we have $S_X^+ = PO_n^+ = PU_n^+$.*

Proof. Consider a linear map $\Phi : C(X) \to C(X) \otimes C(G)$, written as follows, with $\{e_i\}$ being a linear space basis of $C(X)$, which is orthonormal with respect to tr:

$$\Phi(e_j) = \sum_i e_i \otimes u_{ij}$$

It is routine to check, via standard algebraic computations, that Φ is a coaction precisely when u is a unitary corepresentation, satisfying the following conditions:

$$\mu \in Hom(u^{\otimes 2}, u)$$

$$\eta \in Fix(u)$$

But this gives the first assertion. Regarding now the statement about $X = \{1, \ldots, N\}$ is clear. Finally, regarding $X = M_2$, here we have embeddings as followss:

$$PO_n^+ \subset PU_n^+ \subset S_X^+$$

Now since the fusion rules of all these 3 quantum groups are known to be the same as the fusion rules for SO_3, these inclusions are isomorphisms. See [2]. □

Following now [22], we have the following result:

Proposition 16.11. *Given a finite group F, the algebra $C(S_{\widehat{F}}^+)$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g, h \in F$, with the following relations:*

$$x_{1g} = x_{g1} = \delta_{1g}$$

$$x_{s,gh} = \sum_{t \in F} x_{st^{-1},g} x_{th}$$

$$x_{gh,s} = \sum_{t \in F} x_{gt^{-1}} x_{h,ts}$$

The comultiplication, counit and antipode are given by the formulae

$$\Delta(x_{gh}) = \sum_{s \in F} x_{gs} \otimes x_{sh}$$

$$\varepsilon(x_{gh}) = \delta_{gh}$$

$$S(x_{gh}) = x_{h^{-1}g^{-1}}$$

on the standard generators x_{gh}.

Proof. This follows indeed from a direct verification, based either on Theorem 16.10 above, or on its equivalent formulation from Wang's paper [141]. □

Let us discuss now the twisted version of the above result. Consider a 2-cocycle on F, which is by definition a map $\sigma : F \times F \to \mathbb{C}^*$ satisfying:

$$\sigma_{gh,s} \sigma_{gh} = \sigma_{g,hs} \sigma_{hs}$$

$$\sigma_{g1} = \sigma_{1g} = 1$$

Given such a cocycle, we can construct the associated twisted group algebra $C(\widehat{F}_\sigma)$, as being the vector space $C(\widehat{F}) = C^*(F)$, with product as follows:

$$e_g e_h = \sigma_{gh} e_{gh}$$

We have then the following generalization of Proposition 16.11:

Proposition 16.12. *The algebra $C(S^+_{\widehat{F}_\sigma})$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g,h\in G$, with the relations $x_{1g}=x_{g1}=\delta_{1g}$ and:*

$$\sigma_{gh}x_{s,gh}=\sum_{t\in F}\sigma_{st^{-1},t}x_{st^{-1},g}x_{th}$$

$$\sigma_{gh}^{-1}x_{gh,s}=\sum_{t\in F}\sigma^{-1}_{t^{-1},ts}x_{gt^{-1}}x_{h,ts}$$

The comultiplication, counit and antipode are given by the formulae

$$\Delta(x_{gh})=\sum_{s\in F}x_{gs}\otimes x_{sh}$$

$$\varepsilon(x_{gh})=\delta_{gh}$$

$$S(x_{gh})=\sigma_{h^{-1}h}\sigma^{-1}_{g^{-1}g}x_{h^{-1}g^{-1}}$$

on the standard generators x_{gh}.

