QUANTUM ISOMETRIES AND NONCOMMUTATIVE GEOMETRY

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ABSTRACT. The free complex sphere $S^{N-1}_{\mathbb{C},+}$ is the noncommutative manifold defined by the equations $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$. Certain submanifolds $X \subset S^{N-1}_{\mathbb{C},+}$, related to the quantum groups, are known to have Riemannian features, including an integration functional. We review here the known facts on the subject.

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INTRODUCTION

The story of classical mechanics is quite fascinating. Obviously the Earth is flat, the Sun turns around it, and heavy objects fall quicker than light objects. In addition, a bit of thinking tells us that the natural forces, including gravity, cannot just propagate through the void. And also, that the void itself is probably a quite dubious notion.

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All this leads of course nowhere, but years and years of astronomical observations and computations, producing a fairly enormous amount of data, and some clever experiments as well, led to the conclusion that the situation is more complicated than this.

Quite remarkably, Kepler eventually managed to see the truth in all this, and to formulate three simple laws for the motion of planets. A bit later, Newton came up with a mathematical theory of gravity, explaining Kepler's findings, and refining them.

Modesty dictates that at very big scales, and perhaps at very small scales too, which are not exactly our business, our human knowledge is no longer correct. This is indeed the case with gravity, but Einstein was able to formulate the needed corrections.

For real-life situations, philosophy and applications, the theory of Newton remains of course the final scientific saying on the subject. Quite fortunately, a version of the Newton equations applies as well to the other known natural force, namely magnetism.

Quantum mechanics does not seem to be any simpler, and will probably take a long time as well to be understood. Things here are governed by two forces, called weak and strong, which appear to be trickier than gravity and electromagnetism. The problem comes from the fact that the coordinates of the subatomic moving objects, called "particles", do no longer commute. This is quite puzzling, and the idea is to use some kind of matrix coordinates instead. With a bit of luck, the trajectories of these particles, and the whole thing in general, could be described by some kind of "noncommutative geometry".

Developing all this faster than classical mechanics, in a short amount of time, is the objective of a large part of mathematics and physics. It is not clear, however, if this is feasible. We would have to outsmart here everyone, from the Greeks up to Newton.

With respect to them, we have a big disadvantage in regards with data, which comes only from complicated machinery, that we don't really understand. Also, importantly, we have less freedom of thought, with the previous religious hurdles being now replaced with an overall trend of negating everything that is nice, simple and natural.

There are currently several competing "noncommutative geometry" theories, or rather beginnings of such theories, the most serious, by far, being the one of Connes.

Connes' idea is that the noncommutative manifolds that we are interested in should appear from operator algebras, and should be Riemannian. Roughly speaking, such a manifold X is described by an operator algebra A, which can be commutative or not, with the extra data being a Hilbert space H, and a Dirac type operator D.

This theory has been very successful, both in mathematics and physics, and especially in relation with the Standard Model. We refer here to [41], [52].

Our purpose here is to describe certain classes of noncommutative manifolds, which do not fit exactly into Connes' formalism, but which come from the same philosophy. To be more precise, our manifolds will come from operator algebras too, and will have some Riemannian features as well, and more specifically an integration functional.

As a basic example of such manifold, we have the free complex sphere $S^{N-1}_{\mathbb{C},+}$. This sphere is by definition the compact noncommutative space, meaning dual of an operator algebra, whose coordinates $x_1,\ldots,x_N$ are subject to the following relations:

$$\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$$

This sphere has indeed an integration functional, namely the one with respect to the uniform measure, which is by definition the unique probability measure which is invariant under the corresponding action of the free unitary quantum group:

$$U_N^+\curvearrowright S^{N-1}_{\mathbb{C},+}$$

Importantly, this integration functional can be explicitely computed, via a Weingarten type formula. Let $\mathcal{NC}_2(k)$ be the set of noncrossing matching pairings of a colored integer $k=k_1\ldots k_p$, with the colors being exponents $k_i\in\{\emptyset,*\}$, for $\pi,\sigma\in\mathcal{NC}_2(k)$ set $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$, and finally let $W_{kN}=G_{kN}^{-1}$. The integration formula is then:

$$\int_{S^{N-1}_{\mathbb{C},+}}x_{i_1}^{k_1}\ldots x_{i_p}^{k_p}\,dx=\sum_\pi\sum_{\sigma\leq\ker i}W_{kN}(\pi,\sigma)$$

Regarding more standard differential geometry aspects, what is known is that $S^{N-1}_{\mathbb{C},+}$ has a Laplacian filtration, and that eigenvalues for the Laplacian can be constructed as well. However, $S^{N-1}_{\mathbb{C},+}$ does not appear to fit exactly into Connes' theory.

All this is quite interesting, and there are several ways of extending it:

(1) A first idea is that of adding extra relations between the coordinates $x_1,\ldots,x_N$ and their adjoints, such as $ab=ba$, $ab=\pm ba$, $abc=cba$, $abc=\pm cba$, and so on. There is an axiomatization problem here, involving the associated sphere S, noncommutative torus T, unitary quantum group U, and quantum reflection group K.

(2) A second idea, which is independent from (1), but can be combined with it, is that of looking at more general homogeneous spaces, over the quantum unitary group U, or over closed subgroups $G\subset U$. Of particular interest here are the analogues of the homogeneous spaces consisting of partial isometries $\mathbb{C}^N\to\mathbb{C}^M$, having fixed rank L.

(3) In certain situations, as for instance in the classical case, where $ab=ba$, it is possible to go well beyond the "core manifolds" discussed in (2), with the development of a full and broad geometry theory. To a certain extent, this is valid as well in the twisted case, $ab=\pm ba$, half-classical case, $abc=cba$, and twisted half-classical case, $abc=\pm cba$.

Summarizing, we have here many potential examples of noncommutative manifolds, which appear as algebraic submanifolds $X\subset S^{N-1}_{\mathbb{C},+}$, of a very special type, and which can have some Riemannian features, including an integration functional.

We will review here the known facts on the subject. There is of course a lot of work still to be done, in order to reach for instance to an abstract axiomatization of such manifolds. Besides the surveying and simplifying work, we will do as well a bit of original work, with respect to what is known. However, the important problems will remain open.

Getting back now to Connes' geometry, we believe that the examples of noncommutative manifolds studied here are not very far from his manifolds, and could eventually fit into an extension of his theory. To be more precise, one theoretical downside of Connes' theory is the lack of an analogue of the Nash embedding theorem. Assuming that this question will be solved one day, and with the target of the "generalized Nash embeddings" being the free sphere $S^{N-1}_{\mathbb{C},+}$, the unification problem would be probably solvable.

In short, we believe in the existence of a "Nash-Connes Geometry", covering most of the interesting examples of noncommutative Riemannian manifolds known so far.

The present text is organized as follows: 1-3 contain preliminaries and axiomatization work, in 4-6 we discuss classification results, 7-9 are concerned with probabilistic aspects, and in 10-12 we discuss basic homogeneous spaces, and other manifolds.

Acknowledgements.

My first thanks go to Alain Connes, for his enormously inspiring work. Back in the days, when I started to do mathematics, our science used to be something quite abstract, but Alain was passionately lecturing about NCG, physics and quarks. I always wanted to contribute a bit to NCG, and I hope one day to get into quarks, too.

It is a pleasure to thank as well my PhD advisor Georges Skandalis, for patiently guiding me through the field, and its difficulties, and potential clashes.

This book is based on a number of joint research papers on quantum groups and noncommutative geometry, for the most written around 2010–2015, and I am particularly grateful to Julien Bichon, for his heavy involvement in the subject.

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1. Spheres and tori

What is geometry? This depends of course on your knowledge of the subject. In our opinion, you can't really do something interesting without having at least a sphere S, a torus T, a unitary group U, and a reflection group K, as starting objects.

These basic objects should have relations between them, as follows:

$$\begin{tikzcd} S \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ U \arrow[r, leftrightarrow] & K \end{tikzcd}$$

Our idea here will be that of axiomatizing such quadruplets (S,T,U,K). With this axiomatization in hand, and some classification results as well, we will discuss then the development of each of the geometries that we found. This will be our plan.

Let us first discuss the case of the usual geometry, in $\mathbb{R}^N$. Basic common sense would suggest to add $\mathbb{R}^N$ itself to our list of objects, and with this addition done, why not erasing afterwards all the other objects (!), which can be reconstructed anyway from $\mathbb{R}^N$.

Unfortunately, this is something that we cannot do, in view of our noncommutative geometry goals and motivations. To be more precise, it is well-known that $\mathbb{R}^N$ has no interesting noncommutative analogues. Technically speaking, the problem comes from the fact that $\mathbb{R}^N$ is not compact. We will be back later to this important issue.

So, let us go ahead, and construct our quadruplet (S,T,U,K). We have:

Definition 1.1. *The real sphere, torus, unitary group and reflection group are:*

$$\begin{aligned} S^{N-1}_\mathbb{R} &= \left\{x\in\mathbb{R}^N\Big|\sum_i x_i^2=1\right\}\\ T_N &= \left\{x\in\mathbb{R}^N\Big|x_i=\pm\frac{1}{\sqrt{N}}\right\}\\ O_N &= \left\{U\in M_N(\mathbb{R})\Big|U^t=U^{-1}\right\}\\ H_N &= \left\{U\in M_N(-1,0,1)\Big|U^t=U^{-1}\right\} \end{aligned}$$

These are the usual sphere, cube, orthogonal group, and hyperoctahedral group.

In this definition the superscript $N-1$ for the sphere, which does not fit with the rest, but is very standard, stands for the real dimension as manifold, which is $N-1$.

As another comment, the $1/\sqrt{N}$ normalization for the cube/torus is there in order to have an embedding $T_N\subset S^{N-1}_\mathbb{R}$, this being convenient for our purposes.

Regarding now the correspondences between our objects, there are many ways of establishing them, depending on knowledge and taste, but this is not crucial for us.

So, let us be very sloppy here, and formulate things as follows:

Theorem 1.2. *We have a full set of correspondences, as follows,*

$$\begin{tikzcd}
S^{N-1}_\mathbb{R} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T_N \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
O_N \arrow[r, leftrightarrow] & H_N
\end{tikzcd}$$

obtained via various results from basic geometry and group theory.

Proof. As already mentioned, there are several possible solutions to the problem, and all this is not crucial for us. Here is a way of constructing these correspondences:

(1) $S^{N-1}_\mathbb{R} \leftrightarrow T_N$. Here T_N comes from $S^{N-1}_\mathbb{R}$ via $|x_1| = \ldots = |x_N|$, while $S^{N-1}_\mathbb{R}$ appears from $T_N \subset \mathbb{R}^N$ by "deleting" this relation, while still keeping $\sum_i x_i^2 = 1$.

(2) $S^{N-1}_\mathbb{R} \leftrightarrow O_N$. This comes from the fact that O_N is the isometry group of $S^{N-1}_\mathbb{R}$, and that, conversely, $S^{N-1}_\mathbb{R}$ appears as $\{Ux|U \in O_N\}$, where $x = (1, 0, \ldots, 0)$.

(3) $S^{N-1}_\mathbb{R} \leftrightarrow H_N$. This is something trickier, but the passage can definitely be obtained, for instance via T_N, by using the constructions in (1) above and (5) below.

(4) $T_N \leftrightarrow O_N$. Here $T_N \simeq \mathbb{Z}_2^N$ is a maximal torus of O_N, and the group O_N itself can be reconstructed from this maximal torus, by using various methods.

(5) $T_N \leftrightarrow H_N$. Here, similarly, $T_N \simeq \mathbb{Z}_2^N$ is a maximal torus of H_N, and the group H_N itself can be reconstructed from this torus as a wreath product, $H_N = T_N \wr S_N$.

(6) $O_N \leftrightarrow H_N$. This is once again something trickier, but the passage can definitely be obtained, for instance via T_N, by using the constructions in (4) and (5) above. $\square$

The above result is of course something quite non-trivial, and having it understood properly would take some time. However, as already said, we will technically not need all this. Our purpose for the moment is just to explain our (S, T, U, K) philosophy.

As a second basic example of geometry, we have the usual geometry of $\mathbb{C}^N$. Here, as before, we cannot include the space $\mathbb{C}^N$ itself in our formalism, because this space is not compact, and as already said, we would like to deal with compact spaces only.

The corresponding quadruplet (S, T, U, K) can be constructed as follows:

Definition 1.3. *The complex sphere, torus, unitary group and reflection group are:*

$$\begin{aligned}
S^{N-1}_\mathbb{C} &= \left\{x\in\mathbb{C}^N\Big|\sum_i|x_i|^2=1\right\}\\
\mathbb{T}_N &= \left\{x\in\mathbb{C}^N\Big|\ |x_i|=\frac{1}{\sqrt{N}}\right\}\\
U_N &= \left\{U\in M_N(\mathbb{C})\Big|U^*=U^{-1}\right\}\\
K_N &= \left\{U\in M_N(\mathbb{T}\cup\{0\})\Big|U^*=U^{-1}\right\}
\end{aligned}$$

These are the usual complex sphere, torus, unitary group, and complex reflection group.

As before, the superscript $N-1$ for the sphere does not fit with the rest, but is quite standard, somewhat coming from dimension considerations. We will use it as such.

Also, the $1/\sqrt{N}$ factor is there in order to have an embedding $\mathbb{T}_N\subset S^{N-1}_\mathbb{C}$.

Here is now our theorem on the subject, sloppy of course as they go:

Theorem 1.4. *We have a full set of correspondences, as follows,*

$$\begin{tikzcd}
S^{N-1}_\mathbb{C} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \mathbb{T}_N \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
U_N \arrow[r, leftrightarrow] & K_N
\end{tikzcd}$$

obtained via various results from basic geometry and group theory.

Proof. We follow the proof in the real case, by making adjustments where needed, and with of course the reiterated comment that all this is not crucial for us:

(1) $S^{N-1}_\mathbb{C}\leftrightarrow\mathbb{T}_N$. Same proof as before, using $|x_1|=\ldots=|x_N|$.
(2) $S^{N-1}_\mathbb{C}\leftrightarrow U_N$. Here "isometry" must be taken in an affine complex sense.
(3) $S^{N-1}_\mathbb{C}\leftrightarrow K_N$. Trickier as before, best viewed by passing via $\mathbb{T}_N$.
(4) $\mathbb{T}_N\leftrightarrow U_N$. Coming from the fact that $\mathbb{T}_N\simeq\mathbb{T}^N$ is a maximal torus of U_N.
(5) $\mathbb{T}_N\leftrightarrow K_N$. Once again, maximal torus argument, and $K_N=\mathbb{T}_N\wr S_N$.
(6) $U_N\leftrightarrow K_N$. Trickier as before, best viewed by passing via $\mathbb{T}_N$. □

As a conclusion, our (S,T,U,K) philosophy seems to work, in the sense that these 4 objects, and the relations between them, encode interesting facts about $\mathbb{R}^N,\mathbb{C}^N$.

Our plan in what follows will be that of leaving aside the complete understanding of what has been said above, and going directly for the noncommutative case. We will see that in the noncommutative setting things are more rigid, and therefore, simpler.

In order to talk about noncommutative geometry, we need to have some motivations and goals. Without surprise, these motivations come from quantum mechanics.

The idea indeed is that at the subatomic level the "coordinates" of the various moving objects, called particles, do not necessarily commute. In fact, at this level, our ambient space $\mathbb{R}^3$ gets replaced with something not commutative, and infinite dimensional - typically a space of infinite complex matrices. All this comes indeed from the discovery of the radioactivity and to advances in electromagnetism, and to the subsequent work on the atomic theory, from the beginning and first part of the 20th century.

In fact, the need for "noncommutative coordinates" can be traced even further ago. A simple method, indeed, for studying matter is that of burning it, and then decomposing the resulting light with a prism. This method produces certain "spectral lines", intimately related to the fine structure of the material. The behavior of these lines is subject to the Ritz-Rydberg combination principle, which suggests the use of matrices.

Of course, all this remains quite puzzling, and contrary to our usual $\mathbb{R}^3$ intuition. Observe that there is a vague analogy here with what happens at the other end of the spectrum, involving very big objects, and very large scales. Indeed, here our ambient space $\mathbb{R}^3$ is not the good space either, and curved space-time must be used instead.

All this suggests defining our noncommutative spaces X as being abstract manifolds, whose coordinates $x_1,\ldots,x_N$ do not necessarily commute. Thus, we are in need of some good algebraic geometry correspondence, between such abstract spaces X, and the corresponding algebras of coordinates A. Following Heisenberg, von Neumann and many others, we will use here the correspondence coming from operator algebra theory.

A first idea is that of using "continuous coordinates", with each noncommutative space X corresponding to a certain noncommutative algebra $A=C(X)$. With this idea in mind, getting back to our (S,T,U,K) philosophy, we would like to have objects as follows:

$$\begin{tikzcd}
C(S) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & C(T) \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
C(U) \arrow[r, leftrightarrow] & C(K)
\end{tikzcd}$$

A second idea, which is viable as well, and is probably more far-reaching, in view of the loads of probability theory involved with quantum mechanics, but which is technically more complicated to develop, is that of using L^∞ coordinates for our manifolds. Here we would like to have objects as follows:

$$\begin{tikzcd} L^\infty(S) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & L^\infty(T) \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ L^\infty(U) \arrow[r, leftrightarrow] & L^\infty(K) \end{tikzcd}$$

Our plan will be that of developing first the continuous theory, and leaving the more advanced aspects, involving von Neumann algebras and probability, for later.

In order to get started, we will need a number of preliminaries. First, we have:

Definition 1.5. *A Hilbert space is a complex vector space H given with a scalar product $< x, y >$, satisfying the following conditions:*

(1) *$< x, y >$ is linear in x, and antilinear in y.*
(2) *$\overline{< x, y >} =< y, x >$, for any x, y.*
(3) *$< x, x >> 0$, for any $x \neq 0$.*
(4) *H is complete with respect to the norm $||x|| = \sqrt{< x, x >}$.*

Here the fact that $||.||$ is indeed a norm comes from the Cauchy-Schwarz inequality, $| < x, y > | \leq ||x|| \cdot ||y||$, which can be established by using the fact that the degree 2 polynomial $f(t) = ||x + ty||^2$ being positive, its discriminant must be negative.

In finite dimensions, any algebraic basis $\{f_1, \ldots, f_N\}$ can be turned into an orthonormal basis $\{e_1, \ldots, e_N\}$, by using the Gram-Schmidt procedure. Thus, we have $H \simeq \mathbb{C}^N$, with this latter space being endowed with its usual scalar product:

$$< x, y >= \sum_i x_i \bar{y}_i$$

The same happens in infinite dimensions, once again by Gram-Schmidt, coupled if needed with the Zorn lemma, in case our space is really very big. In other words, any Hilbert space has an orthonormal basis $\{e_i\}_{i \in I}$, and we have $H \simeq l^2(I)$.

Of particular interest is the "separable" case, where I is countable. According to the above, there is up to isomorphism only one Hilbert space here, namely $H = l^2(\mathbb{N})$.

All this is, however, quite tricky, and can be a bit misleading. Consider for instance the space $H = L^2[0, 1]$ of square-summable functions $f : [0, 1] \to \mathbb{C}$, with:

$$< f, g >= \int_0^1 f(x) \overline{g(x)} dx$$

This space is of course separable, because we can use the basis $f_n = x^n$ with $n \in \mathbb{N}$, orthogonalized by Gram-Schmidt. However, the orthogonalization procedure is something non-trivial, and so the isomorphism $H \simeq l^2(\mathbb{N})$ that we obtain is something non-trivial as well. Doing some computations here is actually a very good exercise.

In what follows we will be interested in the linear operators $T : H \to H$ which are bounded. Regarding such operators, we have the following result:

Theorem 1.6. *Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that $||T|| = \sup_{||x||\leq 1} ||Tx||$ is finite, form a complex algebra with unit, denoted $B(H)$. This algebra has the following properties:*

1. *$B(H)$ is complete with respect to $||.||$, so we have a Banach algebra.*
2. *$B(H)$ has an involution $T \to T^*$, given by $< Tx, y >=< x, T^*y >$.*

In addition, the norm and involution are related by the formula $||TT^*|| = ||T||^2$.

Proof. The fact that we have indeed an algebra follows from:

$$||S+T|| \leq ||S|| + ||T|| \quad , \quad ||\lambda T|| = |\lambda| \cdot ||T|| \quad , \quad ||ST|| \leq ||S|| \cdot ||T||$$

Regarding now (1), if $\{T_n\} \subset B(H)$ is Cauchy then $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting $Tx = \lim_{n\to\infty} T_n x$.

As for (2), here the existence of T^* comes from the fact that $\varphi(x) =< Tx, y >$ being a linear map $H \to \mathbb{C}$, we must have $\varphi(x) =< x, T^*y >$, for a certain vector $T^*y \in H$. Moreover, since this vector is unique, T^* is unique too, and we have as well:

$$(S+T)^* = S^* + T^* \quad , \quad (\lambda T)^* = \bar{\lambda} T^* \quad , \quad (ST)^* = T^*S^* \quad , \quad (T^*)^* = T$$

Observe also that we have indeed $T^* \in B(H)$, because:

$$||T|| = \sup_{||x||=1} \sup_{||y||=1} < Tx, y >= \sup_{||y||=1} \sup_{||x||=1} < x, T^*y >= ||T^*||$$

Regarding the last assertion, we have $||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$. Also, we have:

$$||T||^2 = \sup_{||x||=1} | < Tx, Tx > | = \sup_{||x||=1} | < x, T^*Tx > | \leq ||T^*T||$$

By replacing $T \to T^*$ we obtain from this $||T||^2 \leq ||TT^*||$, and we are done. $\square$

Observe that when H comes with an orthonormal basis $\{e_i\}_{i\in I}$, the linear map $T \to M$ given by $M_{ij} =< Te_j, e_i >$ produces an embedding $B(H) \subset M_I(\mathbb{C})$. Moreover, in this picture the operation $T \to T^*$ takes a very simple form, namely $(M^*)_{ij} = \overline{M}_{ji}$.

We will be interested in fact in the algebras of operators, rather than in the operators themselves. The basic axioms here, inspired from Theorem 1.6, are as follows:

Definition 1.7. *A unital C^*-algebra is a complex algebra with unit A, having:*

1. *A norm $a \to ||a||$, making it a Banach algebra (the Cauchy sequences converge).*
2. *An involution $a \to a^*$, which satisfies $||aa^*|| = ||a||^2$, for any $a \in A$.*

According to Theorem 1.6, the operator algebra $B(H)$ itself is a C^*-algebra. More generally, we have as examples all the closed $*$-subalgebras $A \subset B(H)$. We will see later on (the "GNS theorem") that any C^*-algebra appears in fact in this way.

Generally speaking, the elements $a \in A$ are best thought of as being some kind of "generalized operators", on some Hilbert space which is not present. By using this idea, one can emulate spectral theory in this setting, in the following way:

Proposition 1.8. *Given $a \in A$, define its spectrum as $\sigma(a) = \{\lambda \in \mathbb{C} | a - \lambda \notin A^{-1}\}$, and its spectral radius $\rho(a)$ as the radius of the smallest centered disk containing $\sigma(a)$.*

1. *The spectrum of a norm one element is in the unit disk.*
2. *The spectrum of a unitary element ($a^* = a^{-1}$) is on the unit circle.*
3. *The spectrum of a self-adjoint element ($a = a^*$) consists of real numbers.*
4. *The spectral radius of a normal element ($aa^* = a^*a$) is equal to its norm.*

Proof. Our first claim is that for any polynomial $f \in \mathbb{C}[X]$, and more generally for any rational function $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$, we have:

$$\sigma(f(a)) = f(\sigma(a))$$

This indeed something well-known for the usual matrices. In the general case, assume first that we have a polynomial, $f \in \mathbb{C}[X]$. If we pick an arbitrary number $\lambda \in \mathbb{C}$, and write $f(X) - \lambda = c(X - r_1) \ldots (X - r_k)$, we have then, as desired:

$$\begin{aligned}
\lambda \notin \sigma(f(a)) &\iff f(a) - \lambda \in A^{-1} \\
&\iff c(a - r_1) \ldots (a - r_k) \in A^{-1} \\
&\iff a - r_1, \ldots, a - r_k \in A^{-1} \\
&\iff r_1, \ldots, r_k \notin \sigma(a) \\
&\iff \lambda \notin f(\sigma(a))
\end{aligned}$$

Assume now that we are in the general case, $f \in \mathbb{C}(X)$. We pick $\lambda \in \mathbb{C}$, we write $f = P/Q$, and we set $F = P - \lambda Q$. By using the above finding, we obtain, as desired:

$$\begin{aligned}
\lambda \in \sigma(f(a)) &\iff F(a) \notin A^{-1} \\
&\iff 0 \in \sigma(F(a)) \\
&\iff 0 \in F(\sigma(a)) \\
&\iff \exists \mu \in \sigma(a), F(\mu) = 0 \\
&\iff \lambda \in f(\sigma(a))
\end{aligned}$$

Regarding now the assertions in the statement, these basically follows from this:
(1) This comes from the following formula, valid when $||a|| < 1$:

$$\frac{1}{1-a} = 1 + a + a^2 + \ldots$$

(2) This follows by using the rational function $f(z) = z^{-1}$. Indeed, we have:

$$\sigma(a)^{-1} = \sigma(a^{-1}) = \sigma(a^*) = \overline{\sigma(a)}$$

(3) This follows by using (2), and the rational function $f(z) = (z + it)/(z - it)$, with $t \in \mathbb{R}$. Indeed, for $t >> 0$ the element $f(a)$ is well-defined, and we have:

$$\left(\frac{a+it}{a-it}\right)^* = \frac{a-it}{a+it} = \left(\frac{a+it}{a-it}\right)^{-1}$$

Thus $f(a)$ is a unitary, and by (2) its spectrum is contained in $\mathbb{T}$. We conclude that we have $f(\sigma(a)) = \sigma(f(a)) \subset \mathbb{T}$, and so $\sigma(a) \subset f^{-1}(\mathbb{T}) = \mathbb{R}$, as desired.

(4) We have $\rho(a) \leq ||a||$ from (1). Conversely, given $\rho > \rho(a)$, we have:

$$\int_{|z|=\rho} \frac{z^n}{z-a}\,dz = \sum_{k=0}^{\infty}\left(\int_{|z|=\rho} z^{n-k-1}dz\right)a^k = a^{n-1}$$

By applying the norm and taking n-th roots we obtain $\rho \geq \lim_{n\to\infty}||a^n||^{1/n}$. In the case $a = a^*$ we have $||a^n|| = ||a||^n$ for any exponent of the form $n = 2^k$, and by taking n-th roots we get $\rho \geq ||a||$. This gives the missing inequality $\rho(a) \geq ||a||$.

In the general case $aa^* = a^*a$ we have $a^n(a^n)^* = (aa^*)^n$, and we get $\rho(a)^2 = \rho(aa^*)$. Now since aa^* is self-adjoint, we get $\rho(aa^*) = ||a||^2$, and we are done. $\square$

With these technical ingredients in hand, we can now formulate a key theorem:

Theorem 1.9 (Gelfand)**.** *If X is a compact space, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a commutative C^*-algebra, with structure as follows:*

1. *The norm is the usual sup norm, $||f|| = \sup_{x\in X}|f(x)|$.*
2. *The involution is the usual involution, $f^*(x) = \overline{f(x)}$.*

Conversely, any commutative C^*-algebra is of the form $C(X)$, with its "spectrum" $X = Spec(A)$ appearing as the space of characters $\chi : A \to \mathbb{C}$.

Proof. In what regards the first assertion, almost everything here is trivial. We have indeed a commutative algebra, with norm and involution, the Cauchy sequences inside are well-known to converge, and the condition $||ff^*|| = ||f||^2$ is satisfied.

Conversely, given a commutative C^*-algebra A, we can define indeed X to be the set of characters $\chi : A \to \mathbb{C}$, with the topology making continuous all the evaluation maps $ev_a : \chi \to \chi(a)$. Then X is a compact space, and $a \to ev_a$ is a morphism of algebras $ev : A \to C(X)$. We first prove that ev is involutive. We use the following formula:

$$a = \frac{a+a^*}{2} - i \cdot \frac{i(a-a^*)}{2}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$.

Since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

As a conclusion, in order to talk about noncommutative spaces, we can simply set $A = C(X)$, for any C^*-algebra A, and call X a "noncommutative compact space".

Let us discuss now the other basic result regarding the C^*-algebras, namely the GNS representation theorem. We will need some more spectral theory, as follows:

Proposition 1.10. *For an element $a \in A$, the following are equivalent:*

(1) *a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.*
(2) *$a = b^2$, for some $b \in A$ satisfying $b = b^*$.*
(3) *$a = cc^*$, for some $c \in A$.*

Proof. This basically follows from the Gelfand theorem, as follows:

(1) $\implies$ (2) Observe that $\sigma(a) \subset \mathbb{R}$ implies $a = a^*$. Thus the algebra $< a >$ is commutative, and by using the Gelfand theorem, we can set $b = \sqrt{a}$.

(2) $\implies$ (3) This is trivial, because we can simply set $c = b$.

(3) $\implies$ (1) We proceed by contradiction. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. By writing $d = x + iy$ with $x = x^*, y = y^*$ we have $dd^* + d^*d = 2(x^2 + y^2) \geq 0$. Thus $d^*d \geq 0$, which contradicts the elementary fact that $\sigma(dd^*), \sigma(d^*d)$ must coincide outside $\{0\}$. □

Here is now the representation theorem, along with the idea of the proof:

Theorem 1.11 (GNS theorem)**.** *Let A be a C^*-algebra.*

(1) *A appears as a closed $*$-subalgebra $A \subset B(H)$, for some Hilbert space H.*
(2) *When A is separable (usually the case), H can be chosen to be separable.*
(3) *When A is finite dimensional, H can be chosen to be finite dimensional.*

Proof. Let us first discuss the commutative case, $A = C(X)$. Our claim here is that if we pick a probability measure on X, we have an embedding as follows:

$$C(X) \subset B(L^2(X)) \quad , \quad f \to (g \to fg)$$

Indeed, given $f \in C(X)$, consider the operator $T_f(g) = fg$, on the Hilbert space $H = L^2(X)$. Observe that T_f is indeed well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 dx} \leq ||f||_\infty ||g||_2$$

The application $f \to T_f$ being linear, involutive, continuous, and injective as well, we obtain in this way a C^*-algebra embedding $C(X) \subset B(H)$, as claimed.

In general, we can use a similar idea, with the algebraic aspects being fine, and with the positivity issues being taken care of by Proposition 1.8 and Proposition 1.10.

Indeed, assuming that a linear form $\varphi : A \to \mathbb{C}$ has some suitable positivity properties, making it analogous to the integration functionals $\int_X : A \to \mathbb{C}$ from the commutative case, we can define a scalar product on A, by the following formula:

$$< a, b >= \varphi(ab^*)$$

By completing we obtain a Hilbert space H, and we have an embedding as follows:

$$A \subset B(H) \quad , \quad a \to (b \to ab)$$

Thus we obtain the assertion (1), and a careful examination of the construction $A \to H$, outlined above, shows that the assertions (2,3) are in fact proved as well. □

The GNS theorem is something powerful and concrete, which perfectly complements the Gelfand theorem, and the resulting noncommutative compact space formalism. The idea indeed is that "once you are lost into noncommutative geometry considerations, coming from abstract C^*-algebras, you can always get back to good old Hilbert spaces".

With the above formalism is hand, we can go ahead, and construct geometric quadruplets (S, T, U, K), as before. We will do this slowly. Let us begin with the spheres:

Definition 1.12. *We have free real and complex spheres, defined via*

$$C(S^{N-1}_{\mathbb{R},+}) = C^*\left(x_1, \ldots, x_N \Big| x_i = x_i^*, \sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C},+}) = C^*\left(x_1, \ldots, x_N \Big| \sum_i x_i x_i^* = \sum_i x_i^* x_i = 1\right)$$

where the symbol C^* stands for universal enveloping C^*-algebra.

All this deserves some explanations. Given an integer $N \in \mathbb{N}$, consider the free complex unital algebra on $2N$ variables, denoted $x_1, \ldots, x_N$ and $x_1^*, \ldots, x_N^*$:

$$A = \left\langle x_1, \ldots, x_N, x_1^*, \ldots, x_N^* \right\rangle$$

In other words, the elements of A are the formal linear combinations, with complex coefficients, of products between our variables x_i, x_i^*, and of the unit 1.

This algebra has an involution $* : A \to A$, given by $x_i \leftrightarrow x_i^*$. Now let us consider the following $*$-algebra quotients of our $*$-algebra A:

$$A_R = A \Big/ \left\langle x_i = x_i^*, \sum_i x_i^2 = 1 \right\rangle$$

$$A_C = A \Big/ \left\langle \sum_i x_i x_i^* = \sum_i x_i^* x_i = 1 \right\rangle$$

Since the first relations imply the second ones, we have quotient maps as follows:

$$A \to A_C \to A_R$$

Our claim now is both A_C, A_R admit enveloping C^*-algebras, in the sense that the biggest C^*-norms on these $*$-algebras are bounded. We only have to check this for the bigger algebra A_C. But here, our claim follows from the following estimate:

$$||x_i||^2 = ||x_i x_i^*|| \leq ||\sum_i x_i x_i^*|| = 1$$

Summarizing, our claim is proved, so we can define $C(S^{N-1}_{\mathbb{R},+}), C(S^{N-1}_{\mathbb{C},+})$ as being the enveloping C^*-algebras of A_R, A_C, and so Definition 1.12 makes sense.

In order to formulate some results, let us introduce as well:

Definition 1.13. *Given a noncommutative compact space X, its classical version is the subspace $X_{class} \subset X$ obtained by dividing $C(X)$ by its commutator ideal:*

$$C(X_{class}) = C(X)/I \quad , \quad I = < [a,b] >$$

In this situation, we also say that X appears as a "liberation" of X.

In other words, the space X_{class} appears as the Gelfand spectrum of the commutative C^*-algebra $C(X)/I$. Observe in particular that X_{class} is indeed a classical space.

As a first result now, regarding the above free spheres, we have:

Theorem 1.14. *We have embeddings of noncommutative spaces, as follows,*

$$\begin{tikzcd}
S^{N-1}_{\mathbb{R},+} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\
S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C}} \arrow[u]
\end{tikzcd}$$

and the spaces on top appear as liberations of the spaces on the bottom.

Proof. The first assertion, regarding the inclusions, comes from the fact that at the level of the associated C^*-algebras, we have surjective maps, as follows:

$$\begin{tikzcd}
C(S^{N-1}_{\mathbb{R},+}) \arrow[d] & C(S^{N-1}_{\mathbb{C},+}) \arrow[l] \arrow[d] \\
C(S^{N-1}_{\mathbb{R}}) & C(S^{N-1}_{\mathbb{C}}) \arrow[l]
\end{tikzcd}$$

For the second assertion, we must establish the following isomorphisms, where the symbol C^*_{comm} stands for "universal commutative C^*-algebra generated by":

$$C(S^{N-1}_\mathbb{R})=C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_ix_i^2=1\right)$$

$$C(S^{N-1}_\mathbb{C})=C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_ix_ix_i^*=\sum_ix_i^*x_i=1\right)$$

As a first observation, it is enough to establish the second isomorphism, because the first one will follow from it, simply by dividing by the relations $x_i=x_i^*$.

So, consider the second universal commutative C^*-algebra A constructed above. Since the standard coordinates on $S^{N-1}_\mathbb{C}$ satisfy the defining relations for A, we have a quotient map of as follows, mapping standard coordinates to standard coordinates:

$$A\to C(S^{N-1}_\mathbb{C})$$

Conversely, let us write $A=C(S)$, by using the Gelfand theorem. The variables $x_1,\ldots,x_N$ become in this way true coordinates, providing us with an embedding $S\subset\mathbb{C}^N$. Also, the quadratic relations become $\sum_i|x_i|^2=1$, so we have $S\subset S^{N-1}_\mathbb{C}$. Thus, we have a quotient map $C(S^{N-1}_\mathbb{C})\to A$ mapping coordinates to coordinates, as desired. $\square$

Summarizing, we are done with the spheres. We will be back to these spheres on several occasions, throughout this book, with various results about them.

Before getting into tori, let us talk about algebraic manifolds. It is quite clear that $\mathbb{R}^N,\mathbb{C}^N$ themselves do not have free analogues, and this because the free $*$-algebra on N variables, and its quotient by the relations $x_i=x_i^*$, do not have enveloping C^*-algebras. Indeed, there is no way of obtaining an upper bound on the quantities $||x_i||$.

However, by using the free spheres constructed above, we can formulate:

Definition 1.15. *A real algebraic submanifold $X\subset S^{N-1}_{\mathbb{C},+}$ is a closed noncommutative space defined, at the level of the corresponding C^*-algebra, by a formula of type*

$$C(X)=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle f_i(x_1,\ldots,x_N)=0\Big\rangle$$

for certain noncommutative polynomials $f_i\in\mathbb{C}<x_1,\ldots,x_N>$. We denote by $\mathcal{C}(X)$ the $*$-subalgebra of $C(X)$ generated by the coordinate functions $x_1,\ldots,x_N$.

Observe that any family of noncommutative polynomials $f_i\in\mathbb{C}<x_1,\ldots,x_N>$ produces such a manifold X, simply by defining an algebra $C(X)$ as above. Observe also that the use of the free complex sphere is essential in all this, because the quadratic condition $\sum_ix_ix_i^*=\sum_ix_i^*x_i=1$ guarantees the fact that the universal C^*-norm is bounded.

As a basic example of such a manifold, we have the free real sphere $S^{N-1}_{\mathbb{R},+}$. The classical spheres $S^{N-1}_\mathbb{C},S^{N-1}_\mathbb{R}$, and their real submanifolds, are covered as well by this formalism.

At the level of the general theory, we have the following version of the Gelfand theorem, which is something very useful, and that we will use many times in what follows:

Theorem 1.16. *If $X \subset S^{N-1}_{\mathbb{C},+}$ is an algebraic manifold, as above, we have*

$$X_{class} = \left\{ x \in S^{N-1}_{\mathbb{C}} \Big| f_i(x_1, \ldots, x_N) = 0 \right\}$$

and X appears as a liberation of X_{class}.

Proof. This is something that already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we denote by X'_{class} the manifold constructed in the statement, then we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class}) \to C(X'_{class})$$

Conversely now, from $X \subset S^{N-1}_{\mathbb{C},+}$ we obtain $X_{class} \subset S^{N-1}_{\mathbb{C}}$, and since the relations defining X'_{class} are satisfied by X_{class}, we obtain an inclusion of subspaces $X_{class} \subset X'_{class}$. Thus, at the level of algebras of continuous functions, we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X'_{class}) \to C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and we are done. □

Finally, once again at the level of the general theory, we have:

Definition 1.17. *We agree to identify two real algebraic submanifolds $X, Y \subset S^{N-1}_{\mathbb{C},+}$ in the case where we have a $*$-algebra isomorphism*

$$f : \mathcal{C}(Y) \to \mathcal{C}(X)$$

mapping standard coordinates to standard coordinates.

This latter definition is something quite subtle, making our formalism to be somehow half-way between the $*$-algebra formalism, and the C^*-algebra formalism. We will be back to this question, coming from amenability issues, a bit later, with details.

Let us go back now to our general (S, T, U, K) program. Now that we are done with the free spheres, we can introduce as well free tori, as follows:

Definition 1.18. *We have free real and complex tori, defined via*

$$C(T_N^+) = C^*\left(x_1, \ldots, x_N \Big| x_i = x_i^*, x_i^2 = \frac{1}{N} \right)$$

$$C(\mathbb{T}_N^+) = C^*\left(x_1, \ldots, x_N \Big| x_i x_i^* = x_i^* x_i = \frac{1}{N} \right)$$

where the symbol C^* stands for universal enveloping C^*-algebra.

The fact that these tori are indeed well-defined comes from the fact that they are noncommutative manifolds, in the sense of Definition 1.15. In fact, we have:

Proposition 1.19. *We have inclusions of algebraic manifolds, as follows:*

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ T_N^+ & \longrightarrow & \mathbb{T}_N^+ \end{array}$$

In addition, we have $T_N^+ = \mathbb{T}_N^+ \cap S^{N-1}_{\mathbb{R},+}$, as submanifolds of $S^{N-1}_{\mathbb{C},+}$.

Proof. All this is clear indeed, by using the equivalence relation in Definition 1.17, in order to get rid of functional analytic issues at the C^*-algebra level. □

In analogy with Theorem 1.14, we have the following result:

Theorem 1.20. *We have inclusions of algebraic manifolds, as follows,*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

and the manifolds on top appear as liberations of those of the bottom.

Proof. This follows exactly as Theorem 1.14, and best here is to invoke Theorem 1.16 above, which is there precisely for dealing with such situations. □

Summarizing, we have free spheres and tori, having quite similar properties.

In order to further advance, we will need the following result:

Theorem 1.21. *Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by the fact that all group elements are unitaries, $g^* = g^{-1}$.*

1. *The maximal C^*-seminorm on $\mathbb{C}[\Gamma]$ is a C^*-norm, and the closure of $\mathbb{C}[\Gamma]$ with respect to this norm is a C^*-algebra, denoted $C^*(\Gamma)$.*
2. *When Γ is abelian, we have an isomorphism $C^*(\Gamma) \simeq C(G)$, where $G = \widehat{\Gamma}$ is its Pontrjagin dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$.*

Proof. All this is very standard, the idea being as follows:

(1) In order to prove the result, we must find a $*$-algebra embedding $\mathbb{C}[\Gamma]\subset B(H)$, with H being a Hilbert space. For this purpose, consider the space $H=l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi:\mathbb{C}[\Gamma]\subset B(H)\quad,\quad \pi(g)(h)=gh$$

Indeed, since $\pi(g)$ maps the basis $\{h\}_{h\in\Gamma}$ into itself, this operator is well-defined, bounded, and is an isometry. It is also clear from the formula $\pi(g)(h)=gh$ that $g\to\pi(g)$ is a morphism of algebras, and since this morphism maps the unitaries $g\in\Gamma$ into isometries, this is a morphism of $*$-algebras. Finally, the faithfulness of π is clear.

(2) Since Γ is abelian, the corresponding group algebra $A=C^*(\Gamma)$ is commutative. Thus, we can apply the Gelfand theorem, and we obtain $A=C(X)$, with $X=Spec(A)$. But the spectrum $X=Spec(A)$, consisting of the characters $\chi:C^*(\Gamma)\to\mathbb{C}$, can be identified with the Pontrjagin dual $G=\widehat{\Gamma}$, and this gives the result. □

The above result suggests the following definition:

Definition 1.22. *Given a discrete group Γ, the noncommutative space G given by*

$$C(G)=C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G=\widehat{\Gamma}$.

This is in fact something which is not fully correct. Indeed, in the context of Theorem 1.21 (1) above, the closure $C^*_{red}(\Gamma)$ of the group algebra $\mathbb{C}[\Gamma]$ in the regular representation is a C^*-algebra as well. We have a quotient map $C^*(\Gamma)\to C^*_{red}(\Gamma)$, and if this map is not an isomorphism, which is something that can happen, we are in trouble.

However, in the case of the finitely generated discrete groups $\Gamma=<g_1,\ldots,g_N>$, which is the one that we are mainly interested in here, the corresponding duals appear as algebraic submanifolds $\widehat{\Gamma}\subset S^{N-1}_{\mathbb{C},+}$, and the notion of equivalence from Definition 1.17 is precisely the one that we need, identifying full and reduced group algebras.

We can now refine our findings about tori, as follows:

Theorem 1.23. *The basic tori are all group duals, as follows,*

$$\begin{array}{ccc}
\widehat{\mathbb{Z}_2^{*N}} & \longrightarrow & \widehat{F_N}\\
\uparrow & & \uparrow\\
\mathbb{Z}_2^N & \longrightarrow & \mathbb{T}^N
\end{array}$$

where F_N is the free group on N generators, and $*$ is a group-theoretical free product.

Proof. The basic tori appear indeed as group duals, follows:

$$\begin{tikzcd}
\widehat{\mathbb{Z}_2^{*N}} \arrow[r] & \widehat{\mathbb{Z}^{*N}} \\
\widehat{\mathbb{Z}_2^N} \arrow[u] \arrow[r] & \widehat{\mathbb{Z}^N} \arrow[u]
\end{tikzcd}$$

Together with the Fourier transform identifications from Theorem 1.21 (2), and with our free group convention $F_N = \mathbb{Z}^{*N}$, this gives the result. □

Summarizing, we have so far a beginning of theory, involving spheres and tori.

2. Quantum groups

We have seen so far that the pairs sphere/torus (S,T) corresponding to the real and complex geometries, of $\mathbb{R}^N,\mathbb{C}^N$, have some natural free analogues. In order to build now a theory, based on this simple fact, we have several natural ideas, as follows:

(1) A first idea would be that of axiomatizing the pairs (S,T), by imposing differential geometry type axioms on S. This is something quite natural, but not obvious.

(2) A second idea, of the same type, would be that of imposing group-theoretic axioms on $\Gamma=\widehat{T}$. Once again, this is something natural, but not exactly obvious.

(3) A third idea, which is perhaps the most straightforward, is that of adding to the picture quantum groups (U,K), as to reach to quadruplets (S,T,U,K).

Making a choice here is a quite delicate task. However, when thinking well, this is in fact a non-issue, because in the end we would like to have all these things understood. In what follows we will use (3), by going along the lines suggested in section 1. Once this done, we will comment on (2), and then we will comment on (1) as well.

So, our objective now will be that of adding a pair of quantum groups (U,K) to the data that we already have, namely the pair formed by sphere and the torus (S,T), as to reach to a quadruplet of objects (S,T,U,K), with relations between them, as follows:

$$\begin{tikzcd} S \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T \arrow[d, leftrightarrow] \\ U \arrow[r, leftrightarrow] \arrow[ur, leftrightarrow] & K \end{tikzcd}$$

For this purpose, we will first recall Woronowicz's compact quantum group formalism from [98], [99]. Then we will construct pairs (O_N^+,H_N^+) and (U_N^+,K_N^+), as to complete the pairs $(S^{N-1}_{\mathbb{R},+},T_N^+)$ and $(S^{N-1}_{\mathbb{C},+},\mathbb{T}_N^+)$ that we already have. And then, once this done, we will talk about general quadruplets (S,T,U,K), and their axiomatization.

In order to discuss the compact quantum groups, let us first look at the classical case. Any compact Lie group is known to appear as a closed subgroup of a unitary group, $G\subset U_N$. In functional analytic terms, this means that the algebra $C(G)$ comes along with N^2 coordinate functions $u_{ij}:g\to g_{ij}$, which must form altogether a unitary matrix $u=(u_{ij})$, satisfying certain conditions, coming from the group structure on G.

In general, we can use the same idea, simply by dropping the assumption that the coordinates u_{ij} commute. The axioms, coming from [98], are as follows:

Definition 2.1. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u\in M_N(A)$ whose coefficients generate A, such that the formulae*

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}\quad,\quad\varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}^*$$

define morphisms of C^*-algebras $\Delta:A\to A\otimes A$, $\varepsilon:A\to\mathbb{C}$, $S:A\to A^{opp}$.

The morphisms Δ,ε,S are called comultiplication, counit and antipode. We will see later on that these morphisms satisfy the usual Hopf algebra axioms.

Observe that these morphisms, if they exist, are unique. This is in analogy with the fact that a closed subset $G\subset U_N$ is either a closed subgroup, or not.

Finally, let us mention that the formalism in [98], [99] is a bit more general, technically allowing deformations with Drinfeld-Jimbo parameter $q\in\mathbb{R}$. In what follows we will only need deformations with $q=\pm1$, and the above formalism is the one that we need.

We say that A is cocommutative when $\Sigma\Delta=\Delta$, where $\Sigma(a\otimes b)=b\otimes a$ is the flip. We have the following result, which justifies the terminology and axioms:

Theorem 2.2. *The following are Woronowicz algebras:*

(1) *$C(G)$, with $G\subset U_N$ compact Lie group. Here the structural maps are:*

$$\begin{aligned}\Delta(\varphi)&=(g,h)\to\varphi(gh)\\ \varepsilon(\varphi)&=\varphi(1)\\ S(\varphi)&=g\to\varphi(g^{-1})\end{aligned}$$

(2) *$C^*(\Gamma)$, with $F_N\to\Gamma$ finitely generated group. Here the structural maps are:*

$$\begin{aligned}\Delta(g)&=g\otimes g\\ \varepsilon(g)&=1\\ S(g)&=g^{-1}\end{aligned}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

Proof. In both cases, we have to exhibit a certain matrix u. For the first assertion, we can use the matrix $u=(u_{ij})$ formed by matrix coordinates of G, given by:

$$g=\begin{pmatrix}u_{11}(g)&\ldots&u_{1N}(g)\\ \vdots&&\vdots\\ u_{N1}(g)&\ldots&u_{NN}(g)\end{pmatrix}$$

For the second assertion, we can use the diagonal matrix formed by generators:

$$u=\begin{pmatrix}g_1&&0\\ &\ddots&\\ 0&&g_N\end{pmatrix}$$

Finally, the last assertion follows from the Gelfand theorem, in the commutative case. In the cocommutative case, this is something more technical, to be explained below. □

In general now, the structural maps Δ, ε, S have the following properties:

Proposition 2.3. *Let (A, u) be a Woronowicz algebra.*

(1) *Δ, ε satisfy the usual axioms for a comultiplication and a counit, namely:*

$$\begin{aligned}(\Delta \otimes id)\Delta &= (id \otimes \Delta)\Delta \\ (\varepsilon \otimes id)\Delta &= (id \otimes \varepsilon)\Delta = id\end{aligned}$$

(2) *S satisfies the antipode axiom, on the $*$-subalgebra generated by entries of u:*

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(.)1$$

(3) *In addition, the square of the antipode is the identity, $S^2 = id$.*

Proof. The two comultiplication axioms follow from:

$$\begin{aligned}(\Delta \otimes id)\Delta(u_{ij}) &= (id \otimes \Delta)\Delta(u_{ij}) = \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj} \\ (\varepsilon \otimes id)\Delta(u_{ij}) &= (id \otimes \varepsilon)\Delta(u_{ij}) = u_{ij}\end{aligned}$$

As for the antipode formulae, the verification here is similar. □

Summarizing, the Woronowicz algebras appear to have very nice properties. In view of Theorem 2.2, we can formulate the following definition:

Definition 2.4. *Given a Woronowicz algebra A, we formally write*

$$A = C(G) = C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

When A is both commutative and cocommutative, G is a compact abelian group, Γ is a discrete abelian group, and these groups are dual to each other, $G = \widehat{\Gamma}, \Gamma = \widehat{G}$. In general, we still agree to write $G = \widehat{\Gamma}, \Gamma = \widehat{G}$, but in a formal sense.

With this picture in mind, let us call now corepresentation of A any unitary matrix $v \in M_n(A)$ satisfying the same conditions are those satisfied by u, namely:

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

These corepresentations can be thought of as corresponding to the unitary representations of the underlying compact quantum group G. As main examples, we have $u = (u_{ij})$ itself, its conjugate $\bar{u} = (u_{ij}^*)$, as well as any tensor product between $u, \bar{u}$.

We have the following key result, due to Woronowicz [98]:

Theorem 2.5. *Any Woronowicz algebra $A=C(G)$ has a Haar integration functional,*

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

which can be constructed by starting with any faithful positive form $\varphi\in A^*$, and setting

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation $v\in M_n(\mathbb C)\otimes A$ we have

$$\left(id\otimes\int_G\right)v=P$$

where P is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb C^n|v\xi=\xi\}$.

Proof. Following [98], this can be done in 3 steps, as follows:

(1) Given $\varphi\in A^*$, our claim is that the following limit converges, for any $a\in A$:

$$\int_\varphi a=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)$$

Indeed, we can assume, by linearity, that a is the coefficient of a corepresentation:

$$a=(\tau\otimes id)v$$

But in this case, an elementary computation shows that we have the following formula, where P_φ is the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi)v$:

$$\left(id\otimes\int_\varphi\right)v=P_\varphi$$

(2) Since $v\xi=\xi$ implies $[(id\otimes\varphi)v]\xi=\xi$, we have $P_\varphi\geq P$, where P is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb C^n|v\xi=\xi\}$. The point now is that when $\varphi\in A^*$ is faithful, by using a positivity trick, one can prove that we have $P_\varphi=P$. Thus our linear form $\int_\varphi$ is independent of φ, and is given on the coefficients $a=(\tau\otimes id)v$ by:

$$\left(id\otimes\int_\varphi\right)v=P$$

(3) With the above formula in hand, the left and right invariance of $\int_G=\int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [98]. □

The above result is something quite fundamental, and as a main application, one can develop in this setting an analogue of the Peter-Weyl theory [95]. Consider indeed the dense $*$-subalgebra $\mathcal A\subset A$ generated by the coefficients of the fundamental corepresentation u, and endow it with the scalar product $<a,b>=\int_G ab^*$. We have then:

Theorem 2.6. *We have the following Peter-Weyl type results:*

(1) *Any corepresentation decomposes as a sum of irreducible corepresentations.*
(2) *Each irreducible corepresentation appears inside a certain $u^{\otimes k}$.*
(3) $\mathcal{A} = \bigoplus_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$, *the summands being pairwise orthogonal.*
(4) *The characters of irreducible corepresentations form an orthonormal system.*

Proof. All these results are from [98], the idea being as follows:

(1) Given $v \in M_n(A)$, its interwiner algebra $End(v) = \{T \in M_n(\mathbb{C}) | Tv = vT\}$ is a finite dimensional C^*-algebra, and so decomposes as $End(v) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C})$. But this gives a decomposition of type $v = v_1 + \ldots + v_r$, as desired.

(2) Consider indeed the Peter-Weyl corepresentations, $u^{\otimes k}$ with k colored integer, defined by $u^{\otimes \emptyset} = 1$, $u^{\otimes \circ} = u$, $u^{\otimes \bullet} = \bar{u}$ and multiplicativity. The coefficients of these corepresentations span the dense algebra $\mathcal{A}$, and by using (1), this gives the result.

(3) Here the direct sum decomposition, which is technically a $*$-coalgebra isomorphism, follows from (2). As for the second assertion, this follows from the fact that $(id \otimes \int_G)v$ is the orthogonal projection P_v onto the space $Fix(v)$, for any corepresentation v.

(4) Let us define indeed the character of $v \in M_n(A)$ to be the matrix trace, $\chi_v = Tr(v)$. Since this character is a coefficient of v, the orthogonality assertion follows from (3). As for the norm 1 claim, this follows once again from $(id \otimes \int_G)v = P_v$. $\square$

We refer to [98] for full details on all the above, and for some applications as well. Let us just record here the fact that in the cocommutative case, we obtain from (4) that the irreducible corepresentations must be all 1-dimensional, and so that we must have $A = C^*(\Gamma)$ for some discrete group Γ, as mentioned in Theorem 2.2 above.

At a more technical level now, we have the following result:

Theorem 2.7. *Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

(1) *The Haar functional of A_{full} is faithful.*
(2) *The projection map $A_{full} \to A_{red}$ is an isomorphism.*
(3) *The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red}.*
(4) *We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

Proof. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial, and conversely, a counit map $\varepsilon : A_{red} \to \mathbb{C}$ produces an isomorphism $A_{red} \to A_{full}$, via a formula of type $(\varepsilon \otimes id)\Phi$. See [82].

(3) $\iff$ (4) Here $\implies$ is clear, coming from $\varepsilon(N - Re(\chi(u))) = 0$, and the converse can be proved by doing some functional analysis. Once again, we refer here to [82]. $\square$

Yet another technical result is Tannakian duality, which is as follows:

Theorem 2.8. *The following operations are inverse to each other:*

(1) *The construction $A \to C$, which associates to any Woronowicz algebra A the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
(2) *The construction $C \to A$, which associates to any tensor category C the Woronowicz algebra A presented by the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

Proof. This is something quite deep, going back to [99] in a slightly different form, and to [75] in the simplified form presented above. The idea is as follows:

(1) We have indeed a construction $A \to C$ as above, whose output is a tensor C^*-subcategory with duals of the tensor C^*-category of Hilbert spaces.