Proof. Once again, this follows from a direct verification. Note that by using the cocycle identities we obtain the formula $\sigma_{gg^{-1}}=\sigma_{g^{-1}g}$, needed in the proof. □

In what follows, we will prove that $S^+_{\widehat{F}}$ and $S^+_{\widehat{F}_\sigma}$ are related by a cocycle twisting operation. Let us begin with some preliminaries. Let H be a Hopf algebra. We recall that a left 2-cocycle is a convolution invertible linear map $\sigma:H\otimes H\to\mathbb C$ satisfying:

$$\sigma_{x_1y_1}\sigma_{x_2y_2,z}=\sigma_{y_1z_1}\sigma_{x,y_2z_2}$$

$$\sigma_{x1}=\sigma_{1x}=\varepsilon(x)$$

Note that σ is a left 2-cocycle if and only if σ^{-1}, the convolution inverse of σ, is a right 2-cocycle, in the sense that we have:

$$\sigma^{-1}_{x_1y_1,z}\sigma^{-1}_{x_1y_2}=\sigma^{-1}_{x,y_1z_1}\sigma^{-1}_{y_2z_2}$$

$$\sigma^{-1}_{x1}=\sigma^{-1}_{1x}=\varepsilon(x)$$

Given a left 2-cocycle σ on H, one can form the 2-cocycle twist H^σ as follows. As a coalgebra, $H^\sigma=H$, and an element $x\in H$, when considered in H^σ, is denoted $[x]$. The product in H^σ is defined, in Sweedler notation, by:

$$[x][y]=\sum\sigma_{x_1y_1}\sigma^{-1}_{x_3y_3}[x_2y_2]$$

Note that the cocycle condition ensures the fact that we have indeed a Hopf algebra. Note also that the coalgebra isomorphism $H\to H^\sigma$ given by $x\to[x]$ commutes with the respective Haar integrals, as soon as H has a Haar integral.

Following [22], we can now state a main twisting theorem, as follows:

Theorem 16.13. *If F is a finite group and σ is a 2-cocycle on F, the Hopf algebras*

$$C(S^+_{\widehat{F}})\quad,\quad C(S^+_{\widehat{F}_\sigma})$$

are 2-cocycle twists of each other, in the above sense.

Proof. In order to prove this result, we use the following Hopf algebra map:

$$\pi:C(S^+_{\widehat{F}})\to C(\widehat{F})$$

$$x_{gh}\to\delta_{gh}e_g$$

Our 2-cocycle $\sigma:F\times F\to\mathbb{C}^*$ can be extended by linearity into a linear map as follows, which is a left and right 2-cocycle in the above sense:

$$\sigma:C(\widehat{F})\otimes C(\widehat{F})\to\mathbb{C}$$

Consider now the following composition:

$$\alpha=\sigma(\pi\otimes\pi):C(S^+_{\widehat{F}})\otimes C(S^+_{\widehat{F}})\to C(\widehat{F})\otimes C(\widehat{F})\to\mathbb{C}$$

Then α is a left and right 2-cocycle, because it is induced by a cocycle on a group algebra, and so is its convolution inverse α^{-1}. Thus we can construct the twisted algebra $C(S^+_{\widehat{F}})^{\alpha^{-1}}$, and inside this algebra we have the following computation:

$$\begin{aligned}[x_{gh}][x_{rs}]&=\alpha^{-1}(x_g,x_r)\alpha(x_h,x_s)[x_{gh}x_{rs}]\\&=\sigma_{gr}^{-1}\sigma_{hs}[x_{gh}x_{rs}]\end{aligned}$$

By using this, we obtain the following formula:

$$\begin{aligned}\sum_{t\in F}\sigma_{st^{-1},t}[x_{st^{-1},g}][x_{th}]&=\sum_{t\in F}\sigma_{st^{-1},t}\sigma^{-1}_{st^{-1},t}\sigma_{gh}[x_{st^{-1},g}x_{th}]\\&=\sigma_{gh}[x_{s,gh}]\end{aligned}$$

Similarly, we have the following formula:

$$\sum_{t\in F}\sigma^{-1}_{t^{-1},ts}[x_{g,t^{-1}}][x_{h,ts}]=\sigma^{-1}_{gh}[x_{gh,s}]$$