(2) We have as well a construction $C \to A$ as above, simply by dividing the free $*$-algebra on N^2 variables by the relations in the statement.

Regarding now the bijection claim, some elementary algebra shows that $C = C_{A_C}$ implies $A = A_{C_A}$, and also that $C \subset C_{A_C}$ is automatic. Thus we are left with proving $C_{A_C} \subset C$. But this latter inclusion can be proved indeed, by doing a lot of algebra, and using von Neumann's bicommutant theorem, in finite dimensions. See [75]. □

As a concrete consequence of the above result, we have:

Theorem 2.9. *We have an embedding as follows, using double indices,*

$$G \subset S^{N^2-1}_{\mathbb{C},+} \quad , \quad x_{ij} = \frac{u_{ij}}{\sqrt{N}}$$

making G an algebraic submanifold of the free sphere.

Proof. The fact that we have an embedding as above follows from the fact that the matrix $u = (u_{ij})$ and its complex conjugate $\bar{u} = (u_{ij}^*)$ are both unitaries.

Regarding now the algebricity claim, which is something non-trivial, this follows from Theorem 2.8. Indeed, assuming that $A = C(G)$ is of the form $A = A_C$, it follows that G is algebraic. But this is always the case, because we can take $C = C_A$. □

The above result is quite interesting for us, because it makes the compact quantum groups fit into our general algebraic manifold formalism. In particular, our usual equivalence relation for manifolds becomes in this setting $G \sim G'$ when we have a $*$-algebra isomorphism $\mathcal{A} \simeq \mathcal{A}'$, mapping standard coordinates to standard coordinates.

Thus, the amenability issues coming from Theorem 2.7 are not a problem, and our standard notation $A = C(G) = C^*(\Gamma)$ from Definition 2.4 perfectly makes sense.

With these preliminaries done, let us get back now to our original objective, namely constructing pairs (O_N^+, H_N^+) and (U_N^+, K_N^+), as to complete the pairs $(S^{N-1}_{\mathbb{R},+}, T_N^+)$ and $(S^{N-1}_{\mathbb{C},+}, \mathbb{T}_N^+)$ that we have. In the continuous case, the construction is as follows:

Proposition 2.10. *The following constructions produce compact quantum groups,*

$$\begin{aligned} C(O_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},u^t=u^{-1}\right) \\ C(U_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right) \end{aligned}$$

which appear respectively as liberations of the groups O_N and U_N.

Proof. This first assertion follows from the elementary fact that if a matrix $u=(u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj} \quad , \quad u_{ij}^\varepsilon=\delta_{ij} \quad , \quad u_{ij}^S=u_{ji}^*$$

As for the second assertion, this follows by applying Theorem 1.16. See [92]. □

As a first result regarding these quantum groups, we have:

Theorem 2.11. *We have embeddings of algebraic manifolds as follows, obtained in double indices by rescaling the coordinates, $x_{ij}=u_{ij}/\sqrt{N}$:*

$$\begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix} \quad \to \quad \begin{matrix} S^{N^2-1}_{\mathbb{R},+} & \longrightarrow & S^{N^2-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N^2-1}_{\mathbb{R}} & \longrightarrow & S^{N^2-1}_{\mathbb{C}} \end{matrix}$$

Moreover, the quantum groups on the left appear from the noncommutative spheres on the right by intersecting them with U_N^+, inside $S^{N^2-1}_{\mathbb{C},+}$.

Proof. As explained in Theorem 2.9 above, the biunitarity of $u=(u_{ij})$ gives an embedding $U_N^+\subset S^{N^2-1}_{\mathbb{C},+}$ as in the statement. Now since the relations defining $O_N,O_N^+,U_N\subset U_N^+$ are the same as those defining $S^{N^2-1}_{\mathbb{R}},S^{N^2-1}_{\mathbb{R},+},S^{N^2-1}_{\mathbb{C}}\subset S^{N^2-1}_{\mathbb{C},+}$, this gives the result. □

As a comment here, the above result seems be related to our (S,T,U,K) philosophy, but it is not. To be more precise, while the last assertion provides us of course with a correspondence $S\to U$, this is not the "correct" correspondence. The correct correspondence involves quantum isometries of the spheres, and we will discuss this, later on.

In the discrete case now, the construction is more tricky, involving quantum permutation groups. Let us recall indeed that in the classical case we have $H_N=\mathbb{Z}_2\wr S_N$ and $K_N=\mathbb{T}\wr S_N$. In the free case, the idea will be that of performing constructions of type $H_N^+=\mathbb{Z}_2\wr_* S_N^+$ and $K_N^+=\mathbb{T}\wr_* S_N^+$, and so we must talk first about S_N^+.

The quantum permutation groups are introduced as follows:

Theorem 2.12. *The following construction, where "magic" means formed of projections, which sum up to* 1 *on each row and column,*

$$C(S_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \text{magic}\right)$$

produces a quantum group liberation of S_N. Moreover, the inclusion $S_N \subset S_N^+$ is an isomorphism at $N \leq 3$, but not at $N \geq 4$, where S_N^+ is not classical, nor finite.

Proof. The quantum group assertion follows as in the proof of Proposition 2.10, because if u is magic, then so are the matrices $u^\Delta, u^\varepsilon, u^S$. Also, we have an embedding $S_N \subset S_N^+$, obtained by using the standard coordinates of S_N, viewed as an algebraic group:

$$u_{ij} = \chi\left(\sigma \in S_N\Big|\sigma(j) = i\right)$$

By using Theorem 1.16 above, $S_N \subset S_N^+$ is indeed a liberation. Finally, regarding the last assertion, this follows from the existence or non-existence of $N \times N$ magic matrices with noncommuting entries, depending on $N \in \mathbb{N}$, and we refer here to [93]. □

With the above result in hand, we can now introduce the quantum reflections:

Proposition 2.13. *The following constructions produce compact quantum groups,*

$$\begin{aligned} C(H_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u_{ij} = u_{ij}^*,\ (u_{ij}^2) = \text{magic}\right) \\ C(K_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|[u_{ij}, u_{ij}^*] = 0,\ (u_{ij}u_{ij}^*) = \text{magic}\right) \end{aligned}$$

which appear respectively as liberations of the reflection groups H_N and K_N.

Proof. This can be proved in the usual way, with the first assertion coming from the fact that if u satisfies the relations in the statement, then so do the matrices $u^\Delta, u^\varepsilon, u^S$, and with the second assertion coming from Theorem 1.16. See [11], [16]. □

Summarizing, we are done with our construction task. Let us record as well the following result, which refines the various liberation statements formulated above:

Theorem 2.14. *The quantum unitary and reflection groups are as follows,*

$$\begin{tikzcd}
& K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
& K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[ur] \arrow[uu] & & O_N \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

and in this diagram, any face $P \subset Q, R \subset S$ has the property $P = Q \cap R$.

Proof. The fact that we have inclusions as in the statement follows from the definition of the various quantum groups involved. As for the various intersection claims, these follow as well from definitions. For some further details on all this, we refer to [10]. □

We recall that our goal is that of having a full set of correspondences between our objects (S, T, U, K). In particular, we must understand the correspondences between (T, U, K), and for this purpose, we are in need of some further quantum group theory.

Let us first discuss the correspondences $U \to K \to T$, which are both, or rather all three, elementary. Regarding $U \to K$, the statement here is as follows:

Proposition 2.15. *Given a closed subgroup $G \subset U_N^+$, define its "reflection subgroup" to be the compact quantum group K given by*

$$K = G \cap K_N^+$$

with the intersection being taken inside U_N^+. Then, the reflection subgroups of the quantum groups O_N, U_N, O_N^+, U_N^+ are the quantum groups H_N, K_N, H_N^+, K_N^+.

Proof. This follows from Theorem 2.14, because the left face of the cube diagram there appears by intersecting the right face with the quantum group K_N^+. □

In order to construct now the correspondences $U \to T$ and $K \to T$, which are elementary as well, we can use the following general notion, from [30]:

Theorem 2.16. *Given a closed subgroup $G \subset U_N^+$, consider its "diagonal torus", which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda = < g_1, \ldots, g_N >$ is the discrete group generated by the elements $g_i = u_{ii}$, which are unitaries inside $C(T)$.

Proof. Since u is unitary, its diagonal entries $g_i = u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^*-completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. See [30]. □

In connection now with our questions, we have the following result:

Proposition 2.17. *The diagonal tori of the basic quantum unitary groups are*

$$\begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix} \quad \to \quad \begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix}$$

and the diagonal tori of the corresponding reflection subgroups are the same.

Proof. As a first observation, for $G = U_N^+$ the diagonal torus is $T = \mathbb{T}_N^+$. In fact, with the convention $\mathbb{T}_N^+ \subset U_N^+$, coming from this, the construction of the diagonal torus can be reformulated as follows, with the intersection being computed inside U_N^+:

$$T = G \cap \mathbb{T}_N^+$$

With this picture in mind, it is clear that for the 4 basic triples (T, U, K), constructed above, the torus T appears as diagonal torus of both U, and K. $\square$

The problem that we would like to solve now, which is also purely quantum group theoretical as well, is that of understanding the correspondences $T \to K \to U$. This is something quite subtle, which will take us into advanced quantum group theory.

To be more precise, we will need some Tannakian duality results, in the spirit of the Brauer theorem [40]. Let us start with the following key definition:

Definition 2.18. *Associated to any partition $\pi \in P(k, l)$ between an upper row of k points and a lower row of l points is the linear map $T_\pi : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$ given by*

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with the Kronecker type symbols $\delta_\pi \in \{0, 1\}$ depending on whether the indices fit or not.

To be more precise, in this definition, we agree to put the two multi-indices on the two rows of points, in the obvious way. The Kronecker symbols are then defined by $\delta_\pi = 1$ when all the strings of π join equal indices, and by $\delta_\pi = 0$ otherwise.

The relation with the Tannakian categories comes from:

Proposition 2.19. *The assignement $\pi \to T_\pi$ is categorical, in the sense that we have*

$$T_\pi \otimes T_\sigma = T_{[\pi\sigma]} \quad , \quad T_\pi T_\sigma = N^{c(\pi,\sigma)} T_{[^\sigma_\pi]} \quad , \quad T_\pi^* = T_{\pi^*}$$

where $c(\pi, \sigma)$ are certain integers, coming from the erased components in the middle.

Proof. The concatenation axiom follows from the following computation:

$$\begin{aligned}
& (T_\pi \otimes T_\sigma)(e_{i_1} \otimes \ldots \otimes e_{i_p} \otimes e_{k_1} \otimes \ldots \otimes e_{k_r}) \\
= & \sum_{j_1 \ldots j_q} \sum_{l_1 \ldots l_s} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_p \\ j_1 & \ldots & j_q \end{pmatrix} \delta_\sigma \begin{pmatrix} k_1 & \ldots & k_r \\ l_1 & \ldots & l_s \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_q} \otimes e_{l_1} \otimes \ldots \otimes e_{l_s} \\
= & \sum_{j_1 \ldots j_q} \sum_{l_1 \ldots l_s} \delta_{[\pi\sigma]} \begin{pmatrix} i_1 & \ldots & i_p & k_1 & \ldots & k_r \\ j_1 & \ldots & j_q & l_1 & \ldots & l_s \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_q} \otimes e_{l_1} \otimes \ldots \otimes e_{l_s} \\
= & T_{[\pi\sigma]}(e_{i_1} \otimes \ldots \otimes e_{i_p} \otimes e_{k_1} \otimes \ldots \otimes e_{k_r})
\end{aligned}$$

The composition axiom follows from the following computation:

$$\begin{aligned}
& T_\pi T_\sigma (e_{i_1} \otimes \ldots \otimes e_{i_p}) \\
= & \sum_{j_1 \ldots j_q} \delta_\sigma \begin{pmatrix} i_1 & \ldots & i_p \\ j_1 & \ldots & j_q \end{pmatrix} \sum_{k_1 \ldots k_r} \delta_\pi \begin{pmatrix} j_1 & \ldots & j_q \\ k_1 & \ldots & k_r \end{pmatrix} e_{k_1} \otimes \ldots \otimes e_{k_r} \\
= & \sum_{k_1 \ldots k_r} N^{c(\pi,\sigma)} \delta_{[^\sigma_\pi]} \begin{pmatrix} i_1 & \ldots & i_p \\ k_1 & \ldots & k_r \end{pmatrix} e_{k_1} \otimes \ldots \otimes e_{k_r} \\
= & N^{c(\pi,\sigma)} T_{[^\sigma_\pi]}(e_{i_1} \otimes \ldots \otimes e_{i_p})
\end{aligned}$$

Finally, the involution axiom follows from the following computation:

$$\begin{aligned}
& T_\pi^*(e_{j_1} \otimes \ldots \otimes e_{j_q}) \\
= & \sum_{i_1 \ldots i_p} < T_\pi^*(e_{j_1} \otimes \ldots \otimes e_{j_q}), e_{i_1} \otimes \ldots \otimes e_{i_p} > e_{i_1} \otimes \ldots \otimes e_{i_p} \\
= & \sum_{i_1 \ldots i_p} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_p \\ j_1 & \ldots & j_q \end{pmatrix} e_{i_1} \otimes \ldots \otimes e_{i_p} \\
= & T_{\pi^*}(e_{j_1} \otimes \ldots \otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. See [29]. □

In order to interpret this, and finish our discussion, let us make the convention that k, l will be from now on colored integers. We have the following notion, from [29], [85]:

Definition 2.20. *A collection of sets $D = \bigsqcup_{k,l} D(k,l)$ with $D(k,l) \subset P(k,l)$ is called a category of partitions when it has the following properties:*

1. *Stability under the horizontal concatenation, $(\pi, \sigma) \to [\pi\sigma]$.*
2. *Stability under vertical concatenation $(\pi, \sigma) \to [^\sigma_\pi]$, with matching middle symbols.*
3. *Stability under the upside-down turning $*$, with switching of colors, $\circ \leftrightarrow \bullet$.*
4. *Each set $P(k,k)$ contains the identity partition $|| \ldots ||$.*
5. *The sets $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ both contain the semicircle $\cap$.*

As a basic example, P itself is a category of partitions. The set of pairings $P_2 \subset P$ is a category of partitions as well. The same goes for the subset $\mathcal{P}_2(k,l) \subset P_2(k,l)$ of "matching" pairings, whose horizontal strings connect $\circ - \circ$ or $\bullet - \bullet$, and whose vertical strings connect $\circ - \bullet$. There are many other examples, and we will discuss this later.

We can now formulate a key result, from [29], as follows:

Theorem 2.21. *Each category of partitions $D = (D(k,l))$ produces a family of compact quantum groups $G = (G_N)$, one for each $N \in \mathbb{N}$, via the formula*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

which produces a Tannakian category, and therefore a closed subgroup $G_N \subset U_N^+$. The quantum groups which appear in this way are called "easy".

Proof. This follows indeed from Woronowicz's Tannakian duality, in its "soft" form from [75], as explained in Theorem 2.8 above. Indeed, let us set:

$$C(k,l) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

By using the axioms in Definition 2.20, and the categorical properties of the operation $\pi \to T_\pi$, from Proposition 2.19, we deduce that $C = (C(k,l))$ is a Tannakian category. Thus the Tannakian duality result applies, and gives the result. $\square$

The easy quantum groups are quite interesting objects. Indeed, the Brauer theorem [40] states that O_N, U_N appear in this way, from the categories $P_2, \mathcal{P}_2$. According to [19], the free versions O_N^+, U_N^+ appear as well in this way, from the categories $NC_2, \mathcal{NC}_2$ obtained by restricting the attention to the noncrossing partitions. In fact, we have:

Theorem 2.22. *The basic quantum unitary and quantum reflection groups, namely*

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[uu] \arrow[ur] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

are all easy. The corresponding categories of partitions form an intersection diagram.

Proof. This is well-known, the corresponding categories being as follows, with P_{even} being the category of partitions having even blocks, and with $\mathcal{P}_{even}(k,l) \subset P_{even}(k,l)$ consisting of the partitions satisfying $\#\circ = \#\bullet$ in each block, when flattening the partition:

$$\begin{tikzcd}[row sep=small, column sep=small]
 & \mathcal{NC}_{even} \arrow[dl] \arrow[dd] & & \mathcal{NC}_2 \arrow[ll] \arrow[dl] \arrow[dd] \\
NC_{even} \arrow[dd] & & NC_2 \arrow[ll] \arrow[dd] & \\
 & \mathcal{P}_{even} \arrow[dl] & & \mathcal{P}_2 \arrow[ll] \arrow[dl] \\
P_{even} & & P_2 \arrow[ll] &
\end{tikzcd}$$

As for the second assertion, which will be of use later on, this is something well-known and standard too. We refer here to [11], [16], [19], and to [10], [29] as well. $\square$

Getting back now to our axiomatization question, namely constructing the correspondences $T \to K \to U$, let us start our discussion here with the following definition:

Definition 2.23. *Consider a closed subgroup $G \subset U_N^+$, and let $T \subset K \subset G$ be its diagonal torus, and its reflection subgroup. The inclusion $G_{class} \subset G$ is called:*

(1) *A soft liberation, when $G = < G_{class}, K >$.*
(2) *A hard liberation, when $G = < G_{class}, T >$.*

Here the diagonal torus is obtained via the usual formula $T = G \cap \mathbb{T}_N^+$, and the reflection subgroup is obtained as in Proposition 2.15, via the formula $K = G \cap K_N^+$.

We should mention that the terminology in the above definition comes from the hardest ever combinatorial problem in the history of mankind, namely the Brexit one. This has dominated the headlines during the preparation of the present book, in 2019. In the hope that, at the time of reading this book, this has not led to a nuclear winter, or so.

As a first remark, given $G \subset U_N^+$, we have a diagram as follows, which is an intersection diagram, in the sense that any subsquare $P \subset Q, R \subset S$ satisfies $P = Q \cap R$:

$$\begin{tikzcd}
T \arrow[r] & K \arrow[r] & G \\
T_{class} \arrow[r] \arrow[u] & K_{class} \arrow[r] \arrow[u] & G_{class} \arrow[u]
\end{tikzcd}$$

With this picture in mind, the soft liberation condition states that the square on the right $P \subset Q, R \subset S$ is a generation diagram, $< Q, R >= S$. As for the hard liberation condition, which is stronger, this states that the whole rectangle has this property.

We will need the following key result, coming from [39], [44], [46]:

Theorem 2.24. *The following happen:*

(1) O_N^+,U_N^+ *appear as soft liberations of* O_N,U_N.
(2) O_N^+,U_N^+ *appear as well as hard liberations of* O_N,U_N.
(3) H_N^+,K_N^+ *appear as soft liberations of* H_N,K_N.
(4) H_N^+,K_N^+ *do not appear as hard liberations of* H_N,K_N.

Proof. This result, while being fundamental for us, is something quite technical. In the lack of a simple proof for all this, here is the idea:

(1) This simply follows from (2) below. Normally there should be a simpler proof for this, by using Tannakian duality, but this is something which is not known yet.

(2) A key result from [44], [46], whose proof is quite technical, not to be explained here, states that we have the following generation formula, valid at any $N\geq 3$:

$$O_N^+=<O_N,O_{N-1}^+>$$

With this in hand, the hard liberation formula $O_N^+=<O_N,T_N^+>$ can be proved by recurrence on N. Indeed, at $N=1$ there is nothing to prove, at $N=2$ this is something well-known, and elementary, as explained for instance in [44], [46], and in general, the recurrence step $N-1\to N$ can be established as follows:

$$\begin{aligned}
O_N^+&=<O_N,O_{N-1}^+>\\
&=<O_N,O_{N-1},T_{N-1}^+>\\
&=<O_N,T_{N-1}^+>\\
&=<O_N,T_N,T_{N-1}^+>\\
&=<O_N,T_N^+>
\end{aligned}$$

Regarding now the hard liberation formula $U_N^+=<U_N,\mathbb{T}_N^+>$, this basically follows from $O_N^+=<O_N,T_N^+>$. Indeed, as explained in [44], [46], the standard isomorphism $PO_N^+=PU_N^+$ shows that we have $U_N^+=<U_N,O_N^+>$, and by using this, we obtain:

$$\begin{aligned}
U_N^+&=<U_N,O_N^+>\\
&=<U_N,O_N,T_N^+>\\
&=<U_N,T_N^+>\\
&=<U_N,\mathbb{T}_N^+>
\end{aligned}$$

All this is of course quite non-trivial, using many technical ingredients. We believe that there should be a simpler proof for this, by using Tannakian duality, but this is something which is not known yet. For the details on all this material, see [44], [46].

(3) This is something trivial, because H_N^+,K_N^+ equal their reflection subgroups.

(4) This result, which is something quite surprising, is well-known, coming from the fact that the quantum group $H_N^{[\infty]}\subset H_N^+$ constructed in [84], and its unitary counterpart

$K_N^{[\infty]} \subset K_N^+$, have the same diagonal subgroups as H_N^+, K_N^+. Thus, the hard liberation procedure "stops" at $H_N^{[\infty]}, K_N^{[\infty]}$, and cannot reach H_N^+, K_N^+. □

All the above is quite subtle and interesting, and is waiting for more study. Without getting into details here, let us just mention that $G =< H, K >$ at the quantum group level corresponds to $C_G = C_H \cap C_K$ at the Tannakian level, so all the above results ultimately correspond to doing some combinatorics. This remains to be understood.

Getting back now to our questions, we have the following result:

Theorem 2.25. *For our basic triples (T, U, K), the correspondences $T \to K \to U$ are given by the "soft" and "hard" generation formulae*

$$U =< O_N, K >$$

$$U =< O_N, T >$$

and by the related generation/intersection formula $K =< O_N, T > \cap K_N^+$.

Proof. The above conditions are very close to the soft and hard liberation conditions, and we can check them with the technology that we have. To be more precise, observe first that $U =< O_N, T >$ implies $U =< O_N, K >$. Thus, it is enough to check that we have $U =< O_N, T >$ for the 4 basic quadruplets, and the situation here is as follows:

(1) In the classical real case the condition is $O_N =< O_N, T_N >$, clear.

(2) In the classical complex case the condition is $U_N =< O_N, \mathbb{T}_N >$. But this is soemthing well-known, coming for instance from the fact that the inclusion of compact Lie groups $\mathbb{T}O_N \subset U_N$ is maximal. For more details on this, we refer to [17].

(3) In the free real case the condition is $O_N^+ =< O_N, T_N^+ >$. But this is exactly the hard liberation property of $O_N \subset O_N^+$, coming from [44], [46], as explained above.

(4) In the free complex case the condition is $U_N^+ =< O_N, \mathbb{T}_N^+ >$. But this comes from the hard liberation formula $U_N^+ =< U_N, T_N^+ >$, as follows:

$$\begin{aligned} U_N^+ &= < U_N, \mathbb{T}_N^+ > \\ &= < O_N, \mathbb{T}_N, \mathbb{T}_N^+ > \\ &= < O_N, \mathbb{T}_N^+ > \end{aligned}$$

Summarizing, the new correspondences are indeed the correct ones, as stated. □

Before going further, let us make some comments on all this. First, in view of Theorem 2.25, a natural idea would be that of constructing the correspondence $T \to K$ by a hard generation type formula as well, namely $K =< H_N, T >$. However, this does not work, the formula being wrong in the free case, due to the negative result from Theorem 2.24 (4), and more specifically to the quantum groups $H_N^{[\infty]}, K_N^{[\infty]}$ used there, in the proof.

As a second comment, all the above is quite interesting in connection with the cube formed by the basic quantum unitary and reflection groups. Let us recall indeed from Theorem 2.14 that these quantum groups form an intersection diagram, as follows:

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[ur] \arrow[uu] & & O_N \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

It is conjectured that this diagram should be a generation diagram too, and the above results prove this conjecture for 5 of the faces. For the remaining face, the one on the left, the generation formula is $K_N^+ =< K_N, H_N^+ >$, and this is not known yet.

3. Axiomatization

We finish here our axiomatization work. We recall that our goal is that of axiomatizing the quadruplets (S,T,U,K) consisting of a noncommutative sphere, torus, unitary group and reflection group, with a full set of correspondences between them, as follows:

$$\begin{tikzcd} S \arrow[rr, leftrightarrow] \arrow[dd, leftrightarrow] \arrow[ddrr, leftrightarrow] & & T \arrow[dd, leftrightarrow] \arrow[ddll, leftrightarrow] \\ & & \\ U \arrow[rr, leftrightarrow] & & K \end{tikzcd}$$

In order to discuss all this, we first need precise definitions for all the objects involved. So, let us start with the following general definition:

Definition 3.1. *We call noncommutative sphere, noncommutative torus, unitary quantum group and quantum reflection group the intermediate objects as follows,*

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C},+}$$

$$T_N \subset T \subset \mathbb{T}_N^+$$

$$O_N \subset U \subset U_N^+$$

$$H_N \subset K \subset K_N^+$$

with S being an algebraic manifold, and T,U,K being compact quantum groups.

Here, as usual, all objects are taken up to the standard equivalence relation for the noncommutative algebraic manifolds, coming from Definition 1.17 and Theorem 2.9.

Observe that this type of definition brings us into the "hybrid" zone, between real and complex. There are several good reasons for doing so, for instance because we would like to deal at the same time with the real and complex cases. Also, at a more advanced level, we will see later on that we have an isomorphism as follows:

$$P^{N-1}_{\mathbb{R},+} = P^{N-1}_{\mathbb{C},+}$$

This isomorphism is quite important, philosophically speaking, the conclusion being that in the free setting, the usual $\mathbb{R}/\mathbb{C}$ dichotomy tends to become "blurred". Thus, it is a good idea to forget about this dichotomy, and formulate things as above.

At the level of the basic examples, the situation is as follows:

Proposition 3.2. *We have "basic" quadruplets (S,T,U,K) as follows,*

$$\begin{tikzcd} S^{N-1}_{\mathbb{R},+} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & T_N^+ \arrow[d, dash] \arrow[dl, dash] \\ O_N^+ \arrow[r, dash] & H_N^+ \end{tikzcd} \qquad \begin{tikzcd} S^{N-1}_{\mathbb{C},+} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbb{T}_N^+ \arrow[d, dash] \arrow[dl, dash] \\ U_N^+ \arrow[r, dash] & K_N^+ \end{tikzcd}$$

called free real and free complex, as well as

$$\begin{tikzcd} S^{N-1}_{\mathbb{R}} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & T_N \arrow[d, dash] \arrow[dl, dash] \\ O_N \arrow[r, dash] & H_N \end{tikzcd} \qquad \begin{tikzcd} S^{N-1}_{\mathbb{C}} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbb{T}_N \arrow[d, dash] \arrow[dl, dash] \\ U_N \arrow[r, dash] & K_N \end{tikzcd}$$

called classical real and classical complex.

Proof. This is more or less an empty statement, with the various objects appearing in the above diagrams being those constructed in sections 1 and 2 above. □

Regarding now the correspondences between our objects (S,T,U,K), we would like to have all 12 of them axiomatized. There is still quite some work to be done here, and in order to get started, let us begin with a summary of what we have so far:

Theorem 3.3. *For the basic quadruplets, we have correspondences as follows,*

$$\begin{tikzcd} S \arrow[r] & T \arrow[dl, leftrightarrow] \arrow[d, leftrightarrow] \\ U \arrow[r, leftrightarrow] & K \end{tikzcd}$$

constructed via the following intersection and generation formulae:

(1) $T = S \cap \mathbb{T}_N^+ = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+$.
(2) $U = < O_N, T > = < O_N, K >$.
(3) $K = U \cap K_N^+ = < O_N, T > \cap K_N^+$.

Proof. This is a summary of what we already have, with the fact that the 4 correspondences in the statement work well for the 4 basic quadruplets, from Proposition 3.2, coming from the various results established in sections 1 and 2 above. □

Regarding now the missing correspondences, we are especially in need of a double arrow at left, connecting $S \leftrightarrow U$. Let us first discuss the construction $S \to U$.

We need to talk here about classical and quantum isometries. Let us start with:

Proposition 3.4. *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C}}$, the formula*

$$G(X) = \left\{ U \in U_N \Big| U(X) = X \right\}$$

defines a compact group of unitary matrices, or isometries, called affine isometry group of X. For the spheres $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$ we obtain in this way the groups O_N, U_N.

Proof. The fact that $G(X)$ as defined above is indeed a group is clear, its compactness is clear as well, and finally the last assertion is clear as well. In fact, all this works for any closed subset $X \subset \mathbb{C}^N$, but we are not interested here in such general spaces. □

In the case of the spheres, $G(X)$ leaves invariant as well the Riemannian metric, simply because this metric is equivalent to the one inherited from $\mathbb{C}^N$, which is preserved by our isometries $U \in U_N$. Thus, we could have constructed as well $G(X)$ as being the group of metric isometries of X, with of course some extra care in relation with the complex structure, as for $X = S^{N-1}_{\mathbb{C}}$ to obtain $G(X) = U_N$ instead of $G(X) = O_{2N}$. However, in the noncommutative setting, all this becomes considerably more complicated.

We have the following quantum analogue of the above construction:

Theorem 3.5. *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, the category of the closed subgroups $G \subset U_N^+$ acting affinely on X, in the sense that the formula*

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

defines a morphism of C^*-algebras $\Phi : C(X) \to C(X) \otimes C(G)$, has a universal object, denoted $G^+(X)$, and called affine quantum isometry group of X.

Proof. Observe first that in the case where the above morphism Φ exists, this morphism is automatically a coaction, in the sense that it satisfies the following conditions:

$$(\Phi \otimes id)\Phi = (id \otimes \Delta)\Phi \quad , \quad (id \otimes \varepsilon)\Phi = id$$

In order to prove now the result, assume that $X \subset S^{N-1}_{\mathbb{C},+}$ comes as follows:

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_\alpha(x_1, \ldots, x_N) = 0 \Big\rangle$$

Our claim is that the universal quantum group $G = G^+(X)$ in the statement appears as follows, where $X_i = \sum_j x_j \otimes u_{ji} \in C(X) \otimes C(U_N^+)$:

$$C(G) = C(U_N^+) \Big/ \Big\langle f_\alpha(X_1, \ldots, X_N) = 0 \Big\rangle$$

In order to prove this claim, we have to clarify how the relations $f_\alpha(X_1,\ldots,X_N)=0$ are interpreted inside $C(U_N^+)$, and then show that G is indeed a quantum group.

So, pick one of the defining polynomials, $f=f_\alpha$, and write it as follows:

$$f(x_1,\ldots,x_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\cdot x_{i_1^r}\ldots x_{i_{s_r}^r}$$

With $X_i=\sum_j x_j\otimes u_{ji}$ as above, we have the following formula:

$$f(X_1,\ldots,X_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\sum_{j_1^r\ldots j_{s_r}^r}x_{j_1^r}\ldots x_{j_{s_r}^r}\otimes u_{j_1^r i_1^r}\ldots u_{j_{s_r}^r i_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f(X_1,\ldots,X_N)=0$ correspond to certain relations between the variables u_{ij}. Thus, we have indeed a subspace $G\subset U_N^+$, with a universal map $\Phi:C(X)\to C(X)\otimes C(G)$.

In order to show now that G is a quantum group, consider the following elements:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}^*$$

If we consider the associated elements $X_i^\gamma=\sum_j x_j\otimes u_{ji}^\gamma$, with $\gamma\in\{\Delta,\varepsilon,S\}$, then from the relations $f(X_1,\ldots,X_N)=0$ we deduce that we have:

$$f(X_1^\gamma,\ldots,X_N^\gamma)=(id\otimes\gamma)f(X_1,\ldots,X_N)=0$$

Thus we can map $u_{ij}\to u_{ij}^\gamma$ for any $\gamma\in\{\Delta,\varepsilon,S\}$, and we are done. $\square$

In connection with our questions, we have the following result:

Theorem 3.6. *We have the following quantum isometry group computations,*

$$\begin{matrix}S^{N-1}_{\mathbb R,+}&\longrightarrow&S^{N-1}_{\mathbb C,+}\\ \uparrow&&\uparrow\\ S^{N-1}_{\mathbb R}&\longrightarrow&S^{N-1}_{\mathbb C}\end{matrix}\qquad\to\qquad\begin{matrix}O_N^+&\longrightarrow&U_N^+\\ \uparrow&&\uparrow\\ O_N&\longrightarrow&U_N\end{matrix}$$

modulo identifying, as usual, the various C^*-algebraic completions.

Proof. We have 4 results to be proved, and we can proceed as follows:

$\underline{S^{N-1}_{\mathbb C,+}}$. Let us first construct an action $U_N^+\curvearrowright S^{N-1}_{\mathbb C,+}$. We must prove here that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb C,+}$, namely:

$$\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$$

By using the biunitarity of u, we have the following computation:

$$\begin{aligned}\sum_i X_iX_i^* &= \sum_{ijk} x_jx_k^* \otimes u_{ji}u_{ki}^*\\ &= \sum_j x_jx_j^* \otimes 1\\ &= 1 \otimes 1\end{aligned}$$

Once again by using the biunitarity of u, we have as well:

$$\begin{aligned}\sum_i X_i^*X_i &= \sum_{ijk} x_j^*x_k \otimes u_{ji}^*u_{ki}\\ &= \sum_j x_j^*x_j \otimes 1\\ &= 1 \otimes 1\end{aligned}$$

Thus we have an action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$, which gives $G^+(S^{N-1}_{\mathbb{C},+}) = U_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{R},+}}$. Let us first construct an action $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$. We already know that the variables $X_i = \sum_j x_j \otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb{C},+}$, so we just have to check that these variables are self-adjoint. But this follows from $u = \bar{u}$, as follows:

$$\begin{aligned}X_i^* &= \sum_j x_j^* \otimes u_{ji}^*\\ &= \sum_j x_j \otimes u_{ji}\\ &= X_i\end{aligned}$$

Conversely, assume that we have an action $G \curvearrowright S^{N-1}_{\mathbb{R},+}$, with $G \subset U_N^+$. The variables $X_i = \sum_j x_j \otimes u_{ji}$ must be then self-adjoint, and the above computation shows that we must have $u = \bar{u}$. Thus our quantum group must satisfy $G \subset O_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{C}}}$. The fact that we have an action $U_N \curvearrowright S^{N-1}_{\mathbb{C}}$ is clear, because we have:

$$U \in U_N, ||x|| = 1 \implies ||Ux|| = 1$$

We can deduce this as well with algebraic computations as above. Indeed, we just need to show here that the variables $X_i = \sum_j x_j \otimes u_{ji}$ commute, and this is clear:

$$\begin{aligned}X_iX_k &= \sum_{jl} x_jx_l \otimes u_{ji}u_{lk}\\ &= \sum_{jl} x_lx_j \otimes u_{lk}u_{ji}\\ &= X_kX_i\end{aligned}$$

Conversely, assume that we have an action $G \curvearrowright S^{N-1}_{\mathbb{C}}$, with $G \subset U_N^+$. We must prove that this implies $G \subset U_N$, and for this purpose, we will use a trick from [35].

The coaction map is given by $\Phi(x_i) = \sum_j x_j \otimes u_{ji}$, which is the same as saying that $\Phi(x_k) = \sum_l x_l \otimes u_{lk}$, and by multiplying these two formulae we obtain:

$$\Phi(x_i x_k) = \sum_{jl} x_j x_l \otimes u_{ji} u_{lk}$$

$$\Phi(x_k x_i) = \sum_{jl} x_l x_j \otimes u_{lk} u_{ji}$$

Since the variables x_i commute, these formulae can be written as:

$$\Phi(x_i x_k) = \sum_{j<l} x_j x_l \otimes (u_{ji} u_{lk} + u_{li} u_{jk}) + \sum_j x_j^2 \otimes u_{ji} u_{jk}$$

$$\Phi(x_i x_k) = \sum_{j<l} x_j x_l \otimes (u_{lk} u_{ji} + u_{jk} u_{li}) + \sum_j x_j^2 \otimes u_{jk} u_{ji}$$

Since the tensors at left are linearly independent, we must have:

$$u_{ji} u_{lk} + u_{li} u_{jk} = u_{lk} u_{ji} + u_{jk} u_{li}$$

By applying the antipode to this formula, then applying the involution, and then relabelling the indices, we succesively obtain:

$$u_{kl}^* u_{ij}^* + u_{kj}^* u_{il}^* = u_{ij}^* u_{kl}^* + u_{il}^* u_{kj}^*$$

$$u_{ij} u_{kl} + u_{il} u_{kj} = u_{kl} u_{ij} + u_{kj} u_{il}$$

$$u_{ji} u_{lk} + u_{jk} u_{li} = u_{lk} u_{ji} + u_{li} u_{jk}$$

Now by comparing with the original formula, we obtain from this:

$$u_{li} u_{jk} = u_{jk} u_{li}$$

In order to finish, it remains to prove that the coordinates u_{ij} commute as well with their adjoints. For this purpose, we use a similar method. We have:

$$\Phi(x_i x_k^*) = \sum_{jl} x_j x_l^* \otimes u_{ji} u_{lk}^*$$

$$\Phi(x_k^* x_i) = \sum_{jl} x_l^* x_j \otimes u_{lk}^* u_{ji}$$

Since the variables on the left are equal, we deduce from this that we have:

$$\sum_{jl} x_j x_l^* \otimes u_{ji} u_{lk}^* = \sum_{jl} x_j x_l^* \otimes u_{lk}^* u_{ji}$$

It follows that $u_{ji} u_{lk}^* = u_{lk}^* u_{ji}$, and so that we have $G \subset U_N$, as claimed.

$S^{N-1}_{\mathbb{R}}$. The fact that we have indeed an action $O_N \curvearrowright S^{N-1}_{\mathbb{R}}$ is clear, exactly as in the complex case, because we have:

$$U \in O_N, ||x|| = 1 \implies ||Ux|| = 1$$

Observe that this follows as well from algebraic computations with the variables $X_i = \sum_j x_j \otimes u_{ji}$, by combining the two facts, that we already know from the above proofs for $S^{N-1}_{\mathbb{R},+}$ and for $S^{N-1}_{\mathbb{C}}$, that these variables must be self-adjoint, and must commute.

Finally, observe that this latter proof can be summarized as follows:

$$O_N \curvearrowright S^{N-1}_{\mathbb{R},+}, S^{N-1}_{\mathbb{C}} \implies O_N \curvearrowright S^{N-1}_{\mathbb{R},+} \cap S^{N-1}_{\mathbb{C}} = S^{N-1}_{\mathbb{R}}$$

In what regards the converse now, our claim is that this follows in a similar way, simply by combining the results that we already have. Indeed, we have:

$$\begin{aligned} G \curvearrowright S^{N-1}_{\mathbb{R}} &\implies G \curvearrowright S^{N-1}_{\mathbb{R},+}, S^{N-1}_{\mathbb{C}} \\ &\implies G \subset O_N^+, U_N \\ &\implies G \subset O_N^+ \cap U_N = O_N \end{aligned}$$

Thus, we conclude that we have $G^+(S^{N-1}_{\mathbb{R}}) = O_N$, as desired. □

We can now update our main result so far, as follows:

Theorem 3.7. *For the basic quadruplets, we have correspondences as follows,*

$$\begin{tikzcd} S \arrow[r] \arrow[d] & T \arrow[dl, leftrightarrow] \arrow[d, leftrightarrow] \\ U \arrow[r, leftrightarrow] & K \end{tikzcd}$$

constructed via the following formulae:

(1) $T = S \cap \mathbb{T}_N^+ = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+$.
(2) $U = G^+(S) =< O_N, T >=< O_N, K >$.
(3) $K = U \cap K_N^+ =< O_N, T > \cap K_N^+$.

Proof. This is an update of Theorem 3.3, taking into account Theorem 3.6. □

Let us discuss now the construction $U \to S$. In the classical case the situation is very simple, because S appears by rotating the point $x = (1, 0, \ldots, 0)$ by the isometries in U. Equivalently, $S = S^{N-1}$ appears from $U = U_N$ as an homogeneous space, as follows:

$$S^{N-1} = U_N / U_{N-1}$$

In functional analytic terms, all this becomes even simpler, the correspondence $U \to S$ being obtained, at the level of algebras of functions, as follows:

$$C(S^{N-1}) \subset C(U_N) \quad , \quad x_i \to u_{1i}$$

In general now, let us start with the following observation:

Proposition 3.8. *For the basic spheres, we have a diagram as follows,*

$$\begin{tikzcd} C(S) \arrow[r, "\Phi"] \arrow[d, "\alpha"] & C(S)\otimes C(U) \arrow[d, "\alpha\otimes id"] \\ C(U) \arrow[r, "\Delta"] & C(U)\otimes C(U) \end{tikzcd}$$

where $\Phi(x_i)=\sum_j x_j\otimes u_{ji}$ is the affine coaction map, and where $\alpha(x_i)=u_{1i}$.

Proof. The diagram commutes on the standard coordinates, the arrows being:

$$\begin{tikzcd} x_i \arrow[r] \arrow[d] & \sum_j x_j\otimes u_{ji} \arrow[d] \\ u_{1i} \arrow[r] & \sum_j u_{1j}\otimes u_{ji} \end{tikzcd}$$

Thus by linearity and multiplicativity the whole the diagram commutes. □

We therefore have the following result:

Theorem 3.9. *We have a quotient map and an inclusion as follows,*

$$U\to S_U\subset S$$

with S_U being the first row space of U, given by

$$C(S_U)=<u_{1i}>\subset C(U)$$

at the level of the corresponding algebras of functions.

Proof. At the algebra level, we have an inclusion and a quotient map as follows:

$$C(S)\to C(S_U)\subset C(U)$$

Thus, we obtain the result, by transposing. □

We will prove in what follows that the inclusion $S_U\subset S$ is an isomorphism. In order to do so, we will use the integration over S. This can be introduced as follows:

Definition 3.10. *We endow each of the algebras $C(S)$ with its integration functional*

$$\int_S : C(S)\to C(U)\to\mathbb{C}$$

obtained by composing the morphism given by $x_i\to u_{1i}$ with the Haar integral of U.

In order to efficiently integrate over S, we need to know how to efficiently integrate over U. And the answer here comes from easiness, as follows:

Theorem 3.11. *Assuming that a compact quantum group $G\subset U_N^+$ is easy, coming from a category of partitions $D\subset P$, we have the Weingarten formula*

$$\int_G u_{i_1j_1}^{e_1}\ldots u_{i_kj_k}^{e_k}=\sum_{\pi,\sigma\in D(k)}\delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

for any indices $i_r,j_r\in\{1,\ldots,N\}$ and exponents $e_r\in\{\emptyset,*\}$, where δ are Kronecker type symbols, and where $W_{kN}=G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

Proof. Let us arrange indeed all the integrals to be computed, at a fixed value of the exponent $k=(e_1\ldots e_k)$, into a single matrix, of size $N^k\times N^k$, as follows:

$$P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{i_1j_1}^{e_1}\ldots u_{i_kj_k}^{e_k}$$

By [98], this matrix P is the orthogonal projection onto the following space:

$$Fix(u^{\otimes k})=span\left(\xi_\pi\Big|\pi\in D(k)\right)$$

By a standard linear algebra computation, it follows that we have $P=WE$, where $E(x)=\sum_{\pi\in D(k)}<x,\xi_\pi>\xi_\pi$, and where W is the inverse on $span(T_\pi|\pi\in D(k))$ of the restriction of E. But this restriction is the linear map corresponding to G_{kN}, so W is the linear map corresponding to W_{kN}, and this gives the result. See [19], [29]. □

We can now explicitely integrate over the spheres S, as follows:

Proposition 3.12. *The integration over the basic spheres is given by*

$$\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\sum_\pi\sum_{\sigma\leq\ker i}W_{kN}(\pi,\sigma)$$

with $\pi,\sigma\in D(k)$, where $W_{kN}=G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

Proof. According to our conventions, the integration over S is a particular case of the integration over U, via $x_i=u_{1i}$. By using the formula in Theorem 3.11, we obtain:

$$\begin{aligned}\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}&=\int_U u_{1i_1}^{e_1}\ldots u_{1i_k}^{e_k}\\&=\sum_{\pi,\sigma\in D(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\&=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Following now [22], we have the following key result:

Theorem 3.13. *The integration functional of S has the ergodicity property*

$$\left(id\otimes\int_U\right)\Phi(x)=\int_S x$$

where $\Phi:C(S)\to C(S)\otimes C(U)$ is the universal affine coaction map.

Proof. In the real case, $x_i=x_i^*$, it is enough to check the equality in the statement on an arbitrary product of coordinates, $x_{i_1}\ldots x_{i_k}$. The left term is as follows:

$$\begin{aligned}\left(id\otimes\int_U\right)\Phi(x_{i_1}\ldots x_{i_k}) &= \sum_{j_1\ldots j_k}x_{j_1}\ldots x_{j_k}\int_U u_{j_1i_1}\ldots u_{j_ki_k}\\ &= \sum_{j_1\ldots j_k}\sum_{\pi,\sigma\in D(k)}\delta_\pi(j)\delta_\sigma(i)W_{kN}(\pi,\sigma)x_{j_1}\ldots x_{j_k}\\ &= \sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\sum_{j_1\ldots j_k}\delta_\pi(j)x_{j_1}\ldots x_{j_k}\end{aligned}$$

Let us look now at the last sum on the right. The situation is as follows:

(1) In the free case we have to sum quantities of type $x_{j_1}\ldots x_{j_k}$, over all choices of multi-indices $j=(j_1,\ldots,j_k)$ which fit into our given noncrossing pairing π, and just by using the condition $\sum_i x_i^2=1$, we conclude that the sum is 1.

(2) The same happens in the classical case. Indeed, our pairing π can now be crossing, but we can use the commutation relations $x_ix_j=x_jx_i$, and the sum is again 1.

Thus the sum on the right is 1, in all cases, and we obtain:

$$\left(id\otimes\int_U\right)\Phi(x_{i_1}\ldots x_{i_k})=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)$$

On the other hand, another application of the Weingarten formula gives:

$$\begin{aligned}\int_S x_{i_1}\ldots x_{i_k} &= \int_U u_{1i_1}\ldots u_{1i_k}\\ &= \sum_{\pi,\sigma\in D(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\ &= \sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\end{aligned}$$

Thus, we are done. In the complex case the proof is similar, by adding exponents. □

We can now formulate an abstract characterization of the integration, as follows:

Theorem 3.14. *There is a unique positive unital trace $tr : C(S) \to \mathbb{C}$ satisfying*

$$(tr \otimes id)\Phi(x) = tr(x)1$$

where Φ is the coaction map of the corresponding quantum isometry group,

$$\Phi : C(S) \to C(S) \otimes C(U)$$

and this is the canonical integration, as constructed in Definition 3.10.

Proof. First of all, it follows from the Haar integral invariance condition for U that the canonical integration has indeed the invariance property in the statement.

In order to prove now the uniqueness, let tr be as in the statement. We have:

$$tr\left(id \otimes \int_U\right)\Phi(x) = \int_U (tr \otimes id)\Phi(x) = \int_U (tr(x)1) = tr(x)$$

On the other hand, according to Theorem 3.13, we have as well:

$$tr\left(id \otimes \int_U\right)\Phi(x) = tr\left(\int_S x\right) = \int_S x$$

We therefore conclude that tr equals the standard integration, as claimed. □

Getting back now to our axiomatization questions, we have:

Theorem 3.15. *We have correspondences as follows,*

$$\begin{array}{ccc} \begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array} & \to & \begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{array} \end{array}$$

obtained via the operation $U \to S_U$.

Proof. We use the ergodicity formula from Theorem 3.13, namely:

$$\left(id \otimes \int_U\right)\Phi = \int_S$$

We know that $\int_U$ is faithful on $\mathcal{C}(U)$, and since we have $(id \otimes \varepsilon)\Phi = id$, the coaction map Φ follows to be faithful as well. Thus for any $x \in \mathcal{C}(S)$ we have:

$$\int_S xx^* = 0 \implies x = 0$$

Thus $\int_S$ is faithful on $\mathcal{C}(S)$. But this shows that we have $S = S_U$, as desired. □

We can now update our main result so far, as follows:

Theorem 3.16. *For the basic quadruplets, we have correspondences as follows,*

$$\begin{tikzcd} S \arrow[r] \arrow[d, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ U \arrow[r, leftrightarrow] & K \end{tikzcd}$$

constructed via the following formulae:

(1) $S = S_U$.
(2) $T = S \cap \mathbb{T}_N^+ = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+$.
(3) $U = G^+(S) =< O_N, T >=< O_N, K >$.
(4) $K = U \cap K_N^+ =< O_N, T > \cap K_N^+$.

Proof. This is an update of Theorem 3.7, taking into account Theorem 3.15. □

As a partial conclusion now, in what regards the 12 correspondences that we want to establish, between our objects (S, T, U, K), the situation is as follows:

(1) We have 8 simple and solid correspondences, described above.
(2) We have as well $K =< O_N, T > \cap K_N^+$, obtained as a composition.
(3) We have 3 missing correspondences, namely $T \to S$ and $S \leftrightarrow K$.

In what regards the missing correspondences, $T \to S$ and $S \leftrightarrow K$, the situation here is quite complicated. The correspondence $T \to S$ seems to require some general noncommutative algebraic geometry theory, of quite basic type, which is however not available yet. As for the correspondence $S \leftrightarrow K$, this is something quite tricky, even in the classical case, and we have so far no good idea for dealing with this question.

Summarizing, we are pretty much done with our axiomatization work, but one thing which still can be done is that of replacing the formula $K =< O_N, T > \cap K_N^+$, obtained as a somewhat unnatural composition, by something more simple and conceptual.

Normally this can be done, because in the classical case, we have:

$$K = G(T)$$

In order to discuss this, we will need the following construction:

Theorem 3.17. *The following constructions produce compact quantum groups,*

$$\begin{aligned} C(\bar{O}_N) &= C(O_N^+)\Big/\Big\langle u_{ij}u_{kl} = \pm u_{kl}u_{ij}\Big\rangle \\ C(\bar{U}_N) &= C(U_N^+)\Big/\Big\langle u_{ij}\dot{u}_{kl} = \pm \dot{u}_{kl}u_{ij}\Big\rangle \end{aligned}$$

with the signs corresponding to anticommutation of different entries on same rows or same columns, and commutation otherwise, and where $\dot{u}$ stands for u or for $\bar{u}$.

Proof. This is indeed well-known, and follows in the usual way, by considering the matrices $u^\Delta, u^\varepsilon, u^S$. We will be back later to this, in section 8 below, with full details. □

We should mention that $\bar{O}_N, \bar{U}_N$, while inspired from the Drinfeld-Jimbo philosophy, are not the compact forms of the $q=-1$ enveloping Lie algebra twists of Drinfeld and Jimbo. This is in fact part of a wider phenomenon, the point being that at $q=-1$, which is the most important value of the parameter, besides of course $q=1$, the Drinfeld-Jimbo theory is not the correct one. We will be back to this in section 8 below.

Now back to the tori, the quantum isometry groups here are as follows:

Theorem 3.18. *We have the following quantum isometry group computations,*

$$\begin{tikzcd} T_N^+ \arrow[r] & \mathbb{T}_N^+ & & H_N^+ \arrow[r] \arrow[d, dotted, no head] & K_N^+ \arrow[d, dotted, no head] \\ & & \to & & \\ T_N \arrow[r] \arrow[uu] & \mathbb{T}_N \arrow[uu] & & \bar{O}_N \arrow[r] & \bar{U}_N \end{tikzcd}$$

where $\bar{O}_N, \bar{U}_N$ are the standard $q=-1$ twists of O_N, U_N.

Proof. In all cases we must find the conditions on a closed subgroup $G \subset O_N^+$ such that $g_i \to \sum_j g_j \otimes u_{ji}$ defines a coaction. Since the coassociativity of such a map is automatic, we are left with checking that the map itself exists, and this is the same as checking that the variables $G_i = \sum_j g_j \otimes u_{ji}$ satisfy the same relations as the generators $g_i \in G$.

(1) For $\Gamma = \mathbb{Z}_2^N$ the relations to be checked are as follows:

$$G_i^2 = 1 \quad , \quad G_iG_j = G_jG_i$$

We have the following formula, for the squares:

$$G_i^2 = \sum_{kl} g_kg_l \otimes u_{ki}u_{li} = 1 + \sum_{k<l} g_kg_l \otimes (u_{ki}u_{li} + u_{li}u_{ki})$$

We have as well the following formula, for the commutants:

$$\begin{aligned} [G_i, G_j] &= \sum_{kl} g_kg_l \otimes (u_{ki}u_{lj} - u_{kj}u_{li}) \\ &= \sum_{k<l} g_kg_l \otimes (u_{ki}u_{lj} - u_{kj}u_{li} + u_{li}u_{kj} - u_{lj}u_{ki}) \end{aligned}$$

From the first relation we obtain $ab = -ba$ for $a \neq b$ on the same column of u, and by using the antipode, the same happens for rows. From the second relation we obtain:

$$[u_{ki}, u_{lj}] = [u_{kj}, u_{li}] \quad , \quad \forall k \neq l$$

Now by applying the antipode we obtain from this:

$$[u_{ik}, u_{jl}] = [u_{jk}, u_{il}] \quad , \quad \forall k \neq l$$

By relabelling, this gives the following formula:

$$[u_{ki},u_{lj}]=[u_{li},u_{kj}]\quad,\quad\forall i\neq j$$

Summing up, we are therefore led to the following conclusion:

$$[u_{ki},u_{lj}]=[u_{kj},u_{li}]=0\quad,\quad\forall i\neq j,k\neq l$$

Thus we must have $G\subset\bar{O}_N$, and this finishes the proof.
(2) For $\Gamma=\mathbb{Z}^N$ the proof is similar, as explained in [4].
(3) For $\Gamma=\mathbb{Z}_2^{*N}$ the only relations to be checked are $G_i^2=1$. We have:

$$G_i^2=\sum_{kl}g_kg_l\otimes u_{ki}u_{li}=1+\sum_{k\neq l}g_kg_l\otimes u_{ki}u_{li}$$

Thus we obtain $G\subset H_N^+$, as claimed.
(4) For $\Gamma=F_N$ the proof is similar, as explained in [4]. □

The above result is a bit bizarre, but we can "recycle" it, as follows:

Theorem 3.19. *We have correspondences as follows,*

$$\begin{matrix}\begin{tikzcd}T_N^+ \arrow[r] & \mathbb{T}_N^+\\ T_N \arrow[u]\arrow[r] & \mathbb{T}_N\arrow[u]\end{tikzcd} & \to & \begin{tikzcd}H_N^+ \arrow[r] & K_N^+\\ H_N \arrow[u]\arrow[r] & K_N\arrow[u]\end{tikzcd}\end{matrix}$$

obtained via the operation $T\to G^+(T)\cap K_N^+$.

Proof. In view of Theorem 3.18, we just need to prove that we have:

$$\begin{matrix}\begin{tikzcd}H_N^+ \arrow[r] & K_N^+\\ \bar{O}_N\cap H_N^+ \arrow[u]\arrow[r] & \bar{U}_N\cap K_N^+\arrow[u]\end{tikzcd} & = & \begin{tikzcd}H_N^+ \arrow[r] & K_N^+\\ H_N \arrow[u]\arrow[r] & K_N\arrow[u]\end{tikzcd}\end{matrix}$$

But this is routine, coming from the fact that commutation + anticommutation means vanishing. For details here, we refer to [4]. We will be back to this, later on. □

We can now update our main result so far, as follows:

Theorem 3.20. *For the basic quadruplets, we have correspondences as follows,*

$$\begin{tikzcd} S \arrow[r] \arrow[d, leftrightarrow] & T \arrow[dl, leftrightarrow] \arrow[d, leftrightarrow] \\ U \arrow[r, leftrightarrow] & K \end{tikzcd}$$

constructed via the following formulae:

(1) $S = S_U$.
(2) $T = S \cap \mathbb{T}_N^+ = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+$.
(3) $U = G^+(S) =< O_N, T >=< O_N, K >$.
(4) $K = U \cap K_N^+ = G^+(T) \cap K_N^+$.

Proof. This is an update of Theorem 3.16, taking into account Theorem 3.19. □

As already mentioned in the comments after Theorem 3.16, in what regards the missing correspondences, $T \to S$ and $S \leftrightarrow K$, the situation here is quite complicated. In short, we have to give up now with our general principle of constructing all the correspondences independently of each other, and compose what we have. So, let us formulate:

Definition 3.21. *A quadruplet (S, T, U, K) is said to produce a noncommutative geometry when one can pass from each object to all the other objects, as follows,*

$$\begin{array}{ccccccc} S & = & S_{<O_N,T>} & = & S_U & = & S_{<O_N,K>} \\ S \cap \mathbb{T}_N^+ & = & T & = & U \cap \mathbb{T}_N^+ & = & K \cap \mathbb{T}_N^+ \\ G^+(S) & = & < O_N, T > & = & U & = & < O_N, K > \\ G^+(S) \cap K_N^+ & = & G^+(T) \cap K_N^+ & = & U \cap K_N^+ & = & K \end{array}$$

with the usual convention that all this is up to the equivalence relation.

As a first remark, if we plug the data from any axiom line into the 3 other lines, we obtain axiomatizations in terms of S, T, U, K alone, that we can try to simplify afterwards. It is of course possible to axiomatize everything in terms of ST, SU, SK, TU, TK, UK as well, and also in terms of STU, STK, SUK, TUK, and try to simplify afterwards.

In what follows we will not bother much with this, and use Definition 3.21 as it is. We will need that 12 correspondences, as results, and whether we call such results "verifications of the axioms" or "basic properties of our geometry" is irrelevant.

As another technical comment, the previous work in [15] was based on (S, T, U) triples, but as explained there, this formalism, missing a lot of restrictions coming from K, is a bit too broad. As for the subsequent work in [9], this was based on sextuplets $(S, \bar{S}, T, U, \bar{U}, K)$, with the bars standing for twists, which is perhaps something quite natural, but which leads to too many correspondences between objects, namely 30.

Regarding now the basic examples, these are of course the classical and free, real and complex geometries. To be more precise, we have the following result:

Theorem 3.22. *We have 4 basic geometries, as follows:*

(1) *Classical real, produced by $(S^{N-1}_\mathbb{R}, T_N, O_N, H_N)$.*
(2) *Classical complex, produced by $(S^{N-1}_\mathbb{C}, \mathbb{T}_N, U_N, K_N)$.*
(3) *Free real, produced by $(S^{N-1}_{\mathbb{R},+}, T_N^+, O_N^+, H_N^+)$.*
(4) *Free complex, produced by $(S^{N-1}_{\mathbb{C},+}, \mathbb{T}_N^+, U_N^+, K_N^+)$.*

Proof. This is something that we already know, and more of a reminder, which follows from Theorem 3.20 above, as explained in the above discussion. □

As a philosophical conclusion, we have so far 4 main examples of noncommutative geometries in our sense, which can be represented as folllows:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N \arrow[u] \arrow[r] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

We will be back to more examples in sections 4-6 below, and to some classification results as well, the idea being that of looking for intermediate geometries on the horizontal, and on the vertical of the above diagram, and then combining these constructions.

Getting back to abstract things, and to the axioms from Definition 3.21 above, let us recall that the correspondences there were partly obtained by composing. Here is now an equivalent formulation of our axioms, cutting some trivial redundancies:

Theorem 3.23. *A quadruplet (S,T,U,K) produces a noncommutative geometry when*

$$\begin{array}{ccccc}
S & = & S_U & & \\
S\cap\mathbb{T}_N^+ & = & T & = & K\cap\mathbb{T}_N^+ \\
G^+(S) & = & <O_N,T> & = & U \\
G^+(T)\cap K_N^+ & = & U\cap K_N^+ & = & K
\end{array}$$

with the usual convention that all this is up to the equivalence relation.

Proof. This follows indeed by examining the axioms in Definition 3.21 above, by cutting some trivial redundancies, and then by rescaling the whole table. □

4. Half-liberation

We have seen so far that the quadruplets of type (S,T,U,K) can be axiomatized, and that at the level of basic examples we have 4 such quadruplets, corresponding to the usual real and complex geometries $\mathbb{R}^N,\mathbb{C}^N$, and to the free versions of these:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

Here the upper symbols $\mathbb{R}^N_+,\mathbb{C}^N_+$ do not stand for the free versions of $\mathbb{R}^N,\mathbb{C}^N$, because such free versions do not exist. However, the free versions of the "geometries" of $\mathbb{R}^N,\mathbb{C}^N$, taken in our sense, do exist, and the symbols $\mathbb{R}^N_+,\mathbb{C}^N_+$ stand for them.

Our purpose in what follows will be that of extending the above diagram, with the construction of some supplementary examples. There are two methods here:
(1) Look for intermediate geometries $\mathbb{R}^N\subset X\subset\mathbb{R}^N_+$, and their complex analogues.
(2) Look for intermediate geometries $\mathbb{R}^N\subset X\subset\mathbb{C}^N$, and their free analogues.
We will see that, in each case, there is a "standard" solution, and that these solutions can be combined. Thus, we will end up with a total of $3\times 3=9$ solutions.

We will discuss all this in this section, and in the next one. The updated 3×3 diagram, refining the above 2×2 one, will be as follows:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{T}\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{C}^N_* \arrow[u] \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

We will see afterwards, in section 6 below, that under certain strong axioms, of combinatorial type, these 9 geometries are conjecturally the only ones.
As for the continuation of all this, to be done later, in sections 7-9 and 10-12 below, this will basically consist in "developing" these 6 main geometries that we found.

Let us focus on the first question to be solved, namely finding the intermediate geometries $\mathbb{R}^N \subset X \subset \mathbb{R}^N_+$. Since such a geometry is given by a quadruplet (S,T,U,K), we are led to 4 different intermediate object questions, as follows:

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{R},+}$$

$$T_N \subset T \subset T_N^+$$

$$O_N \subset U \subset O_N^+$$

$$H_N \subset K \subset H_N^+$$

At the sphere and torus level, there are obviously uncountably many solutions, and it is hard to get beyond this, with bare hands. An idea here would be of course that of throwing some differential geometry considerations into the picture, but we do not know how to do this. Thus, our hopes will basically come from the unitary and reflection quantum groups, where things are more rigid than for spheres and tori.

Let us record, however, the following interesting fact regarding the spheres, from [23], which will appear to be quite relevant, later on:

Theorem 4.1. *The algebraic manifold $S^{(k)} \subset S^{N-1}_{\mathbb{R},+}$ obtained by imposing the relations $a_1 \ldots a_k = a_k \ldots a_1$ to the standard coordinates of $S^{N-1}_{\mathbb{R},+}$ is as follows:*

1. *At $k=1$ we have $S^{(k)} = S^{N-1}_{\mathbb{R},+}$.*
2. *At $k=2,4,6,\ldots$ we have $S^{(k)} = S^{N-1}_{\mathbb{R}}$.*
3. *At $k=3,5,7,\ldots$ we have $S^{(k)} = S^{(3)}$.*

Proof. Since the relations $ab=ba$ imply the relations $a_1 \ldots a_k = a_k \ldots a_1$ for $k \geq 2$, we have $S^{(2)} \subset S^{(k)}$ for $k \geq 2$. It is also elementary to check that the relations $abc = cba$ imply the relations $a_1 \ldots a_k = a_k \ldots a_1$ for $k \geq 3$ odd, so $S^{(3)} \subset S^{(k)}$ for $k \geq 3$ odd.

Our claim now is that we have $S^{(k+2)} \subset S^{(k)}$, for any $k \geq 2$. In order to prove this, we must show that the relations $a_1 \ldots a_{k+2} = a_{k+2} \ldots a_1$ between $x_1, \ldots, x_N$ imply the relations $a_1 \ldots a_k = a_k \ldots a_1$ between $x_1, \ldots, x_N$. But this holds indeed, because:

$$\begin{aligned}
x_{i_1} \ldots x_{i_{k+2}} = x_{i_{k+2}} \ldots x_{i_1} \quad &\implies \quad x_{i_1} \ldots x_{i_k} x_j^2 = x_j^2 x_{i_k} \ldots x_{i_1} \\
&\implies \quad \sum_j x_{i_1} \ldots x_{i_k} x_j^2 = \sum_j x_j^2 x_{i_k} \ldots x_{i_1} \\
&\implies \quad x_{i_1} \ldots x_{i_k} = x_{i_k} \ldots x_{i_1}
\end{aligned}$$

Summing up, we have proved that we have inclusions as follows:

$$S^{(2)} \subset \ldots\ldots \subset S^{(6)} \subset S^{(4)} \subset S^{(2)}$$

$$S^{(3)} \subset \ldots\ldots \subset S^{(7)} \subset S^{(5)} \subset S^{(3)}$$

Thus, we are led to the conclusions in the statement. $\square$

As a conclusion, the sphere $S^{(3)}$, obtained via the relations $abc = cba$, might be the "privileged" intermediate sphere $S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{R},+}$ that we are looking for.

It is possible to go further in this direction, with a study of the spheres given by relations of type $a_1 \ldots a_k = a_{\sigma(1)} \ldots a_{\sigma(k)}$ with $\sigma \in S_k$, which leads to a similar conclusion, and we will discuss this later on, in section 6 below. All this remains, however, quite ad-hoc. So, instead of insisting on spheres and tori, where the solutions to the intermediate object problem are definitely uncountable, let us focus instead on the quantum groups. We will see that there is a lot more rigidity here, which makes things simpler.

At the quantum group level, our goal will be that of finding the intermediate objects $O_N \subset U \subset O_N^+$, and the intermediate objects $H_N \subset K \subset H_N^+$. Quite surprisingly, these two questions are of quite different nature, the situation being as follows:

(1) Regarding $O_N \subset U \subset O_N^+$, there is a solution here, denoted O_N^*, coming via the relations $abc = cba$, and conjecturally nothing more.

(2) Regarding $H_N \subset K \subset H_N^+$, here it is possible to use for instance crossed products, in order to construct uncountably many solutions.

In short, in connection with our intermediate noncommutative geometry question, we do have in principle our solution, coming via the relations $abc = cba$, and this is compatible with our above $S^{(3)}$ guess for the spheres, but all this is quite subtle.

In order to get started, let us recall that we have:

Theorem 4.2. *The basic quantum unitary and reflection groups, namely*

$$
\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[ur] \arrow[uu] & & O_N \arrow[ur] \arrow[uu] &
\end{tikzcd}
$$

are all easy, coming from certain categories of partitions.

Proof. This is something that we already discussed, in section 2 above, the corresponding categories of partitions being as follows:

$$\begin{tikzcd}[row sep=small, column sep=small]
 & \mathcal{NC}_{even} \arrow[dl] \arrow[dd] & & \mathcal{NC}_2 \arrow[ll] \arrow[dl] \arrow[dd] \\
NC_{even} \arrow[dd] & & NC_2 \arrow[ll, crossing over] & \\
 & \mathcal{P}_{even} \arrow[dl] & & \mathcal{P}_2 \arrow[ll] \arrow[dl] \\
P_{even} & & P_2 \arrow[ll] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

To be more precise, P_{even} is the category of partitions whose blocks have even size, and $\mathcal{P}_{even}(k,l) \subset P_{even}(k,l)$ is the category of partitions which are "matching", in the sense that they satisfy $\#\circ = \#\bullet$ in each block, when flattening the partition. $\square$

Getting back now to the half-liberation question, let us start by constructing the solutions. The result here, which is well-known as well, is as follows:

Theorem 4.3. *We have quantum groups as follows, obtained via the "half-commutation" relations $abc = cba$, which fit into the diagram of basic quantum groups:*

$$\begin{tikzcd}
K_N^* \arrow[r] & U_N^* \\
H_N^* \arrow[r] \arrow[u] & O_N^* \arrow[u]
\end{tikzcd}$$

These quantum groups are all easy, and the corresponding categories of partitions fit into the diagram of categories of partitions for the basic quantum groups.

Proof. All this is standard, and known since [29], [30]. The idea indeed is that the half-commutation relations $abc = cba$ come from the operator T_* associated to the half-liberating partition $* \in P(3,3)$, and so the quantum groups in the statement are indeed easy, obtained by adding $*$ to the corresponding categories of noncrossing partitions.

We obtain the following categories, with $*$ standing for the fact that, when relabelling clockwise the legs $\circ\bullet\circ\bullet\ldots$, the formula $\#\circ=\#\bullet$ must hold in each block:

$$\begin{array}{ccc}\mathcal{P}^*_{even} & \leftarrow & \mathcal{P}^*_2\\ \downarrow & & \downarrow\\ P^*_{even} & \leftarrow & P^*_2\end{array}$$

Finally, the fact that our new quantum groups and categories fit well into the previous diagrams of quantum groups and categories is clear from this. See [10]. □

The point now is that we have the following result, from [30]:

Theorem 4.4. *There is only one proper intermediate easy quantum group*

$$O_N\subset G\subset O_N^+$$

namely the half-classical orthogonal group O_N^*.

Proof. We must compute here the categories of pairings $NC_2\subset D\subset P_2$, and this can be done via some standard combinatorics, in three steps, as follows:

(1) Let $\pi\in P_2-NC_2$, having $s\geq 4$ strings. Our claim is that:
– If $\pi\in P_2-P_2^*$, there exists a semicircle capping $\pi'\in P_2-P_2^*$.
– If $\pi\in P_2^*-NC_2$, there exists a semicircle capping $\pi'\in P_2^*-NC_2$.
Indeed, both these assertions can be easily proved, by drawing pictures.

(2) Consider now a partition $\pi\in P_2(k,l)-NC_2(k,l)$. Our claim is that:
– If $\pi\in P_2(k,l)-P_2^*(k,l)$ then $<\pi>=P_2$.
– If $\pi\in P_2^*(k,l)-NC_2(k,l)$ then $<\pi>=P_2^*$.
This can be indeed proved by recurrence on the number of strings, $s=(k+l)/2$, by using (1), which provides us with a descent procedure $s\to s-1$, at any $s\geq 4$.

(3) Finally, assume that we are given an easy quantum group $O_N\subset G\subset O_N^+$, coming from certain sets of pairings $D(k,l)\subset P_2(k,l)$. We have three cases:
– If $D\not\subset P_2^*$, we obtain $G=O_N$.
– If $D\subset P_2,D\not\subset NC_2$, we obtain $G=O_N^*$.
– If $D\subset NC_2$, we obtain $G=O_N^+$.
Thus, we are led to the conclusion in the statement. □

The above result is something quite remarkable, and it is actually believed that the result could still hold, without the easiness assumption. We refer here to [17].

Regarding the related inclusions $H_N\subset H_N^+$ and $U_N\subset U_N^+$, studied in [77] and [84], these are far from being maximal, having uncountably many intermediate objects, and the same is known to hold for $K_N\subset K_N^+$. There are many open questions here.

Summarizing, under a certain natural "easiness" assumption, and perhaps even in general, we can only have an intermediate geometry between classical real and free real, namely half-classical real. In practice now, what we have to do is to construct this geometry, and its complex analogue as well, and check the axioms from section 3.

Let us begin by constructing the corresponding quadruplets. We have:

Proposition 4.5. *We have quadruplets (S,T,U,K) as follows,*

$$\begin{tikzcd}
S^{N-1}_{\mathbb{R},*} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & T_N^* \arrow[d, dash] \arrow[dl, dash] \\
O_N^* \arrow[r, dash] & H_N^*
\end{tikzcd}
\qquad
\begin{tikzcd}
S^{N-1}_{\mathbb{C},*} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbb{T}_N^* \arrow[d, dash] \arrow[dl, dash] \\
U_N^* \arrow[r, dash] & K_N^*
\end{tikzcd}$$

called half-classical real and complex, obtained via the relations $abc=cba$.

Proof. This is more or less an empty statement, with the quantum groups appearing in the above diagrams being those constructed above, and with the corresponding spheres and tori being constructed in a similar way, by imposing the half-commutation relations $abc=cba$ to the standard coordinates, and their adjoints. $\square$

In order to check now our noncommutative geometry axioms, we are in need of a better understanding of the half-liberation operation, via some supplementary results.

We recall that $P^{N-1}_\mathbb{R}$ is the space of lines in $\mathbb{R}^N$ passing through the origin. We have a quotient map $S^{N-1}_\mathbb{R}\to P^{N-1}_\mathbb{R}$, which produces an embedding $C(P^{N-1}_\mathbb{R})\subset C(S^{N-1}_\mathbb{R})$, and the image of this embedding is the algebra generated by the variables $p_{ij}=x_ix_j$.

The complex projective space $P^{N-1}_\mathbb{C}$ has a similar description, and we have an embedding $C(P^{N-1}_\mathbb{C})\subset C(S^{N-1}_\mathbb{C})$, whose image is generated by the variables $p_{ij}=x_i\bar{x}_j$.

The spaces $P^{N-1}_\mathbb{R},P^{N-1}_\mathbb{C}$ have the following functional analytic description:

Theorem 4.6. *We have presentation results as follows,*

$$\begin{aligned}
C(P^{N-1}_\mathbb{C}) &= C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=1\right)\\
C(P^{N-1}_\mathbb{R}) &= C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=\bar{p}=p^*=p^2,Tr(p)=1\right)
\end{aligned}$$

where by C^*_{comm} we mean as usual universal commutative C^*-algebra.

Proof. We use the fact that $P^{N-1}_\mathbb{C},P^{N-1}_\mathbb{R}$ are respectively the spaces of rank one projections in $M_N(\mathbb{C}),M_N(\mathbb{R})$. With this picture in mind, it is clear that we have arrows $\leftarrow$.

In order to construct now arrows $\to$, consider the universal algebras on the right, A_C, A_R. These algebras being both commutative, by the Gelfand theorem we can write $A_C = C(X_C)$ and $A_R = C(X_R)$, with X_C, X_R being certain compact spaces.

Now by using the coordinate functions p_{ij}, we conclude that X_C, X_R are certain spaces of rank one projections in $M_N(\mathbb{C}), M_N(\mathbb{R})$. In other words, we have embeddings $X_C \subset P^{N-1}_{\mathbb{C}}$ and $X_R \subset P^{N-1}_{\mathbb{R}}$, and by transposing we obtain arrows $\to$, as desired. $\square$

The above result suggests constructing free projective spaces $P^{N-1}_{\mathbb{R},+}, P^{N-1}_{\mathbb{C},+}$, simply by lifting the commutativity conditions between the variables p_{ij}. However, there is something wrong with this, and more specifically with $P^{N-1}_{\mathbb{R},+}$, coming from the fact that if certain noncommutative coordinates $x_1, \ldots, x_N$ are self-adjoint, then the corresponding projective coordinates $p_{ij} = x_i x_j$ are not necessarily self-adjoint:

$$x_i = x_i^* \quad \not\Longrightarrow \quad x_i x_j = (x_i x_j)^*$$

In short, our attempt to construct free projective spaces $P^{N-1}_{\mathbb{R},+}, P^{N-1}_{\mathbb{C},+}$ as above is not exactly correct, with the space $P^{N-1}_{\mathbb{R},+}$ being rather "irrelevant", and with the space $P^{N-1}_{\mathbb{C},+}$ being probably the good one, but being at the same time "real and complex".

In view of all this, let us formulate the following definition:

Definition 4.7. *Associated to any $N \in \mathbb{N}$ is the following universal algebra,*

$$C(P^{N-1}_+) = C^*\left((p_{ij})_{i,j=1,\ldots,N}\Big|p = p^* = p^2, Tr(p) = 1\right)$$

whose abstract spectrum is called "free projective space".

Observe that we have embeddings of noncommutative spaces $P^{N-1}_{\mathbb{R}} \subset P^{N-1}_{\mathbb{C}} \subset P^{N-1}_+$, and that the complex projective space $P^{N-1}_{\mathbb{C}}$ is the classical version of P^{N-1}_+.

Let us compute now the projective versions of the noncommutative spheres that we have, including the half-classical ones. We use the following formalism here:

Definition 4.8. *The projective version of $S \subset S^{N-1}_{\mathbb{C},+}$ is the quotient space $S \to PS$ determined by the fact that $C(PS) \subset C(S)$ is the subalgebra generated by $p_{ij} = x_i x_j^*$.*

We have the following result, coming from [1], [22], [23]:

Theorem 4.9. *The projective versions of the basic spheres are as follows,*

$$\begin{tikzcd} S^{N-1}_{\mathbb{R},+} \arrow[r] & S^{N-1}_{\mathbb{C},+} & & P^{N-1}_+ \arrow[r] & P^{N-1}_+ \\ S^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C},*} \arrow[u] & \to & P^{N-1}_{\mathbb{C}} \arrow[r] \arrow[u] & P^{N-1}_{\mathbb{C}} \arrow[u] \\ S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C}} \arrow[u] & & P^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & P^{N-1}_{\mathbb{C}} \arrow[u] \end{tikzcd}$$

modulo, in the free case, a GNS construction with respect to the uniform integration.

Proof. The formulae on the bottom are true by definition. For the formulae on top, we have to prove first that the variables $p_{ij} = x_i x_j^*$ over the free sphere $S^{N-1}_{\mathbb{C},+}$ satisfy the defining relations for $C(P^{N-1}_+)$, and the verification here goes as follows:

$$\begin{aligned} (p^*)_{ij} &= p_{ji}^* = (x_j x_i^*)^* = x_i x_j^* = p_{ij} \\ (p^2)_{ij} &= \sum_k p_{ik} p_{kj} = \sum_k x_i x_k^* x_k x_j^* = x_i x_j^* = p_{ij} \\ Tr(p) &= \sum_k p_{kk} = \sum_k x_k x_k^* = 1 \end{aligned}$$

Thus, we have embeddings of algebraic manifolds, as follows:

$$PS^{N-1}_{\mathbb{R},+} \subset PS^{N-1}_{\mathbb{C},+} \subset P^{N-1}_+$$

Regarding now the GNS construction assertion, this follows by reasoning as in the case of the free spheres, the idea being that the uniform integration on these projective spaces comes from the uniform integration over the quantum group $PO_N^+ = PU_N^+$. All this is quite technical, and we will not need this result, in what follows. See [23].

Finally, regarding the middle assertions, concerning the projective versions of the half-classical spheres, it is enough to prove here that we have inclusions as follows:

$$P^{N-1}_{\mathbb{C}} \subset PS^{N-1}_{\mathbb{R},*} \subset PS^{N-1}_{\mathbb{C},*} \subset P^{N-1}_{\mathbb{C}}$$

(1) $P^{N-1}_{\mathbb{C}} \subset PS^{N-1}_{\mathbb{R},*}$. Our claim here is that we have a morphism of C^*-algebras as follows, where z_i are the standard coordinates of $S^{N-1}_{\mathbb{C}}$:

$$C(S^{N-1}_{\mathbb{R},*}) \to M_2(C(S^{N-1}_{\mathbb{C}})) \quad : \quad x_i \to \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

Indeed, we have to prove that the matrices X_i on the right satisfy the defining relations for $S^{N-1}_{\mathbb{R},*}$. But these matrices are self-adjoint, and we have:

$$\begin{aligned}\sum_i X_i^2 &= \sum_i \begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}^2\\ &= \sum_i \begin{pmatrix}|z_i|^2&0\\ 0&|z_i|^2\end{pmatrix}\\ &= \begin{pmatrix}1&0\\ 0&1\end{pmatrix}\end{aligned}$$

As for the half-commutation relations, these follow from the following formula:

$$\begin{aligned}X_iX_jX_k &= \begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}\begin{pmatrix}0&z_j\\ \bar{z}_j&0\end{pmatrix}\begin{pmatrix}0&z_k\\ \bar{z}_k&0\end{pmatrix}\\ &= \begin{pmatrix}0&z_i\bar{z}_jz_k\\ \bar{z}_iz_j\bar{z}_k&0\end{pmatrix}\end{aligned}$$

Indeed, the quantities on the right being symmetric in i,k, this gives the result.

Thus our claim is proved. Now observe that the model that we constructed maps:

$$p_{ij}\to P_{ij}=diag(z_i\bar{z}_j,\bar{z}_iz_j)$$

Thus, at the level of generated algebras, our model maps:

$$<p_{ij}>\to<P_{ij}>=C(P^{N-1}_{\mathbb{C}})$$

We conclude from this that we have a quotient map $C(PS^{N-1}_{\mathbb{R},*})\to C(P^{N-1}_{\mathbb{C}})$, and so at the level of corresponding spaces, an inclusion $P^{N-1}_{\mathbb{C}}\subset PS^{N-1}_{\mathbb{R},*}$, as desired.

(2) $PS^{N-1}_{\mathbb{R},*}\subset PS^{N-1}_{\mathbb{C},*}$. This is something trivial, coming from the inclusion of spheres $S^{N-1}_{\mathbb{R},*}\subset S^{N-1}_{\mathbb{C},*}$, by functoriality of the operation $S\to PS$.

(3) $PS^{N-1}_{\mathbb{C},*}\subset P^{N-1}_{\mathbb{C}}$. This follows from the half-commutation relations, which imply $ab^*cd^*=cb^*ad^*=cd^*ab^*$. Indeed, this computation shows that the projective version $PS^{N-1}_{\mathbb{C},*}$ is classical, and so that we have $PS^{N-1}_{\mathbb{C},*}\subset(P^{N-1}_+)_{class}=P^{N-1}_{\mathbb{C}}$, as desired. □

Summarizing, we have some projective geometry results regarding the half-classical case, that we will use in what follows. We have as well a number of findings on the free case, but we will not need this, in what follows. We will be back to this, later on.

Theorem 4.9 above deals with the spheres, but the same arguments apply to the tori, and to the quantum groups as well. We are led in this way to the following result:

Theorem 4.10. *The projective versions of the half-classical quadruplets are*

$$\begin{array}{ccc} P^{N-1}_{\mathbb{C}} & - & P\mathbb{T}_N\\ | & \times & |\\ PU_N & - & PK_N \end{array}$$

both in the real and in the complex cases.

Proof. This is something that we already know from the spheres. For the other objects, this follows by suitably adapting the proof of Theorem 4.9 above. □

Let us check now the axioms. We first need some quantum isometry group results:

Theorem 4.11. *The quantum isometry groups of the basic spheres are the basic orthogonal and unitary quantum groups, as follows,*

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \rightarrow & S^{N-1}_{\mathbb{C},+}\\ \uparrow & & \uparrow\\ S^{N-1}_{\mathbb{R},*} & \rightarrow & S^{N-1}_{\mathbb{C},*}\\ \uparrow & & \uparrow\\ S^{N-1}_{\mathbb{R}} & \rightarrow & S^{N-1}_{\mathbb{C}} \end{array} \quad \to \quad \begin{array}{ccc} O_N^+ & \rightarrow & U_N^+\\ \uparrow & & \uparrow\\ O_N^* & \rightarrow & U_N^*\\ \uparrow & & \uparrow\\ O_N & \rightarrow & U_N \end{array}$$

modulo identifying, as usual, the various C^*-algebraic completions.

Proof. We just have to prove the results in the middle. Assume $G \curvearrowright S^{N-1}_{\mathbb{C},*}$. From $\Phi(x_a) = \sum_i x_i \otimes u_{ia}$ we obtain, with $p_{ab} = z_a\bar{z}_b$:

$$\Phi(p_{ab}) = \sum_{ij} p_{ij} \otimes u_{ia}u_{jb}^*$$

By multiplying two such arbitrary formulae, we obtain:

$$\begin{aligned} \Phi(p_{ab}p_{cd}) &= \sum_{ijkl} p_{ij}p_{kl} \otimes u_{ia}u_{jb}^*u_{kc}u_{ld}^*\\ \Phi(p_{ad}p_{cb}) &= \sum_{ijkl} p_{il}p_{kj} \otimes u_{ia}u_{ld}^*u_{kc}u_{jb}^* \end{aligned}$$

The left terms being equal, and the first terms on the right being equal too, we deduce that, with $[a,b,c]=abc-cba$, we must have the following equality:

$$\sum_{ijkl}p_{ij}p_{kl}\otimes u_{ia}[u_{jb}^*,u_{kc},u_{ld}^*]=0$$

Since the variables $p_{ij}p_{kl}=z_i\bar{z}_jz_k\bar{z}_l$ depend only on $|\{i,k\}|,|\{j,l\}|\in\{1,2\}$, and this dependence produces the only relations between them, we are led to 4 equations:

(1) $u_{ia}[u_{jb}^*,u_{ka},u_{lb}^*]=0$, $\forall a,b$.
(2) $u_{ia}[u_{jb}^*,u_{ka},u_{ld}^*]+u_{ia}[u_{jd}^*,u_{ka},u_{lb}^*]=0$, $\forall a$, $\forall b\neq d$.
(3) $u_{ia}[u_{jb}^*,u_{kc},u_{lb}^*]+u_{ic}[u_{jb}^*,u_{ka},u_{lb}^*]=0$, $\forall a\neq c$, $\forall b$.
(4) $u_{ia}([u_{jb}^*,u_{kc},u_{ld}^*]+[u_{jd}^*,u_{kc},u_{lb}^*])+u_{ic}([u_{jb}^*,u_{ka},u_{ld}^*]+[u_{jd}^*,u_{ka},u_{lb}^*])=0,\forall a\neq c,\forall b\neq d$.

From (1,2) we conclude that (2) holds with no restriction on the indices. By multiplying now this formula to the left by u_{ia}^*, and then summing over i, we obtain:

$$[u_{jb}^*,u_{ka},u_{ld}^*]+[u_{jd}^*,u_{ka},u_{lb}^*]=0$$

By applying now the antipode, then the involution, and finally by suitably relabelling all the indices, we successively obtain from this formula:

$$\begin{aligned}
&[u_{dl},u_{ak}^*,u_{bj}]+[u_{bl},u_{ak}^*,u_{dj}]=0\\
\implies\quad&[u_{dl}^*,u_{ak},u_{bj}^*]+[u_{bl}^*,u_{ak},u_{dj}^*]=0\\
\implies\quad&[u_{ld}^*,u_{ka},u_{jb}^*]+[u_{jd}^*,u_{ka},u_{lb}^*]=0
\end{aligned}$$

Now by comparing with the original relation, above, we conclude that we have:

$$[u_{jb}^*,u_{ka},u_{ld}^*]=[u_{jd}^*,u_{ka},u_{lb}^*]=0$$

Thus we have reached to the formulae defining U_N^*, and we are done.

Finally, in what regards the universality of $O_N^*\curvearrowright S^{N-1}_{\mathbb{R},*}$, this follows from the universality of $U_N^*\curvearrowright S^{N-1}_{\mathbb{C},*}$ and of $O_N^+\curvearrowright S^{N-1}_{\mathbb{R},+}$, and from $U_N^*\cap O_N^+=O_N^*$. $\square$

Regarding now the tori, the computation here is as follows:

Theorem 4.12. *The quantum isometry groups of the basic tori are as follows,*

$$\begin{matrix}
T_N^+ & \longrightarrow & \mathbb{T}_N^+ & & H_N^+ & \longrightarrow & K_N^+\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
T_N^* & \longrightarrow & \mathbb{T}_N^* & \to & H_N^* & \longrightarrow & K_N^*\\
\uparrow & & \uparrow & & \vdots & & \vdots\\
T_N & \longrightarrow & \mathbb{T}_N & & \bar{O}_N & \longrightarrow & \bar{U}_N
\end{matrix}$$

with all arrows being inclusions, and with no vertical maps at bottom right.

Proof. We just have to prove the results in the middle. In the real case, we must find the conditions on $G\subset O_N^+$ such that $g_a\to\sum_i g_a\otimes u_{ia}$ defines a coaction.

In order for this map to be a coaction, the variables $G_a=\sum_i g_a\otimes u_{ia}$ must satisfy the following relations, which define the groups in the statement:

$$G_a^2=1\quad,\quad G_aG_bG_c=G_cG_bG_a$$

In what regards the squares, we have the following formula:

$$\begin{aligned}G_a^2&=\sum_{ij}g_ig_j\otimes u_{ia}u_{ja}\\&=1+\sum_{i\neq j}g_ig_j\otimes u_{ia}u_{ja}\end{aligned}$$

As for the products, with the notation $[x,y,z]=xyz-zyx$, we have:

$$[G_a,G_b,G_c]=\sum_{ijk}g_ig_jg_k\otimes[u_{ia},u_{jb},u_{kc}]$$

From the first relations, $G_a^2=1$, we obtain $G\subset H_N^+$. In order to process now the second relations, $G_aG_bG_c=G_cG_bG_a$, we can split the sum over i,j,k, as follows:

$$\begin{aligned}[G_a,G_b,G_c]&=\sum_{i,j,k\ distinct}g_ig_jg_k\otimes[u_{ia},u_{jb},u_{kc}]\\&+\sum_{i\neq j}g_ig_jg_i\otimes[u_{ia},u_{jb},u_{ic}]\\&+\sum_{i\neq j}g_i\otimes[u_{ia},u_{jb},u_{jc}]\\&+\sum_{i\neq k}g_k\otimes[u_{ia},u_{ib},u_{kc}]\\&+\sum_{i}g_i\otimes[u_{ia},u_{ib},u_{ic}]\end{aligned}$$

Our claim is that the last three sums vanish. Indeed, observe that we have:

$$[u_{ia},u_{ib},u_{ic}]=\delta_{abc}u_{ia}-\delta_{abc}u_{ia}=0$$

Thus the last sum vanishes. Regarding now the fourth sum, we have:

$$\begin{aligned}
\sum_{i\neq k}[u_{ia},u_{ib},u_{kc}] &= \sum_{i\neq k} u_{ia}u_{ib}u_{kc}-u_{kc}u_{ib}u_{ia}\\
&= \sum_{i\neq k}\delta_{ab}u_{ia}^2u_{kc}-\delta_{ab}u_{kc}u_{ia}^2\\
&= \delta_{ab}\sum_{i\neq k}[u_{ia}^2,u_{kc}]\\
&= \delta_{ab}\left[\sum_{i\neq k}u_{ia}^2,u_{kc}\right]\\
&= \delta_{ab}[1-u_{ka}^2,u_{kc}]\\
&= 0
\end{aligned}$$

The proof for the third sum is similar. Thus, we are left with the first two sums. By using $g_ig_jg_k=g_kg_jg_i$ for the first sum, the formula becomes:

$$\begin{aligned}
[G_a,G_b,G_c] &= \sum_{i<k,j\neq i,k} g_ig_jg_k\otimes([u_{ia},u_{jb},u_{kc}]+[u_{ka},u_{jb},u_{ic}])\\
&+ \sum_{i\neq j}g_ig_jg_i\otimes[u_{ia},u_{jb},u_{ic}]
\end{aligned}$$

In order to have a coaction, the above coefficients must vanish. Now observe that, when setting $i=k$ in the coefficients of the first sum, we obtain twice the coefficients of the second sum. Thus, our vanishing conditions can be formulated as follows:

$$[u_{ia},u_{jb},u_{kc}]+[u_{ka},u_{jb},u_{ic}]=0,\forall j\neq i,k$$

Now observe that at $a=b$ or $b=c$ this condition reads $0+0=0$. Thus, we can formulate our vanishing conditions in a more symmetric way, as follows:

$$[u_{ia},u_{jb},u_{kc}]+[u_{ka},u_{jb},u_{ic}]=0,\forall j\neq i,k,\forall b\neq a,c$$

We use now the trick from [35]. We apply the antipode to this formula, and then we relabel the indices $i\leftrightarrow c,j\leftrightarrow b,k\leftrightarrow a$. We succesively obtain in this way:

$$[u_{ck},u_{bj},u_{ai}]+[u_{ci},u_{bj},u_{ak}]=0,\forall j\neq i,k,\forall b\neq a,c$$

$$[u_{ia},u_{jb},u_{kc}]+[u_{ic},u_{jb},u_{ka}]=0,\forall b\neq a,c,\forall j\neq i,k$$

Since we have $[x,y,z]=-[z,y,x]$, by comparing the last formula with the original one, we conclude that our vanishing relations reduce to a single formula, as follows:

$$[u_{ia},u_{jb},u_{kc}]=0,\forall j\neq i,k,\forall b\neq a,c$$

Our first claim is that this formula implies $G\subset H_N^{[\infty]}$, where $H_N^{[\infty]}\subset O_N^+$ is defined via the relations $xyz=0$, for any $x\neq z$ on the same row or column of u. In order to prove this, we will just need the $c=a$ particular case of this formula, which reads:

$$u_{ia}u_{jb}u_{ka}=u_{ka}u_{jb}u_{ia},\forall j\neq i,k,\forall a\neq b$$

It is enough to check that the assumptions $j\neq i,k$ and $a\neq b$ can be dropped. But this is what happens indeed, because at $j=i$, $j=k$, $a=b$, we respectively have:

$$\begin{aligned}
[u_{ia},u_{ib},u_{ka}] &= u_{ia}u_{ib}u_{ka}-u_{ka}u_{ib}u_{ia}=\delta_{ab}(u_{ia}^2u_{ka}-u_{ka}u_{ia}^2)=0\\
[u_{ia},u_{kb},u_{ka}] &= u_{ia}u_{kb}u_{ka}-u_{ka}u_{kb}u_{ia}=\delta_{ab}(u_{ia}u_{ka}^2-u_{ka}^2u_{ia})=0\\
[u_{ia},u_{ja},u_{ka}] &= u_{ia}u_{ja}u_{ka}-u_{ka}u_{ja}u_{ia}=\delta_{ijk}(u_{ia}^3-u_{ia}^3)=0
\end{aligned}$$

Our second claim now is that, due to $G\subset H_N^{[\infty]}$, we can drop the assumptions $j\neq i,k$ and $b\neq a,c$ in the original relations $[u_{ia},u_{jb},u_{kc}]=0$. Indeed, at $j=i$ we have:

$$\begin{aligned}
[u_{ia},u_{ib},u_{kc}] &= u_{ia}u_{ib}u_{kc}-u_{kc}u_{ib}u_{ia}\\
&= \delta_{ab}(u_{ia}^2u_{kc}-u_{kc}u_{ia}^2)\\
&= 0
\end{aligned}$$

The proof at $j=k$ and at $b=a$, $b=c$ being similar, this finishes the proof of our claim. We conclude that the half-commutation relations $[u_{ia},u_{jb},u_{kc}]=0$ hold without any assumption on the indices, and so we obtain $G\subset H_N^*$, as claimed.

As for the proof in the complex case, this is similar. See [4]. ☐

By intersecting now with K_N^+, as required by our (S,T,U,K) axioms, we obtain:

Theorem 4.13. *The quantum reflection groups of basic tori are as follows,*

$$\begin{matrix}
T_N^+ & \longrightarrow & \mathbb{T}_N^+\\
\uparrow & & \uparrow\\
T_N^* & \longrightarrow & \mathbb{T}_N^*\\
\uparrow & & \uparrow\\
T_N & \longrightarrow & \mathbb{T}_N
\end{matrix}
\quad\to\quad
\begin{matrix}
H_N^+ & \longrightarrow & K_N^+\\
\uparrow & & \uparrow\\
H_N^* & \longrightarrow & K_N^*\\
\uparrow & & \uparrow\\
H_N & \longrightarrow & K_N
\end{matrix}$$

with all the arrows being inclusions.

Proof. We already know that the results on the left and on the right hold indeed. As for the results in the middle, these follow from Theorem 4.12 above. ☐

We can now formulate our extension result, as follows:

Theorem 4.14. *We have basic noncommutative geometries, as follows,*

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{C}^N_* \arrow[u] \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

with each $\mathbb{K}^N_\times$ symbol standing for the corresponding (S,T,U,K) quadruplet.

Proof. We have to check the axioms from section 3, for the half-classical geometries. The algebraic axioms are all clear, and the quantum isometry axioms follow from the above computations. Next in line, we have to prove the following formulae:

$$O_N^* = < O_N, T_N^* >$$

$$U_N^* = < U_N, \mathbb{T}_N^* >$$

By using standard generation results, it is enough to prove the first formula. Moreover, once again by standard generation results, it is enough to check that:

$$H_N^* = < H_N, T_N^* >$$

The inclusion $\supset$ being clear, we are left with proving the inclusion $\subset$. But this follows from the formula $H_N^* = T_N^* \rtimes S_N$, established in [83], as follows:

$$\begin{aligned} H_N^* &= T_N^* \rtimes S_N \\ &= < S_N, T_N^* > \\ &\subset < H_N, T_N^* > \end{aligned}$$

Alternatively, these formulae can be established by using the technology in [37], or by doing some combinatorial computations, using categories and easiness.

Finally, the axiom $S = S_U$ can be proved as in the classical and free cases, by using the Weingarten formula. Indeed, let us go back to Theorem 3.13, which was the key ingredient of the proof, in the classical and free cases. The statement there was that the integration functional of S has the following ergodicity property:

$$\left(id \otimes \int_U\right)\Phi(x) = \int_S x$$

Our claim, which will finish the proof, is that this holds as well in the half-classical case. Indeed, in the real case, where $x_i = x_i^*$, it is enough to check the above equality on an arbitrary product of coordinates, $x_{i_1}\ldots x_{i_k}$. The left term is as follows:

$$\begin{aligned}\left(id\otimes\int_{O_N^*}\right)\Phi(x_{i_1}\ldots x_{i_k}) &= \sum_{j_1\ldots j_k}x_{j_1}\ldots x_{j_k}\int_{O_N^*}u_{j_1i_1}\ldots u_{j_ki_k}\\ &= \sum_{j_1\ldots j_k}\sum_{\pi,\sigma\in P_2^*(k)}\delta_\pi(j)\delta_\sigma(i)W_{kN}(\pi,\sigma)x_{j_1}\ldots x_{j_k}\\ &= \sum_{\pi,\sigma\in P_2^*(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\sum_{j_1\ldots j_k}\delta_\pi(j)x_{j_1}\ldots x_{j_k}\end{aligned}$$

Let us look now at the last sum on the right. We have to sum there quantities of type $x_{j_1}\ldots x_{j_k}$, over all choices of multi-indices $j=(j_1,\ldots,j_k)$ which fit into our given pairing $\pi\in P_2^*(k)$. But by using the relations $x_ix_jx_k=x_kx_jx_i$, and then $\sum_i x_i^2=1$ in order to simplify, we conclude that the sum of these quantities is 1. Thus, we obtain:

$$\left(id\otimes\int_{O_N^*}\right)\Phi(x_{i_1}\ldots x_{i_k})=\sum_{\pi,\sigma\in P_2^*(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)$$

On the other hand, another application of the Weingarten formula gives:

$$\begin{aligned}\int_{S^{N-1}_{\mathbb{R},*}}x_{i_1}\ldots x_{i_k} &= \int_{O_N^*}u_{1i_1}\ldots u_{1i_k}\\ &= \sum_{\pi,\sigma\in P_2^*(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\ &= \sum_{\pi,\sigma\in P_2^*(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\end{aligned}$$

Thus, we are done. In the complex case the proof is similar, by adding exponents. For further details, we refer to [22] for the real case, and to [1] for the complex case. □

Summarizing, we have done so far half of our extension program.

5. Basic geometries

In order to finish the extension program outlined in the beginning of the previous section, we must discuss now the second question, concerning the "hybrid" case.

We will see that there is one privileged intermediate geometry $\mathbb{R}^N \subset \mathbb{T}\mathbb{R}^N \subset \mathbb{C}^N$. This privileged geometry has a half-classical version $\mathbb{R}^N_* \subset \mathbb{T}\mathbb{R}^N_* \subset \mathbb{C}^N_*$, and a free version $\mathbb{R}^N_+ \subset \mathbb{T}\mathbb{R}^N_+ \subset \mathbb{C}^N_+$, and this will lead to an extension of our diagram, as follows:

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{T}\mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{C}^N_* \arrow[u] \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{T}\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

We will see later on, in section 6 below, that under strong combinatorial axioms, of "easiness" type, these 9 geometries are conjecturally the only ones.

In order to get started, an intermediate geometry $\mathbb{R}^N \subset X \subset \mathbb{C}^N$ is given by a quadruplet (S, T, U, K), whose components are subject to the following conditions:

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C}}$$

$$T_N \subset T \subset \mathbb{T}_N$$

$$O_N \subset U \subset U_N$$

$$H_N \subset K \subset K_N$$

Our plan will be that of investigating first these intermediate object questions. Then, we will discuss the verification of the geometric axioms, for the solutions that we found. And then, afterwards, we will discuss the half-classical and the free cases as well.

In what regards the $S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C}}$ problem, there are obviously infinitely many solutions. However, we have a "privileged" solution, constructed as follows:

Theorem 5.1. *We have an intermediate sphere as follows,*

$$S^{N-1}_{\mathbb{R}} \subset \mathbb{T}S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{C}}$$

which appears as the affine lift of $P^{N-1}_{\mathbb{R}}$, inside the complex sphere $S^{N-1}_{\mathbb{C}}$.

Proof. The projective version of the intermediate sphere $\mathbb{T}S^{N-1}_\mathbb{R}$ is given by:

$$P\mathbb{T}S^{N-1}_\mathbb{R}=PS^{N-1}_\mathbb{R}=P^{N-1}_\mathbb{R}$$

Conversely, assume that $S\subset S^{N-1}_\mathbb{C}$ satisfies $PS\subset P^{N-1}_\mathbb{R}$. For $x\in S$ the projective coordinates $p_{ij}=x_i\bar{x}_j$ must be real, $x_i\bar{x}_j=\bar{x}_ix_j$, and so we must have:

$$\frac{x_1}{\bar{x}_1}=\frac{x_2}{\bar{x}_2}=\ldots=\frac{x_N}{\bar{x}_N}$$

Now if we denote by $\lambda\in\mathbb{T}$ this common number, we succesively have:

$$\begin{aligned}\frac{x_i}{\bar{x}_i}=\lambda\quad&\iff\quad x_i=\lambda\bar{x}_i\\&\iff\quad x_i^2=\lambda|x_i|^2\\&\iff\quad x_i=\pm\sqrt{\lambda}|x_i|\end{aligned}$$

Thus we obtain $x\in\sqrt{\lambda}S^{N-1}_\mathbb{R}$, and this gives the result. □

In the case of the tori, we have a similar result, as follows:

Theorem 5.2. *We have an intermediate torus as follows, which appears as the affine lift of the Clifford torus $PT_N=T_{N-1}$, inside the complex torus $\mathbb{T}_N$:*

$$T_N\subset\mathbb{T}T_N\subset\mathbb{T}_N$$

More generally, we have intermediate tori as follows, with $r\in\mathbb{N}\cup\{\infty\}$,

$$T_N\subset\mathbb{Z}_rT_N\subset\mathbb{T}_N$$

all whose projective versions equal the Clifford torus $PT_N=T_{N-1}$.

Proof. The first assertion, regarding $\mathbb{T}T_N$, follows exactly as for the spheres, as in proof of Theorem 5.1. The second assertion is clear as well, because we have:

$$P\mathbb{Z}_rT_N=PT_N=T_{N-1}$$

Thus, we are led to the conclusion in the statement. □

In connection with the above statement, an interesting question is that of classifying the intermediate tori, which in our case are usual compact groups, $T_N\subset T\subset\mathbb{T}_N$. At the group dual level, we must classify the following intermediate quotients:

$$\mathbb{Z}^N\to\Gamma\to\mathbb{Z}_2^N$$

There are many examples of such groups, and this even when imposing strong supplementary conditions, such as having an action of the symmetric group S_N on the generators. We will not go further in this direction, our main idea being anyway that of basing our study mostly on quantum group theory, and on the related notion of easiness.

At the unitary group level now, the situation is of course much more rigid, and becomes quite interesting. We have the following result from [17], to start with:

Theorem 5.3. *The following inclusions are maximal:*

(1) $\mathbb{T}O_N \subset U_N$.
(2) $PO_N \subset PU_N$.

Proof. In order to prove these results, consider as well the group $\mathbb{T}SO_N$. Observe that we have $\mathbb{T}SO_N = \mathbb{T}O_N$ if N is odd. If N is even the group $\mathbb{T}O_N$ has two connected components, with $\mathbb{T}SO_N$ being the component containing the identity.

Let us denote by $\mathfrak{so}_N, \mathfrak{u}_N$ the Lie algebras of SO_N, U_N. It is well-known that $\mathfrak{u}_N$ consists of the matrices $M \in M_N(\mathbb{C})$ satisfying $M^* = -M$, and that $\mathfrak{so}_N = \mathfrak{u}_N \cap M_N(\mathbb{R})$. Also, it is easy to see that the Lie algebra of $\mathbb{T}SO_N$ is $\mathfrak{so}_N \oplus i\mathbb{R}$.

Step 1. Our first claim is that if $N \geq 2$, the adjoint representation of SO_N on the space of real symmetric matrices of trace zero is irreducible.

Let indeed $X \in M_N(\mathbb{R})$ be symmetric with trace zero. We must prove that the following space consists of all the real symmetric matrices of trace zero:

$$V = span\left\{UXU^t\Big|U \in SO_N\right\}$$

We first prove that V contains all the diagonal matrices of trace zero. Since we may diagonalize X by conjugating with an element of SO_N, our space V contains a nonzero diagonal matrix of trace zero. Consider such a matrix:

$$D = diag(d_1, d_2, \ldots, d_N)$$

We can conjugate this matrix by the following matrix:

$$\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & I_{N-2}\end{pmatrix} \in SO_N$$

We conclude that our space V contains as well the following matrix:

$$D' = diag(d_2, d_1, d_3, \ldots, d_N)$$

More generally, we see that for any $1 \leq i, j \leq N$ the diagonal matrix obtained from D by interchanging d_i and d_j lies in V. Now since S_N is generated by transpositions, it follows that V contains any diagonal matrix obtained by permuting the entries of D. But it is well-known that this representation of S_N on the diagonal matrices of trace zero is irreducible, and hence V contains all such diagonal matrices, as claimed.

In order to conclude now, assume that Y is an arbitrary real symmetric matrix of trace zero. We can find then an element $U \in SO_N$ such that UYU^t is a diagonal matrix of trace zero. But we then have $UYU^t \in V$, and hence also $Y \in V$, as desired.

Step 2. Our claim is that the inclusion $\mathbb{T}SO_N\subset U_N$ is maximal in the category of connected compact groups.

Let indeed G be a connected compact group satisfying $\mathbb{T}SO_N\subset G\subset U_N$. Then G is a Lie group. Let $\mathfrak{g}$ denote its Lie algebra, which satisfies:

$$\mathfrak{so}_N\oplus i\mathbb{R}\subset\mathfrak{g}\subset\mathfrak{u}_N$$

Let ad_G be the action of G on $\mathfrak{g}$ obtained by differentiating the adjoint action of G on itself. This action turns $\mathfrak{g}$ into a G-module. Since $SO_N\subset G$, $\mathfrak{g}$ is also a SO_N-module.

Now if $G\neq\mathbb{T}SO_N$, then since G is connected we must have $\mathfrak{so}_N\oplus i\mathbb{R}\neq\mathfrak{g}$. It follows from the real vector space structure of the Lie algebras $\mathfrak{u}_N$ and $\mathfrak{so}_N$ that there exists a nonzero symmetric real matrix of trace zero X such that:

$$iX\in\mathfrak{g}$$

We know that the space of symmetric real matrices of trace zero is an irreducible representation of SO_N under the adjoint action. Thus $\mathfrak{g}$ must contain all such X, and hence $\mathfrak{g}=\mathfrak{u}_N$. But since U_N is connected, it follows that $G=U_N$.

Step 3. Our claim is that the commutant of SO_N in $M_N(\mathbb{C})$ is as follows:

(1) $SO_2'=\left\{\begin{pmatrix}\alpha&\beta\\-\beta&\alpha\end{pmatrix}\Big|\alpha,\beta\in\mathbb{C}\right\}$.

(2) If $N\geq3$, $SO_N'=\{\alpha I_N|\alpha\in\mathbb{C}\}$.

Indeed, at $N=2$ this is a direct computation. At $N\geq3$, an element in $X\in SO_N'$ commutes with any diagonal matrix having exactly $N-2$ entries equal to 1 and two entries equal to -1. Hence X is a diagonal matrix. Now since X commutes with any even permutation matrix and $N\geq3$, it commutes in particular with the permutation matrix associated with the cycle (i,j,k) for any $1<i<j<k$, and hence all the entries of X are the same. We conclude that X is a scalar matrix, as claimed.

Step 4. Our claim is that the set of matrices with nonzero trace is dense in SO_N.

At $N=2$ this is clear, since the set of elements in SO_2 having a given trace is finite. So assume $N>2$, and let $T\in SO_N\simeq SO(\mathbb{R}^N)$ with $Tr(T)=0$. Let $E\subset\mathbb{R}^N$ be a 2-dimensional subspace preserved by T, such that $T_{|E}\in SO(E)$.

Let $\varepsilon>0$ and let $S_\varepsilon\in SO(E)$ with $||T_{|E}-S_\varepsilon||<\varepsilon$, and with $Tr(T_{|E})\neq Tr(S_\varepsilon)$, in the $N=2$ case. Now define $T_\varepsilon\in SO(\mathbb{R}^N)=SO_N$ by:

$$T_{\varepsilon|E}=S_\varepsilon\quad,\quad T_{\varepsilon|E^\perp}=T_{|E^\perp}$$

It is clear that $||T-T_\varepsilon||\leq||T_{|E}-S_\varepsilon||<\varepsilon$ and that:

$$Tr(T_\varepsilon)=Tr(S_\varepsilon)+Tr(T_{|E^\perp})\neq0$$

Thus, we have proved our claim.

Step 5. Our claim is that $\mathbb{T}O_N$ is the normalizer of $\mathbb{T}SO_N$ in U_N, i.e. is the subgroup of U_N consisting of the unitaries U for which $U^{-1}XU \in \mathbb{T}SO_N$ for all $X \in \mathbb{T}SO_N$.

It is clear that the group $\mathbb{T}O_N$ normalizes $\mathbb{T}SO_N$, so in order to prove the result, we must show that if $U \in U_N$ normalizes $\mathbb{T}SO_N$ then $U \in \mathbb{T}O_N$.

First note that U normalizes SO_N. Indeed if $X \in SO_N$ then $U^{-1}XU \in \mathbb{T}SO_N$, so $U^{-1}XU = \lambda Y$ for some $\lambda \in \mathbb{T}$ and $Y \in SO_N$. If $Tr(X) \neq 0$, we have $\lambda \in \mathbb{R}$ and hence:

$$\lambda Y = U^{-1}XU \in SO_N$$

The set of matrices having nonzero trace being dense in SO_N, we conclude that $U^{-1}XU \in SO_N$ for all $X \in SO_N$. Thus, we have:

$$\begin{aligned} X \in SO_N &\implies (UXU^{-1})^t(UXU^{-1}) = I_N \\ &\implies X^tU^tUX = U^tU \\ &\implies U^tU \in SO_N' \end{aligned}$$

It follows that at $N \geq 3$ we have $U^tU = \alpha I_N$, with $\alpha \in \mathbb{T}$, since U is unitary. Hence we have $U = \alpha^{1/2}(\alpha^{-1/2}U)$ with $\alpha^{-1/2}U \in O_N$, and $U \in \mathbb{T}O_N$. If $N = 2$, $(U^tU)^t = U^tU$ gives again that $U^tU = \alpha I_2$, and we conclude as in the previous case.

Step 6. Our claim is that the inclusion $\mathbb{T}O_N \subset U_N$ is maximal in the category of compact groups.

Suppose indeed that $\mathbb{T}O_N \subset G \subset U_N$ is a compact group such that $G \neq U_N$. It is a well-known fact that the connected component of the identity in G is a normal subgroup, denoted G_0. Since we have $\mathbb{T}SO_N \subset G_0 \subset U_N$, we must have $G_0 = \mathbb{T}SO_N$. But since G_0 is normal in G, the group G normalizes $\mathbb{T}SO_N$, and hence $G \subset \mathbb{T}O_N$.

Step 7. Our claim is that the inclusion $PO_N \subset PU_N$ is maximal in the category of compact groups.

This follows from the above result. Indeed, if $PO_N \subset G \subset PU_N$ is a proper intermediate subgroup, then its preimage under the quotient map $U_N \to PU_N$ would be a proper intermediate subgroup of $\mathbb{T}O_N \subset U_N$, which is a contradiction. $\square$

We refer to [17] for more on the above questions, and for a proof of the fact that the related inclusion $O_N \subset O_N^*$ is maximal as well.