We deduce from this that there exists a Hopf algebra map, as follows:

$$\Phi:C(S^+_{\widehat{F}_\sigma})\to C(S^+_{\widehat{F}})^{\alpha^{-1}}$$

$$x_{gh}\to[x_{g,h}]$$

This map is clearly surjective, and is injective as well, by a standard fusion semiring argument, because both Hopf algebras have the same fusion semiring. ☐

Summarizing, we have proved our main twisting result. Our purpose in what follows will be that of working out versions and particular cases of it. We first have:

Proposition 16.14. *If F is a finite group and σ is a 2-cocycle on F, then*

$$\Phi(x_{g_1h_1}\ldots x_{g_mh_m})=\Omega(g_1,\ldots,g_m)^{-1}\Omega(h_1,\ldots,h_m)x_{g_1h_1}\ldots x_{g_mh_m}$$

with the coefficients on the right being given by the formula

$$\Omega(g_1,\ldots,g_m)=\prod_{k=1}^{m-1}\sigma_{g_1\ldots g_k,g_{k+1}}$$

is a coalgebra isomorphism $C(S^+_{\widehat{F}_\sigma})\to C(S^+_{\widehat{F}})$, commuting with the Haar integrals.

Proof. This is indeed just a technical reformulation of Theorem 16.13. □

Here is another useful result, that we will need in what follows:

Theorem 16.15. *Let $X\subset F$ be such that $\sigma_{gh}=1$ for any $g,h\in X$, and consider the subalgebra*

$$B_X\subset C(S^+_{\widehat{F}_\sigma})$$

generated by the elements x_{gh}, with $g,h\in X$. Then we have an injective algebra map

$$\Phi_0:B_X\to C(S^+_{\widehat{F}})$$

given by $x_{g,h}\to x_{g,h}$.

Proof. With the notations in the proof of Theorem 16.13, we have the following equality in $C(S^+_{\widehat{F}})^{\alpha^{-1}}$, for any $g_i,h_i,r_i,s_i\in X$:

$$[x_{g_1h_1}\ldots x_{g_ph_p}]\cdot[x_{r_1s_1}\ldots x_{r_qs_q}]=[x_{g_1h_1}\ldots x_{g_ph_p}x_{r_1s_1}\ldots x_{r_qs_q}]$$

Now Φ_0 can be defined to be the composition of $\Phi_{|B_X}$ with the linear isomorphism $C(S^+_{\widehat{F}})^{\alpha^{-1}}\to C(S^+_{\widehat{F}})$ given by $[x]\to x$, and is clearly an injective algebra map. □

Let us discuss now some concrete applications of the general results established above. Consider the group $F=\mathbb{Z}_n^2$, let $w=e^{2\pi i/n}$, and consider the following map:

$$\sigma:F\times F\to\mathbb{C}^*$$

$$\sigma_{(ij)(kl)}=w^{jk}$$

It is easy to see that σ is a bicharacter, and hence a 2-cocycle on F. Thus, we can apply our general twisting result, to this situation.

In order to understand what is the formula that we obtain, we must do some computations. Let E_{ij} with $i,j\in\mathbb{Z}_n$ be the standard basis of $M_n(\mathbb{C})$. We have:

Proposition 16.16. *The linear map given by*

$$\psi(e_{(i,j)})=\sum_{k=0}^{n-1}w^{ki}E_{k,k+j}$$

defines an isomorphism of algebras $\psi:C(\widehat{F}_\sigma)\simeq M_n(\mathbb{C})$.

Proof. Consider indeed the following linear map:

$$\psi'(E_{ij}) = \frac{1}{n}\sum_{k=0}^{n-1} w^{-ik} e_{(k,j-i)}$$

It is routine then to check that ψ, ψ' are inverse morphisms of algebras. $\square$

As a consequence, we have the following result:

Proposition 16.17. *The algebra map given by*

$$\varphi(u_{ij}u_{kl}) = \frac{1}{n}\sum_{a,b=0}^{n-1} w^{ai-bj} x_{(a,k-i),(b,l-j)}$$

defines a Hopf algebra isomorphism $\varphi : C(S^+_{M_n}) \simeq C(S^+_{\widehat{F}_\sigma})$.