In connection now with our question, which is that of classifying the intermediate groups $O_N \subset G \subset U_N$, the above result leads to a dichotomy, coming from:

$$PG \in \{PO_N, PU_N\}$$

In the lack of a classification result here, which is probably something known, but that we were unable to find in the literature, here are some basic examples of such intermediate groups, which are of interest for us, for each of the 2 cases that can appear:

Proposition 5.4. *We have compact groups $O_N \subset G \subset U_N$ as follows:*

(1) *The groups $\mathbb{Z}_r O_N$ with $r \in \mathbb{N} \cup \{\infty\}$, whose projective versions equal PO_N, the biggest of which is the group $\mathbb{T}O_N$, which appears as affine lift of PO_N.*
(2) *The groups $U_N^d = \{U \in U_N | \det U \in \mathbb{Z}_d\}$ with $d \in 2\mathbb{N} \cup \{\infty\}$, interpolating between U_N^2 and $U_N^\infty = U_N$, whose projective versions equal PU_N.*

Proof. All the assertions are elementary, the idea being as follows:

(1) We have indeed compact groups $\mathbb{Z}_r O_N$ with $r \in \mathbb{N} \cup \{\infty\}$ as in the statement, whose projective versions are given by $P\mathbb{Z}_r O_N = PO_N$. At $r = \infty$ we obtain the group $\mathbb{T}O_N$, and the fact that this group appears as the affine lift of PO_N follows exactly as in the sphere case, by using the computation from the proof of Theorem 5.1.

(2) The formula $U_N^d = \{U \in U_N | \det U \in \mathbb{Z}_d\}$ with $d \in \mathbb{N} \cup \{\infty\}$ defines indeed a closed subgroup $U_N^d \subset U_N$, and in the case where d is even, this subgroup contains the orthogonal group O_N. As for the last assertion, namely $PU_N^d = PU_N$, this follows either be suitably rescaling the unitary matrices, or by applying the result in Theorem 5.3. □

The above results suggest that the solutions of $O_N \subset G \subset U_N$ should come from O_N, U_N, by succesively applying the constructions $G \to \mathbb{Z}_r G$ and $G \to G \cap U_N^d$. These operations do not exactly commute, but normally we should be led in this way to a 2-parameter series, unifying the two 1-parameter series from (1,2) above. However, some other groups like $\mathbb{Z}_N SO_N$ work too, so all this is probably a bit more complicated.

As already mentioned, all this looks like quite standard group and Lie algebra theory, but we unable to find a good reference here. So, in the lack of something better, the above results will be our final saying on the subject, along with the reference to [17].

In what follows we will be mostly interested in the group $\mathbb{T}O_N$, which fits with the spheres and tori that we already have. This group, and the whole series $\mathbb{Z}_r O_N$ with $r \in \mathbb{N} \cup \{\infty\}$ that it is part of, is easy, the precise result being as follows:

Theorem 5.5. *We have the following results:*

(1) *The group $\mathbb{T}O_N$ is easy, the corresponding category $\bar{P}_2 \subset P_2$ consisting of the pairings having the property that when flatenning, we have $\#\circ = \#\bullet$.*
(2) *More generally, $\mathbb{Z}_r O_N$ is easy, the corresponding category $P_2^r \subset P_2$ consisting of the pairings having the property that when flatenning, we have $\#\circ = \#\bullet\,(r)$.*

Proof. These results are standard and well-known, the proof being as follows:

(1) If we denote the standard corepresentation by $u = zv$, with $z \in \mathbb{T}$ and with $v = \bar{v}$, then in order to have $Hom(u^{\otimes k}, u^{\otimes l}) \neq \emptyset$, the z variabes must cancel, and in the case where they cancel, we obtain the same Hom-space as for O_N.

Now since the cancelling property for the z variables corresponds precisely to the fact that k, l must have the same numbers of $\circ$ symbols minus $\bullet$ symbols, the associated Tannakian category must come from the category of pairings $\bar{P}_2 \subset P_2$, as claimed.

(2) This is something that we already know at $r = 1, \infty$, where the group in question is $O_N, \mathbb{T}O_N$. The proof in general is similar, by writing $u = zv$ as above. □

Quite remarkably, the above result has the following converse:

Theorem 5.6. *The proper intermediate easy compact groups*

$$O_N \subset G \subset U_N$$

are precisely the groups $\mathbb{Z}_r O_N$, with $r \in \{2, 3, \ldots, \infty\}$.

Proof. According to our conventions for the easy quantum groups, which apply of course to the classical case, we must compute the following intermediate categories:

$$\mathcal{P}_2 \subset D \subset P_2$$

So, assume that we have such a category, $D \neq \mathcal{P}_2$, and pick an element $\pi \in D - \mathcal{P}_2$, assumed to be flat. We can modify π, by performing the following operations:

(1) First, we can compose with the basic crossing, in order to assume that π is a partition of type $\cap \ldots\ldots \cap$, consisting of consecutive semicircles. Our assumption $\pi \notin \mathcal{P}_2$ means that at least one semicircle is colored black, or white.

(2) Second, we can use the basic mixed-colored semicircles, and cap with them all the mixed-colored semicircles. Thus, we can assume that π is a nonzero partition of type $\cap \ldots\ldots \cap$, consisting of consecutive black or white semicircles.

(3) Third, we can rotate, as to assume that π is a partition consisting of an upper row of white semicircles, $\cup \ldots\ldots \cup$, and a lower row of white semicircles, $\cap \ldots\ldots \cap$. Our assumption $\pi \notin \mathcal{P}_2$ means that this latter partition is nonzero.

For $a, b \in \mathbb{N}$ consider the partition consisting of an upper row of a white semicircles, and a lower row of b white semicircles, and set:

$$\mathcal{C} = \left\{\pi_{ab}\Big|a, b \in \mathbb{N}\right\} \cap D$$

According to the above we have $\pi \in < \mathcal{C} >$. The point now is that we have:

(1) There exists $r \in \mathbb{N} \cup \{\infty\}$ such that $\mathcal{C}$ equals the following set:

$$\mathcal{C}_r = \left\{\pi_{ab}\Big|a = b(r)\right\}$$

This is indeed standard, by using the categorical axioms.

(2) We have the following formula, with P_2^r being as above:

$$< \mathcal{C}_r >= P_2^r$$

This is standard as well, by doing some diagrammatic work.

With these results in hand, the conclusion now follows. Indeed, with $r \in \mathbb{N} \cup \{\infty\}$ being as above, we know from the beginning of the proof that any $\pi \in D$ satisfies:

$$\pi \in < \mathcal{C} >=< \mathcal{C}_r >= P_2^r$$

Thus we have $D\subset P_2^r$. Conversely, we have as well:

$$P_2^r=<\mathcal{C}_r>=<\mathcal{C}>\subset<D>=D$$

Thus we have $D=P_2^r$, and this finishes the proof. See [85]. □

As a conclusion, $\mathbb{T}O_N$ is definitely the "privileged" unitary group that we were looking for, with the remark that its arithmetic versions $\mathbb{Z}_rO_N$ are interesting as well.

Finally, let us discuss the reflection group case. Here the problem is that of classifying the intermediate compact groups $H_N\subset G\subset K_N$, and this looks of course like something which is probably well-known, as part of the general reflection group theory.

In practice, however, the situation is considerably more complicated than in the continuous group case, with the expected 2-parameter series there being replaced by an expected 3-parameter series. So, instead of getting into this quite technical subject, let us just formulate a basic result, explaining what the 3 parameters are:

Proposition 5.7. *We have compact groups $H_N\subset G\subset K_N$ as follows:*

1. *The groups $\mathbb{Z}_rH_N$, with $r\in\mathbb{N}\cup\{\infty\}$.*
2. *The groups $H_N^s=\mathbb{Z}_s\wr S_N$, with $s\in 2\mathbb{N}$.*
3. *The groups $H_N^{sd}=H_N^s\cap U_N^d$, with $d|s$ and $s\in 2\mathbb{N}$.*

Proof. The constructions in the statement produce indeed closed subgroups $G\subset K_N$, for all the possible values of the parameters. Regarding now the condition $H_N\subset G$, this is automatic for the construction (1), and follows from $s\in 2\mathbb{N}$ in (2), and from $d|s$ and $s\in 2\mathbb{N}$ in (3). Thus, we are led to the conclusion in the statement. □

The same discussion as in the continuous case applies, the idea being that the constructions $G\to\mathbb{Z}_rG$ and $G\to G\cap H_N^{sd}$ can be combined, and that all this leads in principle to a 3-parameter series. All this is, however, quite technical, and we do not really know if it is so. We will actually not need all this, so we will just stop our study here, and recommend to the interested reader the complex reflection group literature.

As in the continuous case, a solution to these classification problems comes from the notion of easiness. We have indeed the following result, coming from [11], [85]:

Theorem 5.8. *The following groups are easy:*

1. *$\mathbb{Z}_rH_N$, the corresponding category $P_{even}^r\subset P_{even}$ consisting of the partitions having the property that when flatenning, we have $\#\circ=\#\bullet\,(r)$.*
2. *$H_N^s=\mathbb{Z}_s\wr S_N$, the corresponding category $P_{even}^{(s)}\subset P_{even}$ consisting of the partitions having the property that we have $\#\circ=\#\bullet\,(s)$, in each block.*

In addition, the easy solutions of $H_N\subset G\subset K_N$ appear by combining these examples.

Proof. All this is well-known, the idea being as follows:

(1) The computation here is similar to the one in the proof of Theorem 5.5, by writing the fundamental representation $u=zv$ as there.

(2) This is something very standard and fundamental, known since the paper [11], and which follows from a long, routine computation, perfomed there.

As for the last assertion, things here are quite technical, and for the precise statement and proof of the classification result, we refer here to paper [85]. □

Summarizing, the situation here is more complicated than in the continuous group case. However, in what regards the "standard" solution, this is definitely $\mathbb{T}H_N$.

With all this preliminary work done, let us turn now to our main question, namely constructing new geometries. We have here the following result:

Theorem 5.9. *We have correspondences as follows,*

$$\begin{tikzcd}
\mathbb{T}S^{N-1}_{\mathbb{R}} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \mathbb{T}T_N \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
\mathbb{T}O_N \arrow[r, leftrightarrow] & \mathbb{T}H_N
\end{tikzcd}$$

which produce a new geometry, in the sense of Definition 3.21.

Proof. We have indeed a quadruplet (S,T,U,K) as in the statement, produced by the various constructions above. Regarding now the verification of the axioms:

(1) We have the following computation:

$$\begin{aligned}
P(\mathbb{T}S^{N-1}_{\mathbb{R}} \cap \mathbb{T}_N^+) &= P(\mathbb{T}S^{N-1}_{\mathbb{R}} \cap \mathbb{T}_N) \\
&\subset P\mathbb{T}S^{N-1}_{\mathbb{R}} \cap P\mathbb{T}_N \\
&= P^{N-1}_{\mathbb{R}} \cap \mathbb{T}_{N-1} \\
&= T_{N-1}
\end{aligned}$$

By lifting, we obtain from this that we have:

$$\mathbb{T}S^{N-1}_{\mathbb{R}} \cap \mathbb{T}_N^+ \subset \mathbb{T}T_N$$

The inclusion "$\supset$" being clear as well, we are done with checking the first axiom.

(2) The verification of the second axiom, namely $\mathbb{T}H_N \cap \mathbb{T}_N^+ = \mathbb{T}T_N$, is similar.

(3) The third axiom, $\mathbb{T}O_N \cap K_N^+ = \mathbb{T}H_N$, can be checked either directly, or by proceeding as above, by taking projective versions, and then lifting.

(4) The verification of the quantum isometry group axiom, namely $G^+(\mathbb{T}S^{N-1}_{\mathbb{R}}) = \mathbb{T}O_N$ is routine, and all this is explained for instance in [5].

(5) The quantum reflection group axiom, namely $G^+(\mathbb{T}T_N) \cap K_N^+ = \mathbb{T}H_N$, can be checked in a similar way, by adapting the computation from the classical real case.

(6) Regarding now the hard liberation axiom, this is clear, because we have:

$$\begin{aligned} < O_N, \mathbb{T}T_N > &= < O_N, \mathbb{T}, T_N > \\ &= < O_N, \mathbb{T} > \\ &= \mathbb{T}O_N \end{aligned}$$

(7) Finally, we have as well $S_{\mathbb{T}O_N} = \mathbb{T}S^{N-1}_\mathbb{R}$, and this completes the proof. $\square$

Let us discuss now the half-classical and free extensions of Theorem 5.9, and of some of the results preceding it. In order to have no redundant discussion and diagrams, we will talk directly about the $\times 9$ extension of the theory that we have so far.

We first need to complete our collection of spheres S, tori T, unitary groups U, and reflection groups K. In what regards the spheres, the result is as follows:

Proposition 5.10. *We have noncommutative spheres as follows,*

$$\begin{array}{ccccc} S^{N-1}_{\mathbb{R},+} & \rightarrow & \mathbb{T}S^{N-1}_{\mathbb{R},+} & \rightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R},*} & \rightarrow & \mathbb{T}S^{N-1}_{\mathbb{R},*} & \rightarrow & S^{N-1}_{\mathbb{C},*} \\ \uparrow & & \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \rightarrow & \mathbb{T}S^{N-1}_{\mathbb{R}} & \rightarrow & S^{N-1}_{\mathbb{C}} \end{array}$$

with the middle vertical objects coming via the relations $ab^* = a^*b$.

Proof. We can indeed construct new spheres via the relations $ab^* = a^*b$, and these fit into previous 6-diagram of spheres as indicated. As for the fact that in the classical case we obtain the previously constructed sphere $\mathbb{T}S^{N-1}_\mathbb{R}$, this follows from Theorem 5.1 and its proof, because the relations used there are precisely those of type $a\bar{b} = \bar{a}b$. $\square$

There are many things that can be done with the above spheres. As a basic result here, let us record the following fact, regarding the corresponding projective spaces:

Theorem 5.11. *The projective spaces associated to the basic spheres are*

$$\begin{array}{ccccc} P^{N-1}_+ & \longrightarrow & P^{N-1}_+ & \longrightarrow & P^{N-1}_+ \\ \uparrow & & \uparrow & & \uparrow \\ P^{N-1}_{\mathbb{C}} & \longrightarrow & P^{N-1}_{\mathbb{C}} & \longrightarrow & P^{N-1}_{\mathbb{C}} \\ \uparrow & & \uparrow & & \uparrow \\ P^{N-1}_{\mathbb{R}} & \longrightarrow & P^{N-1}_{\mathbb{R}} & \longrightarrow & P^{N-1}_{\mathbb{C}} \end{array}$$

via the standard identifications for noncommutative algebraic manifolds.

Proof. This is something that we already know for the 6 previous spheres. As for the 3 new spheres, this follows from the defining relations $ab^* = a^*b$, which tell us that the coordinates of the corresponding projective spaces must be self-adjoint. □

At the torus level now, the construction is similar, as follows:

Proposition 5.12. *We have noncommutative tori as follows,*

$$\begin{array}{ccccc} T_N^+ & \longrightarrow & \mathbb{T}T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ T_N^* & \longrightarrow & \mathbb{T}T_N^* & \longrightarrow & \mathbb{T}_N^* \\ \uparrow & & \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

with the middle vertical objects coming via the relations $ab^* = a^*b$.

Proof. This is clear from Proposition 5.10, by intersecting everything with $\mathbb{T}_N^+$. □

In what regards the unitary quantum groups, the result is as follows:

Theorem 5.13. *We have quantum groups as follows, which are all easy,*

$$\begin{tikzcd}
O_N^+ \arrow[r] & \mathbb{T}O_N^+ \arrow[r] & U_N^+ \\
O_N^* \arrow[r] \arrow[u] & \mathbb{T}O_N^* \arrow[r] \arrow[u] & U_N^* \arrow[u] \\
O_N \arrow[r] \arrow[u] & \mathbb{T}O_N \arrow[r] \arrow[u] & U_N \arrow[u]
\end{tikzcd}$$

with the middle vertical objects coming via the relations $ab^* = a^*b$.

Proof. This is standard, indeed, the categories of partitions being as follows:

$$\begin{tikzcd}
NC_2 \arrow[d] & \bar{NC}_2 \arrow[l] \arrow[d] & \mathcal{NC}_2 \arrow[l] \arrow[d] \\
P_2^* \arrow[d] & \bar{P}_2^* \arrow[l] \arrow[d] & \mathcal{P}_2^* \arrow[l] \arrow[d] \\
P_2 & \bar{P}_2 \arrow[l] & \mathcal{P}_2 \arrow[l]
\end{tikzcd}$$

Observe that our diagrams are both intersection diagrams. □

Regarding the quantum reflection groups, we have here:

Theorem 5.14. *We have quantum groups as follows, which are all easy,*

$$\begin{tikzcd}
H_N^+ \arrow[r] & \mathbb{T}H_N^+ \arrow[r] & K_N^+ \\
H_N^* \arrow[r] \arrow[u] & \mathbb{T}H_N^* \arrow[r] \arrow[u] & K_N^* \arrow[u] \\
H_N \arrow[r] \arrow[u] & \mathbb{T}H_N \arrow[r] \arrow[u] & K_N \arrow[u]
\end{tikzcd}$$

with the middle vertical objects coming via the relations $ab^* = a^*b$.

Proof. This is standard, indeed, the categories of partitions being as follows:

$$\begin{tikzcd}
NC_{even} & \bar{NC}_{even} \arrow[l] \arrow[d] & \mathcal{NC}_{even} \arrow[l] \arrow[d] \\
P^*_{even} \arrow[from=u] & \bar{P}^*_{even} \arrow[l] \arrow[d] & \mathcal{P}^*_{even} \arrow[l] \arrow[d] \\
P_{even} \arrow[from=u] & \bar{P}_{even} \arrow[l] & \mathcal{P}_{even} \arrow[l]
\end{tikzcd}$$

Observe that our diagrams are both intersection diagrams. □

Summarizing, we have new quadruplets (S,T,U,K). Before going further, with the verification of the geometric axioms, let us make however a number of comments on classification issues for the half-classical and free "hybrid" quantum groups. The situation here is a bit different to the one in the classical case, because:

(1) We have $PO_N^+ = PU_N^+$, and we have as well $PO_N^* = PU_N^*$. All this is in contrast with the result from the classical case, stating that $PO_N \subset PU_N$ is maximal. In short, the dichotomy real/complex at the projective level dissapears.

(2) The determinant, which produces many interesting classical groups, has no free analogue, and has no half-classical analogue either. All this is a bit folklore, but results can be obtained by using soft and hard liberation methods.

In short, we have some interesting questions here, regarding the classification of the intermediate compact quantum groups for the following 4 inclusions:

$$\begin{tikzcd}
& K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[ur, Rightarrow] \arrow[rr] & & O_N^+ \arrow[ur, Rightarrow] & \\
& K_N^* \arrow[uu] \arrow[rr] & & U_N^* \arrow[uu] \\
H_N^* \arrow[uu] \arrow[ur, Rightarrow] \arrow[rr] & & O_N^* \arrow[uu] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

In what regards the half-classical questions, these can be in principle fully investigated by using the technology in [37], but we do not know what the final answer is. As for the free questions, these are more delicate, but in the easy case, they are solved by [85].

Let us go back now to our questions, and prove that the new quadruplets (S,T,U,K) that we constructed satisfy indeed the geometric axioms from section 3.

We first need quantum isometry group results. We first have here:

Theorem 5.15. *The quantum isometries of the basic spheres, namely*

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb{R},+} & \longrightarrow & \mathbb{T}S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+}\\
\uparrow & & \uparrow & & \uparrow\\
S^{N-1}_{\mathbb{R},*} & \longrightarrow & \mathbb{T}S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{C},*}\\
\uparrow & & \uparrow & & \uparrow\\
S^{N-1}_{\mathbb{R}} & \longrightarrow & \mathbb{T}S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}}
\end{array}$$

are the basic unitary quantum groups.

Proof. This is routine, by lifting the results that we already have. ☐

Regarding now the tori, we first have here:

Proposition 5.16. *The quantum isometries of the basic tori are*

$$\begin{array}{ccccc}
H_N^+ & \longrightarrow & \mathbb{T}H_N^+ & \longrightarrow & K_N^+\\
\uparrow & & \uparrow & & \uparrow\\
H_N^* & \longrightarrow & \mathbb{T}H_N^* & \longrightarrow & K_N^*\\
\uparrow & & \uparrow & & \uparrow\\
\bar{O}_N & \longrightarrow & \mathbb{T}\bar{O}_N & \longrightarrow & \bar{U}_N
\end{array}$$

with the bars denoting as usual Schur-Weyl twists.

Proof. The result follows by lifting the results that we already have. ☐

By looking now at quantum reflections, we obtain:

Theorem 5.17. *The quantum reflections of the tori,*

$$\begin{tikzcd}
T_N^+ \arrow[r] & \mathbb{T}T_N^+ \arrow[r] & \mathbb{T}_N^+ \\
T_N^* \arrow[r] \arrow[u] & \mathbb{T}T_N^* \arrow[r] \arrow[u] & \mathbb{T}_N^* \arrow[u] \\
T_N \arrow[r] \arrow[u] & \mathbb{T}T_N \arrow[r] \arrow[u] & \mathbb{T}_N \arrow[u]
\end{tikzcd}$$

are the basic quantum reflection groups.

Proof. This is indeed routine, by intersecting. □

Finally, we have hard liberation results, as follows:

Theorem 5.18. *We have hard liberation formulae of type*

$$U = < O_N, T >$$

for all the basic unitary quantum groups.

Proof. We only need to check this for the "hybrid" examples, constructed in this section. But for these hybrid examples, $U = \mathbb{T}O_N^\times$, the results follow from:

$$\begin{aligned}
\mathbb{T}O_N^\times &= < \mathbb{T}, O_N^\times > \\
&= < \mathbb{T}, < O_N, T_N^\times >> \\
&= < O_N, < \mathbb{T}, T_N^\times >> \\
&= < O_N, \mathbb{T}T_N^\times >
\end{aligned}$$

Thus, we have indeed complete hard liberation results, as claimed. □

We can now formulate our main result, as follows:

Theorem 5.19. *We have* 9 *noncommutative geometries, as follows,*

$$\begin{tikzcd}
\mathbb{R}^N_+ \arrow[r] & \mathbb{TR}^N_+ \arrow[r] & \mathbb{C}^N_+ \\
\mathbb{R}^N_* \arrow[r] \arrow[u] & \mathbb{TR}^N_* \arrow[r] \arrow[u] & \mathbb{C}^N_* \arrow[u] \\
\mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{TR}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u]
\end{tikzcd}$$

with each of the $\mathbb{K}^\times$ *symbols standing for the corresponding quadruplet.*

Proof. This follows indeed by putting everything together, a bit as in the proof of Theorem 5.9, the idea being that the intersection axioms are clear, the quantum isometry axioms follow from the above computations, and the remaining axioms are elementary. $\square$

As a comment, we have in principle some arithmetic versions as well, obtained by replacing all the above products by $\mathbb{T}$ by products by an arbitrary cyclic group, $\mathbb{Z}_r$ with $r \in \{1, 2, \ldots, \infty\}$. However, all this requires some new verifications, which are not available yet. We will not get here into this subject, which is too technical anyway.

6. Classification

We have seen that the quadruplets of type (S,T,U,K), generalizing those coming from the usual geometries of $\mathbb{R}^N,\mathbb{C}^N$, can be axiomatized. At the level of main examples, we have 9 such quadruplets, coming from noncommutative geometries as follows:

$$\begin{array}{ccccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{T}\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+\\
\uparrow & & \uparrow & & \uparrow\\
\mathbb{R}^N_* & \longrightarrow & \mathbb{T}\mathbb{R}^N_* & \longrightarrow & \mathbb{C}^N_*\\
\uparrow & & \uparrow & & \uparrow\\
\mathbb{R}^N & \longrightarrow & \mathbb{T}\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

All this is quite nice, and our belief is that, under some extra axioms, probably of rather mild type, these 9 geometries should be the only ones.

The problem, however, is that we don't have yet such a classification result. So, in the lack of something here, we will present some partial results, as follows:

(1) A first idea is that of assuming that S,T are monomial, in the sense that they come from commutation-type relations between the coordinates. We will present some classification results here, and leave the general case open.

(2) A second idea is that of assuming that U,K are easy quantum groups. This is something less restrictive, and once again we are led into some combinatorics, this time involving categories of partitions, that we will discuss here.

We will discuss as well such questions in the projective geometry setting. Here we do have nice classification results, but since we do not have yet axioms for the quadruplets of type (PS,PT,PU,PK), our study here will be something partial as well.

To be more precise, in what regards projective geometry, the problem which is open is that of axiomatizing these geometries, with correspondences as follows:

$$\begin{tikzcd}
PS \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & PT \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
PU \arrow[r, leftrightarrow] & PK
\end{tikzcd}$$

Modulo this issue, things are potentially quite nice, because we seem to have only 3 geometries, namely real, complex and free. But, this is what we have, for the moment.

In order to get started, let us focus on the spheres S. Looking back at the definition of the 9 spheres that we already have, we are led into the following notion:

Definition 6.1. *A monomial sphere is a subset $S\subset S^{N-1}_{\mathbb C,+}$ obtained via relations of type*

$$x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=x_{i_{\sigma(1)}}^{f_1}\ldots x_{i_{\sigma(k)}}^{f_k}\quad,\quad\forall(i_1,\ldots,i_k)\in\{1,\ldots,N\}^k$$

with $\sigma\in S_k$ being certain permutations, and with $e_r,f_r\in\{1,*\}$ being certain exponents.

As a first remark, the relations in the above definition are trivially satisfied for the standard coordinates of $S^{N-1}_{\mathbb R}$. Indeed, here the exponents do not matter, and what we have is a relation of type $x_{i_1}\ldots x_{i_k}=x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$, which does hold, by commutativity. Thus, the monomial spheres appear as intermediate subspaces, as follows:

$$S^{N-1}_{\mathbb R}\subset S\subset S^{N-1}_{\mathbb C,+}$$

Also at the theoretical level, since the exponents can be regarded as elements $e,f\in\mathbb Z_2^k$, a monomial sphere is ultimately a group-theoretical object, coming from a certain subset of the crossed product of S_∞ by two copies of $\mathbb Z_2^\infty$. We will be back later to this.

Finally, we can talk as well, in a similar way, about monomial tori, which appear as certain intermediate group duals $T_N\subset T\subset\mathbb T_N^+$. Dually, when setting $T=\widehat{\Gamma}$, what we are looking at here are certain intermediate discrete group quotients, as follows:

$$F_\infty\to\Gamma\to\mathbb Z_2^{\infty}$$

Summarizing, all this belongs to group theory. In practice now, in order to formulate some results, let us agree to represent the relations in Definition 6.1 by "colored permutations", in the obvious way. With this convention, we first have:

Theorem 6.2. *The 9 basic spheres are all monomial, with the real and hybrid ones appearing via the following relations,*

with the classical and half-classical ones appearing via the following relations, taken with all the possible matching colorings of the diagrams,

and with the remaining spheres being obtained as intersections.

Proof. We know that $S^{N-1}_{\mathbb{R},+}$, $\mathbb{T}S^{N-1}_{\mathbb{R}}$ come respectively from the relations $a = a^*$, $ab^* = a^*b$, applied to the basic coordinates $x_1, \ldots, x_N$, and this gives the first assertion.

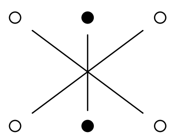
Regarding now $S^{N-1}_{\mathbb{C}}$, $S^{N-1}_{\mathbb{C},*}$, these come respectively from the relations $ab = ba$, $abc = cba$, applied to the basic coordinates $x_1, \ldots, x_N$, and their adjoints. Thus, we are led to the diagrams in the statement, with all the possible matching colorings, as stated.

Finally, the remaining spheres appear by definition as intersections, and according to our conventions in Definition 6.1, they follow to be monomial as well. □

In order to obtain now a uniqueness result for our 9 spheres, we have a quite clear potential method, in three steps, as follows:

(1) $S^{N-1}_{\mathbb{R},*}$ is the only monomial sphere $S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{R},+}$.
(2) $\mathbb{T}S^{N-1}_{\mathbb{R}}$ is the only monomial sphere $S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C}}$.
(3) S depends only on $S_r = S \cap S^{N-1}_{\mathbb{R},+}$ and $S_c = S \cap S^{N-1}_{\mathbb{C}}$.

We will see in what follows that (1,2) do work. However, (3) does not work as stated. In fact, the uniqueness result that we are trying to prove is wrong (!), and there are more than 9 monomial spheres, in our sense. Indeed, we have for instance the sphere $S^{N-1}_{\mathbb{C},\times}$ coming from the relations $ab^*c = cb^*a$, corresponding to the following diagram:



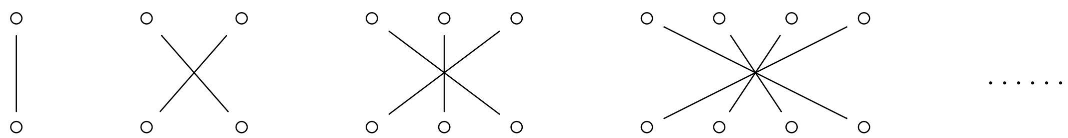
This latter sphere is actually a quite interesting object, coming from the considerations in [33], [34]. However, while being monomial, this sphere does not exactly fit with our noncommutative geometry considerations here. To be more precise:

(1) According to the work in [5], [15], this sphere is part of a triple $(S^{N-1}_{\mathbb{C},\times}, \mathbb{T}^\times_N, U^\times_N)$, satisfying a simplified set of noncommutative geometry axioms.
(2) However, according to the work in [76], [77], suitably adapted to our questions, the quantum group $U^\times_N$ has no reflection group counterpart $K^\times_N$.

Summarizing, we are in a bit of trouble here. We will discuss in what follows the real and classical cases, where we do have results, and leave the general case open.

Let us first discuss the real case. As a first remark, Theorem 4.1 reformulates as:

Theorem 6.3. *The real monomial spheres coming from mirroring permutations,*



are precisely the 3 *main real spheres,* $S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$.

Proof. This follows indeed from Theorem 4.1, because the relations $a_1 \dots a_k = a_k \dots a_1$ used there are precisely those coming from mirroring permutations. □

We will prove in what follows, following [23], that the basic 3 real spheres are the only monomial ones. For this purpose, it is convenient to introduce the inductive limit $S_\infty = \bigcup_{k \geq 0} S_k$, with the inclusions $S_k \subset S_{k+1}$ being given by:

$$\sigma \in S_k \implies \sigma(k+1) = k+1$$

In terms of S_∞, the definition of the monomial spheres reformulates as follows:

Proposition 6.4. *The monomial spheres are the subsets $S \subset S^{N-1}_{\mathbb{R},+}$ obtained via relations*

$$x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}, \ \forall (i_1, \dots, i_k) \in \{1, \dots, N\}^k$$

associated to certain elements $\sigma \in S_\infty$, where $k \in \mathbb{N}$ is such that $\sigma \in S_k$.

Proof. We must prove that the relations $x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}$ are left unchanged when replacing $k \to k+1$. But this follows from $\sum_i x_i^2 = 1$, because:

$$\begin{aligned}
& x_{i_1} \dots x_{i_k} x_{i_{k+1}} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}} x_{i_{k+1}} \\
\implies \quad & x_{i_1} \dots x_{i_k} x_{i_{k+1}}^2 = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}} x_{i_{k+1}}^2 \\
\implies \quad & \sum_{i_{k+1}} x_{i_1} \dots x_{i_k} x_{i_{k+1}}^2 = \sum_{i_{k+1}} x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}} x_{i_{k+1}}^2 \\
\implies \quad & x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}
\end{aligned}$$

Thus we can indeed "simplify at right", and this gives the result. □

In order to prove now the uniqueness result for our 3 spheres, we use group theory methods. We call a subgroup $G \subset S_\infty$ filtered when it is stable under concatenation, in the sense that when writing $G = (G_k)$ with $G_k \subset S_k$, we have:

$$\sigma \in G_k, \pi \in G_l \implies \sigma\pi \in G_{k+l}$$

With this convention, we have the following result:

Theorem 6.5. *The monomial spheres are the subsets $S_G \subset S^{N-1}_{\mathbb{R},+}$ given by*

$$C(S_G) = C(S^{N-1}_{\mathbb{R},+}) \Big/ \Big\langle x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}, \forall (i_1, \dots, i_k) \in \{1, \dots, N\}^k, \forall \sigma \in G_k \Big\rangle$$

where $G = (G_k)$ is a filtered subgroup of $S_\infty = (S_k)$.

Proof. We know from Proposition 6.4 that the construction in the statement produces a monomial sphere. Conversely, given a monomial sphere $S \subset S^{N-1}_{\mathbb{R},+}$, let us set:

$$G_k = \Big\{ \sigma \in S_k \Big| x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}, \forall (i_1, \dots, i_k) \in \{1, \dots, N\}^k \Big\}$$

With $G = (G_k)$ we have $S = S_G$, so it remains to prove that G is a filtered group.

Since the relations $x_{i_1}\ldots x_{i_k}=x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$ can be composed and reversed, each G_k follows to be stable under composition and inversion, and is therefore a group.

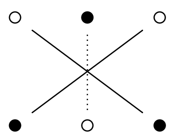
Also, since the relations $x_{i_1}\ldots x_{i_k}=x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$ can be concatenated as well, our group $G=(G_k)$ is stable under concatenation, and we are done. □

At the level of examples, the groups $\{1\}\subset S_\infty$ produce the spheres $S^{N-1}_{\mathbb{R},+}\supset S^{N-1}_{\mathbb{R}}$. In order to discuss now the half-liberated case, we will need:

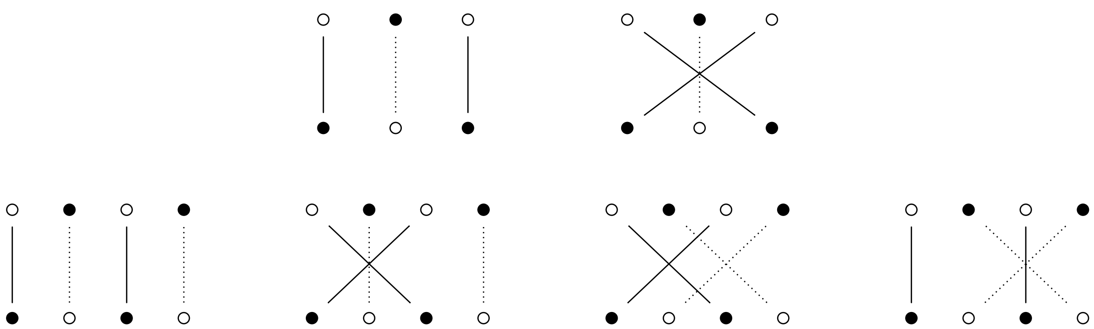
Proposition 6.6. *Let $S_\infty^*\subset S_\infty$ be the set of permutations having the property that when labelling cyclically the legs $\bullet\circ\bullet\circ\ldots$, each string joins a black leg to a white leg.*

1. *S_∞^* is a filtered subgroup of S_∞, generated by the half-liberated crossing.*
2. *We have $S_{2k}^*\simeq S_k\times S_k$, and $S_{2k+1}^*\simeq S_k\times S_{k+1}$, for any $k\in\mathbb{N}$.*

Proof. The fact that S_∞^* is indeed a subgroup of S_∞, which is filtered, is clear. Observe now that the half-liberated crossing has the "black-to-white" joining property:



Thus this crossing belongs to S_3^*, and it is routine to check, by double inclusion, that the filtered subgroup of S_∞ generated by it is the whole S_∞^*. Regarding now the last assertion, observe first that S_3^*,S_4^* consist of the following permutations:



Thus we have $S_3^*=S_1\times S_2$ and $S_4^*=S_2\times S_2$, with the first component coming from dotted permutations, and with the second component coming from the solid line permutations. The same argument works in general, and gives the last assertion. □

Now back to the main 3 real spheres, the result is as follows:

Proposition 6.7. *The basic monomial real spheres, namely*

$$S^{N-1}_{\mathbb{R}}\subset S^{N-1}_{\mathbb{R},*}\subset S^{N-1}_{\mathbb{R},+}$$

come respectively from the filtered groups $S_\infty\supset S_\infty^*\supset\{1\}$ via the above correspondence.

Proof. This is clear by definition in the classical and in the free cases. In the half-liberated case, the result follows from Proposition 6.6 (1) above. □

Now back to the general case, consider a monomial sphere $S_G\subset S^{N-1}_{\mathbb R,+}$, with the filtered group $G\subset S_\infty$ taken to be maximal, as in the proof of Theorem 6.5. We have:

Proposition 6.8. *The filtered group $G\subset S_\infty$ associated to a monomial sphere $S\subset S^{N-1}_{\mathbb R,+}$ is stable under the following operations, on the corresponding diagrams:*

1. *Removing outer strings.*
2. *Removing neighboring strings.*

Proof. Both these results follow by using the quadratic condition:

(1) Regarding the outer strings, by summing over a, we have indeed:

$$Xa=Ya\implies Xa^2=Ya^2\implies X=Y$$

$$aX=aY\implies a^2X=a^2Y\implies X=Y$$

(2) Regarding the neighboring strings, once again by summing over a, we have:

$$XabY=ZabT\implies Xa^2Y=Za^2T\implies XY=ZT$$

$$XabY=ZbaT\implies Xa^2Y=Za^2T\implies XY=ZT$$

Thus $G=(G_k)$ has both the properties in the statement. □

We are now in position of stating and proving a main result, as follows:

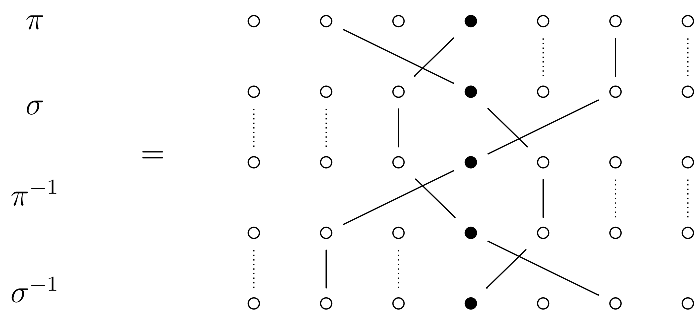
Theorem 6.9. *There is only one intermediate monomial sphere*

$$S^{N-1}_{\mathbb R}\subset S\subset S^{N-1}_{\mathbb R,+}$$

namely the half-classical real sphere $S^{N-1}_{\mathbb R,*}$.

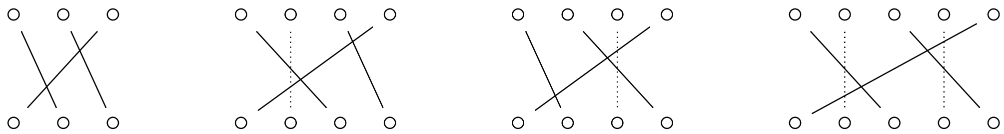
Proof. We will prove that the only filtered groups $G\subset S_\infty$ satisfying the conditions in Proposition 6.8 are $\{1\}\subset S_\infty^*\subset S_\infty$, correspoding to our 3 spheres. In order to do so, consider such a filtered group $G\subset S_\infty$, assumed to be non-trivial, $G\neq\{1\}$.

Step 1. Our first claim is that G contains a 3-cycle. For this purpose, we use a standard trick, stating that if $\pi,\sigma\in S_\infty$ have support overlapping on exactly one point, say $supp(\pi)\cap supp(\sigma)=\{i\}$, then the commutator $\sigma^{-1}\pi^{-1}\sigma\pi$ is a 3-cycle, namely $(i,\sigma^{-1}(i),\pi^{-1}(i))$. Indeed the computation of the commutator goes as follows:



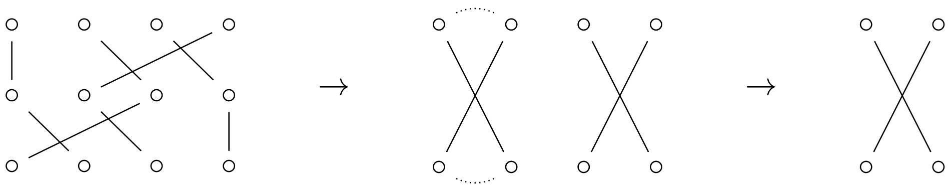
Now let us pick a non-trivial element $\tau \in G$. By removing outer strings at right and at left we obtain permutations $\tau' \in G_k, \tau'' \in G_s$ having a non-trivial action on their right/left leg, and by taking $\pi = \tau' \otimes id_{s-1}, \sigma = id_{k-1} \otimes \tau''$, the trick applies.

Step 2. Our second claim is G must contain one of the following permutations:

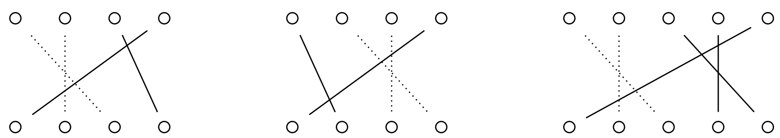


Indeed, consider the 3-cycle that we just constructed. By removing all outer strings, and then all pairs of adjacent vertical strings, we are left with these permutations.

Step 3. Our claim now is that we must have $S_\infty^* \subset G$. Indeed, let us pick one of the permutations that we just constructed, and apply to it our various diagrammatic rules. From the first permutation we can obtain the basic crossing, as follows:



Also, by removing a suitable $\slash\!\!\!\backslash$ shaped configuration, which is represented by dotted lines in the diagrams below, we can obtain the basic crossing from the second and third permutation, and the half-liberated crossing from the fourth permutation:



Thus, in all cases we have a basic or half-liberated crossing, and so $S_\infty^* \subset G$.

Step 4. Our last claim, which will finish the proof, is that there is no proper intermediate subgroup $S_\infty^* \subset G \subset S_\infty$. In order to prove this, observe that $S_\infty^* \subset S_\infty$ is the subgroup of parity-preserving permutations, in the sense that "i even $\implies \sigma(i)$ even".

Now let us pick an element $\sigma \in S_k - S_k^*$, with $k \in \mathbb{N}$. We must prove that the group $G =< S_\infty^*, \sigma >$ equals the whole S_∞. In order to do so, we use the fact that σ is not parity preserving. Thus, we can find i even such that $\sigma(i)$ is odd.

In addition, up to passing to $\sigma|$, we can assume that $\sigma(k) = k$, and then, up to passing one more time to $\sigma|$, we can further assume that k is even.

Since both i, k are even we have $(i, k) \in S_k^*$, and so $\sigma(i, k)\sigma^{-1} = (\sigma(i), k)$ belongs to G. But, since $\sigma(i)$ is odd, by deleting an appropriate number of vertical strings, $(\sigma(i), k)$ reduces to the basic crossing $(1, 2)$. Thus $G = S_\infty$, and we are done. $\square$

Regarding now the hybrid case, we have here the following result:

Theorem 6.10. *There is only one intermediate monomial sphere*

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C}}$$

namely the hybrid classical sphere $\mathbb{T}S^{N-1}_{\mathbb{R}}$.

Proof. Assume indeed that we have a sphere as in the statement, obtained via monomial relations. Since the variables commute, the monomial relations can be written as:

$$x_{i_1}^{e_1}\ldots x_{i_k}^{e_k} = x_{i_1}^{f_{\sigma^{-1}(1)}}\ldots x_{i_k}^{f_{\sigma^{-1}(k)}}$$

Assuming that we have non-trivial exponents here, we are in need of a relation of type $x_i^* = \lambda x_i$, connecting the coordinates and their adjoints, and this gives the result. □

As mentioned before, the classification of the monomial spheres in general, in connection with our noncommutative geometry considerations, is a quite tricky question.

We will be back to this later, after discussing our second idea for the general classification question of our noncommutative geometries, based on the notion of easiness.

Let us discuss now our second approach to classification problems, using this time easiness. In order to get started, we will need some preliminaries. We first have:

Proposition 6.11. *The intersection and generation operation for closed subgroups of* U_N^+ *are given, at the Tannakian level, by the following formulae:*

(1) $C_{G\cap H} =< C_G, C_H >$.
(2) $C_{<G,H>} = C_G \cap C_H$.

Proof. This is standard, coming from the functoriality properties of the Tannakian duality correspondence $G \to C_G$. For full details on this, we refer for instance in [44]. □

In the easy case now, the Tannakian categories come by definition from partitions, so we would like to know how Proposition 6.11 reformulates, in a purely combinatorial way. In what regards the intersection operation, the statement here is very simple:

Proposition 6.12. *Assuming that* G, H *are easy, then so is* $G \cap H$*, and we have*

$$D_{G\cap H} =< D_G, D_H >$$

at the level of the corresponding categories of partitions.

Proof. We have indeed the following computation:

$$\begin{aligned} C_{G\cap H} &= < C_G, C_H > \\ &= < span(D_G), span(D_H) > \\ &= span(< D_G, D_H >) \end{aligned}$$

Thus, by Tannakian duality we obtain the result. □

Regarding the generation operation, the situation is more complicated. With the convention that $G \to G'$ denotes the easy envelope operation, which consists in considering the smallest easy quantum group G' containing G, the result is as follows:

Proposition 6.13. *Assuming that G, H are easy, we have inclusions*

$$< G, H > \subset < G, H >' \subset \{G, H\}$$

coming from inclusions of Tannakian categories as follows,

$$C_G \cap C_H \supset span\left(T_\pi \middle| T_\pi \in C_G \cap C_H\right) \supset span(D_G \cap D_H)$$

where $\{G, H\}$ is the easy quantum group having as category of partitions $D_G \cap D_H$.

Proof. We have indeed the following computation, with the last inclusion being clear:

$$\begin{aligned} C_{<G,H>} &= C_G \cap C_H \\ &= span(D_G) \cap span(D_H) \\ &\supset span(D_G \cap D_H) \end{aligned}$$

By Tannakian duality we obtain from this all the assertions. □

It is not clear if the inclusions in Proposition 6.13 are isomorphisms or not, and this even with a $N >> 0$ assumption added. Technically speaking, the problem comes from the fact that the operation $\pi \to T_\pi$ does not produce linearly independent maps.

Our belief is that there should be a notion of "asymptotic easy quantum group", based on the theory in [29], [100], making the formula $D_{<G,H>} = D_G \cap D_H$ work. This is not known yet, and in the lack of such a theory, we will use Proposition 6.13 as it is.

As a conclusion to all these considerations, we have:

Theorem 6.14. *The intersection and easy generation operations $\cap$ and $\{\,,\}$ can be constructed via the Tannakian correspondence $G \to D_G$, as follows:*

1. *Intersection: defined via $D_{G \cap H} = < D_G, D_H >$.*
2. *Easy generation: defined via $D_{\{G,H\}} = D_G \cap D_H$.*

Proof. Here (1) is an true and honest result, coming from Proposition 6.12, and (2) is an empty statement, related to the difficulties that we met in Proposition 6.13. □

Let us go back now to our questions, regarding the axiomatization of the "easy" geometries in our (S, T, U, K) sense, and more specifically to the pairs (U, K) satisfying $U = < O_N, K >$ and $K = U \cap K_N^+$. In view of Theorem 6.14, we can formulate:

Definition 6.15. *A geometry (S, T, U, K) is called easy when U, K are easy, and*

$$U = \{O_N, K\}$$

with the operation on the right being the easy generation operation.

In other words, the easiness condition asks of course for U,K to be easy, and asks as well for $<O_N,K>=\{O_N,K\}$ to be satisfied. All this is perhaps not very elegant, but in view of the difficulties with $<,>$ explained above, we must proceed in this way.

The easy geometries in the above sense can be investigated by using:

Theorem 6.16. *An easy geometry is uniquely determined by a pair (D,E) of categories of partitions, which must be as follows,*

$$\mathcal{NC}_2\subset D\subset P_2$$

$$\mathcal{NC}_{even}\subset E\subset P_{even}$$

and which are subject to the following intersection and generation conditions,

$$D=E\cap P_2$$

$$E=<D,\mathcal{NC}_{even}>$$

and to the usual axioms for the associated quadruplet (S,T,U,K), where U,K are respectively the easy quantum groups associated to the categories D,E.

Proof. This simply comes from the conditions $U=\{O_N,K\}$ and $K=U\cap K_N^+$, reformulated via Theorem 6.14. To be more precise, let us look at Definition 6.15. The main condition there tells us that U,K must be easy, coming from certain categories D,E. It is clear that D,E must appear as intermediate categories, as in the statement, and the fact that the intersection and generation conditions must be satisfied follows from:

$$\begin{array}{rcl}U=\{O_N,K\}&\iff&D=E\cap P_2\\ K=U\cap K_N^+&\iff&E=<D,\mathcal{NC}_{even}>\end{array}$$

Thus, we are led to the conclusion in the statement. $\square$

Generally speaking, the idea now is that everything can be reformulated in terms of (D,E), which must satisfy the conditions in Theorem 6.16. Instead of discussing the full reformulation, let us work out at least the construction of the quadruplet (S,T,U,K).

In what regards the quantum groups, these come from Tannakian duality, in its "soft" form, the precise result being as follows:

Theorem 6.17. *The easy quantum group $G\subset U_N^+$ associated to a category of partitions $D\subset P$ appears as follows,*

$$C(G)=C(U_N^+)\Big/\left\langle T_\pi\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall k,l,\forall T\in D(k,l)\right\rangle$$

modulo the usual equivalence relation for the compact quantum groups.

Proof. This follows from Tannakian duality, in its "soft" form, worked out in [75], and explained in section 2 above. Indeed, given a tensor category $C = (C(k,l))$, the corresponding Woronowicz algebra $A = C(G)$ appears as follows:

$$C(G) = C(U_N^+) \Big/ \left\langle T \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall k, l, \forall T \in C(k,l) \right\rangle$$

In the case of a category of the form $C = span(D)$, with $D \subset P$ being a category of partitions, this gives the formula in the statement. □

In connection now with our questions, we have:

Theorem 6.18. *In the context of an easy geometry (S,T,U,K), we have:*

$$C(U) = C(U_N^+) \Big/ \left\langle T_\pi \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall k, l, \forall T \in D(k,l) \right\rangle$$

$$C(K) = C(K_N^+) \Big/ \left\langle T_\pi \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall k, l, \forall T \in D(k,l) \right\rangle$$

In fact, these formulae simply follow from the fact that U is easy.

Proof. This is clear indeed by applying Theorem 6.17 above. □

Regarding now the associated torus T, the result here is as follows:

Theorem 6.19. *In the context of an easy geometry (S,T,U,K), we have:*

$$\Gamma = F_N \Big/ \left\langle g_{i_1} \ldots g_{i_k} = g_{j_1} \ldots g_{j_l} \Big| \forall i, j, k, l, \exists \pi \in D(k,l), \delta_\pi \begin{pmatrix} i \\ j \end{pmatrix} \neq 0 \right\rangle$$

In fact, this formula simply follows from the fact that U is easy.

Proof. If we denote by $g_i = u_{ii}$ the standard coordinates on the associated torus T, then we have the following computation, with $g = diag(g_1, \ldots, g_N)$:

$$\begin{aligned} C(T) &= \left[C(U_N^+) \Big/ \left\langle T_\pi \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall \pi \in D \right\rangle \right] \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle \\ &= \left[C(U_N^+) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle \right] \Big/ \left\langle T_\pi \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall \pi \in D \right\rangle \\ &= C^*(F_N) \Big/ \left\langle T_\pi \in Hom(g^{\otimes k}, g^{\otimes l}) \Big| \forall \pi \in D \right\rangle \end{aligned}$$

Now observe that, with $g = diag(g_1, \ldots, g_N)$, we have:

$$\begin{aligned} T_\pi g^{\otimes k}(e_{i_1} \otimes \ldots \otimes e_{i_k}) &= \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l} \cdot g_{i_1} \ldots g_{i_k} \\ g^{\otimes l} T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) &= \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l} \cdot g_{j_1} \ldots g_{j_l} \end{aligned}$$

Thus we obtain the formula in the statement. Finally, the last assertion is clear. □

Finally, regarding the sphere S, the result here is as follows:

Theorem 6.20. *In the context of an easy geometry (S,T,U,K), we have*

$$C(S)=C(S^{N-1}_{\mathbb{C},+})\Big/\left\langle x_{i_1}\ldots x_{i_k}=x_{j_1}\ldots x_{j_k}\Big|\forall i,j,k,l,\exists\pi\in D(k)\cap I_k,\delta_\pi\begin{pmatrix}i\\ j\end{pmatrix}\neq 0\right\rangle$$

where $I_k\subset P_2(k,k)$ is the set of colored permutations.

Proof. This follows indeed from Theorem 6.18 above, by applying the construction $U\to S$, which amounts in taking the first column space. □

We should mention that all this can be subject to some further discussion. First, it is our feeling that the above formula for S, constructed in terms of $D\cap I_\infty$, should be rather part of our axioms, for the easy noncommutative geometries. Also, in connection with the monomial spheres, the corresponding category of partitions $D\subset P_2$ and filtered group $L\subset I_\infty$ should be related by the following formulae:

$$L=D\cap H_\infty\quad,\quad D=<L>$$

Leaving now aside this discussion, and going ahead, with the formalism that we have, the classification result that we can hope for is roughly as follows:

Theorem 6.21. *We have the following classification results for the easy geometries:*

(1) *In the real case, the 3 geometries that we have are the only ones.*
(2) *In the classical case, we have once again uniqueness, under an extra axiom.*
(3) *More generally, in the "pure" case we have uniqueness, under an extra axiom.*
(4) *In general, we have uniqueness as well, under an extra "slicing" axiom.*

Proof. This is of course something quite informal, with still work to be done here.

The idea indeed is that this should follow by using the conditions in Theorem 6.16 alone, by doing some combinatorics. Indeed, in terms of D, the main equation is:

$$D=<D,\mathcal{NC}_{even}>\cap P_2$$

But this equation can be solved by using the classification results in [76], [77], [84], [85], and we are led to the conclusion in the statement. To be more precise:

(1) This is something that we already know, simply coming from the fact that O_N^* is the unique intermediate easy quantum group $O_N\subset U\subset U_N^+$.

(2) This follows from the classification results in [85], when adding a mild extra axiom, in order to take care of the arithmetic versions of the hybrid geometries.

(3) Here by "pure" we mean real, classical, complex or free, and the proof is quite long and technical, using the various classification results from [76], [77], [84], [85].

(4) This is something which follows from (1,2), or from (3), with the slicing axiom being something quite technical, formulated by using the method in [10]. □

Summarizing, all this is quite technical. It is certainly possible to clarify all the above, but there is no hurry here. Indeed, our feeling is that the classification program of Weber and al. could soon solve the general $O_N \subset U \subset U_N^+$ case, and why not the $H_N \subset K \subset K_N^+$ case as well, and having such ingredients would enormously simplify our task.

Finally, let us discuss now classification results in the projective geometry setting. Following [23], let us first formulate the following definition:

Definition 6.22. *A monomial space is a subset $P \subset P_+^{N-1}$ obtained via relations of type*

$$p_{i_1 i_2} \ldots p_{i_{k-1} i_k} = p_{i_{\sigma(1)} i_{\sigma(2)}} \ldots p_{i_{\sigma(k-1)} i_{\sigma(k)}}, \ \forall (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$$

with σ ranging over a certain subset of $\bigcup_{k \in 2\mathbb{N}} S_k$, stable under $\sigma \to |\sigma|$.

Observe the similarity with Definition 6.1. The only subtlety in the projective case is the stability under $\sigma \to |\sigma|$, which in practice means that if the above relation associated to σ holds, then the following relation, associated to $|\sigma|$, must hold as well:

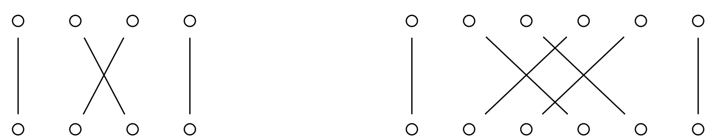
$$p_{i_0 i_1} \ldots p_{i_k i_{k+1}} = p_{i_0 i_{\sigma(1)}} p_{i_{\sigma(2)} i_{\sigma(3)}} \ldots p_{i_{\sigma(k-2)} i_{\sigma(k-1)}} p_{i_{\sigma(k)} i_{k+1}}$$

As an illustration, the basic projective spaces are all monomial:

Proposition 6.23. *The 3 projective spaces are all monomial, with the permutations*

producing respectively the spaces $P_{\mathbb{R}}^{N-1}, P_{\mathbb{C}}^{N-1}$.

Proof. We must divide the algebra $C(P_+^{N-1})$ by the relations associated to the diagrams in the statement, as well as those associated to their shifted versions, given by:



(1) The basic crossing, and its shifted version, produce the relations $p_{ab} = p_{ba}$ and $p_{ab} p_{cd} = p_{ac} p_{bd}$. Now by using these relations several times, we obtain:

$$p_{ab} p_{cd} = p_{ac} p_{bd} = p_{ca} p_{db} = p_{cd} p_{ab}$$

Thus, the space produced by the basic crossing is classical, $P \subset P_{\mathbb{C}}^{N-1}$, and by using one more time the relations $p_{ab} = p_{ba}$ we conclude that we have $P = P_{\mathbb{R}}^{N-1}$, as claimed.

(2) The fattened crossing, and its shifted version, produce the relations $p_{ab}p_{cd} = p_{cd}p_{ab}$ and $p_{ab}p_{cd}p_{ef} = p_{ad}p_{eb}p_{cf}$. The first relations tell us that the projective space must be classical, $P \subset P^{N-1}_{\mathbb{C}}$. Now observe that with $p_{ij} = z_i \bar{z}_j$, the second relations read:

$$z_a \bar{z}_b z_c \bar{z}_d z_e \bar{z}_f = z_a \bar{z}_d z_e \bar{z}_b z_c \bar{z}_f$$

Since these relations are automatic, we have $P = P^{N-1}_{\mathbb{C}}$, and we are done. □

We can now formulate our projective classification result, as follows:

Theorem 6.24. *The basic projective spaces, namely*

$$P^{N-1}_{\mathbb{R}} \subset P^{N-1}_{\mathbb{C}} \subset P^{N-1}_{+}$$

are the only monomial ones.

Proof. We follow the proof from the affine case. Let $\mathcal{R}_\sigma$ be the collection of relations associated to a permutation $\sigma \in S_k$ with $k \in 2\mathbb{N}$, as in Definition 6.22. We fix a monomial projective space $P \subset P^{N-1}_{+}$, and we associate to it subsets $G_k \subset S_k$, as follows:

$$G_k = \begin{cases} \{\sigma \in S_k | \mathcal{R}_\sigma \text{ hold over } P\} & (k \text{ even}) \\ \{\sigma \in S_k | \mathcal{R}_{|\sigma} \text{ hold over } P\} & (k \text{ odd}) \end{cases}$$

As in the affine case, we obtain in this way a filtered group $G = (G_k)$, which is stable under removing outer strings, and under removing neighboring strings. Thus the computations in the proof of Theorem 6.9 apply, and show that we have only 3 possible situations, corresponding to the 3 projective spaces in Proposition 6.23 above. □

In the quantum group case now, we have the following definition:

Definition 6.25. *A projective category of pairings is a collection of subsets*

$$NC_2(2k, 2l) \subset E(k, l) \subset P_2(2k, 2l)$$

stable under the usual categorical operations, and satisfying $\sigma \in E \implies |\sigma| \in E$.

As basic examples here, we have the categories $NC_2 \subset P_2^* \subset P_2$, where P_2^* is the category of matching pairings. This follows indeed from definitions.

Now with the above notion in hand, we can formulate:

Definition 6.26. *A quantum group $PO_N \subset H \subset PO_N^+$ is called projectively easy when*

$$span(NC_2(2k, 2l)) \subset Hom(v^{\otimes k}, v^{\otimes l}) \subset span(P_2(2k, 2l))$$

comes via $Hom(v^{\otimes k}, v^{\otimes l}) = span(E(k, l))$, for a certain projective category $E = (E(k, l))$.

Observe that, given any easy quantum group $O_N \subset G \subset O_N^+$, its projective version $PO_N \subset PG \subset PO_N^+$ is projectively easy in our sense. In particular the quantum groups $PO_N \subset PU_N \subset PO_N^+$ are all projectively easy, coming from $NC_2 \subset P_2^* \subset P_2$.

We have in fact the following general result, from [23]:

Theorem 6.27. *We have a bijective correspondence between the affine and projective categories of partitions, given by*

$$G\to PG$$

at the quantum group level.

Proof. The construction of correspondence $D\to E$ is clear, simply by setting:

$$E(k,l)=D(2k,2l)$$

Conversely, given $E=(E(k,l))$ as in Definition 6.26, we can set:

$$D(k,l)=\begin{cases}E(k,l)&(k,l\text{ even})\\ \{\sigma:|\sigma\in E(k+1,l+1)\}&(k,l\text{ odd})\end{cases}$$

Our claim is that $D=(D(k,l))$ is a category of partitions. Indeed:

(1) The composition action is clear. Indeed, when looking at the numbers of legs involved, in the even case this is clear, and in the odd case, this follows from:

$$\begin{aligned}|\sigma,|\sigma'\in E&\implies |^\sigma_\tau\in E\\ &\implies {}^\sigma_\tau\in D\end{aligned}$$

(2) For the tensor product axiom, we have 4 cases to be investigated. The even/even case is clear. The odd/even case follows from the following computation:

$$\begin{aligned}|\sigma,\tau\in E&\implies |\sigma\tau\in E\\ &\implies \sigma\tau\in D\end{aligned}$$

Regarding now the even/odd case, this can be solved as follows:

$$\begin{aligned}\sigma,|\tau\in E&\implies |\sigma|,|\tau\in E\\ &\implies |\sigma||\tau\in E\\ &\implies |\sigma\tau\in E\\ &\implies \sigma\tau\in D\end{aligned}$$

As for the remaining odd/odd case, here the computation is as follows:

$$\begin{aligned}|\sigma,|\tau\in E&\implies ||\sigma|,|\tau\in E\\ &\implies ||\sigma||\tau\in E\\ &\implies \sigma\tau\in E\\ &\implies \sigma\tau\in D\end{aligned}$$

(3) Finally, the conjugation axiom is clear from definitions.

Now with these definitions in hand, both compositions $D\to E\to D$ and $E\to D\to E$ follow to be the identities, and the quantum group assertion is clear as well. $\square$

Now back to the uniqueness issues, we have here:

Theorem 6.28. *We have the following results:*

(1) O_N^* *is the only intermediate easy quantum group* $O_N \subset G \subset O_N^+$.
(2) PU_N *is the only intermediate projectively easy quantum group* $PO_N \subset G \subset PO_N^+$.

Proof. The assertion regarding $O_N \subset O_N^* \subset O_N^+$ is from [30], and the assertion regarding $PO_N \subset PU_N \subset PO_N^+$ follows from it, and from the duality in Theorem 6.27. □

All this is quite nice, but there is one big issue with it, coming from the fact that we do not have yet axioms for the quadruplets of type (PS, PT, PU, PK), in the spirit of those from section 3 above. A lot of work in this direction still remains to be done.

7. Haar integration

We have seen so far that the two basic geometries, namely those of $\mathbb{R}^N, \mathbb{C}^N$, have free analogues, namely those of $\mathbb{R}^N_+, \mathbb{C}^N_+$. Moreover, a number of supplementary geometries can be constructed, and some classification results are available as well.

The question of "developing" the geometries that we found appears. For the moment we have nothing much. To be more precise, each of our geometries consists so far of 4 objects, namely a sphere S, a torus T, a unitary group U, and a reflection group K.

So, what comes next? This is a quite complicated question, of rather philosophical type. We will attempt to solve it now, with some general discussion.

There are many types of geometry. In relation with our questions, the few manifolds that we have so far come by definition from algebraic equations. So, without any doubt, what we are doing here is "basic algebraic geometry", and this should be the way.

For the moment we have spheres and tori, and some isometry and reflection groups as well. This looks a bit like the knowledge of the Greeks, or perhaps a bit more advanced than that, say in the spirit of Descartes. Thus, we should simply keep building, without much fuss. That is, our goal now should be that of extending our class of examples, with manifolds which are a bit more complicated, say basic homogeneous spaces.

Another potential starting point comes from the fact that our manifolds, in the classical case at least, are smooth. So, we should get into smoothness questions as well.

Whether things are smooth or not, in the noncommutative world, is a deep question, related to many things. There is no agreement so far on this subject.

Mathematically speaking, and from a very naive point of view, a classical real algebraic manifold can have many types of singularities, and the smoothness property corresponds somehow to a "point" in the space of all possible singularities. And, by some kind of miracle, this point is in fact quite often attained, many manifolds being smooth.

In the noncommutative setting, the situation is quite unclear. There have been many attempts here, and, at least in our opinion, the conclusions seem to be rather negative. One scenario is that the space of all possible singularities gets really big, say uncountable, and the smoothness point "gets lost" in it, and is generically not attainable.

This of course quite debatable. In fact, our knowledge of the subject mostly comes from the free case, and what has been said above should be taken in this sense.

As a third and last starting point now, the classical spheres have the obvious property of being "round". In fact, mathematical knowledge left aside, this is something more basic than what has been said before. Ask any kid in the street about what a sphere is (nerds excluded) and the answer will be "a sphere is something round".

Thus, we are led to Riemannian geometry. As a first remark, we are a bit into tricky territory here, because in the classical case, being Riemannian assumes smoothness.

However, is this really the case? After all, Riemannian manifolds are so famous and interesting basically because you can integrate over them. So, getting now to the noncommutative world, in the lack of a clear idea in what regards the smoothness, we can simply declare that "a noncommutative manifold is Riemannian if you can integrate over it, and the better you can integrate over it, the more it is Riemannian".

This might sound of course a bit shocking, going against the spirit of modern mathematics, and of modern civilisation in general. However, at this stage at least, we have nothing better instead, nice and simple, so we will just use this general principle.

To summarize what we have so far, mathematics suggests to develop our noncommutative geometry theories as "basic algebraic geometry", and with the smoothness and Riemannianity aspects being taken care of by the above "integration principle".

In practice now, what we have to do is to develop an algebraic geometric and probabilistic theory of basic homogeneous spaces, generalizing the spheres.

This seems to agree indeed with what has been said above, and in fact, if we truly believe all the above, this is the only way forward. So, this is what we will do.

Before starting, we should do a check with physics as well. At first glance, there is nothing much smooth going on at small scales, and the available data, say basic formulae, fancy pictures for electron distributions, and so on, clearly looks like "Hilbert spaces and probability theory". So, at least from this very naive point of view, we are fine.

At a more advanced level now, things get fairly ununderstandable, with a fair amount of advanced differential geometry involved, but at the same time with lots of apparent singularities a bit everywhere. After all, the word "quantum", or rather its precise Greek origin, is there for reminding us this. So, it is not clear what to conclude here.

Getting now to very large scales, which by some kind of magic are usually related to the small scales, the situation here is a bit similar, but somehow opposite. Indeed, at a very basic level, things are governed by the general principle that "N moving bodies will generically not collide", and so the physics is smooth, with the objects tending to arrange themselves in hierarchic cluster-type structures, a bit like at the microscopic level.

However, at a more advanced level, singularities like black holes appear. Moreover, the interesting feature of the big celestial objects is that they are always hot, with their stability brought by various complicated mechanisms, rather of non-smooth type.

In short, no clear conclusion coming from all this, besides perhaps the fact that there is a great deal of probability theory involved in all this, and so that our noncommutative algebraic manifolds should have an integration functional, indeed.

In practice now, our first task will be that of explaining how to integrate over S, T, U, K. We will first discuss the integration over U, K, and then over T, S.

In order to integrate over U, K, we can use the Weingarten formula [49], [94], whose quantum group formulation, from [19], [29], is as follows:

Theorem 7.1. *Assuming that a compact quantum group $G \subset U_N^+$ is easy, coming from a category of partitions $D \subset P$, we have the Weingarten formula*

$$\int_G u_{i_1 j_1}^{e_1} \ldots u_{i_k j_k}^{e_k} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

for any indices $i_r, j_r \in \{1, \ldots, N\}$ and exponents $e_r \in \{\emptyset, *\}$, where δ are Kronecker type symbols, and where $W_{kN} = G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma) = N^{|\pi \vee \sigma|}$.

Proof. This is a formula that we already know. Let us arrange indeed all the integrals to be computed, at a fixed value of $k = (e_1 \ldots e_k)$, into a single matrix, of size $N^k \times N^k$:

$$P_{i_1 \ldots i_k, j_1 \ldots j_k} = \int_G u_{i_1 j_1}^{e_1} \ldots u_{i_k j_k}^{e_k}$$

By [98], this matrix P is the orthogonal projection onto the following space:

$$Fix(u^{\otimes k}) = span\left(\xi_\pi \Big| \pi \in D(k)\right)$$

By a standard linear algebra computation, it follows that we have $P = WE$, where $E(x) = \sum_{\pi \in D(k)} < x, \xi_\pi > \xi_\pi$, and where W is the inverse on $span(T_\pi | \pi \in D(k))$ of the restriction of E. But this restriction is the linear map corresponding to G_{kN}, so W is the linear map corresponding to W_{kN}, and this gives the result. See [19], [29]. $\square$

Regarding now the integration over the torus T, this is something very simple, because we can use here the following fact, coming again from [98]:

Theorem 7.2. *Given a finitely generated discrete group $\Gamma =< g_1, \ldots, g_N >$, the integrals over the corresponding torus $T = \widehat{\Gamma}$ are given by*

$$\int_T g_{i_1}^{e_1} \ldots g_{i_k}^{e_k} = \delta_{g_{i_1}^{e_1} \ldots g_{i_k}^{e_k}, 1}$$

for any indices $i_r \in \{1, \ldots, N\}$ and any exponents $e_r \in \{\emptyset, *\}$, with the Kronecker symbol on the right being a usual one, computed inside the group Γ.

Proof. This is something standard, coming from the fact that the formula $\int_T g = \delta_{g1}$ defines a functional on the algebra $C(T) = C^*(\Gamma)$, having the correct left and right invariance properties. For details on all this, we refer to [98]. $\square$

In connection with our questions, it is of course possible to refine a bit the above result, in the easy case, by expressing T in terms of the associated categories of partitions D, E. However, in practice, the tori that we are interested in appear as duals of very simple groups, so the integration problematics over T remains something quite elementary.

Finally, regarding the associated spheres S, here the integrals appear as particular cases of the integrals over the corresponding unitary groups U, as explained in section 3 above, and in the easy case, we have a Weingarten formula, as follows:

Theorem 7.3. *The integration over a noncommutative sphere S, coming from a category of pairings D, is given by the Weingarten formula*

$$\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\sum_\pi\sum_{\sigma\leq\ker i}W_{kN}(\pi,\sigma)$$

with $\pi,\sigma\in D(k)$, where $W_{kN}=G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

Proof. This follows from the definition of the integration functional over S, as being the composition of the morphism $C(S)\to C(U)$ with the Haar integration over U:

$$\int_S:C(S)\to C(U)\to\mathbb C$$

Indeed, with this description of the integration functional in mind, we can compute this functional via the Weingarten formula for U, from Theorem 7.1, as follows:

$$\begin{aligned}\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}&=\int_U u_{1i_1}^{e_1}\ldots u_{1i_k}^{e_k}\\&=\sum_{\pi,\sigma\in D(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\&=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\\&=\sum_\pi\sum_{\sigma\leq\ker i}W_{kN}(\pi,\sigma)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Let us discuss now the explicit computation of the various integrals over our manifolds, with respect to the uniform measure. In order to formulate our results in a conceptual form, we use the modern measure theory language, namely probability theory.

In the noncommutative setting, the starting definition is as follows:

Definition 7.4. *Let A be a C^*-algebra, given with a trace tr.*

1. *The elements $a\in A$ are called random variables.*
2. *The moments of such a variable are the numbers $M_k(a)=tr(a^k)$.*
3. *The law of such a variable is the functional $\mu:P\to tr(P(a))$.*

Here $k=\circ\bullet\bullet\circ\ldots$ is as usual a colored integer, and the powers a^k are defined by the usual formulae, namely $a^\emptyset=1,a^\circ=a,a^\bullet=a^*$ and multiplicativity. As for the polynomial P, this is by definition a noncommuting $*$-polynomial in one variable:

$$P\in\mathbb C<X,X^*>$$

Observe that the law is uniquely determined by the moments, because:

$$P(X)=\sum_k\lambda_kX^k\implies\mu(P)=\sum_k\lambda_kM_k(a)$$

In the self-adjoint case, the law is a usual probability measure, supported by the spectrum of a. This follows indeed from the Gelfand theorem, and the Riesz theorem.

There are many things that can be said, at this general level, so as a more concrete objective, let us try to understand how the main result in classical probability, namely the Central Limit Theorem (CLT), can be extended in the noncommutative setting.

Let us start with the usual formulation of the CLT, which is as follows:

Theorem 7.5 (CLT). *Given real random variables $x_1,x_2,x_3,\ldots$, which are i.i.d., centered, and with variance $t>0$, we have, with $n\to\infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^nx_i\sim g_t$$

where g_t is the Gaussian law of parameter t, having as density $\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dx$.

Proof. This is something standard, the proof being in two steps, as follows:

(1) Linearization of the convolution. It well-known that the log of the Fourier transform $F_x(\xi)=\mathbb{E}(e^{i\xi x})$ does the job, in the sense that if x,y are independent, then:

$$F_{x+y}=F_xF_y$$

(2) Study of the limit. We have the following formula for F_x, in terms of moments:

$$F_x(\xi)=\sum_{k=0}^\infty\frac{i^kM_k(x)}{k!}\,\xi^k$$

It follows that the Fourier transform of the variable in the statement is:

$$F(\xi)=\left[F_x\left(\frac{\xi}{\sqrt{n}}\right)\right]^n=\left[1-\frac{t\xi^2}{2n}+O(n^{-2})\right]^n\simeq e^{-t\xi^2/2}$$

But this being the Fourier transform of g_t, we obtain the result. □

In order to extend the CLT, our starting point will be the following definition:

Definition 7.6. *Given a pair (A,tr), two subalgebras $B,C\subset A$ are called free when the following condition is satisfied, for any $x_i\in B$ and $y_i\in C$:*

$$tr(x_i)=tr(y_i)=0\implies tr(x_1y_1x_2y_2\ldots)=0$$

Also, two noncommutative random variables $b,c\in A$ are called free when the C^*-algebras $B=<b>,C=<c>$ that they generate inside A are free, in the above sense.

As a first observation, there is a similarity here with the classical notion of independence. Indeed, modulo some standard identifications, two subalgebras $B, C \subset L^\infty(X)$ are independent when the following condition is satisfied, for any $x \in B$ and $y \in C$:

$$tr(x) = tr(y) = 0 \implies tr(xy) = 0$$

Thus, freeness appears as a kind of "free analogue" of independence.

As a basic result now regarding the notion of freeness, which provides us with a useful class of examples, which can be used for various modelling purposes, we have:

Theorem 7.7. *We have the following results, valid for group algebras:*

(1) $C^*(\Gamma), C^*(\Lambda)$ *are independent inside* $C^*(\Gamma \times \Lambda)$.
(2) $C^*(\Gamma), C^*(\Lambda)$ *are free inside* $C^*(\Gamma * \Lambda)$.

Proof. In order to prove these results, we can use the fact that each group algebra is spanned by the corresponding group elements. Thus, it is enough to check the independence and freeness formulae on group elements, and the proof goes as follows:

(1) Here the computation is trivial, and the result itself follows as well from the fact that $C^*(\Gamma \times \Lambda)$ appears as tensor product of the algebras $C^*(\Gamma), C^*(\Lambda)$.

(2) This is something elementary too, because the notion of freeness, from Definition 7.6 above, is just a translation of the usual, algebraic notion of freeness. □

There are many things that can be said about the analogy between independence and freeness. We have in particular the following result, due to Voiculescu [87]:

Theorem 7.8. *Given a real probability measure μ, consider its Cauchy transform*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t}$$

and define its R-transform as being the solution of the following equation:

$$G_\mu\left(R_\mu(\xi) + \frac{1}{\xi}\right) = \xi$$

The operation $\mu \to R_\mu$ linearizes then the free convolution operation.

Proof. In order to prove this, we need a good model for the free convolution. The best here is to use the monoid algebra of the free monoid on two generators:

$$A = C^*(\mathbb{N} * \mathbb{N})$$

Indeed, we have some freeness in the monoid setting, a bit in the same way as for the group algebras $C^*(\Gamma * \Lambda)$, from Theorem 7.7 (2), and in addition to this fact, and to what happens in the group algebra case, the following two key things happen:

(1) The variables of type $S^*+f(S)$, with $S\in C^*(\mathbb N)$ being the shift, and with $f\in\mathbb C[X]$ being a polynomial, model in moments all the distributions $\mu:\mathbb C[X]\to\mathbb C$. This is indeed something elementary, which can be checked via a direct algebraic computation.

(2) Given $f,g\in\mathbb C[X]$, the variables $S^*+f(S)$ and $T^*+g(T)$, where $S,T\in C^*(\mathbb N*\mathbb N)$ are the shifts corresponding to the generators of $\mathbb N*\mathbb N$, are free, and their sum has the same law as $S^*+(f+g)(S)$. This follows indeed by using a 45° argument.

With these results in hand, we can see that the operation $\mu\to f$ linearizes the free convolution. We are therefore left with a computation inside $C^*(\mathbb N)$, whose conclusion is that $R_\mu=f$ can be recaptured from μ via the Cauchy transform G_μ, as stated. □

We can now state and prove a free analogue of the CLT, as follows:

Theorem 7.9 (Free CLT). *Given self-adjoint variables $x_1,x_2,x_3,\ldots$, which are f.i.d., centered, with variance $t>0$, we have, with $n\to\infty$, in moments,*

$$\frac{1}{\sqrt n}\sum_{i=1}^n x_i\sim\gamma_t$$

where γ_t is the Wigner semicircle law of parameter t, having density $\frac{1}{2\pi t}\sqrt{4t^2-x^2}dx$.

Proof. We follow the same idea as in the proof of Theorem 7.5 above. At $t=1$, the R-transform of the variable in the statement can be computed by using the linearization property with respect to the free convolution, and is given by:

$$R(\xi)=nR_x\left(\frac{\xi}{\sqrt n}\right)\simeq\xi$$

On the other hand, some elementary computations show that the Cauchy transform of the Wigner law γ_1 satisfies the following equation:

$$G_{\gamma_1}\left(\xi+\frac{1}{\xi}\right)=\xi$$

Thus we have $R_{\gamma_1}(\xi)=\xi$, which by the way follows as well from $\frac{S^*+S}{2}\sim\gamma_1$, and this gives the result. The passage to the general case, $t>0$, is routine, by dilation. □

Summarizing, what we know so far is that free probability theory appears by definition as a natural "free analogue" of the usual classical probability theory. We also know that this analogy can be quickly pushed into something non-trivial and interesting, namely an analogy between the classical CLT, and the free CLT. And finally, we know that the limiting measure appearing in the free CLT is something fundamental in quantum physics, namely Wigner's semicircle law. All this is, of course, extremely interesting.

We will be back to theoretical aspects of the analogy between classical and free probability, on several occasions, in what follows, by explaining things here gradually.

With these ingredients in hand, let us go back now to our quantum groups. In what regards the unitary ones, we can formulate right away a nice result, as follows:

Theorem 7.10. *With $N\to\infty$, the main characters*
$$\chi=\sum_{i=1}^N u_{ii}$$
for the basic unitary quantum groups are as follows:

(1) *O_N: real Gaussian, following g_1.*
(2) *O_N^+: semicircular, following γ_1.*
(3) *U_N: complex Gaussian, following G_1.*
(4) *U_N^+: circular, following Γ_1.*

Proof. Following [19], we use the moment method. For an arbitrary closed subgroup $G_N\subset U_N^+$, we have, according to the general Peter-Weyl type results from [98]:
$$\int_{G_N}\chi^k=\dim(Fix(u^{\otimes k}))$$

In the easy case now, where $G=(G_N)$ comes from a certain category of partitions D, the fixed point space on the right is spanned by the vectors T_π with $\pi\in D(k)$. Now since with $N\to\infty$ these vectors can be shown to be linearly independent,we have:
$$\lim_{N\to\infty}\int_{G_N}\chi^k=\#D(k)$$

Thus, we are led into some combinatorics, and the continuation is as follows:

(1) For O_N we have $D=P_2$, and so we obtain as asymptotic moments the numbers $\#P_2(k)=k!!$, which are well-known to be the moments of the Gaussian law.

(2) For O_N^+ we have $D=NC_2$, and so we obtain as asymptotic moments the Catalan numbers $\#NC_2(k)=C_{k/2}$, which are the moments of the Wigner semicircle law.

(3) For U_N we have $D=\mathcal{P}_2$, and we can conclude as in the real case, involving O_N, by using this time moments with respect to colored integers, as in Definition 7.4.

(4) For U_N^+ we have $D=\mathcal{NC}_2$, and once again we can conclude as in the real case, involving O_N^+, by using moments with respect to colored integers, as in Definition 7.4. $\square$

Summarizing, we have seen so far that for O_N,O_N^+,U_N,U_N^+, the asymptotic laws of the main characters are the laws $g_1,\gamma_1,G_1,\Gamma_1$ coming from the various CLT.

This is certainly nice, but there is still one conceptual problem, coming from:

Proposition 7.11. *The above convergences $law(\chi_u)\to g_1,\gamma_1,G_1,\Gamma_1$ are as follows:*

(1) *They are non-stationary in the classical case.*
(2) *They are stationary in the free case, starting from $N=2$.*

Proof. This is something quite subtle, which can be proved as follows:

(1) Here we can use an amenability argument, based on the Kesten criterion. Indeed, O_N, U_N being coamenable, the upper bound of the support of the law of $Re(\chi_u)$ is precisely N, and we obtain from this that the law of χ_u itself depends on $N \in \mathbb{N}$.

(2) Here the result follows from the fact that the linear maps T_π associated to the noncrossing pairings are linearly independent, at any $N \geq 2$. $\square$

Fortunately, the solution to the convergence question is quite simple. The idea will be that of improving our $g_1, \gamma_1, G_1, \Gamma_1$ results with certain $g_t, \gamma_t, G_t, \Gamma_t$ results, which will require $N \to \infty$ in both the classical and free cases, in order to hold at any t.

In practice, the definition that we will need is as follows:

Definition 7.12. *Given a Woronowicz algebra (A, u), the variable*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

is called truncation of the main character, with parameter $t \in (0, 1]$.

Our purpose in what follows will be that of proving that for O_N, O_N^+, U_N, U_N^+, the asymptotic laws of the truncated characters χ_t with $t \in (0, 1]$ are the laws $g_t, \gamma_t, G_t, \Gamma_t$. This is something quite technical, motivated by the findings in Proposition 7.11 above, and also by a number of more advanced considerations, to become clear later on.

In order to study the truncated characters, we can use:

Theorem 7.13. *The moments of truncated characters are given by the formula*

$$\int_G (u_{11} + \ldots + u_{ss})^k = Tr(W_{kN} G_{ks})$$

and with $N \to \infty$ this quantity equals $(s/N)^k \# D(k)$.

Proof. The first assertion follows from the following computation:

$$\begin{aligned}
\int_G (u_{11} + \ldots + u_{ss})^k &= \sum_{i_1=1}^{s} \ldots \sum_{i_k=1}^{s} \int u_{i_1 i_1} \ldots u_{i_k i_k} \\
&= \sum_{\pi, \sigma \in D(k)} W_{kN}(\pi, \sigma) \sum_{i_1=1}^{s} \ldots \sum_{i_k=1}^{s} \delta_\pi(i) \delta_\sigma(i) \\
&= \sum_{\pi, \sigma \in D(k)} W_{kN}(\pi, \sigma) G_{ks}(\sigma, \pi) \\
&= Tr(W_{kN} G_{ks})
\end{aligned}$$

We have $G_{kN}(\pi,\sigma)=N^k$ for $\pi=\sigma$, and $G_{kN}(\pi,\sigma)\leq N^{k-1}$ for $\pi\neq\sigma$. Thus with $N\to\infty$ we have $G_{kN}\sim N^k1$, which gives:

$$\begin{aligned}\int_G(u_{11}+\ldots+u_{ss})^k &= Tr(G_{kN}^{-1}G_{ks})\\ &\sim Tr((N^k1)^{-1}G_{ks})\\ &= N^{-k}Tr(G_{ks})\\ &= N^{-k}s^k\#D(k)\end{aligned}$$

Thus, we have obtained the formula in the statement. See [19]. □

In order to process the above moment formula, we will need some more classical and free probability theory. Given a random variable a, we write:

$$\log F_a(\xi)=\sum_n k_n(a)\xi^n\quad,\quad R_a(\xi)=\sum_n\kappa_n(a)\xi^n$$

We call the coefficients $k_n(a),\kappa_n(a)$ cumulants, respectively free cumulants of a. With this notion in hand, we can define then more general quantities $k_\pi(a),\kappa_\pi(a)$, depending on arbitrary partitions $\pi\in P(k)$, by multiplicativity over the blocks.

With these conventions, we have then the following result:

Theorem 7.14. *We have the classical and free moment-cumulant formulae*

$$M_k(a)=\sum_{\pi\in P(k)}k_\pi(a)\quad,\quad M_k(a)=\sum_{\pi\in NC(k)}\kappa_\pi(a)$$

where $k_\pi(a),\kappa_\pi(a)$ are the generalized cumulants and free cumulants of a.

Proof. This is standard, either by using the formulae of F_a,R_a, or by doing some direct combinatorics, based on the Möbius inversion formula. □

We can now improve our results about characters, as follows:

Theorem 7.15. *With $N\to\infty$, the laws of truncated characters are as follows:*

(1) *For O_N we obtain the Gaussian law g_t.*
(2) *For O_N^+ we obtain the Wigner semicircle law γ_t.*
(3) *For U_N we obtain the complex Gaussian law G_t.*
(4) *For U_N^+ we obtain the Voiculescu circular law Γ_t.*

Proof. With $s=[tN]$ and $N\to\infty$, the formula in Theorem 7.13 above gives:

$$\lim_{N\to\infty}\int_{G_N}\chi_t^k=\sum_{\pi\in D(k)}t^{|\pi|}$$

By using now the formulae in Theorem 7.14, this gives the results. See [19]. □

As an interesting consequence, related to [31], let us formulate as well:

Theorem 7.16. *The asymptotic laws of truncated characters for the liberation operations*

$$O_N \to O_N^+ \quad , \quad U_N \to U_N^+$$

are in Bercovici-Pata bijection, in the sense that the classical cumulants in the classical case equal the free cumulants in the free case.

Proof. This follows indeed from the computations in the proof of Theorem 7.15, and from the combinatorial interpretation of the Bercovici-Pata bijection [31]. □

Summarizing, our geometric liberation operations are compatible with the standard liberation operation from free probability theory.

Let us discuss now the integration over the spheres. A basic probabilistic question regarding the spheres concerns the computation of the associated hyperspherical laws. We have here the following result, from [4], [22]:

Theorem 7.17. *With $N \to \infty$, the variables $\sqrt{N} x_i \in C(S^{N-1}_\times)$ are as follows:*

(1) $S^{N-1}_\mathbb{R}$: *real Gaussian.*
(2) $S^{N-1}_{\mathbb{R},+}$: *semicircular.*
(3) $S^{N-1}_\mathbb{C}$: *complex Gaussian.*
(4) $S^{N-1}_{\mathbb{C},+}$: *circular.*

Proof. This follows from Theorem 7.10, but we can use as well the Weingarten formula for the spheres, from Theorem 7.3 above. Indeed, since with $N \to \infty$ the Gram matrix $G_{kN}(\pi,\sigma) = N^{|\pi\vee\sigma|}$ is asymptotically constant, $G_{kN}(\pi,\sigma) \simeq \delta_{\pi,\sigma} N^{k/2}$, its inverse is asymptotically constant as well, $W_{kN}(\pi,\sigma) \simeq \delta_{\pi,\sigma} N^{-k/2}$, and so:

$$\int_{S^{N-1}_\times} x_{i_1} \ldots x_{i_k} \, dx \simeq N^{-k/2} \sum_{\sigma \in P_2^\times(k)} \delta_\sigma(i)$$

With this formula in hand, we can compute the asymptotic moments of each coordinate x_i. Indeed, by setting $i_1 = \ldots = i_k = i$, all Kronecker symbols are 1, and we obtain:

$$\int_{S^{N-1}_\times} x_i^k \, dx \simeq N^{-k/2} \# P_2^\times(k)$$

Thus, in the real case, the even asymptotic moments of $\sqrt{N} x_i$ are the numbers $\# P_2^\times(2l)$, which are equal respectively to $(2l)!!$, $\frac{1}{l+1}\binom{2l}{l}$, and this gives the result. In the complex case the proof is similar, by adding colored exponents everywhere. See [4], [22]. □

In order to discuss now the quantum reflection groups, we will need some more theory, namely Poisson limit theorems. In the classical case, we have the following result:

Theorem 7.18 (PLT). *We have the following convergence, in moments,*

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{*n}\to p_t$$

the limiting measure being

$$p_t=\frac{1}{e^t}\sum_{k=0}^\infty\frac{t^k\delta_k}{k!}$$

which is the Poisson law of parameter $t>0$.

Proof. This is standard, indeed, by using the Fourier transform. We will actually reprove this in a moment, in a more general setting, that of the compound Poisson laws. □

In the free case, the result is as follows:

Theorem 7.19 (Free PLT). *We have the following convergence, in moments,*

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{\boxplus n}\to \pi_t$$

the limiting measure being the Marchenko-Pastur law of parameter $t>0$,

$$\pi_t=\max(1-t,0)\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter $t>0$.

Proof. This is standard as well, by using the R-transform. Once again, we will reprove this in a moment, in a more general setting, that of the compound Poisson laws. □

Summarizing, the analogy between CLT and free CLT has a discrete counterpart, involving the PLT and free PLT, and once again with the free case being related to random matrix theory, this time via Wishart matrices, and the Marchenko-Pastur law.

In order to get further beyond, let us introduce the following notions:

Definition 7.20. *Associated to any compactly supported positive measure ρ on $\mathbb{R}$ are the probability measures*

$$p_\rho=\lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0+\frac{1}{n}\rho\right)^{*n}\quad,\quad \pi_\rho=\lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0+\frac{1}{n}\rho\right)^{\boxplus n}$$

where $c=mass(\rho)$, called compound Poisson and compound free Poisson laws.

In what follows we will be interested in the case where ρ is discrete, as is for instance the case for $\rho=t\delta_1$ with $t>0$, which produces the Poisson and free Poisson laws.

The following result allows one to detect compound Poisson/free Poisson laws:

Theorem 7.21. *For $\rho=\sum_{i=1}^s c_i\delta_{z_i}$ with $c_i>0$ and $z_i\in\mathbb R$, we have*

$$F_{p_\rho}(y)=\exp\left(\sum_{i=1}^s c_i(e^{iyz_i}-1)\right)\quad,\quad R_{\pi_\rho}(y)=\sum_{i=1}^s\frac{c_iz_i}{1-yz_i}$$

where F,R denote respectively the Fourier transform, and Voiculescu's R-transform.

Proof. Let μ_n be the measure appearing in Definition 7.20, under the convolution signs. In the classical case, we have the following computation:

$$\begin{aligned}
&& F_{\mu_n}(y)&=\left(1-\frac{c}{n}\right)+\frac{1}{n}\sum_{i=1}^s c_ie^{iyz_i}\\
&\implies & F_{\mu_n^{*n}}(y)&=\left(\left(1-\frac{c}{n}\right)+\frac{1}{n}\sum_{i=1}^s c_ie^{iyz_i}\right)^n\\
&\implies & F_{p_\rho}(y)&=\exp\left(\sum_{i=1}^s c_i(e^{iyz_i}-1)\right)
\end{aligned}$$

In the free case now, we use a similar method. The Cauchy transform of μ_n is:

$$G_{\mu_n}(\xi)=\left(1-\frac{c}{n}\right)\frac{1}{\xi}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{\xi-z_i}$$

Consider now the R-transform of the measure $\mu_n^{\boxplus n}$, which is given by:

$$R_{\mu_n^{\boxplus n}}(y)=nR_{\mu_n}(y)$$

The above formula of G_{μ_n} shows that the equation for $R=R_{\mu_n^{\boxplus n}}$ is as follows:

$$\begin{aligned}
&& \left(1-\frac{c}{n}\right)\frac{1}{y^{-1}+R/n}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{y^{-1}+R/n-z_i}&=y\\
&\implies & \left(1-\frac{c}{n}\right)\frac{1}{1+yR/n}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{1+yR/n-yz_i}&=1
\end{aligned}$$

Now multiplying by n, rearranging the terms, and letting $n\to\infty$, we get:

$$\begin{aligned}
&& \frac{c+yR}{1+yR/n}&=\sum_{i=1}^s\frac{c_i}{1+yR/n-yz_i}\\
&\implies & c+yR_{\pi_\rho}(y)&=\sum_{i=1}^s\frac{c_i}{1-yz_i}\\
&\implies & R_{\pi_\rho}(y)&=\sum_{i=1}^s\frac{c_iz_i}{1-yz_i}
\end{aligned}$$

This finishes the proof in the free case, and we are done. □

We also have the following result, providing an alternative to Definition 7.20:

Theorem 7.22. *For $\rho = \sum_{i=1}^s c_i\delta_{z_i}$ with $c_i > 0$ and $z_i \in \mathbb{R}$, we have*

$$p_\rho/\pi_\rho = \text{law}\left(\sum_{i=1}^s z_i\alpha_i\right)$$

where the variables α_i are Poisson/free Poisson(c_i), independent/free.

Proof. Let α be the sum of Poisson/free Poisson variables in the statement.

By using some well-known Fourier transform formulae, we have:

$$\begin{aligned} F_{\alpha_i}(y) = \exp(c_i(e^{iy}-1)) \quad &\implies \quad F_{z_i\alpha_i}(y) = \exp(c_i(e^{iyz_i}-1)) \\ &\implies \quad F_\alpha(y) = \exp\left(\sum_{i=1}^s c_i(e^{iyz_i}-1)\right) \end{aligned}$$

Also, by using some well-known R-transform formulae, we have:

$$\begin{aligned} R_{\alpha_i}(y) = \frac{c_i}{1-y} \quad &\implies \quad R_{z_i\alpha_i}(y) = \frac{c_iz_i}{1-yz_i} \\ &\implies \quad R_\alpha(y) = \sum_{i=1}^s \frac{c_iz_i}{1-yz_i} \end{aligned}$$

Thus we have indeed the same formulae as those which are needed. □

We refer to [31], [87], [89] for the general theory here, to [19], [21], [49] for representation theory aspects, and to [78], [88], [96] for random matrix aspects.

In what follows we will only need the main examples of classical and free compound Poisson laws, which are the classical and free Bessel laws, constructed as follows:

Definition 7.23. *The Bessel and free Bessel laws are the compound Poisson laws*

$$b_t^s = p_{t\varepsilon_s} \quad , \quad \beta_t^s = \pi_{t\varepsilon_s}$$

where ε_s is the uniform measure on the s-th roots unity. In particular:

1. *At $s = 1$ we obtain the usual Poisson and free Poisson laws, p_t, π_t.*
2. *At $s = 2$ we obtain the "real" Bessel and free Bessel laws, denoted b_t, β_t.*
3. *At $s = \infty$ we obtain the "complex" Bessel and free Bessel laws, denoted $B_t, \mathfrak{B}_t$.*

There is a lot of interesting theory regarding these laws, involving classical and quantum reflection groups, subfactors and planar algebras, plus of course free probability and random matrices. We refer here to [11], where these laws were introduced, and studied.

Getting back now to our quantum reflection groups, we first have:

Theorem 7.24. *With $N \to \infty$, the laws of characters are as follows:*
(1) *For H_N we obtain the Bessel law b_1.*
(2) *For H_N^+ we obtain the free Bessel law β_1.*
(3) *For K_N we obtain the complex Bessel law B_1.*
(4) *For K_N^+ we obtain the complex free Bessel law $\mathfrak{B}_1$.*

Proof. This is routine indeed, by counting the partitions, a bit as in the continuous case, in the proof of Theorem 7.10 above. For the full proof here, we refer to [11]. □

At the level of truncated characters, we have:

Theorem 7.25. *With $N \to \infty$, the laws of truncated characters are as follows:*
(1) *For H_N we obtain the Bessel law b_t.*
(2) *For H_N^+ we obtain the free Bessel law β_t.*
(3) *For K_N we obtain the complex Bessel law B_t.*
(4) *For K_N^+ we obtain the complex free Bessel law $\mathfrak{B}_t$.*
Also, we have the Bercovici-Pata bijection for truncated characters.

Proof. Once again this is routine, by using the Weingarten formula, as in the continuous case, in the proof of Theorem 7.15 above. For the full proof here, we refer to [11]. □

Summarizing, we have nice liberation results for S, U, K at the probabilistic level, with Bercovici-Pata bijection for the laws of truncated characters, in the $N \to \infty$ limit.

Regarding now the tori, the situation here is more complicated, no longer involving the Bercovici-Pata bijection. Let us recall indeed that the basic tori are as follows:

$$\begin{tikzcd} T_N^+ \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[r] \arrow[u] & \mathbb{T}_N \arrow[u] \end{tikzcd}$$

These tori appear by definiton as duals of the following discrete groups:

$$\begin{tikzcd} \mathbb{Z}_2^{*N} \arrow[d] & F_N \arrow[l] \arrow[d] \\ \mathbb{Z}_2^N & \mathbb{Z}^N \arrow[l] \end{tikzcd}$$

We are interested in the computation of the laws of the associated truncated characters, $\chi_t = g_1 + g_2 + \ldots + g_{[tN]}$. For this purpose, we can use the moment formula, and as a first conclusion here, we can assume by dilation that we are dealing with the $t = 1$ case.

For the complex tori, $\mathbb{T}_N\subset\mathbb{T}_N^+$, we are led into the computation of the Kesten measures for $F_N\to\mathbb{Z}^N$, and so into the Meixner/free Meixner correspondence.

As for the real tori, $T_N\subset T_N^+$, here we are led into the computation of the Kesten measures for $\mathbb{Z}_2^{*N}\to\mathbb{Z}_2^N$, and so into a real version of this correspondence.

This is the idea here, and all this remains to be worked out in detail.

Summarizing, we have some nice liberation results for S,T,U,K, with a technical problem, however, coming from the fact that those for S,U,K come from the Bercovici-Pata bijection, while those for T come from the Meixner/free Meixner correspondence.

8. Twisting results

The theory that we have so far takes place at $q = 1$. Our purpose now is to extend this theory, by twisting it at $q = -1$. We will see that the spheres and unitary groups are twistable, while the tori and reflection groups are equal to their own twists.

As a consequence of this work, our quadruplets (S, T, U, K) will have twisted counterparts $(\bar{S}, T, \bar{U}, K)$. One interesting theoretical question, that we will discuss as well, is that of modifying our geometric axioms, as to cover both the $q = \pm 1$ cases.

Before starting, let us mention as well that in the free case the twisting operation is trivial. Thus, most of what we will do below will basically concern the classical and half-classical cases. However, this will shed some new light on the free case as well.

The twisting philosophy goes back to the papers of Drinfeld [64] and Jimbo [69]. Their idea was to deform the compact Lie groups with the help of a parameter $q \in \mathbb{C}$, the interesting case being $q \in \mathbb{T}$. However, as explained by Woronowicz in [98], in the operator algebra setting the parameter needs to be real, $q \in \mathbb{R}$. We are therefore led to:

$$q \in \mathbb{T} \cap \mathbb{R} = \{\pm 1\}$$

In practice now, we can think for instance of the easy quantum groups as corresponding to the case $q = 1$, and we are led to the question of "twisting" them, at $q = -1$.

We will discuss in what follows the twisting procedure in the easy case, by using a Schur-Weyl type approach. Before starting, we should mention that what we will be doing here is, technically speaking, not related to [64], [69] at $q = -1$. In fact, the Drinfeld-Jimbo theory is "wrong" at $q = -1$, and our purpose will be that of fixing this.

We use the standard embedding $S_k \subset P_2(k, k)$, via the pairings having only up-to-down strings. Given $\tau \in P(k, l)$, we call "switch" the operation which consists in switching two neighbors, belonging to different blocks, in the upper row, or in the lower row.

With these conventions, we have the following result:

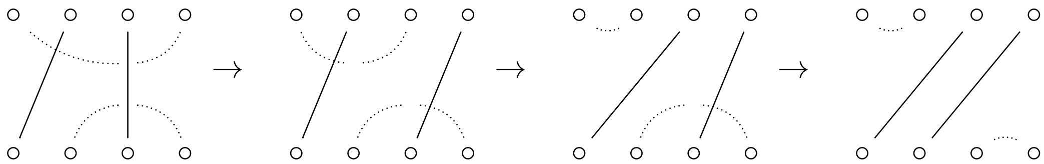
Proposition 8.1. *There is a signature map $\varepsilon : P_{even} \to \{-1, 1\}$, given by $\varepsilon(\tau) = (-1)^c$, where c is the number of switches needed to make τ noncrossing. In addition:*

1. *For $\tau \in S_k$, this is the usual signature.*
2. *For $\tau \in P_2$ we have $(-1)^c$, where c is the number of crossings.*
3. *For $\tau \leq \pi \in NC_{even}$, the signature is 1.*

Proof. In order to show that ε is well-defined, we must prove that the number c in the statement is well-defined modulo 2. It is enough to perform the verification for the noncrossing partitions. More precisely, given $\tau, \tau' \in NC_{even}$ having the same block structure, we must prove that the number of switches c required for the passage $\tau \to \tau'$ is even.

In order to do so, observe that any partition $\tau \in P(k, l)$ can be put in "standard form", by ordering its blocks according to the appearence of the first leg in each block, counting clockwise from top left, and then by performing the switches as for block 1 to be at left, then for block 2 to be at left, and so on. Here the required switches are also uniquely determined, by the order coming from counting clockwise from top left.

Here is an example of such an algorithmic switching operation, with block 1 being first put at left, by using two switches, then with block 2 left unchanged, and then with block 3 being put at left as well, but at right of blocks 1 and 2, with one switch:



The point now is that, under the assumption $\tau \in NC_{even}(k,l)$, each of the moves required for putting a leg at left, and hence for putting a whole block at left, requires an even number of switches. Thus, putting τ is standard form requires an even number of switches. Now given $\tau, \tau' \in NC_{even}$ having the same block structure, the standard form coincides, so the number of switches c required for the passage $\tau \to \tau'$ is indeed even.

Regarding now the remaining assertions, these are all elementary:

(1) For $\tau \in S_k$ the standard form is $\tau' = id$, and the passage $\tau \to id$ comes by composing with a number of transpositions, which gives the signature.

(2) For a general $\tau \in P_2$, the standard form is of type $\tau' = |\ldots|^{\cup\ldots\cup}_{\cap\ldots\cap}$, and the passage $\tau \to \tau'$ requires $c \bmod 2$ switches, where c is the number of crossings.

(3) Assuming that $\tau \in P_{even}$ comes from $\pi \in NC_{even}$ by merging a certain number of blocks, we can prove that the signature is 1 by proceeding by recurrence. □

We define the kernel of a multi-index $\binom{i}{j}$ to be the partition obtained by joining the equal indices. Also, we write $\pi \leq \sigma$ if each block of π is contained in a block of σ.

With these conventions, and the above result in hand, we can now formulate:

Definition 8.2. *Associated to a partition $\pi \in P_{even}(k,l)$ is the linear map*

$$\bar{T}_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \bar{\delta}_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where $\bar{\delta}_\pi \in \{-1,0,1\}$ is $\bar{\delta}_\pi = \varepsilon(\tau)$ if $\tau \geq \pi$, and $\bar{\delta}_\pi = 0$ otherwise, with $\tau = \ker\binom{i}{j}$.

In other words, what we are doing here is to add signatures to the usual formula of T_π. Indeed, observe that the usual formula for T_π can be written as folllows:

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j:\ker\binom{i}{j} \geq \pi} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Now by inserting signs, coming from the signature map $\varepsilon : P_{even} \to \{\pm 1\}$, we are led to the following formula, which coincides with the one from Definition 8.2:

$$\bar{T}_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{\tau \geq \pi} \varepsilon(\tau) \sum_{j:\ker(^i_j)=\tau} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

We will be back later to this analogy, with more details on what can be done with it. For the moment, we must first prove a key categorical result, as follows:

Proposition 8.3. *The assignement $\pi \to \bar{T}_\pi$ is categorical, in the sense that*

$$\bar{T}_\pi \otimes \bar{T}_\sigma = \bar{T}_{[\pi\sigma]}, \qquad \bar{T}_\pi \bar{T}_\sigma = N^{c(\pi,\sigma)} \bar{T}_{[^\sigma_\pi]}, \qquad \bar{T}^*_\pi = \bar{T}_{\pi^*}$$

where $c(\pi, \sigma)$ are certain positive integers.

Proof. In order to prove this result we can go back to the proof from the easy case, and insert signs, where needed. We have to check three conditions, as follows:

1. Concatenation. In the untwisted case, this was based on the following formula:

$$\delta_\pi \begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_q \end{pmatrix} \delta_\sigma \begin{pmatrix} k_1 \ldots k_r \\ l_1 \ldots l_s \end{pmatrix} = \delta_{[\pi\sigma]} \begin{pmatrix} i_1 \ldots i_p & k_1 \ldots k_r \\ j_1 \ldots j_q & l_1 \ldots l_s \end{pmatrix}$$

In the twisted case, it is enough to check the following formula:

$$\varepsilon \left(\ker \begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_q \end{pmatrix} \right) \varepsilon \left(\ker \begin{pmatrix} k_1 \ldots k_r \\ l_1 \ldots l_s \end{pmatrix} \right) = \varepsilon \left(\ker \begin{pmatrix} i_1 \ldots i_p & k_1 \ldots k_r \\ j_1 \ldots j_q & l_1 \ldots l_s \end{pmatrix} \right)$$

Let us denote by τ, ν the partitions on the left, so that the partition on the right is of the form $\rho \leq [\tau\nu]$. Now by switching to the noncrossing form, $\tau \to \tau'$ and $\nu \to \nu'$, the partition on the right transforms into $\rho \to \rho' \leq [\tau'\nu']$. Now since $[\tau'\nu']$ is noncrossing, we can use Proposition 8.1 (3), and we obtain the result.

2. Composition. In the untwisted case, this was based on the following formula:

$$\sum_{j_1 \ldots j_q} \delta_\pi \begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_q \end{pmatrix} \delta_\sigma \begin{pmatrix} j_1 \ldots j_q \\ k_1 \ldots k_r \end{pmatrix} = N^{c(\pi,\sigma)} \delta_{[^\pi_\sigma]} \begin{pmatrix} i_1 \ldots i_p \\ k_1 \ldots k_r \end{pmatrix}$$

In order to prove now the result in the twisted case, it is enough to check that the signs match. More precisely, we must establish the following formula:

$$\varepsilon \left(\ker \begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_q \end{pmatrix} \right) \varepsilon \left(\ker \begin{pmatrix} j_1 \ldots j_q \\ k_1 \ldots k_r \end{pmatrix} \right) = \varepsilon \left(\ker \begin{pmatrix} i_1 \ldots i_p \\ k_1 \ldots k_r \end{pmatrix} \right)$$

Let τ, ν be the partitions on the left, so that the partition on the right is of the form $\rho \leq [^\tau_\nu]$. Our claim is that we can jointly switch τ, ν to the noncrossing form. Indeed, we can first switch as for $\ker(j_1 \ldots j_q)$ to become noncrossing, and then switch the upper legs of τ, and the lower legs of ν, as for both these partitions to become noncrossing.

Now observe that when switching in this way to the noncrossing form, $\tau \to \tau'$ and $\nu \to \nu'$, the partition on the right transforms into $\rho \to \rho' \leq [^{\tau'}_{\nu'}]$. Now since $[^{\tau'}_{\nu'}]$ is noncrossing, we can apply Proposition 8.1 (3), and we obtain the result.

3. Involution. Here we must prove the following formula:

$$\bar{\delta}_\pi\begin{pmatrix}i_1\dots i_p\\ j_1\dots j_q\end{pmatrix}=\bar{\delta}_{\pi^*}\begin{pmatrix}j_1\dots j_q\\ i_1\dots i_p\end{pmatrix}$$

But this is clear from the definition of $\bar{\delta}_\pi$, and we are done. □

As a conclusion, our construction $\pi\to\bar{T}_\pi$ has all the needed properties for producing quantum groups, via Tannakian duality. So, we can now formulate:

Theorem 8.4. *Given a category of partitions $D\subset P_{even}$, the construction*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\bar{T}_\pi\Big|\pi\in D(k,l)\right)$$

produces via Tannakian duality a quantum group $\bar{G}_N\subset U_N^+$, for any $N\in\mathbb{N}$.

Proof. This follows indeed from the Tannakian results from section 2 above, exactly as in the easy case, by using this time Proposition 8.3 as technical ingredient. □

We can unify the easy quantum groups, or at least the examples coming from categories $D\subset P_{even}$, with the quantum groups constructed above, as follows:

Definition 8.5. *A closed subgroup $G\subset U_N^+$ is called q-easy, or quizzy, with deformation parameter $q=\pm1$, when its tensor category appears as follows,*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\dot{T}_\pi\Big|\pi\in D(k,l)\right)$$

for a certain category of partitions $D\subset P_{even}$, where $\dot{T}=\bar{T},T$ for $q=-1,1$. The Schur-Weyl twist of G is the quizzy quantum group $\bar{G}\subset U_N^+$ obtained via $q\to-q$.

Summarizing, we have a quite conceptual twisting method for the easy quantum groups containing H_N, which leads to a notion of q-easiness, with $q=\pm1$.

Getting back now to our noncommutative geometry questions, a first problem is that of computing the twists of the quantum groups U,K that we have.

Let us begin with the unitary case. The result here is as follows:

Theorem 8.6. *The twists of the basic unitary quantum groups are as follows, obtained by replacing the relations $ab = ba$, $abc = cba$ with $ab = \pm ba$, $abc = \pm cba$,*

$$\begin{tikzcd}
O_N^+ \arrow[r] & \mathbb{T}O_N^+ \arrow[r] & U_N^+ \\
\bar{O}_N^* \arrow[r] \arrow[u] & \mathbb{T}\bar{O}_N^* \arrow[r] \arrow[u] & \bar{U}_N^* \arrow[u] \\
\bar{O}_N \arrow[r] \arrow[u] & \mathbb{T}\bar{O}_N \arrow[r] \arrow[u] & \bar{U}_N \arrow[u]
\end{tikzcd}$$

with the signs for $\bar{U}_N$ corresponding to anticommutation on rows and columns, and commutation otherwise, and with the other signs coming from functoriality.

Proof. Given coordinates $a, b, c, \ldots \in \{u_{ij}\}$, we set $span(a, b, c, \ldots) = (r, c)$, where $r, c \in \{1, 2, 3, \ldots\}$ are the numbers of rows and columns spanned by $a, b, c, \ldots$, inside the matrix $u = (u_{ij})$. Also, we make the conventions $\alpha = a, a^*$, $\beta = b, b^*$, and so on.

With these conventions, the relations for the quantum groups on the bottom, appearing as subgroups of $\bar{U}_N$, are those indicated in the statement, namely:

$$\alpha\beta = \begin{cases} -\beta\alpha & \text{for } a, b \in \{u_{ij}\} \text{ with } span(a, b) = (1, 2) \text{ or } (2, 1) \\ \beta\alpha & \text{otherwise} \end{cases}$$

Regarding now the quantum groups in the middle, these must be all subgroups of $\bar{U}_N^*$. The point now is that, if we want $\bar{U}_N^*$ to be constructed via relations of type $abc = \pm cba$, the signs in these defining relations are uniquely determined by the fact that we must have $\bar{U}_N \subset \bar{U}_N^*$. Skipping some routine details here, we are led in this way to the following relations for $\bar{U}_N^*$, and for the other quantum groups in the middle:

$$\alpha\beta\gamma = \begin{cases} -\gamma\beta\alpha & \text{for } a, b, c \in \{u_{ij}\} \text{ with } span(a, b, c) = (\leq 2, 3) \text{ or } (3, \leq 2) \\ \gamma\beta\alpha & \text{otherwise} \end{cases}$$

Summarizing, we have constructed quantum groups as in the statement. It remains to prove that these quantum groups are the twists of the basic unitary quantum groups.