Proof. We use the identification $C(\widehat{F}_\sigma) \simeq M_n(\mathbb{C})$ from Proposition 16.16. This identification produces a coaction map, as follows:

$$\gamma : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes C(S^+_{\widehat{F}_\sigma})$$

Now observe that this map is given by the following formula:

$$\gamma(E_{ij}) = \frac{1}{n}\sum_{ab} E_{ab} \otimes \sum_{kr} w^{ar-ik} x_{(r,b-a),(k,j-i)}$$

Thus, we obtain the isomorphism in the statement. $\square$

We will need one more result of this type, as follows:

Proposition 16.18. *The algebra map given by*

$$\rho(x_{(a,b),(i,j)}) = \frac{1}{n^2}\sum_{klrs} w^{ki+lj-ra-sb} p_{(r,s),(k,l)}$$

defines a Hopf algebra isomorphism $\rho : C(S^+_{\widehat{F}}) \simeq C(S^+_F)$.

Proof. This follows by using the Fourier transform isomorphism $C(\widehat{F}) \simeq C(F)$. $\square$

We can now formulate a concrete twisting result, from [22], as follows:

Theorem 16.19. *Let $n \geq 2$ and $w = e^{2\pi i/n}$. Then*

$$\Theta(u_{ij}u_{kl}) = \frac{1}{n}\sum_{ab=0}^{n-1} w^{-a(k-i)+b(l-j)} p_{ia,jb}$$

defines a coalgebra isomorphism

$$C(PO^+_n) \to C(S^+_{n^2})$$

commuting with the Haar integrals.

Proof. The result follows from Theorem 16.13 and Proposition 16.14, by combining them with the various isomorphisms established above. □

Here is a useful version of the above result:

Theorem 16.20. *The following two algebras are isomorphic, via $u_{ij}^2 \to X_{ij}$:*

(1) *The algebra generated by the variables $u_{ij}^2 \in C(O_n^+)$.*
(2) *The algebra generated by $X_{ij} = \frac{1}{n}\sum_{a,b=1}^n p_{ia,jb} \in C(S_{n^2}^+)$*

Proof. This follows by using Theorem 16.15, via the above identifications. □

As a probabilistic consequence now, we have:

Theorem 16.21. *The following families of variables have the same joint law,*

(1) $\{u_{ij}^2\} \in C(O_n^+)$,
(2) $\{X_{ij} = \frac{1}{n}\sum_{ab} p_{ia,jb}\} \in C(S_{n^2}^+)$,

where $u = (u_{ij})$ and $p = (p_{ia,jb})$ are the corresponding fundamental corepresentations.

Proof. This follows indeed from Theorem 16.20 above. □

In particular, we have the following result:

Theorem 16.22. *The free hypergeometric variable*

$$X_{ij} = \frac{1}{n}\sum_{a,b=1}^n u_{ia,jb} \in C(S_{n^2}^+)$$

has the same law as the squared free hyperspherical variable $x_i^2 \in C(S^{N-1}_{\mathbb{R},+})$.

Proof. This follows from Theorem 16.21. See [22]. □

As pointed out in [22], it is possible to derive as well this result directly, by using the Weingarten formula, and manipulations on the partitions. We refer to [22] and subsequent papers for more details on all this. We refer as well to [79], [80], [81] and [27] and related papers for further computations of this type, involving this time Gram matrix determinants, and for comments, regarding the relevance of such questions.

Summarizing, there is a lot of interesting mathematics in relation with the free spheres and orthogonal groups, and with the quantum permutations and quantum reflections as well. This tends to confirm our intial thought, from the beginning of this book, that the study and axiomatization of the quadruplets (S,T,U,K) is a good question.

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