The basic crossing, $\ker\binom{ij}{ji}$ with $i \neq j$, comes from the transposition $\tau \in S_2$, so its signature is -1. As for its degenerated version $\ker\binom{ii}{ii}$, this is noncrossing, so here the signature is 1. We conclude that the linear map associated to the basic crossing is:

$$\bar{T}_{\chi}(e_i \otimes e_j) = \begin{cases} -e_j \otimes e_i & \text{for } i \neq j \\ e_j \otimes e_i & \text{otherwise} \end{cases}$$

For the half-classical crossing, here the signature is once again -1, and by examining the signatures of its various degenerations, we are led to the following formula:

$$\bar{T}_{\not\times}(e_i \otimes e_j \otimes e_k) = \begin{cases} -e_k \otimes e_j \otimes e_i & \text{for } i,j,k \text{ distinct} \\ e_k \otimes e_j \otimes e_i & \text{otherwise} \end{cases}$$

We can proceed now as in the untwisted case, and since the intertwining relations coming from $\bar{T}_{\times}, \bar{T}_{\not\times}$ correspond to the relations defining $\bar{U}_N, \bar{U}_N^*$, we obtain the result. $\square$

Summarizing, our various unitary quantum groups U have twisted counterparts $\bar{U}$.

Our purpose now will be that of showing that the quantum reflection groups equal their own Schur-Weyl twists. It is convenient to include in our discussion two more quantum groups, coming from [84] and denoted $H_N^{[\infty]}, K_N^{[\infty]}$, constructed as follows:

Theorem 8.7. *We have intermediate liberations $H_N^{[\infty]}, K_N^{[\infty]}$ as follows, constructed by using the relations $\alpha\beta\gamma = 0$, for any $a \neq c$ on the same row or column of u,*

$$\begin{tikzcd} K_N \arrow[r] & K_N^* \arrow[r] & K_N^{[\infty]} \arrow[r] & K_N^+ \\ H_N \arrow[r] \arrow[u] & H_N^* \arrow[r] \arrow[u] & H_N^{[\infty]} \arrow[r] \arrow[u] & H_N^+ \arrow[u] \end{tikzcd}$$

with the convention $\alpha = a, a^*$, and so on. These quantum groups are easy, the corresponding categories $P_{even}^{[\infty]} \subset P_{even}$ and $\mathcal{P}_{even}^{[\infty]} \subset \mathcal{P}_{even}$ being generated by $\eta = \ker\binom{iij}{jii}$.

Proof. This is routine, by using the fact that the relations $\alpha\beta\gamma = 0$ in the statement are equivalent to the condition $\eta \in End(u^{\otimes k})$, with $|k| = 3$. We refer here to [84]. $\square$

In order to discuss the twisting, we will need the following technical result:

Proposition 8.8. *We have the following equalities,*

$$\begin{aligned} P_{even}^* &= \left\{\pi \in P_{even} \middle| \varepsilon(\tau) = 1, \forall \tau \leq \pi, |\tau| = 2\right\} \\ P_{even}^{[\infty]} &= \left\{\pi \in P_{even} \middle| \sigma \in P_{even}^*, \forall \sigma \subset \pi\right\} \\ P_{even}^{[\infty]} &= \left\{\pi \in P_{even} \middle| \varepsilon(\tau) = 1, \forall \tau \leq \pi\right\} \end{aligned}$$

where $\varepsilon : P_{even} \to \{\pm 1\}$ is the signature of even permutations.

Proof. This is routine combinatorics, from [5], [84], the idea being as follows:

(1) Given $\pi\in P_{even}$, we have $\tau\leq\pi,|\tau|=2$ precisely when $\tau=\pi^\beta$ is the partition obtained from π by merging all the legs of a certain subpartition $\beta\subset\pi$, and by merging as well all the other blocks. Now observe that π^β does not depend on π, but only on β, and that the number of switches required for making π^β noncrossing is $c=N_\bullet-N_\circ$ modulo 2, where $N_\bullet/N_\circ$ is the number of black/white legs of β, when labelling the legs of π counterclockwise $\circ\bullet\circ\bullet\ldots$ Thus $\varepsilon(\pi^\beta)=1$ holds precisely when $\beta\in\pi$ has the same number of black and white legs, and this gives the result.

(2) This simply follows from the equality $P_{even}^{[\infty]}=<\eta>$ coming from Theorem 8.7, by computing $<\eta>$, and for the complete proof here we refer to [84].

(3) We use here the fact, also from [84], that the relations $g_ig_ig_j=g_jg_ig_i$ are trivially satisfied for real reflections. This leads to the following conclusion:

$$P_{even}^{[\infty]}(k,l)=\left\{\ker\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}\Big|g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l}\text{ inside }\mathbb Z_2^{*N}\right\}$$

In other words, the partitions in $P_{even}^{[\infty]}$ are those describing the relations between free variables, subject to the conditions $g_i^2=1$. We conclude that $P_{even}^{[\infty]}$ appears from NC_{even} by "inflating blocks", in the sense that each $\pi\in P_{even}^{[\infty]}$ can be transformed into a partition $\pi'\in NC_{even}$ by deleting pairs of consecutive legs, belonging to the same block.

Now since this inflation operation leaves invariant modulo 2 the number $c\in\mathbb N$ of switches in the definition of the signature, it leaves invariant the signature $\varepsilon=(-1)^c$ itself, and we obtain in this way the inclusion "$\subset$" in the statement.

Conversely, given $\pi\in P_{even}$ satisfying $\varepsilon(\tau)=1$, $\forall\tau\leq\pi$, our claim is that:

$$\rho\leq\sigma\subset\pi,|\rho|=2\implies\varepsilon(\rho)=1$$

Indeed, let us denote by α,β the two blocks of ρ, and by γ the remaining blocks of π, merged altogether. We know that the partitions $\tau_1=(\alpha\wedge\gamma,\beta)$, $\tau_2=(\beta\wedge\gamma,\alpha)$, $\tau_3=(\alpha,\beta,\gamma)$ are all even. On the other hand, putting these partitions in noncrossing form requires respectively $s+t,s'+t,s+s'+t$ switches, where t is the number of switches needed for putting $\rho=(\alpha,\beta)$ in noncrossing form. Thus t is even, and we are done.

With the above claim in hand, we conclude, by using the second equality in the statement, that we have $\sigma\in P_{even}^*$. Thus we have $\pi\in P_{even}^{[\infty]}$, which ends the proof of "$\supset$". $\square$

With the above result in hand, we can now prove:

Theorem 8.9. *The basic quantum reflection groups, namely*

$$\begin{array}{ccccc}
H_N^+ & \longrightarrow & \mathbb{T}H_N^+ & \longrightarrow & K_N^+\\
\uparrow & & \uparrow & & \uparrow\\
H_N^* & \longrightarrow & \mathbb{T}H_N^* & \longrightarrow & K_N^*\\
\uparrow & & \uparrow & & \uparrow\\
H_N & \longrightarrow & \mathbb{T}H_N & \longrightarrow & K_N
\end{array}$$

equal their own Schur-Weyl twists.

Proof. This result, established in [5], basically comes from the results that we have:

(1) In the real case, the verifications are as follows:

– H_N^+. We know from Proposition 8.1 above that for $\pi \in NC_{even}$ we have $\bar{T}_\pi = T_\pi$, and since we are in the situation $D \subset NC_{even}$, the definitions of $G, \bar{G}$ coincide.

– $H_N^{[\infty]}$. Here we can use the same argument as in (1), based this time on the description of $P_{even}^{[\infty]}$ involving the signatures found in Proposition 8.8.

– H_N^*. We have $H_N^* = H_N^{[\infty]} \cap O_N^*$, so $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the defining relations for $\bar{O}_N^*$. But all the $abc = -cba$ relations defining $\bar{H}_N^*$ are automatic, of type $0 = 0$, and it follows that $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the relations $abc = cba$, for any $a, b, c \in \{u_{ij}\}$. Thus we have $\bar{H}_N^* = H_N^{[\infty]} \cap O_N^* = H_N^*$, as claimed.

– H_N. We have $H_N = H_N^* \cap O_N$, and by functoriality, $\bar{H}_N = \bar{H}_N^* \cap \bar{O}_N = H_N^* \cap \bar{O}_N$. But this latter intersection is easily seen to be equal to H_N, as claimed.

In the complex case the proof is similar, and we refer here to [5]. $\square$

Summarizing, we have so far a twisting theory for the pairs (U, K), leading to twisted pairs $(\bar{U}, K)$. In relation now with the tori, we have the following result:

Theorem 8.10. *The diagonal tori of the twisted quantum groups are*

$$\begin{array}{ccccc}
T_N^+ & \longrightarrow & \mathbb{T}T_N^+ & \longrightarrow & \mathbb{T}_N^+\\
\uparrow & & \uparrow & & \uparrow\\
T_N^* & \longrightarrow & \mathbb{T}T_N^* & \longrightarrow & \mathbb{T}_N^*\\
\uparrow & & \uparrow & & \uparrow\\
T_N & \longrightarrow & \mathbb{T}T_N & \longrightarrow & \mathbb{T}_N
\end{array}$$

exactly as in the untwisted case.

Proof. This is clear for the quantum reflection groups, which are not twistable, and for the quantum unitary groups this is elementary as well. □

Before getting into the spheres, let us discuss as well integration questions. With respect to the untwisted case, we only need a Weingarten integration formula for the twisted unitary groups $\bar{U}$. And the formula here comes as particular case of the following general result, valid for any Schur-Weyl twists in our sense:

Theorem 8.11. *We have the Weingarten type formula*

$$\int_{\dot{U}_N^\times} u_{i_1j_1}^{e_1}\ldots u_{i_kj_k}^{e_k}=\sum_{\pi,\sigma\in P_\times(\alpha)}\dot{\delta}_\pi(i_1\ldots i_k)\dot{\delta}_\sigma(j_1\ldots j_k)W_{kN}(\pi,\sigma)$$

where $W_{kN}=G_{kN}^{-1}$*, with* $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$*, for* $\pi,\sigma\in P_\times(k)$*.*

Proof. This follows indeed as in [19], by using the above results. Observe that the Weingarten matrix is the same in the twisted and the untwisted cases. □

It remains to discuss the spheres. Here the situation is a bit more complicated, and it is convenient to start from zero, and construct the twisted spheres as follows:

Theorem 8.12. *We have noncommutative spheres as follows, obtained via the twisted commutation relations $ab=\pm ba$, and twisted half-commutation relations $abc=\pm cba$,*

$$\begin{tikzcd}
S^{N-1}_{\mathbb{R},+} \arrow[r] & \mathbb{T}S^{N-1}_{\mathbb{R},+} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\
\bar{S}^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[u] & \mathbb{T}\bar{S}^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{C},*} \arrow[u] \\
\bar{S}^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & \mathbb{T}\bar{S}^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{C}} \arrow[u]
\end{tikzcd}$$

where the signs at left correspond to the anticommutation of distinct coordinates, and their adjoints, and the other signs come from functoriality.

Proof. For the spheres on the left, if we want to replace some of the commutation relations $z_iz_j=z_jz_i$ by anticommutation relations $z_iz_j=-z_jz_i$, the one and only natural choice is $z_iz_j=-z_jz_i$ for $i\neq j$. In other words, with the notation $\varepsilon_{ij}=1-\delta_{ij}$, we must have:

$$z_iz_j=(-1)^{\varepsilon_{ij}}z_jz_i$$

Regarding now the spheres in the middle, the situation is a priori a bit more tricky, because we have to take into account the various possible collapsings of $\{i,j,k\}$. However, if we want to have embeddings as above, there is only one choice, namely:

$$z_iz_jz_k=(-1)^{\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}}z_kz_jz_i$$

Thus, we have constructed our spheres, and embeddings, as needed. □

In relation with the quantum groups, we have the following result:

Theorem 8.13. *The twisted spheres have the following properties:*

(1) *They have affine actions of the twisted unitary quantum groups.*
(2) *They have unique invariant Haar functionals, which are ergodic.*
(3) *Their Haar functionals are given by Weingarten type formulae.*
(4) *They appear, via the GNS construction, as first row spaces.*

Proof. The proofs here are similar to those from the untwisted case:

(1) This is clear from definitions.

(2) Our claim here is that the integration functional of $\bar{S}$ has the following ergodicity property, where $\Phi:C(\bar{S})\to C(\bar{S})\otimes C(\bar{U})$ is the affine coaction map:

$$\left(id\otimes\int_{\bar{U}}\right)\Phi(x)=\int_{\bar{S}}x$$

Indeed, in the real case, $x_i=x_i^*$, it is enough to check this on an arbitrary product of coordinates, $x_{i_1}\ldots x_{i_k}$. The left term is as follows:

$$\begin{aligned}\left(id\otimes\int_{\bar{U}}\right)\Phi(x_{i_1}\ldots x_{i_k})&=\sum_{j_1\ldots j_k}x_{j_1}\ldots x_{j_k}\int_{\bar{U}}u_{j_1i_1}\ldots u_{j_ki_k}\\&=\sum_{j_1\ldots j_k}\sum_{\pi,\sigma\in D(k)}\bar{\delta}_\pi(j)\bar{\delta}_\sigma(i)W_{kN}(\pi,\sigma)x_{j_1}\ldots x_{j_k}\\&=\sum_{\pi,\sigma\in D(k)}\bar{\delta}_\sigma(i)W_{kN}(\pi,\sigma)\sum_{j_1\ldots j_k}\bar{\delta}_\pi(j)x_{j_1}\ldots x_{j_k}\end{aligned}$$

Let us look now at the last sum on the right. The situation is as follows:

(1) In the free case we have to sum quantities of type $x_{j_1}\ldots x_{j_k}$, over all choices of multi-indices $j=(j_1,\ldots,j_k)$ which fit into our given noncrossing pairing π, and just by using the condition $\sum_i x_i^2=1$, we conclude that the sum is 1.

(2) The same happens in the classical case. Indeed, our pairing π can now be crossing, but we can use the commutation relations $x_ix_j=x_jx_i$, and the sum is again 1.

Thus the sum on the right is 1, in all cases, and we obtain:

$$\left(id\otimes\int_{\bar{U}}\right)\Phi(x_{i_1}\ldots x_{i_k})=\sum_{\pi,\sigma\in D(k)}\bar{\delta}_\sigma(i)W_{kN}(\pi,\sigma)$$

On the other hand, another application of the Weingarten formula gives:

$$\begin{aligned}\int_{\bar{S}} x_{i_1}\ldots x_{i_k} &= \int_{\bar{U}} u_{1i_1}\ldots u_{1i_k}\\ &= \sum_{\pi,\sigma\in D(k)} \bar{\delta}_\pi(1)\bar{\delta}_\sigma(i)W_{kN}(\pi,\sigma)\\ &= \sum_{\pi,\sigma\in D(k)} \bar{\delta}_\sigma(i)W_{kN}(\pi,\sigma)\end{aligned}$$

In the complex case the proof is similar, by adding exponents. See [4].

We can now formulate an abstract characterization of the integration, as being the unique positive unital trace $tr : C(\bar{S}) \to \mathbb{C}$ satisfying the following condition:

$$(tr \otimes id)\Phi(x) = tr(x)1$$

Indeed, it follows from the Haar integral invariance condition for $\bar{U}$ that the canonical integration has indeed the invariance property in the statement.

In order to prove now the uniqueness, let tr be as in the statement. We have:

$$\begin{aligned}tr\left(id\otimes\int_{\bar{U}}\right)\Phi(x) &= \int_{\bar{U}}(tr\otimes id)\Phi(x)\\ &= \int_{\bar{U}}(tr(x)1)\\ &= tr(x)\end{aligned}$$

On the other hand, according to the ergodicity formula, we have as well:

$$\begin{aligned}tr\left(id\otimes\int_{\bar{U}}\right)\Phi(x) &= tr\left(\int_{\bar{S}} x\right)\\ &= \int_{\bar{S}} x\end{aligned}$$

We therefore conclude that tr equals the standard integration, as claimed.

(3) This is clear from (2).

(4) This is clear as well from (2). □

Summarizing, we have twisted versions of all our objects.

As a conclusion, we have shown that the various quadruplets (S,T,U,K) constructed in sections 1-6 above have twisted counterparts $(\bar{S},T,\bar{U},K)$. The question that we would like to solve now is that of finding correspondences, as follows:

$$\begin{tikzcd}
\bar{S} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
\bar{U} \arrow[r, leftrightarrow] & K
\end{tikzcd}$$

To be more precise, we would like to understand if the twisted quadruplets $(\bar{S}, T, \bar{U}, K)$ satisfy the axioms for the quadruplets (S, T, U, K), or a modification of these axioms.

This latter question is quite tricky. In order to explain all this, let us get back to the axiomatics from section 3. We have seen there that the 12 correspondences between our objects (S, T, U, K) come in fact from 7 main correspondences, as follows:

$$\begin{tikzcd}
S \arrow[r] \arrow[d, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl] \\
U \arrow[r] & K
\end{tikzcd}$$

In the twisted case, 6 of these correspondences seem to hold as well, but the remaining one, namely $S \to T$, is definitely wrong as stated, and must be modified.

Let us begin our discussion with quantum isometry group results. For this purpose, we will need some linear independence results for the products of coordinates:

Proposition 8.14. *The linear relations satisfied by the variables $r_{ij} = z_i z_j$ are:*

(1) *For $S^{N-1}_\mathbb{R}, \bar{S}^{N-1}_\mathbb{R}$ we have $r_{ij} = \pm r_{ji}$, and no other relations.*
(2) *For the remaining 8 spheres, these elements are linearly independent.*

In addition, a similar result holds for the variables $c_{ij} = z_i z_j^*$.

Proof. We first prove the assertion regarding the variables $r_{ij} = z_i z_j$. We have 10 spheres to be investigated, and the proof goes as follows:

<u>1-2. $S^{N-1}_\mathbb{R}, S^{N-1}_\mathbb{C}$.</u> The results here are clear.

<u>3-4. $\bar{S}^{N-1}_\mathbb{R}, \bar{S}^{N-1}_\mathbb{C}$.</u> We prove first the result for $\bar{S}^{N-1}_\mathbb{R}$. We use the model $z_i \to Z_i = u_{1i}$, where u_{ij} are the standard coordinates on $\bar{O}_N$. We have:

$$\begin{aligned}
< Z_i Z_j, Z_k Z_l > &= \int_{\bar{O}_N} u_{1i} u_{1j} u_{1l} u_{1k} \\
&= \sum_{\pi,\sigma \in P_2(4)} \bar{\delta}_\sigma(i,j,l,k) W_{4N}(\pi,\sigma)
\end{aligned}$$

Since $P_2(4) = \{\cap\cap, \Cap, \sqcap\!\!\!\sqcap\}$, the Weingarten matrix on the right is given by:

$$\begin{aligned} W_{4N} &= \begin{pmatrix} N^2 & N & N \\ N & N^2 & N \\ N & N & N^2 \end{pmatrix}^{-1} \\ &= \frac{1}{N(N-1)(N+2)} \begin{pmatrix} N+1 & -1 & -1 \\ -1 & N+1 & -1 \\ -1 & -1 & N+1 \end{pmatrix} \end{aligned}$$

We conclude that we have the following formula:

$$< Z_iZ_j, Z_kZ_l >= \frac{1}{N(N+2)} \sum_{\sigma \in P_2(4)} \bar{\delta}_\sigma(i,j,l,k)$$

The matrix on the right, taken with indices $i \leq j$ and $k \leq l$, is then invertible. Thus the variables Z_iZ_j are linearly independent, and so must be the variables z_iz_j.

For the sphere $\bar{S}^{N-1}_{\mathbb{C}}$, a similar computation, using now a $\bar{U}_N$ model, gives:

$$\begin{aligned} < Z_iZ_j, Z_kZ_l > &= \int_{\bar{U}_N} u_{1i}u_{1j}u_{1l}^*u_{1k}^* \\ &= \sum_{\pi,\sigma \in P_2(11**)} \bar{\delta}_\sigma(i,j,l,k) W_{4N}^{11**}(\pi,\sigma) \end{aligned}$$

We have $P_2(11**) = \{\Cap, \sqcap\!\!\!\sqcap\}$, and the corresponding Weingarten matrix is:

$$\begin{aligned} W_{4N}^{11**} &= \begin{pmatrix} N^2 & N \\ N & N^2 \end{pmatrix}^{-1} \\ &= \frac{1}{N(N^2-1)} \begin{pmatrix} N & -1 \\ -1 & N \end{pmatrix} \end{aligned}$$

We therefore obtain the following formula:

$$< Z_iZ_j, Z_kZ_l >= \frac{1}{N(N+1)} \sum_{\sigma \in P_2(11**)} \bar{\delta}_\sigma(i,j,l,k)$$

Once again, since the matrix on the right is invertible, we obtain the result.

5-6. $S^{N-1}_{\mathbb{R},*}, \bar{S}^{N-1}_{\mathbb{R},*}$. We can use here a 2×2 matrix trick from [37]. Consider indeed one of the spheres $S^{N-1}_{\mathbb{C}}/\bar{S}^{N-1}_{\mathbb{C}}$, with coordinates denoted $y_1, \ldots, y_N$, and let us set:

$$Z_i = \begin{pmatrix} 0 & y_i \\ y_i^* & 0 \end{pmatrix}$$

As in the untwisted case, discussed in section 4, these matrices produce models for $S^{N-1}_{\mathbb{R},*},\bar{S}^{N-1}_{\mathbb{R},*}$. Now observe that the elements $r_{ij}=z_iz_j$ map in this way to:

$$\begin{aligned}R_{ij}&=&Z_iZ_j\\&=&\begin{pmatrix}0&y_i\\y_i^*&0\end{pmatrix}\begin{pmatrix}0&y_j\\y_j^*&0\end{pmatrix}\\&=&\begin{pmatrix}y_iy_j^*&0\\0&y_i^*y_j\end{pmatrix}\end{aligned}$$

Thus, the result follows from the result for $\bar{S}^{N-1}_{\mathbb{R}},\bar{S}^{N-1}_{\mathbb{C}}$, established above.

7-10. $S^{N-1}_{\mathbb{R},+},S^{N-1}_{\mathbb{C},+},S^{N-1}_{\mathbb{C},**},\bar{S}^{N-1}_{\mathbb{C},**}$. The results here follow simply by functoriality, from those established above, for the smaller spheres $S^{N-1}_{\mathbb{R},*},\bar{S}^{N-1}_{\mathbb{R},*}$.

Finally, the proof of the last assertion is similar, with no new computations needed in the real case, where $r_{ij}=c_{ij}$, and with the same Weingarten matrix, this time coming from the set $P_2(1*1*)=\{\cap\cap,⋒\}$, appearing in the complex case. □

In order to deal with the half-liberated cases, we will need as well:

Proposition 8.15. *Consider one of the spheres $S^{N-1}_{\mathbb{R},*},S^{N-1}_{\mathbb{C},**},\bar{S}^{N-1}_{\mathbb{R},*},\bar{S}^{N-1}_{\mathbb{C},**}$.*

1. *The variables $z_az_bz_c$ with $a<c$ and a,b,c distinct are linearly independent.*
2. *These variables are independent as well from any $z_az_bz_c$ with a,b,c not distinct.*

In addition, a similar result holds for the variables of type $z_az_b^*z_c$.

Proof. This follows by using the same method as in the proof of Proposition 8.14, with models coming from the quantum groups $O_N^*,U_N^{**},\bar{O}_N^*,\bar{U}_N^{**}$. □

We can state and prove the following result:

Theorem 8.16. *We have the quantum isometry group formula*

$$\bar{U}=G^+(\bar{S})$$

in all the 9 main cases, twisted and untwisted.

Proof. The proof in the classical twisted cases is similar to the proof in the classical untwisted cases, by adding signs where needed, and by using Proposition 8.14.

Indeed, for the twisted real sphere $\bar{S}^{N-1}_{\mathbb{R}}$ we have:

$$\begin{aligned}\Phi(z_iz_j)&=&\sum_k z_k^2\otimes u_{ki}u_{kj}\\&+&\sum_{k<l}z_kz_l\otimes(u_{ki}u_{lj}-u_{li}u_{kj})\end{aligned}$$

We deduce that with $[[a,b]] = ab+ba$ we have the following formula:

$$\begin{aligned}\Phi([[z_i,z_j]]) &= \sum_k z_k^2 \otimes [[u_{ki},u_{kj}]] \\ &+ \sum_{k<l} z_k z_l \otimes ([u_{ki},u_{lj}] - [u_{li},u_{kj}])\end{aligned}$$

Now assuming $i \neq j$, we have $[[z_i,z_j]] = 0$, and we therefore obtain:

$$[[u_{ki},u_{kj}]] = 0 \quad , \quad \forall k$$

$$[u_{ki},u_{lj}] = [u_{li},u_{kj}] \quad , \quad \forall k<l$$

By using now the standard trick, namely applying the antipode and then relabelling, the latter relation gives $[u_{ki},u_{lj}] = 0$, and we are done.

The proof for $\bar{S}^{N-1}_{\mathbb{C}}$ is similar, by using the above-mentioned categorical trick, in order to deduce from the relations $ab = \pm ba$ the remaining relations $ab^* = \pm b^* a$.

Finally, the proof in the half-classical twisted cases is similar to the proof in the half-classical untwisted cases, by adding signs where needed, and by using Proposition 8.15 as a main technical ingredient, allowing us to write the relevant equations. See [1]. $\square$

Regarding now the $K = G^+(T) \cap K_N^+$ axiom, this is something that we already know. However, regarding the correspondence $S \to T$, things here fail in the twisted case. Our "fix" for this, or at least the best fix that we could find, is as follows:

Theorem 8.17. *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, define its toral isometry group as being the biggest subgroup of $\mathbb{T}_N^+$ acting affinely on X:*

$$\mathcal{G}^+(X) = G^+(X) \cap \mathbb{T}_N^+$$

With this convention, for the 9 basic spheres S, and for their twists as well, the toral isometry group equals the torus T.

Proof. We recall from section 3 that the affine quantum isometry group $G^+(X) \subset U_N^+$ of a noncommutative manifold $X \subset S^{N-1}_{\mathbb{C},+}$ coming from certain polynomial relations P is constructed according to the following procedure:

$$P(x_i) = 0 \implies P\left(\sum_j x_j \otimes u_{ji}\right) = 0$$

Similarly, the toral isometry group $\mathcal{G}^+(X) \subset \mathbb{T}_N^+$ is constructed as follows:

$$P(x_i) = 0 \implies P(x_i \otimes u_i) = 0$$

In the monomial case one can prove that the following formula holds:

$$G^+(\bar{S}) = \overline{G^+(S)}$$

By intersecting with $\mathbb{T}_N^+$, we obtain from this that we have:

$$\mathcal{G}^+(\bar{S}) = \mathcal{G}^+(S)$$

The result can be of course be proved as well directly. For $\bar{S}^{N-1}_{\mathbb{R}}$ we have:

$$\Phi(x_ix_j) = x_ix_j \otimes u_iu_j$$

$$\Phi(x_jx_i) = x_jx_i \otimes u_ju_i$$

Thus we obtain $u_iu_j = -u_ju_i$ for $i \neq j$, and so the quantum group is T_N.

The proof in the complex, half-liberated and hybrid cases is similar. □

Finally, regarding the hard liberation axiom, this seems to hold indeed in all the cases under consideration, but this is non-trivial, and not known yet. As a conclusion, we conjecturally have an extension of our (S,T,U,K) formalism, with the $S \to T$ axiom needing a modification as above, which covers the twisted objects $(\bar{S},T,\bar{U},K)$ as well.

There are many other interesting questions here, in relation with the various generalizations of the easy quantum group theory, coming from [25], [26] and from [65].

9. Free coordinates

We have seen that, according to our philosophy here, a noncommutative geometry should come from a quadruplet (S,T,U,K) consisting of a sphere S, a torus T, a unitary group U, and a reflection group K, having relations between them, as follows:

$$\begin{tikzcd}
S \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
U \arrow[r, leftrightarrow] & K
\end{tikzcd}$$

The quadruplets (S,T,U,K) producing geometries can be axiomatized. With such an axiomatization in hand, some classification results can be worked out. The 9 main geometries can be twisted, with the twisting being trivial in the free case.

We would like to discuss now a few more results on the subject, of more specialized nature. For simplicity we will restrict the attention to the real case. Here we have 3 main geometries, whose associated spheres are as follows:

$$S^{N-1}_{\mathbb R}\subset S^{N-1}_{\mathbb R,*}\subset S^{N-1}_{\mathbb R,+}$$

Our purpose will be that of going beyond the basic level, where we are now, with a number of results regarding the coordinates $x_1,\ldots,x_N$ of such spheres:

(1) A first question, which is algebraic, is that of understanding the precise relations satisfied by these coordinates. We will see that this is related to the question of unifying the twisted and untwisted geometries, via intersection.
(2) A second question, which is analytic, is that of understanding the fixed N behavior of these coordinates. This can be done via deformation methods. We will see as well that there is an unexpected link here with quantum permutations.

Let us begin by discussing algebraic aspects. This is something quite fundamental. Indeed, in the classical case, the algebraic manifolds X can be identified with the corresponding ideals of vanishing polynomials J, and the correspondence $X\leftrightarrow J$ is the foundation for all the known algebraic geometric theory, ancient or more modern.

In the free setting, things are in a quite primitive status, and a suitable theory of "noncommutative algebra", useful in connection with our present considerations, is so far missing. Computing J for the free spheres, and perhaps for some other spheres as well, is a problem which is difficult enough for us, and that we will investigate here.

We first have the following result, dealing with the real case:

Proposition 9.1. *The* 5 *real spheres, and the intersections between them, are*

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+}\\
\uparrow & & \uparrow & & \uparrow\\
S^{N-1,1}_{\mathbb{R}} & \longrightarrow & S^{N-1,1}_{\mathbb{R},*} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R},*}\\
\uparrow & & \uparrow & & \uparrow\\
S^{N-1,0}_{\mathbb{R}} & \longrightarrow & \bar{S}^{N-1,1}_{\mathbb{R}} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R}}
\end{array}$$

where $\dot{S}^{N-1,d-1}_{\mathbb{R},\times}\subset\dot{S}^{N-1}_{\mathbb{R},\times}$ *is obtained by assuming* $x_{i_0}\ldots x_{i_d}=0$*, for* $i_0,\ldots,i_d$ *distinct.*

Proof. Consider the following 4-diagram, obtained by intersecting:

$$\begin{array}{ccc}
S^{N-1}_{\mathbb{R}}\cap\bar{S}^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},*}\cap\bar{S}^{N-1}_{\mathbb{R},*}\\
\uparrow & & \uparrow\\
S^{N-1}_{\mathbb{R}}\cap\bar{S}^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*}\cap\bar{S}^{N-1}_{\mathbb{R}}
\end{array}$$

We must prove that this coincides with the 4-diagram at bottom left, in the statement. But this is clear, because combining the commutation and anticommutation relations leads to the vanishing relations defining spheres of type $\dot{S}^{N-1,d-1}_{\mathbb{R},\times}$. More precisely:

(1) $S^{N-1}_{\mathbb{R}}\cap\bar{S}^{N-1}_{\mathbb{R}}$ consists of the points $x\in S^{N-1}_{\mathbb{R}}$ satisfying $x_ix_j=-x_jx_i$ for $i\neq j$. Since $x_ix_j=x_jx_i$, this relation reads $x_ix_j=0$ for $i\neq j$, which means $x\in S^{N-1,0}_{\mathbb{R}}$.

(2) $S^{N-1}_{\mathbb{R}}\cap\bar{S}^{N-1}_{\mathbb{R},*}$ consists of the points $x\in S^{N-1}_{\mathbb{R}}$ satisfying $x_ix_jx_k=-x_kx_jx_i$ for i,j,k distinct. Once again by commutativity, this relation is equivalent to $x\in S^{N-1,1}_{\mathbb{R}}$.

(3) $S^{N-1}_{\mathbb{R},*}\cap\bar{S}^{N-1}_{\mathbb{R}}$ is obtained from $\bar{S}^{N-1}_{\mathbb{R}}$ by imposing to the standard coordinates the half-commutation relations $abc=cba$. On the other hand, we know from $\bar{S}^{N-1}_{\mathbb{R}}\subset\bar{S}^{N-1}_{\mathbb{R},*}$ that the standard coordinates on $\bar{S}^{N-1}_{\mathbb{R}}$ satisfy $abc=-cba$ for a,b,c distinct, and $abc=cba$ otherwise. Thus, the relations brought by intersecting with $S^{N-1}_{\mathbb{R},*}$ reduce to the relations $abc=0$ for a,b,c distinct, and so we are led to the sphere $\bar{S}^{N-1,1}_{\mathbb{R}}$.

(4) $S^{N-1}_{\mathbb{R},*}\cap\bar{S}^{N-1}_{\mathbb{R},*}$ is obtained from $\bar{S}^{N-1}_{\mathbb{R},*}$ by imposing the relations $abc=-cba$ for a,b,c distinct, and $abc=cba$ otherwise. Since we know that $abc=cba$ for any a,b,c, the extra relations reduce to $abc=0$ for a,b,c distinct, and so we are led to $S^{N-1,1}_{\mathbb{R},*}$. $\square$

In order to find now a suitable axiomatic framework for the 9 spheres, we use the following definition, coming from the various formulae in sections 6 and 8:

Definition 9.2. *Given variables $x_1,\ldots,x_N$, any permutation $\sigma\in S_k$ produces two collections of relations between these variables, as follows:*

(1) *Untwisted relations: $x_{i_1}\ldots x_{i_k}=x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$, for any $i_1,\ldots,i_k$.*
(2) *Twisted relations: $x_{i_1}\ldots x_{i_k}=\varepsilon\left(\ker\binom{i_1\ \ldots\ i_k}{i_{\sigma(1)}\ldots i_{\sigma(k)}}\right)x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$, for any $i_1,\ldots,i_k$.*

The untwisted relations are denoted $\mathcal{R}_\sigma$, and the twisted ones are denoted $\bar{\mathcal{R}}_\sigma$.

Observe that the relations $\mathcal{R}_\sigma$ are trivially satisfied for the standard coordinates on $S^{N-1}_\mathbb{R}$, for any $\sigma\in S_k$. A twisted analogue of this fact holds, in the sense that the standard coordinates on $\bar{S}^{N-1}_\mathbb{R}$ satisfy the relations $\bar{\mathcal{R}}_\sigma$, for any $\sigma\in S_k$. Indeed, by anticommutation we must have a formula of type $x_{i_1}\ldots x_{i_k}=\pm x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$, and the sign $\pm$ obtained in this way is precisely the one given above, namely:

$$\pm=\varepsilon\left(\ker\binom{i_1\ \ldots\ i_k}{i_{\sigma(1)}\ldots i_{\sigma(k)}}\right)$$

We have now all the needed ingredients for axiomatizing the various spheres:

Definition 9.3. *We have 3 types of noncommutative spheres $S\subset S^{N-1}_{\mathbb{R},+}$, as follows:*

(1) *Untwisted: $S^{N-1}_{\mathbb{R},E}$, with $E\subset S_\infty$, obtained via the relations $\{\mathcal{R}_\sigma|\sigma\in E\}$.*
(2) *Twisted: $\bar{S}^{N-1}_{\mathbb{R},F}$, with $F\subset S_\infty$, obtained via the relations $\{\bar{\mathcal{R}}_\sigma|\sigma\in F\}$.*
(3) *Polygonal: $S^{N-1}_{\mathbb{R},E,F}=S^{N-1}_{\mathbb{R},E}\cap\bar{S}^{N-1}_{\mathbb{R},F}$, with $E,F\subset S_\infty$.*

Observe that "untwisted" means precisely "monomial", in the sense of section 6 above. As examples, $S^{N-1}_\mathbb{R},S^{N-1}_{\mathbb{R},*},S^{N-1}_{\mathbb{R},+}$ are untwisted, $\bar{S}^{N-1}_\mathbb{R},\bar{S}^{N-1}_{\mathbb{R},*},S^{N-1}_{\mathbb{R},+}$ are twisted, and the 9 spheres in Proposition 9.1 above are all polygonal. Observe also that the set of polygonal spheres is closed under intersections, due to the following formula:

$$S^{N-1}_{\mathbb{R},E,F}\cap S^{N-1}_{\mathbb{R},E',F'}=S^{N-1}_{\mathbb{R},E\cup E',F\cup F'}$$

Let us try now to understand the structure of the various types of spheres:

Proposition 9.4. *The various spheres can be parametrized by groups, as follows:*

(1) *Untwisted case: $S^{N-1}_{\mathbb{R},G}$, with $G\subset S_\infty$ filtered group.*
(2) *Twisted case: $\bar{S}^{N-1}_{\mathbb{R},H}$, with $H\subset S_\infty$ filtered group.*
(3) *Polygonal case: $S^{N-1}_{\mathbb{R},G,H}$, with $G,H\subset S_\infty$ filtered groups.*

Proof. Here (1) is from section 6 above, (2) follows similarly, by taking $H\subset S_\infty$ to be the set of permutations $\sigma\in S_\infty$ having the property that the relations $\bar{\mathcal{R}}_\sigma$ hold for the standard coordinates, and (3) follows from (1,2), by taking intersections. $\square$

Let us write now the 9 main polygonal spheres as in Proposition 9.4 (3). We say that a polygonal sphere parametrization $S=S^{N-1}_{\mathbb{R},G,H}$ is "standard" when both filtered groups $G,H\subset S_\infty$ are chosen to be maximal. In this case, Proposition 9.4 (3) and its proof tell us that G,H encode all the monomial relations which hold in S.

We have the following result, extending some previous findings from section 6:

Theorem 9.5. *The standard parametrization of the* 9 *main spheres is*

$$\begin{tikzcd}
S_\infty & S_\infty^* & \{1\} & G/H \\
S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u, dotted, no head] & S^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[u, dotted, no head] & S^{N-1}_{\mathbb{R},+} \arrow[r, dotted, no head] \arrow[u, dotted, no head] & \{1\} \\
S^{N-1,1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{N-1,1}_{\mathbb{R},*} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{R},*} \arrow[r, dotted, no head] \arrow[u] & S_\infty^* \\
S^{N-1,0}_{\mathbb{R}} \arrow[r] \arrow[u] & \bar{S}^{N-1,1}_{\mathbb{R}} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{R}} \arrow[r, dotted, no head] \arrow[u] & S_\infty
\end{tikzcd}$$

so these spheres come from the $3 \times 3 = 9$ *pairs of groups among* $\{1\} \subset S_\infty^* \subset S_\infty$.

Proof. The fact that we have parametrizations as above is known to hold for the 5 untwisted and twisted spheres, and for the remaining 4 spheres, this follows by intersecting. In order to prove now that the parametrizations are standard, we must compute the following two filtered groups, and show that we get the groups in the statement:

$$G = \{\sigma \in S_\infty | \text{the relations } \mathcal{R}_\sigma \text{ hold over } S\}$$

$$H = \{\sigma \in S_\infty | \text{the relations } \bar{\mathcal{R}}_\sigma \text{ hold over } S\}$$

As a first observation, by using the various inclusions between spheres, we just have to compute G for the spheres on the bottom, and H for the spheres on the left:

$$X = S^{N-1,0}_{\mathbb{R}}, \bar{S}^{N-1,1}_{\mathbb{R}}, \bar{S}^{N-1}_{\mathbb{R}} \implies G = S_\infty, S_\infty^*, \{1\}$$

$$X = S^{N-1,0}_{\mathbb{R}}, S^{N-1,1}_{\mathbb{R}}, S^{N-1}_{\mathbb{R}} \implies H = S_\infty, S_\infty^*, \{1\}$$

The results for $S^{N-1,0}_{\mathbb{R}}$ being clear, we are left with computing the remaining 4 groups, for the spheres $S^{N-1}_{\mathbb{R}}, \bar{S}^{N-1}_{\mathbb{R}}, S^{N-1,1}_{\mathbb{R}}, \bar{S}^{N-1,1}_{\mathbb{R}}$. The proof here goes as follows:

(1) $S^{N-1}_{\mathbb{R}}$. According to the definition of $H = (H_k)$, we have:

$$\begin{aligned}
H_k &= \left\{\sigma \in S_k \middle| x_{i_1} \ldots x_{i_k} = \varepsilon\left(\ker\left(\begin{smallmatrix} i_1 & \ldots & i_k \\ i_{\sigma(1)} & \ldots & i_{\sigma(k)} \end{smallmatrix}\right)\right) x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}, \forall i_1, \ldots, i_k\right\} \\
&= \left\{\sigma \in S_k \middle| \varepsilon\left(\ker\left(\begin{smallmatrix} i_1 & \ldots & i_k \\ i_{\sigma(1)} & \ldots & i_{\sigma(k)} \end{smallmatrix}\right)\right) = 1, \forall i_1, \ldots, i_k\right\} \\
&= \left\{\sigma \in S_k \middle| \varepsilon(\tau) = 1, \forall \tau \leq \sigma\right\}
\end{aligned}$$

Now since for any $\sigma \in S_k, \sigma \neq 1_k$, we can always find a partition $\tau \leq \sigma$ satisfying $\varepsilon(\tau) = -1$, we deduce that we have $H_k = \{1_k\}$, and so $H = \{1\}$, as desired.

(2) $\bar{S}^{N-1}_{\mathbb{R}}$. The proof of $G = \{1\}$ here is similar to the proof of $H = \{1\}$ in (1) above, by using the same combinatorial ingredient at the end.

(3) $S^{N-1,1}_{\mathbb{R}}$. By definition of $H = (H_k)$, a permutation $\sigma \in S_k$ belongs to H_k when the following condition is satisfied, for any choice of the indices $i_1, \ldots, i_k$:

$$x_{i_1} \ldots x_{i_k} = \varepsilon\left(\ker\left(\begin{smallmatrix} i_1 & \ldots & i_k \\ i_{\sigma(1)} & \ldots & i_{\sigma(k)} \end{smallmatrix}\right)\right) x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}$$

When $|\ker i| = 1$ this formula reads $x_r^k = x_r^k$, which is true. When $|\ker i| \geq 3$ this formula is automatically satisfied as well, because by using the relations $ab = ba$, and $abc = 0$ for a, b, c distinct, which both hold over $S^{N-1,1}_{\mathbb{R}}$, this formula reduces to $0 = 0$. Thus, we are left with studying the case $|\ker i| = 2$. Here the quantities on the left $x_{i_1} \ldots x_{i_k}$ will not vanish, so the sign on the right must be 1, and we therefore have:

$$H_k = \left\{\sigma \in S_k \middle| \varepsilon(\tau) = 1, \forall \tau \leq \sigma, |\tau| = 2\right\}$$

Now by coloring the legs of σ clockwise $\circ \bullet \circ \bullet \ldots$, the above condition is satisfied when each string of σ joins a white leg to a black leg. Thus $H_k = S_k^*$, as desired.

(4) $\bar{S}^{N-1,1}_{\mathbb{R}}$. The proof of $G = S_\infty^*$ here is similar to the proof of $H = S_\infty^*$ in (3) above, by using the same combinatorial ingredient at the end. ☐

We can now formulate a classification result, as follows:

Theorem 9.6. *The following hold:*

(1) $S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$ *are the only untwisted monomial spheres.*
(2) $\bar{S}^{N-1}_{\mathbb{R}} \subset \bar{S}^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$ *are the only twisted monomial spheres.*
(3) *The* 9 *spheres in Theorem 9.5 are the only polygonal ones.*

Proof. By using standard parametrizations, the above 3 statements are equivalent. Now since (1) was proved in section 6 above, all the results hold true. ☐

We have as well the following result, extending some previous findings:

Theorem 9.7. *The quantum isometry groups of the* 9 *polygonal spheres are*

$$\begin{tikzcd}
O_N \arrow[r] & O_N^* \arrow[r] & O_N^+ \\
H_N \arrow[u] \arrow[r] & H_N^{[\infty]} \arrow[u] \arrow[r] & \bar{O}_N^* \arrow[u] \\
H_N^+ \arrow[u] \arrow[r] & H_N \arrow[u] \arrow[r] & \bar{O}_N \arrow[u]
\end{tikzcd}$$

where $H_N^+, H_N^{[\infty]}$ *and* $\bar{O}_N, O_N^*, \bar{O}_N^*, O_N^*$ *are noncommutative versions of* H_N, O_N.

Proof. This is indeed routine, and we refer here to the literature [2]. □

All this is of course just a beginning, and there are many questions left, regarding the extension of our (S,T,U,K) formalism, as to cover these intersections.

Let us turn now to analytic questions. Following [22], we first have:

Proposition 9.8. *The classical integral of $x_{i_1}\ldots x_{i_k}$ vanishes, unless each $a\in\{1,\ldots,N\}$ appears an even number of times in the sequence $i_1,\ldots,i_k$. We have*

$$\int_{S^{N-1}_\mathbb{R}} x_{i_1}\ldots x_{i_k}\,dx=\frac{(N-1)!!l_1!!\ldots l_N!!}{(N+\sum l_i-1)!!}$$

where $m!!=(m-1)(m-1)(m-5)\ldots$, and l_a is this number of occurrences.

Proof. First, the result holds indeed at $N=2$, due to the following well-known formula, where $\varepsilon(p)=1$ when $p\in\mathbb{N}$ is even, and $\varepsilon(p)=0$ when p is odd:

$$\int_0^{\pi/2}\cos^p t\sin^q t\,dt=\left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)}\frac{p!!q!!}{(p+q+1)!!}$$

In general, we can restrict attention to the case $l_a\in 2\mathbb{N}$, since the other integrals vanish. The integral in the statement can be written in spherical coordinates, as follows:

$$I=\frac{2^N}{V}\int_0^{\pi/2}\ldots\int_0^{\pi/2}x_1^{l_1}\ldots x_N^{l_N}\,J\,dt_1\ldots dt_{N-1}$$

Here V is the volume of the sphere, J is the Jacobian, and the 2^N factor comes from the restriction to the $1/2^N$ part of the sphere where all the coordinates are positive.

The normalization constant in front of the integral is:

$$\frac{2^N}{V}=\frac{2^N}{N\pi^{N/2}}\cdot\Gamma\left(\frac{N}{2}+1\right)=\left(\frac{2}{\pi}\right)^{[N/2]}(N-1)!!$$

As for the unnormalized integral, this is given by:

$$\begin{aligned}I'=\int_0^{\pi/2}\ldots\int_0^{\pi/2}\quad &(\cos t_1)^{l_1}(\sin t_1\cos t_2)^{l_2}\\ &\ldots\ldots\ldots\\ &(\sin t_1\sin t_2\ldots\sin t_{N-2}\cos t_{N-1})^{l_{N-1}}\\ &(\sin t_1\sin t_2\ldots\sin t_{N-2}\sin t_{N-1})^{l_N}\\ &\sin^{N-2}t_1\sin^{N-3}t_2\ldots\sin^2 t_{N-3}\sin t_{N-2}\\ &dt_1\ldots dt_{N-1}\end{aligned}$$

By rearranging the terms, we obtain:

$$\begin{aligned}
I' \quad = \quad & \int_0^{\pi/2} \cos^{l_1} t_1 \sin^{l_2+\ldots+l_N+N-2} t_1 \, dt_1 \\
& \int_0^{\pi/2} \cos^{l_2} t_2 \sin^{l_3+\ldots+l_N+N-3} t_2 \, dt_2 \\
& \ldots\ldots\ldots \\
& \int_0^{\pi/2} \cos^{l_{N-2}} t_{N-2} \sin^{l_{N-1}+l_N+1} t_{N-2} \, dt_{N-2} \\
& \int_0^{\pi/2} \cos^{l_{N-1}} t_{N-1} \sin^{l_N} t_{N-1} \, dt_{N-1}
\end{aligned}$$

Now by using the above-mentioned formula at $N = 2$, this gives:

$$\begin{aligned}
I' \quad = \quad & \frac{l_1!!(l_2+\ldots+l_N+N-2)!!}{(l_1+\ldots+l_N+N-1)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(N-2)} \\
& \frac{l_2!!(l_3+\ldots+l_N+N-3)!!}{(l_2+\ldots+l_N+N-2)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(N-3)} \\
& \ldots\ldots\ldots \\
& \frac{l_{N-2}!!(l_{N-1}+l_N+1)!!}{(l_{N-2}+l_{N-1}+l_N+2)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(1)} \\
& \frac{l_{N-1}!!l_N!!}{(l_{N-1}+l_N+1)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(0)}
\end{aligned}$$

Now observe that the various double factorials multiply up to quantity in the statement, modulo a $(N-1)!!$ factor, and that the $\frac{\pi}{2}$ factors multiply up to $(\frac{\pi}{2})^{[N/2]}$. Thus by multiplying with the normalization constant, we obtain the result. □

In the case of the half-liberated sphere, we have the following result:

Proposition 9.9. *The half-liberated integral of $x_{i_1}\ldots x_{i_k}$ vanishes, unless each index a appears the same number of times at odd and even positions in $i_1,\ldots,i_k$. We have*

$$\int_{S^{N-1}_{\mathbb{R},*}} x_{i_1}\ldots x_{i_k}\, dx = 4^{\sum l_i}\frac{(2N-1)!l_1!\ldots l_n!}{(2N+\sum l_i-1)!}$$

where l_a denotes this number of common occurrences.

Proof. As before, we can assume that k is even, $k = 2l$. The corresponding integral can be viewed as an integral over $S^{N-1}_{\mathbb{C}}$, as follows:

$$I = \int_{S^{N-1}_{\mathbb{C}}} z_{i_1}\bar{z}_{i_2}\ldots z_{i_{2l-1}}\bar{z}_{i_{2l}}\, dz$$

Now by using transformations of type $p \to \lambda p$ with $|\lambda| = 1$, we see that I vanishes, unless each z_a appears as many times as $\bar{z}_a$ does, and this gives the first assertion.

Assume now that we are in the non-vanishing case. Then the l_a copies of z_a and the l_a copies of $\bar{z}_a$ produce by multiplication a factor $|z_a|^{2l_a}$, so we have:

$$I=\int_{S^{N-1}_\mathbb{C}}|z_1|^{2l_1}\ldots|z_N|^{2l_N}\,dz$$

Now by using the standard identification $S^{N-1}_\mathbb{C}\simeq S^{2N-1}_\mathbb{R}$, we obtain:

$$\begin{aligned}I&=\int_{S^{2N-1}_\mathbb{R}}(x_1^2+y_1^2)^{l_1}\ldots(x_N^2+y_N^2)^{l_N}\,d(x,y)\\&=\sum_{r_1\ldots r_N}\binom{l_1}{r_1}\ldots\binom{l_N}{r_N}\int_{S^{2N-1}_\mathbb{R}}x_1^{2l_1-2r_1}y_1^{2r_1}\ldots x_N^{2l_N-2r_N}y_N^{2r_N}\,d(x,y)\end{aligned}$$

By using the formula in Proposition 9.8, we obtain:

$$\begin{aligned}I&=\sum_{r_1\ldots r_N}\binom{l_1}{r_1}\ldots\binom{l_N}{r_N}\frac{(2N-1)!!(2r_1)!!\ldots(2r_N)!!(2l_1-2r_1)!!\ldots(2l_N-2r_N)!!}{(2N+2\sum l_i-1)!!}\\&=\sum_{r_1\ldots r_N}\binom{l_1}{r_1}\ldots\binom{l_N}{r_N}\frac{(2N-1)!(2r_1)!\ldots(2r_N)!(2l_1-2r_1)!\ldots(2l_N-2r_N)!}{(2N+\sum l_i-1)!r_1!\ldots r_N!(l_1-r_1)!\ldots(l_N-r_N)!}\end{aligned}$$

We can rewrite the sum on the right in the following way:

$$\begin{aligned}I&=\sum_{r_1\ldots r_N}\frac{l_1!\ldots l_N!(2N-1)!(2r_1)!\ldots(2r_N)!(2l_1-2r_1)!\ldots(2l_N-2r_N)!}{(2N+\sum l_i-1)!(r_1!\ldots r_N!(l_1-r_1)!\ldots(l_N-r_N)!)^2}\\&=\sum_{r_1}\binom{2r_1}{r_1}\binom{2l_1-2r_1}{l_1-r_1}\ldots\sum_{r_N}\binom{2r_N}{r_N}\binom{2l_N-2r_N}{l_N-r_N}\frac{(2N-1)!l_1!\ldots l_N!}{(2N+\sum l_i-1)!}\end{aligned}$$

The sums on the right being $4^{l_1},\ldots,4^{l_N}$, this gives the formula in the statement. □

Finally, in the case of the free sphere, we have the following result, from [20]:

Theorem 9.10. *The moments of the free hyperspherical law are given by*

$$\int_{S^{N-1}_{\mathbb{R},+}}x_1^{2l}\,dx=\frac{1}{(N+1)^l}\cdot\frac{q+1}{q-1}\cdot\frac{1}{l+1}\sum_{r=-l-1}^{l+1}(-1)^r\binom{2l+2}{l+r+1}\frac{r}{1+q^r}$$

where $q\in[-1,0)$ is such that $q+q^{-1}=-N$.

Proof. The idea is that $x_1\in C(S^{N-1}_{\mathbb{R},+})$ has the same law as $u_{11}\in C(O_N^+)$, which has the same law as a certain variable $w\in C(SU^q_2)$, which can be in turn modelled by an explicit operator on $l^2(\mathbb{N})$, whose law can be computed by using advanced calculus.

Let us first explain the relation between O_N^+ and SU_2^q. To any matrix $F\in GL_N(\mathbb R)$ satisfying $F^2=1$ we associate the following universal algebra:

$$C(O_F^+)=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=F\bar{u}F=\text{unitary}\right)$$

Observe that $O_{I_N}^+=O_N^+$. In general, the above algebra satisfies Woronowicz's generalized axioms in [98], which do not include the strong antipode axiom $S^2=id$.

At $N=2$, up to a trivial equivalence relation on the matrices F, and on the quantum groups O_F^+, we can assume that F is as follows, with $q\in[-1,0)$:

$$F=\begin{pmatrix}0&\sqrt{-q}\\1/\sqrt{-q}&0\end{pmatrix}$$

Our claim is that for this matrix we have $O_F^+=SU_2^q$. Indeed, the relations $u=F\bar{u}F$ tell us that u must be of the following special form:

$$u=\begin{pmatrix}\alpha&-q\gamma^*\\ \gamma&\alpha^*\end{pmatrix}$$

Thus $C(O_F^+)$ is the universal algebra generated by two elements α,γ, with the relations making the above matrix u unitary. But these unitarity conditions are:

$$\alpha\gamma=q\gamma\alpha\quad,\quad\alpha\gamma^*=q\gamma^*\alpha\quad,\quad\gamma\gamma^*=\gamma^*\gamma$$

$$\alpha^*\alpha+\gamma^*\gamma=1\quad,\quad\alpha\alpha^*+q^2\gamma\gamma^*=1$$

We recognize here the relations in [98] defining the algebra $C(SU_2^q)$, and it follows that we have an isomorphism of Hopf C^*-algebras:

$$C(O_F^+)\simeq C(SU_2^q)$$

Now back to the general case, let us try to understand the integration over O_F^+. Given $\pi\in NC_2(2k)$ and $i=(i_1,\ldots,i_{2k})$, we set:

$$\delta_\pi^F(i)=\prod_{s\in\pi}F_{i_{s_l}i_{s_r}}$$

Here the product is over all strings $s=\{s_l\curvearrowright s_r\}$ of π. Our claim is that the following family of vectors, with $\pi\in NC_2(2k)$, spans the space of fixed vectors of $u^{\otimes 2k}$:

$$\xi_\pi=\sum_i\delta_\pi^F(i)e_{i_1}\otimes\ldots\otimes e_{i_{2k}}$$

Indeed, having $\xi_\cap$ fixed by $u^{\otimes 2}$ is equivalent to assuming that $u=F\bar{u}F$ is unitary.

By using now the above vectors, we obtain the following Weingarten formula:

$$\int_{O_F^+}u_{i_1j_1}\ldots u_{i_{2k}j_{2k}}=\sum_{\pi\sigma}\delta_\pi^F(i)\delta_\sigma^F(j)W_{kN}(\pi,\sigma)$$

With these preliminaries in hand, let us start the computation. Let $N\in\mathbb N$, and consider the number $q\in[-1,0)$ satisfying $q+q^{-1}=-N$. Our claim is that we have:

$$\int_{O_N^+}\varphi(\sqrt{N+2}\,u_{ij})=\int_{SU_2^q}\varphi(\alpha+\alpha^*+\gamma-q\gamma^*)$$

Indeed, the moments of the variable on the left are given by:

$$\int_{O_N^+}u_{ij}^{2k}=\sum_{\pi\sigma}W_{kN}(\pi,\sigma)$$

On the other hand, the moments of the variable on the right, which in terms of the fundamental corepresentation $v=(v_{ij})$ is given by $w=\sum_{ij}v_{ij}$, are given by:

$$\int_{SU_2^q}w^{2k}=\sum_{ij}\sum_{\pi\sigma}\delta_\pi^F(i)\delta_\sigma^F(j)W_{kN}(\pi,\sigma)$$

We deduce that $w/\sqrt{N+2}$ has the same moments as u_{ij}, which proves our claim.

In order to do now the computation over SU_2^q, we can use a matrix model due to Woronowicz [98], where the standard generators α,γ are mapped as follows:

$$\begin{aligned}\pi_u(\alpha)e_k&=\sqrt{1-q^{2k}}e_{k-1}\\ \pi_u(\gamma)e_k&=uq^ke_k\end{aligned}$$

Here $u\in\mathbb T$ is a parameter, and (e_k) is the standard basis of $l^2(\mathbb N)$. The point with this representation is that it allows the computation of the Haar functional. Indeed, if D is the diagonal operator given by $D(e_k)=q^{2k}e_k$, then the formula is as follows:

$$\int_{SU_2^q}x=(1-q^2)\int_{\mathbb T}tr(D\pi_u(x))\frac{du}{2\pi iu}$$

With the above model in hand, the law of the variable that we are interested in is as follows, where $M(e_k)=e_{k+1}+q^k(u-qu^{-1})e_k+(1-q^{2k})e_{k-1}$:

$$\int_{SU_2^q}\varphi(\alpha+\alpha^*+\gamma-q\gamma^*)=(1-q^2)\int_{\mathbb T}tr(D\varphi(M))\frac{du}{2\pi iu}$$

The point now is that the integral on the right can be computed, by using advanced calculus methods, and this gives the result. We refer here to [20]. $\square$

The computation of the joint free hyperspherical laws remains an open problem. Open as well is the question of finding a more conceptual proof for the above formula.

Finally, following [18], let us discuss an interesting relation all this with the quantum permutations, and with the free hypergeometric laws. The idea will be that of working out some abstract algebraic results, regarding twists of quantum automorphism groups, which will particularize into results relating quantum rotations and permutations

In order to explain this material, from [18], which is quite technical, requiring good algebraic knowledge, let us begin with some generalities. We first have:

Definition 9.11. *A finite noncommutative space X is the abstract dual of a finite dimensional C^*-algebra B, according to the following formula:*

$$C(X) = B$$

The number of elements of such a space is by definition $|X| = \dim B$. By decomposing the algebra B, we have a formula of the following type:

$$C(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

With $n_1 = \ldots = n_k = 1$ we obtain in this way the space $X = \{1, \ldots, k\}$. Also, when $k = 1$ the equation is $C(X) = M_n(\mathbb{C})$, and the solution will be denoted $X = M_n$.

We endow each finite noncommutative space X with its counting measure, corresponding as the algebraic level to the integration functional obtained by applying the regular representation, and then the normalized matrix trace:

$$tr : C(X) \to B(l^2(X)) \to \mathbb{C}$$

As basic examples, for both $X = \{1, \ldots, k\}$ and $X = M_n$ we obtain the usual trace. In general, with $C(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$, the weights of tr are $n_i^2 / \sum_i n_i^2$.

With these conventions, we have the following result, from [93]:

Theorem 9.12. *Given a finite noncommutative space X, there is a universal compact quantum group S_X^+ acting on X, leaving the counting measure invariant. We have*

$$C(S_X^+) = C(U_N^+) \Big/ \Big\langle \mu \in Hom(u^{\otimes 2}, u), \eta \in Fix(u) \Big\rangle$$

where $N = |X|$ and where μ, η are the multiplication and unit maps of $C(X)$. For $X = \{1, \ldots, N\}$ we have $S_X^+ = S_N^+$. Also, for $X = M_n$ we have $S_X^+ = PO_n^+ = PU_n^+$.

Proof. Consider a linear map $\Phi : C(X) \to C(X) \otimes C(G)$, written as follows, with $\{e_i\}$ being a linear space basis of $C(X)$, which is orthonormal with respect to tr:

$$\Phi(e_j) = \sum_i e_i \otimes u_{ij}$$

It is routine to check that Φ is a coaction precisely when u is a unitary corepresentation, satisfying $\mu \in Hom(u^{\otimes 2}, u)$ and $\eta \in Fix(u)$, and this gives the first assertion.

The statement regarding $X = \{1, \ldots, N\}$ is clear. Finally, regarding $X = M_2$, here we have embeddings $PO_n^+ \subset PU_n^+ \subset S_X^+$, and since the fusion rules of these quantum groups can be shown to be the same as for SO_3, these inclusions are isomorphisms. See [93]. $\square$

Following now [18], we have the following result:

Proposition 9.13. *Given a finite group F, the algebra $C(S_{\widehat{F}}^+)$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g,h\in F$, with the following relations:*

$$x_{1g}=x_{g1}=\delta_{1g}\quad,\quad x_{s,gh}=\sum_{t\in F}x_{st^{-1},g}x_{th}\quad,\quad x_{gh,s}=\sum_{t\in F}x_{gt^{-1}}x_{h,ts}$$

The comultiplication, counit and antipode are given by the formulae

$$\Delta(x_{gh})=\sum_{s\in F}x_{gs}\otimes x_{sh}\quad,\quad\varepsilon(x_{gh})=\delta_{gh}\quad,\quad S(x_{gh})=x_{h^{-1}g^{-1}}$$

on the standard generators x_{gh}.

Proof. This follows indeed from a direct verification, based either on Theorem 9.12 above, or on its equivalent formulation from Wang's paper [92]. □

Let us discuss now the twisted version of the above result. Consider a 2-cocycle on F, which is by definition a map $\sigma:F\times F\to\mathbb{C}^*$ satisfying:

$$\sigma_{gh,s}\sigma_{gh}=\sigma_{g,hs}\sigma_{hs}\quad,\quad\sigma_{g1}=\sigma_{1g}=1$$

Given such a cocycle, we can construct the associated twisted group algebra $C(\widehat{F}_\sigma)$, as being the vector space $C(\widehat{F})=C^*(F)$, with product as follows:

$$e_ge_h=\sigma_{gh}e_{gh}$$

We have then the following generalization of Proposition 9.13:

Proposition 9.14. *The algebra $C(S_{\widehat{F}_\sigma}^+)$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g,h\in G$, with the relations $x_{1g}=x_{g1}=\delta_{1g}$ and:*

$$\sigma_{gh}x_{s,gh}=\sum_{t\in F}\sigma_{st^{-1},t}x_{st^{-1},g}x_{th}\quad,\quad\sigma_{gh}^{-1}x_{gh,s}=\sum_{t\in F}\sigma_{t^{-1},ts}^{-1}x_{gt^{-1}}x_{h,ts}$$

The comultiplication, counit and antipode are given by the formulae

$$\Delta(x_{gh})=\sum_{s\in F}x_{gs}\otimes x_{sh}\quad,\quad\varepsilon(x_{gh})=\delta_{gh}\quad,\quad S(x_{gh})=\sigma_{h^{-1}h}\sigma_{g^{-1}g}^{-1}x_{h^{-1}g^{-1}}$$

on the standard generators x_{gh}.

Proof. Once again, this follows from a direct verification. Note that by using the cocycle identities we obtain $\sigma_{gg^{-1}}=\sigma_{g^{-1}g}$, needed in the proof. □

In what follows, we will prove that the quantum groups $S_{\widehat{F}}^+$ and $S_{\widehat{F}_\sigma}^+$ are related by a cocycle twisting operation. Let us begin with some preliminaries.

Let H be a Hopf algebra. We use the Sweedler notation $\Delta(x)=\sum x_1\otimes x_2$. Recall that a left 2-cocycle is a convolution invertible linear map $\sigma:H\otimes H\to\mathbb{C}$ satisfying:

$$\sigma_{x_1y_1}\sigma_{x_2y_2,z}=\sigma_{y_1z_1}\sigma_{x,y_2z_2}\quad,\quad\sigma_{x1}=\sigma_{1x}=\varepsilon(x)$$

Note that σ is a left 2-cocycle if and only if σ^{-1}, the convolution inverse of σ, is a right 2-cocycle, in the sense that we have:

$$\sigma^{-1}_{x_1y_1,z}\sigma^{-1}_{x_1y_2}=\sigma^{-1}_{x,y_1z_1}\sigma^{-1}_{y_2z_2}\quad,\quad\sigma^{-1}_{x1}=\sigma^{-1}_{1x}=\varepsilon(x)$$

Given a left 2-cocycle σ on H, one can form the 2-cocycle twist H^σ as follows. As a coalgebra, $H^\sigma=H$, and an element $x\in H$, when considered in H^σ, is denoted $[x]$. The product in H^σ is defined, in Sweedler notation, by:

$$[x][y]=\sum\sigma_{x_1y_1}\sigma^{-1}_{x_3y_3}[x_2y_2]$$

Note that the cocycle condition ensures the fact that we have indeed a Hopf algebra. Note also that the coalgebra isomorphism $H\to H^\sigma$ given by $x\to[x]$ commutes with the respective Haar integrals, as soon as H has a Haar integral.

We are now in position to state and prove our main theorem:

Theorem 9.15. *If F is a finite group and σ is a 2-cocycle on F, the Hopf algebras*

$$C(S^+_{\widehat{F}})\quad,\quad C(S^+_{\widehat{F}_\sigma})$$

are 2-cocycle twists of each other, in the above sense.

Proof. In order to prove this result, we use the following Hopf algebra map:

$$\pi:C(S^+_{\widehat{F}})\to C(\widehat{F})\quad,\quad x_{gh}\to\delta_{gh}e_g$$

Our 2-cocycle $\sigma:F\times F\to\mathbb{C}^*$ can be extended by linearity into a linear map as follows, which is a left and right 2-cocycle in the above sense:

$$\sigma:C(\widehat{F})\otimes C(\widehat{F})\to\mathbb{C}$$

Consider now the following composition:

$$\alpha=\sigma(\pi\otimes\pi):C(S^+_{\widehat{F}})\otimes C(S^+_{\widehat{F}})\to C(\widehat{F})\otimes C(\widehat{F})\to\mathbb{C}$$

Then α is a left and right 2-cocycle, because it is induced by a cocycle on a group algebra, and so is its convolution inverse α^{-1}. Thus we can construct the twisted algebra $C(S^+_{\widehat{F}})^{\alpha^{-1}}$, and inside this algebra we have the following computation:

$$\begin{aligned}[x_{gh}][x_{rs}]&=\alpha^{-1}(x_g,x_r)\alpha(x_h,x_s)[x_{gh}x_{rs}]\\&=\sigma^{-1}_{gr}\sigma_{hs}[x_{gh}x_{rs}]\end{aligned}$$

By using this, we obtain the following formula:

$$\begin{aligned}\sum_{t\in F}\sigma_{st^{-1},t}[x_{st^{-1},g}][x_{th}]&=\sum_{t\in F}\sigma_{st^{-1},t}\sigma^{-1}_{st^{-1},t}\sigma_{gh}[x_{st^{-1},g}x_{th}]\\&=\sigma_{gh}[x_{s,gh}]\end{aligned}$$

Similarly, we have the following formula:

$$\sum_{t\in F}\sigma^{-1}_{t^{-1},ts}[x_{g,t^{-1}}][x_{h,ts}]=\sigma^{-1}_{gh}[x_{gh,s}]$$

We deduce from this that there exists a Hopf algebra map, as follows:

$$\Phi:C(S^+_{\widehat{F}_\sigma})\to C(S^+_{\widehat{F}})^{\alpha^{-1}}\quad,\quad x_{gh}\to[x_{g,h}]$$

This map is clearly surjective, and is injective as well, by a standard fusion semiring argument, because both Hopf algebras have the same fusion semiring. □

Summarizing, we have proved our main twisting result. Our purpose in what follows will be that of working out versions and particular cases of it. We first have:

Proposition 9.16. *If F is a finite group and σ is a 2-cocycle on F, then*

$$\Phi(x_{g_1h_1}\ldots x_{g_mh_m})=\Omega(g_1,\ldots,g_m)^{-1}\Omega(h_1,\ldots,h_m)x_{g_1h_1}\ldots x_{g_mh_m}$$

with the coefficients on the right being given by the formula

$$\Omega(g_1,\ldots,g_m)=\prod_{k=1}^{m-1}\sigma_{g_1\ldots g_k,g_{k+1}}$$

is a coalgebra isomorphism $C(S^+_{\widehat{F}_\sigma})\to C(S^+_{\widehat{F}})$, commuting with the Haar integrals.

Proof. This is indeed just a technical reformulation of Theorem 9.15. □

Here is another useful result, that we will need in what follows:

Theorem 9.17. *Let $X\subset F$ be such that $\sigma_{gh}=1$ for any $g,h\in X$, and consider the subalgebra $B_X\subset C(S^+_{\widehat{F}_\sigma})$ generated by the elements x_{gh}, with $g,h\in X$. Then we have an injective algebra map $\Phi_0:B_X\to C(S^+_{\widehat{F}})$, given by $x_{g,h}\to x_{g,h}$.*

Proof. With the notations in the proof of Theorem 9.15, we have the following equality in $C(S^+_{\widehat{F}})^{\alpha^{-1}}$, for any $g_i,h_i,r_i,s_i\in X$:

$$[x_{g_1h_1}\ldots x_{g_ph_p}]\cdot[x_{r_1s_1}\ldots x_{r_qs_q}]=[x_{g_1h_1}\ldots x_{g_ph_p}x_{r_1s_1}\ldots x_{r_qs_q}]$$

Now Φ_0 can be defined to be the composition of $\Phi_{|B_X}$ with the linear isomorphism $C(S^+_{\widehat{F}})^{\alpha^{-1}}\to C(S^+_{\widehat{F}})$ given by $[x]\to x$, and is clearly an injective algebra map. □

Let us discuss now some concrete applications of the general results established above. Consider the group $F=\mathbb{Z}_n^2$, let $w=e^{2\pi i/n}$, and consider the following map:

$$\sigma:F\times F\to\mathbb{C}^*\quad,\quad\sigma_{(ij)(kl)}=w^{jk}$$

It is easy to see that σ is a bicharacter, and hence a 2-cocycle on F. Thus, we can apply our general twisting result, to this situation.

In order to understand what is the formula that we obtain, we must do some computations. Let E_{ij} with $i,j\in\mathbb{Z}_n$ be the standard basis of $M_n(\mathbb{C})$. We have:

Proposition 9.18. *The linear map given by*

$$\psi(e_{(i,j)})=\sum_{k=0}^{n-1}w^{ki}E_{k,k+j}$$

defines an isomorphism of algebras $\psi:C(\widehat{F_\sigma})\simeq M_n(\mathbb{C})$.

Proof. Consider indeed the following linear map:

$$\psi'(E_{ij})=\frac{1}{n}\sum_{k=0}^{n-1}w^{-ik}e_{(k,j-i)}$$

It is routine then to check that ψ,ψ' are inverse morphisms of algebras. □

As a consequence, we have the following result:

Proposition 9.19. *The algebra map given by*

$$\varphi(u_{ij}u_{kl})=\frac{1}{n}\sum_{a,b=0}^{n-1}w^{ai-bj}x_{(a,k-i),(b,l-j)}$$

defines a Hopf algebra isomorphism $\varphi:C(S_{M_n}^+)\simeq C(S_{\widehat{F}_\sigma}^+)$.

Proof. We use the identification $C(\widehat{F_\sigma})\simeq M_n(\mathbb{C})$ from Proposition 9.18. This identification produces a coaction map, as follows:

$$\gamma:M_n(\mathbb{C})\to M_n(\mathbb{C})\otimes C(S_{\widehat{F}_\sigma}^+)$$

Now observe that this map is given by the following formula:

$$\gamma(E_{ij})=\frac{1}{n}\sum_{ab}E_{ab}\otimes\sum_{kr}w^{ar-ik}x_{(r,b-a),(k,j-i)}$$

Thus, we obtain the isomorphism in the statement. □

We will need one more result of this type, as follows:

Proposition 9.20. *The algebra map given by*

$$\rho(x_{(a,b),(i,j)})=\frac{1}{n^2}\sum_{klrs}w^{ki+lj-ra-sb}p_{(r,s),(k,l)}$$

defines a Hopf algebra isomorphism $\rho:C(S_{\widehat{F}}^+)\simeq C(S_F^+)$.

Proof. This follows by using the Fourier transform isomorphism $C(\widehat{F})\simeq C(F)$. □

We can now formulate our twisting result, as follows:

Theorem 9.21. *Let $n \geq 2$ and $w = e^{2\pi i/n}$. Then*

$$\Theta(u_{ij}u_{kl}) = \frac{1}{n}\sum_{ab=0}^{n-1} w^{-a(k-i)+b(l-j)} p_{ia,jb}$$

defines a coalgebra isomorphism

$$C(PO_n^+) \to C(S_{n^2}^+)$$

commuting with the Haar integrals.

Proof. The result follows from Theorem 9.15 and Proposition 9.16, by combining them with the various isomorphisms established above. $\square$

Here is a useful version of the above result:

Theorem 9.22. *The following two algebras are isomorphic, via $u_{ij}^2 \to X_{ij}$:*

1. *The algebra generated by the variables $u_{ij}^2 \in C(O_n^+)$.*
2. *The algebra generated by $X_{ij} = \frac{1}{n}\sum_{a,b=1}^n p_{ia,jb} \in C(S_{n^2}^+)$*

Proof. This follows by using Theorem 9.17, via the above identifications. $\square$

As a probabilistic consequence now, we have:

Theorem 9.23. *The following families of variables have the same joint law,*

1. *$\{u_{ij}^2\} \in C(O_n^+)$,*
2. *$\{X_{ij} = \frac{1}{n}\sum_{ab} p_{ia,jb}\} \in C(S_{n^2}^+)$,*

where $u = (u_{ij})$ and $p = (p_{ia,jb})$ are the corresponding fundamental corepresentations.

Proof. This follows indeed from Theorem 9.22 above. $\square$

In particular, we have the following result:

Theorem 9.24. *The free hypergeometric variable*

$$X_{ij} = \frac{1}{n}\sum_{a,b=1}^n u_{ia,jb} \in C(S_{n^2}^+)$$

has the same law as the squared free hyperspherical variable $x_i^2 \in C(S^{N-1}_{\mathbb{R},+})$.

Proof. This follows from Theorem 9.23. See [18]. $\square$

We refer to [18] and various related papers, for more on these topics.

10. Partial isometries

We discuss here the liberation operation, $X \to X^+$. We have seen so far that this operation can be performed for the 4 basic examples of manifolds that we have, namely $X = S, T, U, K$, with the remark however that the case $X = T$ is quite special.

So, our purpose here will be that of unifying and extending the constructions of type $X \to X^+$, in the cases $X = S, U, K$. For this purpose, we will use a suitable class of homogeneous spaces, generalizing at the same time the groups, and the spheres.

Besides being of theoretical interest, in connection with the liberation operation, this will bring as well some advances in relation with our general program of "developing" the geometries that we found, in the free real and complex cases.

In order to unify the unitary and reflection groups U, K with the spheres S, the idea will be of course that of looking at certain special classes of homogeneous spaces.

This can be done at several levels of generality, and there has been quite some work here, starting with [28], and then going further with [6], and even further with [7].

In what follows we discuss the formalism in [6], which is quite broad, while remaining not very abstract. We will study the spaces of the following type:

$$X = (G_M \times G_N)\big/(G_L \times G_{M-L} \times G_{N-L})$$

These spaces cover indeed the quantum groups and the spheres. And also, they are quite concrete and useful objects, consisting of certain classes of "partial isometries".

Our main result will be a verification of the Bercovici-Pata liberation criterion, for certain variables associated $\chi \in C(X)$, in a suitable $L, M, N \to \infty$ limit.

We begin with some study in the classical case. Our starting point will be:

Definition 10.1. *Associated to any integers $L \leq M, N$ are the spaces*

$$O_{MN}^L = \left\{T : E \to F \text{ isometry}\Big| E \subset \mathbb{R}^N, F \subset \mathbb{R}^M, \dim_{\mathbb{R}} E = L\right\}$$

$$U_{MN}^L = \left\{T : E \to F \text{ isometry}\Big| E \subset \mathbb{C}^N, F \subset \mathbb{C}^M, \dim_{\mathbb{C}} E = L\right\}$$

where the notion of isometry is with respect to the usual real/complex scalar products.

As a first observation, at $L = M = N$ we obtain the groups O_N, U_N:

$$O_{NN}^N = O_N \quad , \quad U_{NN}^N = U_N$$

Another interesting specialization is $L = M = 1$. Here the elements of O_{1N}^1 are the isometries $T : E \to \mathbb{R}$, with $E \subset \mathbb{R}^N$ one-dimensional, and such an isometry is uniquely determined by the element $T^{-1}(1) \in \mathbb{R}^N$, which must belong to the sphere $S_{\mathbb{R}}^{N-1}$. Thus, we have $O_{1N}^1 = S_{\mathbb{R}}^{N-1}$. Similarly, in the complex case we have $U_{1N}^1 = S_{\mathbb{C}}^{N-1}$:

$$O_{1N}^1 = S_{\mathbb{R}}^{N-1} \quad , \quad U_{1N}^1 = S_{\mathbb{C}}^{N-1}$$

Yet another interesting specialization is $L = N = 1$. Here the elements of O_{1N}^1 are the isometries $T : \mathbb{R} \to F$, with $F \subset \mathbb{R}^M$ one-dimensional, and such an isometry is uniquely determined by the element $T(1) \in \mathbb{R}^M$, which must belong to the sphere $S_{\mathbb{R}}^{M-1}$. Thus, we have $O_{M1}^1 = S_{\mathbb{R}}^{M-1}$. Similarly, in the complex case we have $U_{M1}^1 = S_{\mathbb{C}}^{M-1}$:

$$O_{M1}^1 = S_{\mathbb{R}}^{M-1} \quad , \quad U_{M1}^1 = S_{\mathbb{C}}^{M-1}$$

Summarizing, our formalism so far covers well the unitary groups, and the spheres.

In general, the most convenient is to view the elements of O_{MN}^L, U_{MN}^L as rectangular matrices, and to use matrix calculus for their study. We have indeed:

Proposition 10.2. *We have identifications of compact spaces*

$$O_{MN}^L \simeq \left\{ U \in M_{M\times N}(\mathbb{R}) \Big| UU^t = \text{projection of trace } L \right\}$$

$$U_{MN}^L \simeq \left\{ U \in M_{M\times N}(\mathbb{C}) \Big| UU^* = \text{projection of trace } L \right\}$$

with each partial isometry being identified with the corresponding rectangular matrix.

Proof. We can indeed identify the partial isometries $T : E \to F$ with their corresponding extensions $U : \mathbb{R}^N \to \mathbb{R}^M$, $U : \mathbb{C}^N \to \mathbb{C}^M$, obtained by setting $U_{E^\perp} = 0$, and then identify these latter linear maps U with the corresponding rectangular matrices. $\square$

As an illustration, at $L = M = N$ we recover in this way the usual matrix description of O_N, U_N. Also, at $L = M = 1$ we obtain the usual description of $S_{\mathbb{R}}^{N-1}, S_{\mathbb{C}}^{N-1}$, as row spaces over the corresponding groups O_N, U_N. Finally, at $L = N = 1$ we obtain the usual description of $S_{\mathbb{R}}^{N-1}, S_{\mathbb{C}}^{N-1}$, as column spaces over the corresponding groups O_N, U_N.

Now back to the general case, observe that the isometries $T : E \to F$, or rather their extensions $U : \mathbb{K}^N \to \mathbb{K}^M$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, obtained by setting $U_{E^\perp} = 0$, can be composed with the isometries of $\mathbb{K}^M, \mathbb{K}^N$, according to the following scheme:

$$\begin{tikzcd}
\mathbb{K}^N \arrow[r, "B^*"] & \mathbb{K}^N \arrow[r, dotted, "U"] & \mathbb{K}^M \arrow[r, "A"] & \mathbb{K}^M \\
B(E) \arrow[u] \arrow[r, dotted] & E \arrow[u] \arrow[r, "T"] & F \arrow[u] \arrow[r, dotted] & A(F) \arrow[u]
\end{tikzcd}$$

In other words, the groups $O_M \times O_N, U_M \times U_N$ act respectively on O_{MN}^L, U_{MN}^L.

With the identifications in Proposition 10.2 made, the statement here is:

Proposition 10.3. *We have action maps as follows, which are transitive,*

$$O_M \times O_N \curvearrowright O_{MN}^L \quad : \quad (A,B)U = AUB^t$$
$$U_M \times U_N \curvearrowright U_{MN}^L \quad : \quad (A,B)U = AUB^*$$

whose stabilizers are respectively $O_L \times O_{M-L} \times O_{N-L}$ and $U_L \times U_{M-L} \times U_{N-L}$.

Proof. We have indeed action maps as in the statement, which are transitive. Let us compute now the stabilizer G of the point $U = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. Since the elements $(A,B) \in G$ satisfy $AU = UB$, their components must be of the following form:

$$A = \begin{pmatrix}x&*\\0&a\end{pmatrix} \quad , \quad B = \begin{pmatrix}x&0\\ *&b\end{pmatrix}$$

Now since A, B are both unitaries, these matrices follow to be block-diagonal, and so:

$$G = \left\{(A,B)\Big|A = \begin{pmatrix}x&0\\0&a\end{pmatrix}, B = \begin{pmatrix}x&0\\0&b\end{pmatrix}\right\}$$

We conclude that the stabilizer of $U = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ is parametrized by triples (x,a,b) belonging respectively to $O_L \times O_{M-L} \times O_{N-L}$ and $U_L \times U_{M-L} \times U_{N-L}$, as claimed. □

Finally, let us work out the quotient space description of O_{MN}^L, U_{MN}^L:

Theorem 10.4. *We have isomorphisms of homogeneous spaces as follows,*

$$O_{MN}^L = (O_M \times O_N)/(O_L \times O_{M-L} \times O_{N-L})$$
$$U_{MN}^L = (U_M \times U_N)/(U_L \times U_{M-L} \times U_{N-L})$$

with the quotient maps being given by $(A,B) \to AUB^*$, where $U = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$.

Proof. This is just a reformulation of Proposition 10.3 above, by taking into account the fact that the fixed point used in the proof there was $U = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. □

Once again, the basic examples here come from the cases $L = M = N$ and $L = M = 1$, where the quotient spaces at right are respectively O_N, U_N and $O_N/O_{N-1}, U_N/U_{N-1}$. In fact, in the general $L = M$ case we obtain the following spaces, considered in [28]:

$$O_{MN}^M = (O_M \times O_N)/(O_M \times O_{N-M}) = O_N/O_{N-M}$$
$$U_{MN}^M = (U_M \times U_N)/(U_M \times U_{N-M}) = U_N/U_{N-M}$$

Similarly, the examples coming from the cases $L = M = N$ and $L = N = 1$ are particular cases of the general $L = N$ case, where we obtain the following spaces:

$$O_{MN}^N = (O_M \times O_N)/(O_M \times O_{M-N}) = O_N/O_{M-N}$$
$$U_{MN}^N = (U_M \times U_N)/(U_M \times U_{M-N}) = U_N/U_{M-N}$$

For some further information on these spaces, we refer to [6], [28].

We can liberate the spaces O_{MN}^L, U_{MN}^L, as follows:

Definition 10.5. *Associated to any integers $L\leq M,N$ are the algebras*

$$\begin{aligned}C(O_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|u=\bar{u},uu^t=\text{projection of trace } L\right)\\ C(U_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|uu^*,\bar{u}u^t=\text{projections of trace } L\right)\end{aligned}$$

with the trace being by definition the sum of the diagonal entries.

Observe that the above universal algebras are indeed well-defined, as it was previously the case for the free spheres, and this due to the trace conditions, which read:

$$\sum_{ij}u_{ij}u_{ij}^*=\sum_{ij}u_{ij}^*u_{ij}=L$$

Indeed, these conditions show that we have $||u_{ij}||\leq\sqrt{L}$, for any i,j.

We have inclusions between the various spaces constructed so far, as follows:

$$\begin{tikzcd}O_{MN}^{L+} \arrow[r] & U_{MN}^{L+}\\ O_{MN}^{L} \arrow[r]\arrow[u] & U_{MN}^{L}\arrow[u]\end{tikzcd}$$

Indeed, these inclusions come from Proposition 10.2, from Definition 10.5, and from the fact that the spaces O_{MN}^L,U_{MN}^L are stable under conjugation.

At the level of basic examples now, we first have the following result:

Proposition 10.6. *At $L=M=1$ and $L=N=1$ we obtain the diagrams*

$$\begin{tikzcd}S^{N-1}_{\mathbb{R},+} \arrow[r] & S^{N-1}_{\mathbb{C},+}\\ S^{N-1}_{\mathbb{R}} \arrow[r]\arrow[u] & S^{N-1}_{\mathbb{C}}\arrow[u]\end{tikzcd}\qquad\begin{tikzcd}S^{M-1}_{\mathbb{R},+} \arrow[r] & S^{M-1}_{\mathbb{C},+}\\ S^{M-1}_{\mathbb{R}} \arrow[r]\arrow[u] & S^{M-1}_{\mathbb{C}}\arrow[u]\end{tikzcd}$$

via some standard identifications.

Proof. We recall that the various spheres involved are constructed as follows, with the symbol $\times$ standing for "commutative" and "free", respectively:

$$\begin{aligned}C(S^{N-1}_{\mathbb{R},\times}) &= C^*_\times\left(z_1,\ldots,z_N\Big|z_i=z_i^*,\sum_i z_i^2=1\right)\\ C(S^{N-1}_{\mathbb{C},\times}) &= C^*_\times\left(z_1,\ldots,z_N\Big|\sum_i z_iz_i^*=\sum_i z_i^*z_i=1\right)\end{aligned}$$

Now by comparing with the definition of $O_{1N}^{1\times}, U_{1N}^{1\times}$, this proves our first claim.

As for the proof of the second claim, this is similar, via standard identifications. $\square$

We have as well the following result:

Proposition 10.7. *At $L = M = N$ we obtain the diagram*

$$\begin{tikzcd}
O_N^+ \arrow[r] & U_N^+ \\
O_N \arrow[r] \arrow[u] & U_N \arrow[u]
\end{tikzcd}$$

consisting of the groups O_N, U_N, and their liberations.

Proof. We recall that the various quantum groups are constructed as follows, with the symbol $\times$ standing once again for "commutative" and "free":

$$\begin{aligned}
C(O_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big| u = \bar{u}, uu^t = u^t u = 1\right)\\
C(U_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big| uu^* = u^* u = 1, \bar{u}u^t = u^t\bar{u} = 1\right)
\end{aligned}$$

On the other hand, according to Proposition 10.2 and to Definition 10.5 above, we have the following presentation results:

$$\begin{aligned}
C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big| u = \bar{u}, uu^t = \text{projection of trace } N\right)\\
C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big| uu^*, \bar{u}u^t = \text{projections of trace } N\right)
\end{aligned}$$

We use now the standard fact that if $p = aa^*$ is a projection then $q = a^*a$ is a projection too. Together with $Tr(uu^*) = Tr(u^t\bar{u})$ and $Tr(\bar{u}u^t) = Tr(u^*u)$, this gives:

$$\begin{aligned}
C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big| u = \bar{u},\ uu^t, u^t u = \text{projections of trace } N\right)\\
C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big| uu^*, u^*u, \bar{u}u^t, u^t\bar{u} = \text{projections of trace } N\right)
\end{aligned}$$

Now observe that, in tensor product notation, and by using the normalized trace, the conditions at right are all of the form $(tr \otimes id)p = 1$, with $p = uu^*, u^*u, \bar{u}u^t, u^t\bar{u}$. We therefore obtain, for any faithful state φ:

$$(tr \otimes \varphi)(1 - p) = 0$$

It follows from this that the projections $p = uu^*, u^*u, \bar{u}u^t, u^t\bar{u}$ must be all equal to the identity, as desired, and this finishes the proof. $\square$

Regarding now the homogeneous space structure of $O_{MN}^{L\times}, U_{MN}^{L\times}$, the situation here is more complicated in the free case than in the classical case. We have:

Proposition 10.8. *The spaces $U_{MN}^{L\times}$ have the following properties:*

(1) *We have an action $U_M^\times\times U_N^\times\curvearrowright U_{MN}^{L\times}$, given by $u_{ij}\to\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$.*
(2) *We have a map $U_M^\times\times U_N^\times\to U_{MN}^{L\times}$, given by $u_{ij}\to\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$.*

Similar results hold for the spaces $O_{MN}^{L\times}$, with all the $*$ exponents removed.

Proof. In the classical case, the transpose of the action map $U_M\times U_N\curvearrowright U_{MN}^L$ and of the quotient map $U_M\times U_N\to U_{MN}^L$ are as follows, where $J=\begin{pmatrix}1&0\\0&0\end{pmatrix}$:

$$\begin{aligned}\varphi&\to((U,A,B)\to\varphi(AUB^*))\\ \varphi&\to((A,B)\to\varphi(AJB^*))\end{aligned}$$

But with $\varphi=u_{ij}$ we obtain precisely the formulae in the statement. The proof in the orthogonal case is similar. Regarding now the free case, the proof goes as follows:

(1) Assuming $uu^*u=u$, with $U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$ we have:

$$\begin{aligned}(UU^*U)_{ij}&=\sum_{pq}\sum_{klmnst}u_{kl}u_{mn}^*u_{st}\otimes a_{ki}a_{mq}^*a_{sq}\otimes b_{lp}^*b_{np}b_{tj}^*\\ &=\sum_{klmt}u_{kl}u_{ml}^*u_{mt}\otimes a_{ki}\otimes b_{tj}^*\\ &=\sum_{kt}u_{kt}\otimes a_{ki}\otimes b_{tj}^*\\ &=U_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij}u_{ij}u_{ij}^*=L$, we obtain:

$$\begin{aligned}\sum_{ij}U_{ij}U_{ij}^*&=\sum_{ij}\sum_{klst}u_{kl}u_{st}^*\otimes a_{ki}a_{si}^*\otimes b_{lj}^*b_{tj}\\ &=\sum_{kl}u_{kl}u_{kl}^*\otimes 1\otimes 1\\ &=L\end{aligned}$$

(2) Assuming $uu^*u=u$, with $V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$ we have:

$$\begin{aligned}(VV^*V)_{ij}&=\sum_{pq}\sum_{x,y,z\leq L}a_{xi}a_{yq}^*a_{zq}\otimes b_{xp}^*b_{yp}b_{zj}^*\\ &=\sum_{x\leq L}a_{xi}\otimes b_{xj}^*\\ &=V_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij} u_{ij}u_{ij}^* = L$, we obtain:

$$\begin{aligned}\sum_{ij} V_{ij}V_{ij}^* &= \sum_{ij}\sum_{r,s\leq L} a_{ri}a_{si}^*\otimes b_{rj}^*b_{sj}\\ &= \sum_{l\leq L} 1\\ &= L\end{aligned}$$

By removing all the $*$ exponents, we obtain as well the orthogonal results. □

Let us examine now the relation between the above maps. In the classical case, given a quotient space $X = G/H$, the associated action and quotient maps are given by:

$$\begin{cases} a: X\times G\to X & : \quad (Hg,h)\to Hgh\\ p: G\to X & : \quad g\to Hg\end{cases}$$

Thus we have $a(p(g),h) = p(gh)$. In our context, a similar result holds:

Theorem 10.9. *With $G = G_M\times G_N$ and $X = G_{MN}^L$, where $G_N = O_N^\times, U_N^\times$, we have*

$$\begin{tikzcd} G\times G \arrow[r, "m"] \arrow[d, "p\times id"'] & G \arrow[d, "p"] \\ X\times G \arrow[r, "a"] & X \end{tikzcd}$$

where a, p are the action map and the map constructed in Proposition 10.8.

Proof. At the level of the associated algebras of functions, we must prove that the following diagram commutes, where Φ,α are morphisms of algebras induced by a, p:

$$\begin{tikzcd} C(X) \arrow[r, "\Phi"] \arrow[d, "\alpha"'] & C(X\times G) \arrow[d, "\alpha\otimes id"] \\ C(G) \arrow[r, "\Delta"] & C(G\times G) \end{tikzcd}$$

When going right, and then down, the composition is as follows:

$$\begin{aligned}(\alpha\otimes id)\Phi(u_{ij}) &= (\alpha\otimes id)\sum_{kl} u_{kl}\otimes a_{ki}\otimes b_{lj}^*\\ &= \sum_{kl}\sum_{r\leq L} a_{rk}\otimes b_{rl}^*\otimes a_{ki}\otimes b_{lj}^*\end{aligned}$$

On the other hand, when going down, and then right, the composition is as follows, where F_{23} is the flip between the second and the third components:

$$\begin{aligned}\Delta\pi(u_{ij}) &= F_{23}(\Delta\otimes\Delta)\sum_{r\leq L}a_{ri}\otimes b_{rj}^*\\ &= F_{23}\left(\sum_{r\leq L}\sum_{kl}a_{rk}\otimes a_{ki}\otimes b_{rl}^*\otimes b_{lj}^*\right)\end{aligned}$$

Thus the above diagram commutes indeed, and this gives the result. □

In general, going beyond Theorem 10.9 leads to some non-trivial questions. A first issue comes from the fact that the inclusions $G_L\times G_{M-L}\times G_{N-L}\subset G_M\times G_N$ are not well-defined, in the free case. There are as well some analytic issues, coming from the fact that the maps in Proposition 10.8 (2) are in general not surjective. See [28].

Let us discuss now some extensions of the above constructions, by using other classes of quantum groups. We will be mostly interested in the quantum reflection groups, so let us first discuss, with full details, the case of the quantum groups H_N^s,H_N^{s+}.

We use the following notion:

Definition 10.10. *Associated to any partial permutation, $\sigma:I\simeq J$ with $I\subset\{1,\ldots,N\}$ and $J\subset\{1,\ldots,M\}$, is the real/complex partial isometry*

$$T_\sigma:span\left(e_i\Big|i\in I\right)\to span\left(e_j\Big|j\in J\right)$$

given on the standard basis elements by $T_\sigma(e_i)=e_{\sigma(i)}$.

We denote by S_{MN}^L the set of partial permutations $\sigma:I\simeq J$ as above, with range $I\subset\{1,\ldots,N\}$ and target $J\subset\{1,\ldots,M\}$, and with $L=|I|=|J|$.

In analogy with the decomposition result $H_N^s=\mathbb Z_s\wr S_N$, we have:

Proposition 10.11. *The space of partial permutations signed by elements of $\mathbb Z_s$,*

$$H_{MN}^{sL}=\left\{T(e_i)=w_ie_{\sigma(i)}\Big|\sigma\in S_{MN}^L,w_i\in\mathbb Z_s\right\}$$

is isomorphic to the quotient space

$$(H_M^s\times H_N^s)/(H_L^s\times H_{M-L}^s\times H_{N-L}^s)$$

via a standard isomorphism.

Proof. This follows by adapting the computations in the proof of Proposition 10.3 above. Indeed, we have an action map as follows, which is transitive:

$$H_M^s\times H_N^s\curvearrowright H_{MN}^{sL}\quad:\quad(A,B)U=AUB^*$$

The stabilizer of the point $U=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ follows to be the group $H_L^s\times H_{M-L}^s\times H_{N-L}^s$, embedded via $(x,a,b)\to[\begin{pmatrix}x&0\\0&a\end{pmatrix},\begin{pmatrix}x&0\\0&b\end{pmatrix}]$, and this gives the result. □

In the free case now, the idea is similar, by using inspiration from the construction of the quantum group $H_N^{s+}=\mathbb{Z}_s\wr_* S_N^+$ in [11]. The result here is as follows:

Proposition 10.12. *The noncommutative space H_{MN}^{sL+} associated to the algebra*

$$C(H_{MN}^{sL+})=C(U_{MN}^{L+})\Big/\left\langle u_{ij}u_{ij}^*=u_{ij}^*u_{ij}=p_{ij}=\text{projections},u_{ij}^s=p_{ij}\right\rangle$$

has an action map, and is the target of a quotient map, as in Theorem 10.9 above.

Proof. We must show that if the variables u_{ij} satisfy the relations in the statement, then these relations are satisfied as well for the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

$$V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

Since the standard coordinates a_{ij},b_{ij} on the quantum groups H_M^{s+},H_N^{s+} satisfy the relations $xy=xy^*=0$, for any $x\neq y$ on the same row or column of a,b, we obtain:

$$\begin{aligned}U_{ij}U_{ij}^* &= \sum_{klmn}u_{kl}u_{mn}^*\otimes a_{ki}a_{mi}^*\otimes b_{lj}^*b_{mj}\\ &= \sum_{kl}u_{kl}u_{kl}^*\otimes a_{ki}a_{ki}^*\otimes b_{lj}^*b_{lj}\end{aligned}$$

We have as well the following formula:

$$\begin{aligned}V_{ij}V_{ij}^* &= \sum_{r,t\leq L}a_{ri}a_{ti}^*\otimes b_{rj}^*b_{tj}\\ &= \sum_{r\leq L}a_{ri}a_{ri}^*\otimes b_{rj}^*b_{rj}\end{aligned}$$

Thus, in terms of the projections $x_{ij}=a_{ij}a_{ij}^*,y_{ij}=b_{ij}b_{ij}^*,p_{ij}=u_{ij}u_{ij}^*$, we have:

$$U_{ij}U_{ij}^*=\sum_{kl}p_{kl}\otimes x_{ki}\otimes y_{lj}$$

$$V_{ij}V_{ij}^*=\sum_{r\leq L}x_{ri}\otimes y_{rj}$$

By repeating the computation, we conclude that these elements are projections. Also, a similar computation shows that $U_{ij}^*U_{ij},V_{ij}^*V_{ij}$ are given by the same formulae.

Finally, once again by using the relations of type $xy=xy^*=0$, we have:

$$\begin{aligned}U_{ij}^s &= \sum_{k_rl_r}u_{k_1l_1}\ldots u_{k_sl_s}\otimes a_{k_1i}\ldots a_{k_si}\otimes b_{l_1j}^*\ldots b_{l_sj}^*\\ &= \sum_{kl}u_{kl}^s\otimes a_{ki}^s\otimes(b_{lj}^*)^s\end{aligned}$$

We have as well the following formula:

$$\begin{aligned} V_{ij}^s &= \sum_{r_l\leq L} a_{r_1i}\ldots a_{r_si}\otimes b_{r_1j}^*\ldots b_{r_sj}^* \\ &= \sum_{r\leq L} a_{ri}^s\otimes (b_{rj}^*)^s \end{aligned}$$

Thus the conditions of type $u_{ij}^s=p_{ij}$ are satisfied as well, and we are done. $\square$

Let us discuss now the general case. We have the following result:

Proposition 10.13. *The various spaces G_{MN}^L constructed so far appear by imposing to the standard coordinates of U_{MN}^{L+} the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

with $s=(e_1,\ldots,e_s)$ ranging over all the colored integers, and with $\pi,\sigma\in D(0,s)$.

Proof. According to the various constructions above, the relations defining G_{MN}^L can be written as follows, with σ ranging over a family of generators, with no upper legs, of the corresponding category of partitions D:

$$\sum_{j_1\ldots j_s}\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=\delta_\sigma(i)$$

We therefore obtain the relations in the statement, as follows:

$$\begin{aligned} \sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s} &= \sum_{i_1\ldots i_s}\delta_\pi(i)\sum_{j_1\ldots j_s}\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s} \\ &= \sum_{i_1\ldots i_s}\delta_\pi(i)\delta_\sigma(i) \\ &= L^{|\pi\vee\sigma|} \end{aligned}$$

As for the converse, this follows by using the relations in the statement, by keeping π fixed, and by making σ vary over all the partitions in the category. $\square$

In the general case now, where $G=(G_N)$ is an arbitary uniform easy quantum group, we can construct spaces G_{MN}^L by using the above relations, and we have:

Theorem 10.14. *The spaces $G_{MN}^L\subset U_{MN}^{L+}$ constructed by imposing the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

with π,σ ranging over all the partitions in the associated category, having no upper legs, are subject to an action map/quotient map diagram, as in Theorem 10.9.

Proof. We proceed as in the proof of Proposition 10.8. We must prove that, if the variables u_{ij} satisfy the relations in the statement, then so do the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^* \quad , \quad V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

Regarding the variables U_{ij}, the computation here goes as follows:

$$\begin{aligned}
&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)U_{i_1j_1}^{e_1}\ldots U_{i_sj_s}^{e_s}\\
=&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}\otimes\delta_\pi(i)\delta_\sigma(j)a_{k_1i_1}^{e_1}\ldots a_{k_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\
=&\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\sigma(l)u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}=L^{|\pi\vee\sigma|}
\end{aligned}$$

For the variables V_{ij} the proof is similar, as follows:

$$\begin{aligned}
&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)V_{i_1j_1}^{e_1}\ldots V_{i_sj_s}^{e_s}\\
=&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(i)\delta_\sigma(j)a_{l_1i_1}^{e_1}\ldots a_{l_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\
=&\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(l)\delta_\sigma(l)=L^{|\pi\vee\sigma|}
\end{aligned}$$

Thus we have constructed an action map, and a quotient map, as in Proposition 10.8 above, and the commutation of the diagram in Theorem 10.9 is then trivial. □

The above results generalize some of the constructions in [3]. As explained in [3], there are many interesting questions regarding such spaces, and their quantum isometry groups. In what follows we will focus on some related topics, of probabilistic nature.

In the remainder of this section we discuss the integration over G_{MN}^L, with a number of explicit formulae. Our main result will be the fact that the operations of type $G_{MN}^L\to G_{MN}^{L+}$ are indeed "liberations", in the sense of the Bercovici-Pata bijection [24].

The integration over G_{MN}^L is best introduced as follows:

Definition 10.15. *The integration functional of G_{MN}^L is the composition*

$$\int_{G_{MN}^L}:C(G_{MN}^L)\to C(G_M\times G_N)\to\mathbb{C}$$

of the representation $u_{ij}\to\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$ with the Haar functional of $G_M\times G_N$.

Observe that in the case $L=M=N$ we obtain the integration over G_N. Also, at $L=M=1$, or at $L=N=1$, we obtain the integration over the sphere.

In the general case now, we first have the following result:

Proposition 10.16. *The integration functional of G_{MN}^L has the invariance property*

$$\left(\int_{G_{MN}^L}\otimes\, id\right)\Phi(x)=\int_{G_{MN}^L}x$$

with respect to the coaction map, $\Phi(u_{ij})=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$.

Proof. We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must check the following formula:

$$\left(\int_{G_{MN}^L}\otimes\, id\right)\Phi(u_{i_1j_1}\ldots u_{i_sj_s})=\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}$$

Let us compute the left term. This is given by:

$$\begin{aligned}
X &= \left(\int_{G_{MN}^L}\otimes\, id\right)\sum_{k_xl_x}u_{k_1l_1}\ldots u_{k_sl_s}\otimes a_{k_1i_1}\ldots a_{k_si_s}\otimes b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\
&= \sum_{k_xl_x}\sum_{r_x\leq L}a_{k_1i_1}\ldots a_{k_si_s}\otimes b_{l_1j_1}^*\ldots b_{l_sj_s}^*\int_{G_M}a_{r_1k_1}\ldots a_{r_sk_s}\int_{G_N}b_{r_1l_1}^*\ldots b_{r_sl_s}^*\\
&= \sum_{r_x\leq L}\sum_{k_x}a_{k_1i_1}\ldots a_{k_si_s}\int_{G_M}a_{r_1k_1}\ldots a_{r_sk_s}\otimes\sum_{l_x}b_{l_1j_1}^*\ldots b_{l_sj_s}^*\int_{G_N}b_{r_1l_1}^*\ldots b_{r_sl_s}^*
\end{aligned}$$

By using now the invariance property of the Haar functionals of G_M,G_N, we obtain:

$$\begin{aligned}
X &= \sum_{r_x\leq L}\left(\int_{G_M}\otimes\, id\right)\Delta(a_{r_1i_1}\ldots a_{r_si_s})\otimes\left(\int_{G_N}\otimes\, id\right)\Delta(b_{r_1j_1}^*\ldots b_{r_sj_s}^*)\\
&= \sum_{r_x\leq L}\int_{G_M}a_{r_1i_1}\ldots a_{r_si_s}\int_{G_N}b_{r_1j_1}^*\ldots b_{r_sj_s}^*\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)\sum_{r_x\leq L}a_{r_1i_1}\ldots a_{r_si_s}\otimes b_{r_1j_1}^*\ldots b_{r_sj_s}^*
\end{aligned}$$

But this gives the formula in the statement, and we are done. □

We will prove now that the above functional is in fact the unique positive unital invariant trace on $C(G_{MN}^L)$. For this purpose, we will need the Weingarten formula:

Theorem 10.17. *We have the Weingarten type formula*

$$\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}=\sum_{\pi\sigma\tau\nu}L^{|\pi\vee\tau|}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

where $W_{sM}=G_{sM}^{-1}$, with $G_{sM}(\pi,\sigma)=M^{|\pi\vee\sigma|}$.

Proof. We make use of the usual quantum group Weingarten formula, for which we refer to [19], [29]. By using this formula for G_M, G_N, we obtain:

$$\begin{aligned}
\int_{G_{MN}^L} u_{i_1j_1}\ldots u_{i_sj_s} &= \sum_{l_1\ldots l_s\leq L}\int_{G_M} a_{l_1i_1}\ldots a_{l_si_s}\int_{G_N} b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\
&= \sum_{l_1\ldots l_s\leq L}\sum_{\pi\sigma}\delta_\pi(l)\delta_\sigma(i)W_{sM}(\pi,\sigma)\sum_{\tau\nu}\delta_\tau(l)\delta_\nu(j)W_{sN}(\tau,\nu)\\
&= \sum_{\pi\sigma\tau\nu}\left(\sum_{l_1\ldots l_s\leq L}\delta_\pi(l)\delta_\tau(l)\right)\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)
\end{aligned}$$

The coefficient being $L^{|\pi\vee\tau|}$, we obtain the formula in the statement. □

We can now derive an abstract characterization of the integration, as follows:

Theorem 10.18. *The integration of G_{MN}^L is the unique positive unital trace*

$$C(G_{MN}^L)\to\mathbb{C}$$

which is invariant under the action of the quantum group $G_M\times G_N$.

Proof. We use a standard method, from [22], [28], the point being to show that we have the following ergodicity formula:

$$\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)=\int_{G_{MN}^L}x$$

We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must verify that the following holds:

$$\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(u_{i_1j_1}\ldots u_{i_sj_s})=\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}$$

By using the Weingarten formula, the left term can be written as follows:

$$\begin{aligned}
X &= \sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}\ldots u_{k_sl_s}\int_{G_M}a_{k_1i_1}\ldots a_{k_si_s}\int_{G_N}b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\
&= \sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}\ldots u_{k_sl_s}\sum_{\pi\sigma}\delta_\pi(k)\delta_\sigma(i)W_{sM}(\pi,\sigma)\sum_{\tau\nu}\delta_\tau(l)\delta_\nu(j)W_{sN}(\tau,\nu)\\
&= \sum_{\pi\sigma\tau\nu}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\tau(l)u_{k_1l_1}\ldots u_{k_sl_s}
\end{aligned}$$

By using now the summing formula in Theorem 10.14, we obtain:

$$X=\sum_{\pi\sigma\tau\nu}L^{|\pi\vee\tau|}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

Now by comparing with the Weingarten formula for G_{MN}^L, this proves our claim.

Assume now that $\tau : C(G_{MN}^L) \to \mathbb{C}$ satisfies the invariance condition. We have:

$$\begin{aligned}
\tau\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x) &= \left(\tau\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)(\tau\otimes id)\Phi(x)\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)(\tau(x)1)\\
&= \tau(x)
\end{aligned}$$

On the other hand, according to the formula established above, we have as well:

$$\tau\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)=\tau(tr(x)1)=tr(x)$$

Thus we obtain $\tau = tr$, and this finishes the proof. □

Let us discuss now the precise computation of the laws of certain linear combinations of coordinates. A set of coordinates $\{u_{ij}\}$ is called "non-overlapping" if each horizontal index i and each vertical index j appears at most once. With this convention, we have:

Proposition 10.19. *For a sum of non-overlapping coordinates, of type*

$$\chi_E=\sum_{(ij)\in E}u_{ij}$$

we have the moment formula

$$\int_{G_{MN}^L}\chi_E^s=\sum_{\pi\sigma\tau\nu}K^{|\pi\vee\tau|}L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

where $K = |E|$ is the cardinality of the indexing set.

Proof. In terms of $K = |E|$, we can write $E = \{(\alpha(i), \beta(i))\}$, for certain embeddings $\alpha : \{1, \ldots, K\} \subset \{1, \ldots, M\}$ and $\beta : \{1, \ldots, K\} \subset \{1, \ldots, N\}$. In terms of these maps α, β, the moment in the statement is given by:

$$M_s=\int_{G_{MN}^L}\left(\sum_{i\le K}u_{\alpha(i)\beta(i)}\right)^s$$

By using the Weingarten formula, we can write this quantity as follows:

$$\begin{aligned} M_s &= \int_{G^L_{MN}} \sum_{i_1\ldots i_s\leq K} u_{\alpha(i_1)\beta(i_1)}\ldots u_{\alpha(i_s)\beta(i_s)} \\ &= \sum_{i_1\ldots i_s\leq K}\sum_{\pi\sigma\tau\nu} L^{|\sigma\vee\nu|}\delta_\pi(\alpha(i_1),\ldots,\alpha(i_s))\delta_\tau(\beta(i_1),\ldots,\beta(i_s))W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu) \\ &= \sum_{\pi\sigma\tau\nu}\left(\sum_{i_1\ldots i_s\leq K}\delta_\pi(i)\delta_\tau(i)\right)L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu) \end{aligned}$$

But, as explained before, the coefficient on the left in the last formula equals $K^{|\pi\vee\tau|}$. We therefore obtain the formula in the statement. □

We can further advance in the classical/twisted and free cases, where the Weingarten theory for the corresponding quantum groups is available from [11], [29]. The result here, which justifies our various "liberation" claims, is as follows:

Theorem 10.20. *In the context of the liberation operations $O^L_{MN}\to O^{L+}_{MN}$, $U^L_{MN}\to U^{L+}_{MN}$, $H^{sL}_{MN}\to H^{sL+}_{MN}$, the laws of the sums of non-overlapping coordinates,*

$$\chi_E=\sum_{(ij)\in E}u_{ij}$$

are in Bercovici-Pata bijection, in the $|E|=\kappa N, L=\lambda N, M=\mu N, N\to\infty$ limit.

Proof. We use the general theory in [11], [29]. According to Proposition 10.19 above, in terms of $K=|E|$, the moments of the variables in the statement are given by:

$$M_s=\sum_{\pi\sigma\tau\nu}K^{|\pi\vee\tau|}L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

We use now two standard facts, namely the fact that in the $N\to\infty$ limit the Weingarten matrix W_{sN} is concentrated on the diagonal, and the fact that we have $|\pi\vee\sigma|\leq\frac{|\pi|+|\sigma|}{2}$, with equality precisely when $\pi=\sigma$. See [29]. In the regime $K=\kappa N, L=\lambda N, M=\mu N, N\to\infty$ from the statement, we therefore obtain:

$$\begin{aligned} M_s &\simeq \sum_{\pi\tau}K^{|\pi\vee\tau|}L^{|\pi\vee\tau|}M^{-|\pi|}N^{-|\tau|} \\ &\simeq \sum_{\pi}K^{|\pi|}L^{|\pi|}M^{-|\pi|}N^{-|\pi|} \\ &= \sum_\pi\left(\frac{\kappa\lambda}{\mu}\right)^{|\pi|} \end{aligned}$$

In order to interpret this formula, we use general theory from [11], [29]:

(1) For $G_N=O_N,\bar{O}_N/O_N^+$, the above variables χ_E follow to be asymptotically Gaussian/semicircular, of parameter $\frac{\kappa\lambda}{\mu}$, and hence in Bercovici-Pata bijection.

(2) For $G_N=U_N,\bar{U}_N/U_N^+$ the situation is similar, with χ_E being asymptotically complex Gaussian/circular, of parameter $\frac{\kappa\lambda}{\mu}$, and in Bercovici-Pata bijection.

(3) Finally, for $G_N=H_N^s/H_N^{s+}$, the variables χ_E are asymptotically Bessel/free Bessel of parameter $\frac{\kappa\lambda}{\mu}$, and once again in Bercovici-Pata bijection. $\square$

The convergence in the above result is of course in moments, and we do not know whether some stronger convergence results can be formulated. Nor do we know whether one can use linear combinations of coordinates which are more general than the sums χ_E that we consider. These are interesting questions, that we would like to raise here.

11. Higher manifolds

We discuss in this section an abstract extension of the constructions of noncommutative algebraic manifolds that we have so far. The idea will be that of looking at certain classes of algebraic manifolds $X \subset S^{N-1}_{\mathbb{C},+}$, which are homogeneous spaces, of a very special type. Our main results will be an axiomatization of such spaces, that we will call "affine homogeneous spaces", along with a study of the main examples, and a number of algebraic and probabilistic results, notably including a Weingarten integration formula.

Following [7], [8], let us formulate the following definition:

Definition 11.1. *An affine homogeneous space over a closed subgroup $G \subset U_N^+$ is a closed subset $X \subset S^{N-1}_{\mathbb{C},+}$, such that there exists an index set $I \subset \{1, \ldots, N\}$ such that*

$$\alpha(x_i) = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ji} \quad , \quad \Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

define morphisms of C^*-algebras, satisfying $(id \otimes \int_G)\Phi = \int_G \alpha(.)1$.

Here, and in what follows, a closed subspace $Y \subset Z$ corresponds by definition to a quotient map $C(Z) \to C(Y)$. As for $\int_G$, this is the Haar integration. See [98].

Observe that $U_N^+ \to S^{N-1}_{\mathbb{C},+}$ is indeed affine in this sense, with $I = \{1\}$.

Also, the $1/\sqrt{|I|}$ constant appearing above is the correct one, because:

$$\begin{aligned} \sum_i \left(\sum_{j \in I} u_{ji} \right) \left(\sum_{k \in I} u_{ki} \right)^* &= \sum_i \sum_{j,k \in I} u_{ji} u_{ki}^* \\ &= \sum_{j,k \in I} (uu^*)_{jk} \\ &= |I| \end{aligned}$$

Generally speaking, the above definition is quite tricky, coming from a long series of papers, dealing with very explicit examples. As a first general result, we have:

Theorem 11.2. *Consider an affine homogeneous space X, as above.*

1. *The coaction condition $(\Phi \otimes id)\Phi = (id \otimes \Delta)\Phi$ is satisfied.*
2. *We have as well the formula $(\alpha \otimes id)\Phi = \Delta\alpha$.*

Proof. The coaction condition is clear. For the second formula, we first have:

$$\begin{aligned} (\alpha \otimes id)\Phi(x_i) &= \sum_k \alpha(x_k) \otimes u_{ki} \\ &= \frac{1}{\sqrt{|I|}} \sum_k \sum_{j \in I} u_{jk} \otimes u_{ki} \end{aligned}$$

On the other hand, we have as well:

$$\begin{aligned}\Delta\alpha(x_i) &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\Delta(u_{ji})\\ &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{jk}\otimes u_{ki}\end{aligned}$$

Thus, by linearity, multiplicativity and continuity, we obtain the result. □

Summarizing, the terminology in Definition 11.1 is justified, in the sense that what we have there are indeed certain homogeneous spaces, of very special, "affine" type.

As a second result regarding such spaces, which closes the discussion in the case where α is injective, which is something that happens in many cases, we have:

Theorem 11.3. *When α is injective we must have $X = X_{G,I}^{min}$, where:*

$$C(X_{G,I}^{min}) = \left\langle \frac{1}{\sqrt{|I|}}\sum_{j\in I} u_{ji}\Big| i = 1,\ldots,N\right\rangle \subset C(G)$$

Moreover, $X_{G,I}^{min}$ is affine homogeneous, for any $G \subset U_N^+$, and any $I \subset \{1,\ldots,N\}$.

Proof. The first assertion is clear from definitions. Regarding now the second assertion, consider the variables $X_i = \frac{1}{\sqrt{|I|}}\sum_{j\in I} u_{ji} \in C(G)$ in the statement.

In order to prove that we have $X_{G,I}^{min} \subset S^{N-1}_{\mathbb{C},+}$, observe first that we have:

$$\begin{aligned}\sum_i X_iX_i^* &= \frac{1}{|I|}\sum_i\sum_{j,k\in I} u_{ji}u_{ki}^*\\ &= \frac{1}{|I|}\sum_{j,k\in I}(uu^*)_{jk}\\ &= 1\end{aligned}$$

We have as well the following computation:

$$\begin{aligned}\sum_i X_i^*X_i &= \frac{1}{|I|}\sum_i\sum_{j,k\in I} u_{ji}^*u_{ki}\\ &= \frac{1}{|I|}\sum_{j,k\in I}(\bar{u}u^t)_{jk}\\ &= 1\end{aligned}$$

Thus $X_{G,I}^{min} \subset S^{N-1}_{\mathbb{C},+}$. Finally, observe that we have:

$$\begin{aligned}\Delta(X_i) &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{jk}\otimes u_{ki}\\ &= \sum_k X_k\otimes u_{ki}\end{aligned}$$

Thus we have a coaction map as in Definition 11.1, given by $\Phi = \Delta$, and the ergodicity condition, namely $(id\otimes\int_G)\Delta = \int_G(.)1$, holds as well, by definition of $\int_G$. □

In general, we cannot assume that α is injective, due to certain analytic issues, appearing for instance in the free case. Our purpose will be to show that the affine homogeneous spaces appear as follows, a bit in the same way as the discrete group algebras:

$$X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$$

We make the standard convention that all the tensor exponents k are "colored integers", that is, $k = e_1 \ldots e_k$ with $e_i \in \{\circ, \bullet\}$, with $\circ$ corresponding to the usual variables, and with $\bullet$ corresponding to their adjoints. With this convention, we have:

Proposition 11.4. *The ergodicity condition $(id\otimes\int_G)\Phi = \int_G\alpha(.)1$ is equivalent to*

$$(Px^{\otimes k})_{i_1\ldots i_k} = \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\quad,\quad \forall k,\forall i_1,\ldots,i_k$$

where $P_{i_1\ldots i_k,j_1\ldots j_k} = \int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$, and where $(x^{\otimes k})_{i_1\ldots i_k} = x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}$.

Proof. We have the following computation:

$$\begin{aligned}\left(id\otimes\int_G\right)\Phi(x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}) &= \sum_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\ &= \sum_{j_1\ldots j_k}P_{i_1\ldots i_k,j_1\ldots j_k}(x^{\otimes k})_{j_1\ldots j_k}\\ &= (Px^{\otimes k})_{i_1\ldots i_k}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\int_G\alpha(x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}) &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\ &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\end{aligned}$$

But this gives the formula in the statement, and we are done. □

As a consequence, we have the following result:

Theorem 11.5. *We must have $X\subset X_{G,I}^{max}$, as subsets of $S^{N-1}_{\mathbb C,+}$, where:*

$$C(X_{G,I}^{max})=C(S^{N-1}_{\mathbb C,+})\Big/\left\langle(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\Big|\forall k,\forall i_1,\ldots i_k\right\rangle$$

Moreover, $X_{G,I}^{max}$ is affine homogeneous, for any $G\subset U_N^+$, and any $I\subset\{1,\ldots,N\}$.

Proof. Let us first prove that we have an action $G\curvearrowright X_{G,I}^{max}$. We must show here that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $X_{G,I}^{max}$. We have:

$$\begin{aligned}
(PX^{\otimes k})_{i_1\ldots i_k}&=\sum_{l_1\ldots l_k}P_{i_1\ldots i_k,l_1\ldots l_k}(X^{\otimes k})_{l_1\ldots l_k}\\
&=\sum_{l_1\ldots l_k}P_{i_1\ldots i_k,l_1\ldots l_k}\sum_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\otimes u_{j_1l_1}^{e_1}\ldots u_{j_kl_k}^{e_k}\\
&=\sum_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\otimes(u^{\otimes k}P^t)_{j_1\ldots j_k,i_1\ldots i_k}
\end{aligned}$$

Since by Peter-Weyl the transpose of $P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$ is the orthogonal projection onto $Fix(u^{\otimes k})$, we have $u^{\otimes k}P^t=P^t$, and we therefore obtain:

$$\begin{aligned}
(PX^{\otimes k})_{i_1\ldots i_k}&=\sum_{j_1\ldots j_k}P_{i_1\ldots i_k,j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\\
&=(Px^{\otimes k})_{i_1\ldots i_k}\\
&=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}
\end{aligned}$$

Thus we have an action $G\curvearrowright X_{G,I}^{max}$, and since this action is ergodic by Proposition 11.4, we have an affine homogeneous space, as claimed. $\square$

We can now merge the results that we have, and we obtain:

Theorem 11.6. *Given a closed quantum subgroup $G\subset U_N^+$, and a set $I\subset\{1,\ldots,N\}$, if we consider the following C^*-subalgebra and the following quotient C^*-algebra,*

$$\begin{aligned}
C(X_{G,I}^{min})&=\left\langle\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}\Big|i=1,\ldots,N\right\rangle\subset C(G)\\
C(X_{G,I}^{max})&=C(S^{N-1}_{\mathbb C,+})\Big/\left\langle(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\Big|\forall k,\forall i_1,\ldots i_k\right\rangle
\end{aligned}$$

then we have maps $G\to X_{G,I}^{min}\subset X_{G,I}^{max}\subset S^{N-1}_{\mathbb C,+}$, the space $G\to X_{G,I}^{max}$ is affine homogeneous, and any affine homogeneous space $G\to X$ appears as $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$.

Proof. This follows indeed from Theorem 11.3 and Theorem 11.5 above. $\square$

As a conclusion, the affine homogeneous spaces over a given closed subgroup $G \subset U_N^+$, in the sense of Definition 11.1, are the intermediate spaces $X_{G,I}^{min} \subset X \subset X_{G,I}^{max}$ having an action of G, with the maximal space $X_{G,I}^{max}$ known to be affine homogeneous.

We will need one more general result from [7], namely an extension of the Weingarten integration formula [19], [49], [94], to the affine homogeneous space setting:

Theorem 11.7. *Assuming that $G \to X$ is an affine homogeneous space, with index set $I \subset \{1, \ldots, N\}$, the Haar integration functional $\int_X = \int_G \alpha$ is given by*

$$\int_X x_{i_1}^{e_1} \ldots x_{i_k}^{e_k} = \sum_{\pi,\sigma \in D} K_I(\pi) \overline{(\xi_\sigma)}_{i_1 \ldots i_k} W_{kN}(\pi, \sigma)$$

where $\{\xi_\pi | \pi \in D\}$ is a basis of $Fix(u^{\otimes k})$, $W_{kN} = G_{kN}^{-1}$ with $G_{kN}(\pi, \sigma) = < \xi_\pi, \xi_\sigma >$ is the associated Weingarten matrix, and $K_I(\pi) = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \ldots j_k \in I} (\xi_\pi)_{j_1 \ldots j_k}$.

Proof. By using the Weingarten formula for the quantum group G, we have:

$$\begin{aligned}
\int_X x_{i_1}^{e_1} \ldots x_{i_k}^{e_k} &= \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \ldots j_k \in I} \int_G u_{j_1 i_1}^{e_1} \ldots u_{j_k i_k}^{e_k} \\
&= \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \ldots j_k \in I} \sum_{\pi,\sigma \in D} (\xi_\pi)_{j_1 \ldots j_k} \overline{(\xi_\sigma)}_{i_1 \ldots i_k} W_{kN}(\pi, \sigma)
\end{aligned}$$

But this gives the formula in the statement, and we are done. □

Let us go back now to the "minimal vs maximal" discussion, in analogy with the group algebras. Here is a natural example of an intermediate space $X_{G,I}^{min} \subset X \subset X_{G,I}^{max}$:

Theorem 11.8. *Given a closed quantum subgroup $G \subset U_N^+$, and a set $I \subset \{1, \ldots, N\}$, if we consider the following quotient algebra*

$$C(X_{G,I}^{med}) = C(S_{\mathbb{C},+}^{N-1}) \Big/ \left\langle \sum_{j_1 \ldots j_k} \xi_{j_1 \ldots j_k} x_{j_1}^{e_1} \ldots x_{j_k}^{e_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \ldots j_k \in I} \xi_{j_1 \ldots j_k} \Big| \forall k, \forall \xi \in Fix(u^{\otimes k}) \right\rangle$$

we obtain in this way an affine homogeneous space $G \to X_{G,I}$.

Proof. We know from Theorem 11.5 above that $X_{G,I}^{max} \subset S_{\mathbb{C},+}^{N-1}$ is constructed by imposing to the standard coordinates the conditions $Px^{\otimes k} = P^I$, where:

$$P_{i_1 \ldots i_k, j_1 \ldots j_k} = \int_G u_{j_1 i_1}^{e_1} \ldots u_{j_k i_k}^{e_k}$$

$$P_{i_1 \ldots i_k}^I = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \ldots j_k \in I} P_{i_1 \ldots i_k, j_1 \ldots j_k}$$

According to the Weingarten integration formula for G, we have:

$$\begin{aligned}(Px^{\otimes k})_{i_1\ldots i_k} &= \sum_{j_1\ldots j_k}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\\ P^I_{i_1\ldots i_k} &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)\end{aligned}$$

Thus $X_{G,I}^{med}\subset X_{G,I}^{max}$, and the other assertions are standard as well. ☐

We can now put everything together, as follows:

Theorem 11.9. *Given a closed subgroup $G\subset U_N^+$, and a subset $I\subset\{1,\ldots,N\}$, the affine homogeneous spaces over G, with index set I, have the following properties:*

1. *These are exactly the intermediate subspaces $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$ on which G acts affinely, with the action being ergodic.*
2. *For the minimal and maximal spaces $X_{G,I}^{min}$ and $X_{G,I}^{max}$, as well as for the intermediate space $X_{G,I}^{med}$ constructed above, these conditions are satisfied.*
3. *By performing the GNS construction with respect to the Haar integration functional $\int_X=\int_G\alpha$ we obtain the minimal space $X_{G,I}^{min}$.*

We agree to identify all these spaces, via the GNS construction, and denote them $X_{G,I}$.

Proof. This follows indeed by combining the various results and observations formulated above. Once again, for full details on all these facts, we refer to [7]. ☐

Observe the similarity with what happens for the C^*-algebras of the discrete groups, where the various intermediate algebras $C^*(\Gamma)\to A\to C^*_{red}(\Gamma)$ must be identified as well, in order to reach to a unique noncommutative space $\widehat{\Gamma}$. For details here, see [98].

Let us discuss now some basic examples of affine homogeneous spaces, namely those coming from the classical groups, and those coming from the group duals.

We will need the following technical result:

Proposition 11.10. *Assuming that a closed subset $X\subset S^{N-1}_{\mathbb C,+}$ is affine homogeneous over a classical group, $G\subset U_N$, then X itself must be classical, $X\subset S^{N-1}_{\mathbb C}$.*

Proof. We use the well-known fact that, since the standard coordinates $u_{ij}\in C(G)$ commute, the corepresentation $u^{\circ\circ\bullet\bullet}=u^{\otimes 2}\otimes\bar{u}^{\otimes 2}$ has the following fixed vector:

$$\xi=\sum_{ij}e_i\otimes e_j\otimes e_i\otimes e_j$$

With $k=\circ\circ\bullet\bullet$ and with this vector ξ, the ergodicity formula reads:

$$\begin{aligned}\sum_{ij}x_ix_jx_i^*x_j^* &= \frac{1}{\sqrt{|I|^4}}\sum_{i,j\in I}1\\ &= 1\end{aligned}$$

By using this formula, along with $\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$, we obtain:

$$\begin{aligned}
&\sum_{ij}(x_ix_j-x_jx_i)(x_j^*x_i^*-x_i^*x_j^*)\\
=&\sum_{ij}x_ix_jx_j^*x_i^*-x_ix_jx_i^*x_j^*-x_jx_ix_j^*x_i^*+x_jx_ix_i^*x_j^*\\
=&\ 1-1-1+1\\
=&\ 0
\end{aligned}$$

We conclude that we have $[x_i,x_j]=0$, for any i,j. By using now this commutation relation, plus once again the relations defining $S^{N-1}_{\mathbb{C},+}$, we have as well:

$$\begin{aligned}
&\sum_{ij}(x_ix_j^*-x_j^*x_i)(x_jx_i^*-x_i^*x_j)\\
=&\sum_{ij}x_ix_j^*x_jx_i^*-x_ix_j^*x_i^*x_j-x_j^*x_ix_jx_i^*+x_j^*x_ix_i^*x_j\\
=&\sum_{ij}x_ix_j^*x_jx_i^*-x_ix_i^*x_j^*x_j-x_j^*x_jx_ix_i^*+x_j^*x_ix_i^*x_j\\
=&\ 1-1-1+1\\
=&\ 0
\end{aligned}$$

Thus we have $[x_i,x_j^*]=0$ as well, and so $X\subset S^{N-1}_{\mathbb{C}}$, as claimed. □

We can now formulate the result in the classical case, as follows:

Theorem 11.11. *In the classical case, $G\subset U_N$, there is only one affine homogeneous space, for each index set $I=\{1,\ldots,N\}$, namely the quotient space*

$$X=G/(G\cap C_N^I)$$

where $C_N^I\subset U_N$ is the group of unitaries fixing the vector $\xi_I=\frac{1}{\sqrt{|I|}}(\delta_{i\in I})_i$.

Proof. Consider an affine homogeneous space $G\to X$. We already know from Proposition 11.10 above that X is classical. We will first prove that we have $X=X_{G,I}^{min}$, and then we will prove that $X_{G,I}^{min}$ equals the quotient space in the statement.

(1) We use the well-known fact that the functional $E=(id\otimes\int_G)\Phi$ is the projection onto the fixed point algebra of the action, given by:

$$C(X)^\Phi=\{f\in C(X)|\Phi(f)=f\otimes 1\}$$

Thus our ergodicity condition, namely $E=\int_G\alpha(.)1$, shows that we must have:

$$C(X)^\Phi=\mathbb{C}1$$

Now since in the classical case the condition $\Phi(f)=f\otimes 1$ reads $f(gx)=f(x)$ for any $g\in G$ and $x\in X$, we recover in this way the usual ergodicity condition, stating that whenever a function $f \in C(X)$ is constant on the orbits of the action, it must be constant. Now observe that for an affine action, the orbits are closed. Thus an affine action which is ergodic must be transitive, and we deduce from this that we have $X = X_{G,I}^{min}$.

(2) We know that the inclusion $C(X) \subset C(G)$ comes via:

$$x_i = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ji}$$

Thus, the quotient map $p : G \to X \subset S^{N-1}_{\mathbb{C}}$ is given by the following formula:

$$p(g) = \left(\frac{1}{\sqrt{|I|}} \sum_{j \in I} g_{ji} \right)_i$$

In particular, the image of the unit matrix $1 \in G$ is the following vector:

$$\begin{aligned} p(1) &= \left(\frac{1}{\sqrt{|I|}} \sum_{j \in I} \delta_{ji} \right)_i \\ &= \left(\frac{1}{\sqrt{|I|}} \delta_{i \in I} \right)_i \\ &= \xi_I \end{aligned}$$

But this gives the result, and we are done. $\square$

Let us discuss now the group dual case. Given a discrete group $\Gamma =< g_1, \ldots, g_N >$, we can consider the embedding $\widehat{\Gamma} \subset U_N^+$ given by $u_{ij} = \delta_{ij} g_i$. We have then:

Theorem 11.12. *In the group dual case, $G = \widehat{\Gamma}$ with $\Gamma =< g_1, \ldots, g_N >$, we have*

$$X = \widehat{\Gamma}_I \quad , \quad \Gamma_I =< g_i | i \in I > \subset \Gamma$$

for any affine homogeneous space X, when identifying full and reduced group algebras.

Proof. Assume indeed that we have an affine homogeneous space $G \to X$. In terms of the rescaled coordinates $h_i = \sqrt{|I|} x_i$, our axioms for α, Φ read:

$$\alpha(h_i) = \delta_{i \in I} g_i \quad , \quad \Phi(h_i) = h_i \otimes g_i$$

As for the ergodicity condition, this translates as follows:

$$\begin{aligned}
&\left(id\otimes\int_G\right)\Phi(h_{i_1}^{e_1}\ldots h_{i_p}^{e_p})=\int_G\alpha(h_{i_1}^{e_p}\ldots h_{i_p}^{e_p})\\
\iff\quad&\left(id\otimes\int_G\right)(h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}\otimes g_{i_1}^{e_1}\ldots g_{i_p}^{e_p})=\int_G\delta_{i_1\in I}\ldots\delta_{i_p\in I}g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}\\
\iff\quad&\delta_{g_{i_1}^{e_1}\ldots g_{i_p}^{e_p},1}h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=\delta_{g_{i_1}^{e_1}\ldots g_{i_p}^{e_p},1}\delta_{i_1\in I}\ldots\delta_{i_p\in I}\\
\iff\quad&\left[g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}=1\implies h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=\delta_{i_1\in I}\ldots\delta_{i_p\in I}\right]
\end{aligned}$$

Now observe that from $g_ig_i^*=g_i^*g_i=1$ we obtain in this way:

$$h_ih_i^*=h_i^*h_i=\delta_{i\in I}$$

Thus the elements h_i vanish for $i\notin I$, and are unitaries for $i\in I$. We conclude that we have $X=\widehat{\Lambda}$, where $\Lambda=<h_i|i\in I>$ is the group generated by these unitaries.

In order to finish the proof, our claim is that for indices $i_x\in I$ we have:

$$g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}=1\iff h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=1$$

Indeed, $\implies$ comes from the ergodicity condition, as processed above, and $\Longleftarrow$ comes from the existence of the morphism α, which is given by $\alpha(h_i)=g_i$, for $i\in I$. $\square$

Let us go back now to the general case, and discuss a number of further axiomatization issues, based on the examples that we have. We will need:

Proposition 11.13. *The closed subspace $C_N^{I+}\subset U_N^+$ defined via*

$$C(C_N^{I+})=C(U_N^+)\Big/\left<u\xi_I=\xi_I\right>$$

where $\xi_I=\frac{1}{\sqrt{|I|}}(\delta_{i\in I})_i$, is a compact quantum group.

Proof. We must check Woronowicz's axioms, and the proof goes as follows:

(1) Let us set $U_{ij}=\sum_k u_{ik}\otimes u_{kj}$. We have then:

$$\begin{aligned}
(U\xi_I)_i&=\frac{1}{\sqrt{|I|}}\sum_{j\in I}U_{ij}\\
&=\frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{ik}\otimes u_{kj}\\
&=\sum_k u_{ik}\otimes(u\xi_I)_k
\end{aligned}$$

Since the vector ξ_I is by definition fixed by u, we obtain:

$$\begin{aligned}(U\xi_I)_i &= \sum_k u_{ik}\otimes(\xi_I)_k\\ &= \frac{1}{\sqrt{|I|}}\sum_{k\in I}u_{ik}\otimes 1\\ &= (u\xi_I)_i\otimes 1\\ &= (\xi_I)_i\otimes 1\end{aligned}$$

Thus we can define indeed a comultiplication map, by $\Delta(u_{ij})=U_{ij}$.

(2) In order to construct the counit map, $\varepsilon(u_{ij})=\delta_{ij}$, we must prove that the identity matrix $1=(\delta_{ij})_{ij}$ satisfies $1\xi_I=\xi_I$. But this is clear.

(3) In order to construct the antipode, $S(u_{ij})=u_{ji}^*$, we must prove that the adjoint matrix $u^*=(u_{ji}^*)_{ij}$ satisfies $u^*\xi_I=\xi_I$. But this is clear from $u\xi_I=\xi_I$. □

Based on the computations that we have so far, we can formulate:

Theorem 11.14. *Given a closed quantum subgroup $G\subset U_N^+$ and a set $I\subset\{1,\ldots,N\}$, we have a quotient map and an inclusion map as follows:*

$$G/(G\cap C_N^{I+})\to X_{G,I}^{min}\subset X_{G,I}^{max}$$

These maps are both isomorphisms in the classical case. In general, they are both proper.

Proof. Consider the quantum group $H=G\cap C_N^{I+}$, which is by definition such that at the level of the corresponding algebras, we have:

$$C(H)=C(G)\Big/\left<u\xi_I=\xi_I\right>$$

In order to construct a quotient map $G/H\to X_{G,I}^{min}$, we must check that the defining relations for $C(G/H)$ hold for the standard generators $x_i\in C(X_{G,I}^{min})$. But if we denote by $\rho:C(G)\to C(H)$ the quotient map, then we have, as desired:

$$\begin{aligned}(id\otimes\rho)\Delta x_i &= (id\otimes\rho)\left(\frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{ik}\otimes u_{kj}\right)\\ &= \sum_k u_{ik}\otimes(\xi_I)_k\\ &= x_i\otimes 1\end{aligned}$$

In the classical case, Theorem 11.11 shows that both the maps in the statement are isomorphisms. For the group duals, however, these maps are not isomorphisms, in general. This follows indeed from Theorem 11.12, and from the general theory in [28]. □

We discuss now a number of further examples. We will need:

Proposition 11.15. *Given a compact matrix quantum group $G=(G,u)$, the pair $G^t=(G,u^t)$, where $(u^t)_{ij}=u_{ji}$, is a compact matrix quantum group as well.*

Proof. The construction of the comultiplication is as follows, where Σ is the flip map:

$$\begin{aligned}
&\Delta^t[(u^t)_{ij}]=\sum_k (u^t)_{ik}\otimes(u^t)_{kj}\\
\iff\quad&\Delta^t(u_{ji})=\sum_k u_{ki}\otimes u_{jk}\\
\iff\quad&\Delta^t=\Sigma\Delta
\end{aligned}$$

As for the corresponding counit and antipode, these can be simply taken to be (ε,S), and the axioms are satisfied. □

We will need as well the following result, which is standard as well:

Proposition 11.16. *Given two closed subgroups $G\subset U_N^+$ and $H\subset U_M^+$, with fundamental corepresentations denoted $u=(u_{ij})$ and $v=(v_{ab})$, their product is a closed subgroup $G\times H\subset U_{NM}^+$, with fundamental corepresentation $w_{ia,jb}=u_{ij}\otimes v_{ab}$.*

Proof. The corresponding structural maps are $\Delta(\alpha\otimes\beta)=\Delta(\alpha)_{13}\Delta(\beta)_{24}$, $\varepsilon(\alpha\otimes\beta)=\varepsilon(\alpha)\varepsilon(\beta)$ and $S(\alpha\otimes\beta)=S(\alpha)S(\beta)$, the verifications being as follows:

$$\begin{aligned}
\Delta(w_{ia,jb})&=\Delta(u_{ij})_{13}\Delta(v_{ab})_{24}\\
&=\sum_{kc}u_{ik}\otimes v_{ac}\otimes u_{kj}\otimes v_{cb}\\
&=\sum_{kc}w_{ia,kc}\otimes w_{kc,jb}
\end{aligned}$$

For the counit, we have:

$$\begin{aligned}
\varepsilon(w_{ia,jb})&=\varepsilon(u_{ij})\varepsilon(v_{ab})\\
&=\delta_{ij}\delta_{ab}\\
&=\delta_{ia,jb}
\end{aligned}$$

For the antipode, we have:

$$\begin{aligned}
S(w_{ia,jb})&=S(u_{ij})S(v_{ab})\\
&=v_{ba}^*u_{ji}^*\\
&=(u_{ji}v_{ba})^*\\
&=w_{jb,ia}^*
\end{aligned}$$

We refer to Wang's paper [92] for more details regarding this construction. □

Let us call a closed quantum subgroup $G \subset U_N^+$ self-transpose when we have an automorphism $T : C(G) \to C(G)$ given by $T(u_{ij}) = u_{ji}$. Observe that in the classical case, this amounts in $G \subset U_N$ to be closed under the transposition operation $g \to g^t$.

With these notions in hand, let us go back to the affine homogeneous spaces.

As a first result here, any closed subgroup $G \subset U_N^+$ appears as an affine homogeneous space over an appropriate quantum group, as follows:

Theorem 11.17. *Given a closed subgroup $G \subset U_N^+$, we have an identification $X_{\mathcal{G},I}^{min} \simeq G$, given at the level of standard coordinates by $x_{ij} = \frac{1}{\sqrt{N}} u_{ij}$, where:*

1. *$\mathcal{G} = G^t \times G \subset U_{N^2}^+$, with coordinates $w_{ia,jb} = u_{ji} \otimes u_{ab}$.*
2. *$I \subset \{1, \ldots, N\}^2$ is the diagonal set, $I = \{(k,k) | k = 1, \ldots, N\}$.*

In the self-transpose case we can choose as well $\mathcal{G} = G \times G$, with $w_{ia,jb} = u_{ij} \otimes u_{ab}$.

Proof. As a first observation, our closed subgroup $G \subset U_N^+$ appears as an algebraic submanifold of the free complex sphere on N^2 variables, as follows:

$$G \subset S_{\mathbb{C},+}^{N^2-1} \quad , \quad x_{ij} = \frac{1}{\sqrt{N}} u_{ij}$$

Let us construct now the affine homogeneous space structure. Our claim is that, with $\mathcal{G} = G^t \times G$ and $I = \{(k,k)\}$ as in the statement, the structural maps are:

$$\alpha = \Delta \quad , \quad \Phi = (\Sigma \otimes id)\Delta^{(2)}$$

Indeed, in what regards $\alpha = \Delta$, this is given by the following formula:

$$\begin{aligned} \alpha(u_{ij}) &= \sum_k u_{ik} \otimes u_{kj} \\ &= \sum_k w_{kk,ij} \end{aligned}$$

Thus, by dividing by $\sqrt{N}$, we obtain the usual affine homogeneous space formula:

$$\alpha(x_{ij}) = \frac{1}{\sqrt{|I|}} \sum_k w_{kk,ij}$$

Regarding now $\Phi = (\Sigma \otimes id)\Delta^{(2)}$, the formula here is as follows:

$$\begin{aligned} \Phi(u_{ij}) &= (\Sigma \otimes id) \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj} \\ &= \sum_{kl} u_{kl} \otimes u_{ik} \otimes u_{lj} \\ &= \sum_{kl} u_{kl} \otimes w_{kl,ij} \end{aligned}$$

Thus, by dividing by $\sqrt{N}$, we obtain the usual affine homogeneous space formula:

$$\Phi(x_{ij})=\sum_{kl}x_{kl}\otimes w_{kl,ij}$$

The ergodicity condition being clear as well, this gives the first assertion.

Regarding now the second assertion, assume that we are in the self-transpose case, and so that we have an automorphism $T:C(G)\to C(G)$ given by $T(u_{ij})=u_{ji}$.

With $w_{ia,jb}=u_{ij}\otimes u_{ab}$, the modified map $\alpha=(T\otimes id)\Delta$ is then given by:

$$\begin{aligned}\alpha(u_{ij}) &= (T\otimes id)\sum_k u_{ik}\otimes u_{kj}\\ &= \sum_k u_{ki}\otimes u_{kj}\\ &= \sum_k w_{kk,ij}\end{aligned}$$

As for the modified map $\Phi=(id\otimes T\otimes id)(\Sigma\otimes id)\Delta^{(2)}$, this is given by:

$$\begin{aligned}\Phi(u_{ij}) &= (id\otimes T\otimes id)\sum_{kl} u_{kl}\otimes u_{ik}\otimes u_{lj}\\ &= \sum_{kl} u_{kl}\otimes u_{ki}\otimes u_{lj}\\ &= \sum_{kl} u_{kl}\otimes w_{kl,ij}\end{aligned}$$

Thus we have the correct affine homogeneous space formulae, and once again the ergodicity condition being clear as well, this gives the result. □

Let us discuss now the generalization of the above result, to the context of the spaces introduced in [28]. We recall from there that we have the following construction:

Definition 11.18. *Given a closed subgroup $G\subset U_N^+$ and an integer $M\leq N$ we set*

$$C(G_{MN})=\left\langle u_{ij}\Big|i\in\{1,\ldots,M\},j\in\{1,\ldots,N\}\right\rangle\subset C(G)$$

and we call row space of G the underlying quotient space $G\to G_{MN}$.

As a basic example here, at $M=N$ we obtain G itself. Also, at $M=1$ we obtain the space whose coordinates are those on the first row of coordinates on G. See [28].

Given $G_N\subset U_N^+$ and an integer $M\leq N$, we can consider the quantum group $G_M=G_N\cap U_M^+$, with the intersection taken inside U_N^+, and with $U_M^+\subset U_N^+$ given by:

$$u=diag(v,1_{N-M})$$

Observe that we have a quotient map $C(G_N)\to C(G_M)$, given by $u_{ij}\to v_{ij}$.

We have the following extension of Theorem 11.17:

Theorem 11.19. *Given a closed subgroup $G_N \subset U_N^+$, we have an identification $X_{\mathcal{G},I}^{min} \simeq G_{MN}$, given at the level of standard coordinates by $x_{ij} = \frac{1}{\sqrt{M}} u_{ij}$, where:*

(1) *$\mathcal{G} = G_M^t \times G_N \subset U_{NM}^+$, where $G_M = G_N \cap U_M^+$, with coordinates $w_{ia,jb} = u_{ji} \otimes v_{ab}$.*
(2) *$I \subset \{1, \ldots, M\} \times \{1, \ldots, N\}$ is the diagonal set, $I = \{(k,k) | k = 1, \ldots, M\}$.*

In the self-transpose case we can choose as well $\mathcal{G} = G_M \times G_N$, with $w_{ia,jb} = u_{ij} \otimes v_{ab}$.

Proof. We will prove that the space $X = G_{MN}$, with coordinates $x_{ij} = \frac{1}{\sqrt{M}} u_{ij}$, coincides with the space $X_{\mathcal{G},I}^{min}$ constructed in the statement, with its standard coordinates.

For this purpose, consider the following composition of morphisms, where in the middle we have the comultiplication, and at left and right we have the canonical maps:

$$C(X) \subset C(G_N) \to C(G_N) \otimes C(G_N) \to C(G_M) \otimes C(G_N)$$

The standard coordinates are then mapped as follows:

$$\begin{aligned} x_{ij} &= \frac{1}{\sqrt{M}} u_{ij} \\ &\to \frac{1}{\sqrt{M}} \sum_k u_{ik} \otimes u_{kj} \\ &\to \frac{1}{\sqrt{M}} \sum_{k \leq M} u_{ik} \otimes v_{kj} \\ &= \frac{1}{\sqrt{M}} \sum_{k \leq M} w_{kk,ij} \end{aligned}$$

Thus we obtain the standard coordinates on the space $X_{\mathcal{G},I}^{min}$, as claimed. Finally, the last assertion is standard as well, by suitably modifying the above morphism. □

Let us discuss now the liberation operation, in the context of the affine homogeneous spaces, and probabilistic aspects. In the easy case, we have the following result:

Proposition 11.20. *When $G \subset U_N^+$ is easy, coming from a category of partitions D, the space $X_{G,I} \subset S_{\mathbb{C},+}^{N-1}$ appears by imposing the relations*

$$\sum_{i_1 \ldots i_k} \delta_\pi(i_1 \ldots i_k) x_{i_1}^{e_1} \ldots x_{i_k}^{e_k} = |I|^{|\pi| - k/2}, \quad \forall k, \forall \pi \in D(k)$$

where $D(k) = D(0,k)$, and where $|.|$ denotes the number of blocks.

Proof. We know by easiness that $Fix(u^{\otimes k})$ is spanned by the vectors $\xi_\pi = T_\pi$, with $\pi \in D(k)$. But these latter vectors are given by:

$$\xi_\pi = \sum_{i_1 \ldots i_k} \delta_\pi(i_1 \ldots i_k) e_{i_1} \otimes \ldots \otimes e_{i_k}$$

We deduce that $X_{G,I}\subset S^{N-1}_{\mathbb{C},+}$ appears by imposing the following relations:

$$\sum_{i_1\ldots i_k}\delta_\pi(i_1\ldots i_k)x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\delta_\pi(j_1\ldots j_k),\quad\forall k,\forall\pi\in D(k)$$

Now since the sum on the right equals $|I|^{|\pi|}$, this gives the result. □

More generally now, in view of the examples given above, making the link with [28], it is interesting to work out what happens when G is a product of easy quantum groups, and the index set I appears as $I=\{(c,\ldots,c)|c\in J\}$, for a certain set J.

The result here, in its most general form, is as follows:

Theorem 11.21. *For a product of easy quantum groups, $G=G^{(1)}_{N_1}\times\ldots\times G^{(s)}_{N_s}$, and with $I=\{(c,\ldots,c)|c\in J\}$, the space $X_{G,I}\subset S^{N-1}_{\mathbb{C},+}$ appears by imposing the relations*

$$\sum_{i_1\ldots i_k}\delta_\pi(i_1\ldots i_k)x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=|J|^{|\pi_1\vee\ldots\vee\pi_s|-k/2},\quad\forall k,\forall\pi\in D^{(1)}(k)\times\ldots\times D^{(s)}(k)$$

where $D^{(r)}\subset P$ is the category of partitions associated to $G^{(r)}_{N_r}\subset U^+_{N_r}$, and where the partition $\pi_1\vee\ldots\vee\pi_s\in P(k)$ is the one obtained by superposing $\pi_1,\ldots,\pi_s$.

Proof. Since we are in a direct product situation, $G=G^{(1)}_{N_1}\times\ldots\times G^{(s)}_{N_s}$, the general theory in [92] applies, and shows that a basis for $Fix(u^{\otimes k})$ is provided by the vectors $\rho_\pi=\xi_{\pi_1}\otimes\ldots\otimes\xi_{\pi_s}$ associated to the following partitions:

$$\pi=(\pi_1,\ldots,\pi_s)\in D^{(1)}(k)\times\ldots\times D^{(s)}(k)$$

We conclude that the space $X_{G,I}\subset S^{N-1}_{\mathbb{C},+}$ appears by imposing the following relations to the standard coordinates:

$$\sum_{i_1\ldots i_k}\delta_\pi(i_1\ldots i_k)x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\delta_\pi(j_1\ldots j_k),\ \forall k,\forall\pi\in D^{(1)}(k)\times\ldots\times D^{(s)}(k)$$

Since the conditions $j_1,\ldots,j_k\in I$ read $j_1=(l_1,\ldots,l_1),\ldots,j_k=(l_k,\ldots,l_k)$, for certain elements $l_1,\ldots l_k\in J$, the sums on the right are given by:

$$\begin{aligned}\sum_{j_1\ldots j_k\in I}\delta_\pi(j_1\ldots j_k)&=\sum_{l_1\ldots l_k\in J}\delta_\pi(l_1,\ldots,l_1,\ldots\ldots,l_k,\ldots,l_k)\\&=\sum_{l_1\ldots l_k\in J}\delta_{\pi_1}(l_1\ldots l_k)\ldots\delta_{\pi_s}(l_1\ldots l_k)\\&=\sum_{l_1\ldots l_k\in J}\delta_{\pi_1\vee\ldots\vee\pi_s}(l_1\ldots l_k)\end{aligned}$$

Now since the sum on the right equals $|J|^{|\pi_1\vee\ldots\vee\pi_s|}$, this gives the result. □

Finally, let us discuss probabilistic aspects. Following [7], we first have:

Proposition 11.22. *The moments of the variable $\chi_T=\sum_{i\leq T}x_{i\ldots i}$ are given by*

$$\int_X\chi_T^k\simeq\frac{1}{\sqrt{M^k}}\sum_{\pi\in D^{(1)}(k)\cap\ldots\cap D^{(s)}(k)}\left(\frac{TM}{N}\right)^{|\pi|}$$

in the $N_i\to\infty$ limit, $\forall i$, where $M=|I|$, and $N=N_1\ldots N_s$.

Proof. We have the following formula:

$$\pi(x_{i_1\ldots i_s})=\frac{1}{\sqrt{M}}\sum_{j\in J}u_{ji_1}\otimes\ldots\otimes u_{ji_s}$$

For the variable in the statement, we therefore obtain:

$$\pi(\chi_T)=\frac{1}{\sqrt{M}}\sum_{i\leq T}\sum_{j\in J}u_{ji}\otimes\ldots\otimes u_{ji}$$

Now by raising to the power k and integrating, we obtain:

$$\begin{aligned}\int_X\chi_T^k&=\frac{1}{\sqrt{M^k}}\sum_{ij}\sum_{\pi\sigma}\delta_{\pi_1}(j)\delta_{\sigma_1}(i)W_{kN_1}^{(1)}(\pi_1,\sigma_1)\ldots\delta_{\pi_s}(j)\delta_{\sigma_s}(i)W_{kN_s}^{(s)}(\pi_s,\sigma_s)\\&=\frac{1}{\sqrt{M^k}}\sum_{\pi\sigma}T^{|\pi_1\vee\ldots\vee\pi_s|}M^{|\sigma_1\vee\ldots\vee\sigma_s|}W_{kN_1}^{(1)}(\pi_1,\sigma_1)\ldots W_{kN_s}^{(s)}(\pi_s,\sigma_s)\end{aligned}$$

Now since the Weingarten functions are diagonal with $N\to\infty$, this gives the result. $\square$

As a consequence, we have the following result:

Theorem 11.23. *In the context of a liberation operation $G^{(i)}\to G^{(i)+}$, the laws of the variables $\sqrt{M}\chi_T$ are in Bercovici-Pata bijection, in the $N_i\to\infty$ limit.*

Proof. Assume indeed that we have easy quantum groups $G^{(1)},\ldots,G^{(s)}$, with free versions $G^{(1)+},\ldots,G^{(s)+}$. At the level of the categories of partitions, we have:

$$\bigcap_i\left(D^{(i)}\cap NC\right)=\left(\bigcap_i D^{(i)}\right)\cap NC$$

Since the intersection of Hom-spaces is the Hom-space for the generated quantum group, we deduce that at the quantum group level, we have:

$$<G^{(1)+},\ldots,G^{(s)+}>=<G^{(1)},\ldots,G^{(s)}>^+$$

Thus the result follows from Proposition 11.22, and from the Bercovici-Pata bijection result for truncated characters for this latter liberation operation [29], [85]. $\square$

12. Matrix models

We discuss in this final section a number of more specialized topics, notably in connection with the important question of finding matrix models for the manifolds that we have. Let us begin, however, with a summary of the things that we have seen so far:

1. Our starting idea was to axiomatize the abstract "noncommutative geometries", as coming from quadruplets of type (S,T,U,K), consisting of a sphere S, torus T, unitary group U, and reflection group K, with relations between them, as follows:

$$\begin{tikzcd} S \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & T \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ U \arrow[r, leftrightarrow] & K \end{tikzcd}$$

2. This idea was mainly supported by the fact that such quadruplets (S,T,U,K) exist indeed for the main 4 examples of noncommutative geometries, namely the usual real and complex ones, and their free analogues, which can be represented as follows:

$$\begin{tikzcd} \mathbb{R}^N_+ \arrow[r] & \mathbb{C}^N_+ \\ \mathbb{R}^N \arrow[r] \arrow[u] & \mathbb{C}^N \arrow[u] \end{tikzcd}$$

3. In order to axiomatize the quadruplets (S,T,U,K), our idea was to use a uniform approach, which can be at the same time real and complex, and classical and free. To be more precise, let us start with intermediate objects, as follows:

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C},+}$$

$$T_N \subset T \subset \mathbb{T}_N^+$$

$$O_N \subset U \subset U_N^+$$

$$H_N \subset K \subset K_N^+$$

4. The problem is that of working out axioms for the 12 possible correspondences between our objects. We have seen here that things can be quite subtle in the noncommutative setting, and our final axioms, simplified, turned to be as follows:

$$\begin{array}{ccccc} S & = & S_U & & \\ S \cap \mathbb{T}_N^+ & = & T & = & K \cap \mathbb{T}_N^+ \\ G^+(S) & = & < O_N, T > & = & U \\ G^+(T) \cap K_N^+ & = & U \cap K_N^+ & = & K \end{array}$$

5. With these axioms in hand, we started looking for further examples. The conclusion here was that we have some natural intermediate geometries both on the horizontal and the vertical, and so a 3×3 diagram, refining the above 2×2 one, as follows:

$$\begin{matrix}
\mathbb R^N_+&\to&\mathbb{TR}^N_+&\to&\mathbb C^N_+\\
\uparrow&&\uparrow&&\uparrow\\
\mathbb R^N_*&\to&\mathbb{TR}^N_*&\to&\mathbb C^N_*\\
\uparrow&&\uparrow&&\uparrow\\
\mathbb R^N&\to&\mathbb{TR}^N&\to&\mathbb C^N
\end{matrix}$$

6. These geometries are all "easy", and the problem of classifying the easy geometries appears. We have seen here that, under mild extra axioms, the above 9 geometries are the only ones. Technically speaking, all this is about categories as follows:

$$\mathcal{NC}_2\subset D\subset P_2\quad,\quad \mathcal{NC}_{even}\subset E\subset P_{even}$$

$$D=E\cap P_2\quad,\quad E=<D,\mathcal{NC}_{even}>$$

7. All this is a priori "noncommutative algebraic geometry". However, with Riemannian geometry motivations in mind, we have developed integration methods for S,T,U,K, based on the Weingarten formula, which in the case of the spheres is:

$$\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}\,dx=\sum_\pi\sum_{\sigma\leq\ker i}W_{kN}(\pi,\sigma)$$

8. We have developed then the geometries that we found, notably with the study of a whole class of homogeneous spaces, of "affine" type. The simplest examples of such homogeneous spaces are the following spaces of partial isometries, with $G=U,K$:

$$X=(G_M\times G_N)\big/(G_L\times G_{M-L}\times G_{N-L})$$

9. As a main result, we have seen that T is subject to the Meixner/free Meixner correspondence, while S,U,K and related homogeneous spaces are subject to the Bercovici-Pata bijection, for sums of non-overlapping coordinates, as follows:

$$\chi_E=\sum_{(ij)\in E}u_{ij}$$

10. Finally, we have explored a number of more technical aspects, namely the extension of our formalism via twisting, the question of axiomatizing the "easy algebraic manifolds", and the exact computation of the integrals, at fixed values of $N\in\mathbb N$.

All this is certainly nice, but we are still far away from something that can be called "noncommutative geometry". There are still many important problems left, namely:

(1) Axiomatization problems, notably in connection with the extension by twisting and intersection, and other methods, such as super-easiness.
(2) At the classification level things are fairly technical, and what we did here is rather some "minimal" work on the subject, waiting to be fine-tuned.
(3) We have the key problem of axiomatizing the "easy algebraic manifolds", and working out further results regarding the integration over such manifolds.

In addition to all these questions, which seem to require a massive amount of work, we have the obvious question of unifying what we are doing with Connes' work. Once again this seems to require a massive amount of work, and our question here is as follows:

Question 12.1. *Is there a Nash-Connes Geometry (NCG) covering all the known interesting examples of noncommutative Riemannian manifolds?*

To be more precise here, we are of course physicists, disguised as pure mathematicians, and what we've been doing here is certainly not algebraic geometry (!) but rather Riemannian geometry, with coordinates, a la Nash [81]. So the question is that of unifying our Nash geometry with the Connes geometry, and the situation here is as follows:

(1) From Nash to Connes, the problem is that of understanding what are the precise Riemannian features of our manifolds, passed of course our rock-solid way of integrating over them. In the case of the spheres, it is known that these have a Laplacian filtration. According to the work of Franz et al. [48], [61], the eigenvalues of the Laplacian can be constructed as well. However, in the free case there does not seem to be a Dirac operator, in the precise sense of Connes.
(2) From Connes to Nash, the problem is that of understanding which Riemannian manifolds in the sense of Connes have "coordinates", in the spirit of the Nash embedding theorem. Ideally, we would like to have embeddings into the free complex sphere $S^{N-1}_{\mathbb{C},+}$, which would make a direct link with our Nash type geometry. However, things here are quite flexible, and many other interesting examples of noncommutative spheres exist [57], [59], and can be probably used.

All this looks quite non-trivial, and will probably take a long time to be done. Perhaps the most distressing thing in all this is that not many people are currently working on this subject, which, in our opinion, is one of the most interesting questions around.

Finally, as a last general comment on the subject, let us mention that, in addition to the above-mentioned papers, we have as well [12], [27], [32], [45], [62], [66], [67], [68], [86], [90] and related papers, dealing with various other considerations relating the compact quantum groups to Connes' noncommutative geometry theory.

Let us get away now from all these difficult questions, and discuss one more topic, which is independent of all this, and is probably of interest as well: matrix modelling.

Let us first recall the GNS representation theorem, in a detailed form:

Theorem 12.2. *Any C^*-algebra A appears as closed $*$-algebra of operators on a Hilbert space, $A \subset B(H)$, in the following way:*

(1) *In the commutative case, where $A = C(X)$, we can set $H = L^2(X)$, with respect to some probability measure on X, and use the embedding $g \to (g \to fg)$.*
(2) *In general, we can set $H = L^2(A)$, with respect to some faithful positive trace $tr : A \to \mathbb{C}$, and then use a similar embedding, $a \to (b \to ab)$.*

Proof. This is something that we already know, from section 1 above, the idea being that (1) is quite elementary, and that the subtle point in (2) is the construction of the trace $tr : A \to \mathbb{C}$, which can be done via abstract functional analysis methods. □

The above result is something quite fundamental, allowing us to study the C^*-algebras via their representations $A \to B(H)$, having suitable faithfulness properties.

In the case of the algebras $A = C(X)$ with $X \subset S^{N-1}_{\mathbb{C},+}$ that we are interested in, this philosophy amounts in looking for looking for operators $T_i \in B(H)$ which model the standard coordinates $x_i \in C(X)$. To be more precise, assuming that we have found a family of such operators $T_i \in B(H)$, which satisfy the polynomial relations which relate the standard coordinates $x_i \in C(X)$, we have a representation $C(X) \to B(H)$.

In practice, all this is a bit too general, and not very useful. However, and here comes our point, by replacing the operator algebra models $C(X) \to B(H)$ by suitable models of type $C(X) \to B$, with B being a C^*-algebra, not necessarily equal to a full operator algebra over a Hilbert space, we are led to some interesting and useful theory.

In order to discuss this, we need a good family of target algebras B, that can we can say that we understand very well. And here, we can use:

Definition 12.3. *A random matrix algebra is a C^*-algebra of type*

$$B = M_K(C(T))$$

with T being a compact space, and $K \in \mathbb{N}$ being an integer.

The terminology here comes from the fact that, in practice, the space T usually comes with a probability measure on it, which makes the elements of B "random matrices".

Observe that we can write $B = M_K(\mathbb{C}) \otimes C(T)$. Thus, the random matrix algebras appear by definition as tensor products of the simplest types of C^*-algebras that we know, namely (1) the full matrix algebras, $M_K(\mathbb{C})$ with $K \in \mathbb{N}$, and (2) the commutative algebras, which are of the form $C(T)$, with T being a compact space.

Getting back now to our modelling questions, we can formulate:

Definition 12.4. *A matrix model for a noncommutative algebraic manifold*

$$X \subset S^{N-1}_{\mathbb{C},+}$$

is a morphism of C^*-algebras of the following type,

$$\pi : C(X) \to M_K(C(T))$$

having its values in a random matrix algebra.

As a first observation, when X happens to be classical, we can take $K = 1$ and $T = X$, and we have a faithful model for our manifold, namely $id : C(X) \to M_1(C(X))$.

In general, we will be looking of course for faithful models for our manifolds, or at least for models having some suitable, weaker faithfulness properties. For this purpose we cannot use of course $K = 1$, and the smallest value $K \in \mathbb{N}$ doing the job, if any, will correspond somehow to the "degree of noncommutativity" of our manifold.

Before getting into all this, we would like to clarify a few more abstract issues. As mentioned above, the C^*-algebras of type $B = M_K(C(T))$ are called "random matrix algebras". The reason for this is the fact that most of the interesting compact spaces T come by definition with a natural probability measure of them. Thus, B is a subalgebra of the algebra $B' = M_K(L^\infty(T))$, usually known as a "random matrix algebra".

This perspective is quite interesting for us, because most of our examples of manifolds $X \subset X^{N-1}_{\mathbb{C},+}$ appear as homogeneous spaces, and so are measured spaces too. Thus, we can further ask for our models $C(X) \to M_K(C(T))$ to extend into models of type $L^\infty(X) \to M_K(L^\infty(T))$, which can help in connection with integration problems.

In short, time now to talk about L^∞-functions, in the noncommutative setting.

In order to discuss all this, we will need some basic von Neumann algebra theory, coming as a complement to the basic C^*-algebra theory from section 1 above:

Theorem 12.5. *The von Neumann algebras, which are the $*$-algebras $A \subset B(H)$ closed under the weak topology, making each $T \to Tx$ continuous, are as follows:*

1. *They are C^*-algebras. Also, they are exactly the $*$-algebras of operators $A \subset B(H)$ which are equal to their bicommutant, $A = A''$.*
2. *In the commutative case, these are the algebras of type $A = L^\infty(X)$, with X measured space, represented on $H = L^2(X)$, up to a multiplicity.*
3. *If we write the center as $Z(A) = L^\infty(X)$, then we have a decomposition of type $A = \int_X A_x \, dx$, with the fibers A_x having trivial center, $Z(A_x) = \mathbb{C}$.*
4. *The factors, $Z(A) = \mathbb{C}$, can be fully classified in terms of* II_1 *factors, which are those satisfying* $\dim A = \infty$*, and having a faithful trace $tr : A \to \mathbb{C}$.*

Proof. This is something quite heavy, the idea being as follows:

(1) The first assertion is clear, and the second one is von Neumann's bicommutant theorem, whose proof uses elementary Hilbert space theory.

(2) It is clear, via basic measure theory, that $L^\infty(X)$ is indeed a von Neumann algebra on $H = L^2(X)$. The converse can be proved as well, by using spectral theory.

(3) This is von Neumann's reduction theory main result, whose statement is already quite hard to understand, and whose proof uses advanced functional analysis.

(4) This is heavy, due to Murray-von Neumann and Connes, the idea being that the other factors can be basically obtained via crossed product constructions. □

All the above is of course very brief. We recommend here the original papers of von Neumann and Connes, starting for instance with [79], [91], and then [50], [51].

We can now extend our noncommutative space setting, as follows:

Theorem 12.6. *Consider the category of "noncommutative measure spaces", having as objects the pairs (A, tr) consisting of a von Neumann algebra with a faithful trace, and with the arrows reversed, which amounts in writing $A = L^\infty(X)$ and $tr = \int_X$.*

1. *The category of usual measured spaces embeds into this category, and we obtain in this way the objects whose associated von Neumann algebra is commutative.*
2. *Each C^*-algebra given with a trace produces as well a noncommutative measure space, by performing the GNS construction, and taking the weak closure.*
3. *In what regards the finitely generated group duals, or more generally the compact matrix quantum groups, the corresponding identification is injective.*
4. *Even more generally, for noncommutative algebraic manifolds having an integratiuon functional, like the spheres, the identification is injective.*

Proof. This is clear indeed from the basic properties of the GNS construction, from Theorem 12.2, and from the general theory from Theorem 12.5. □

Before getting into matrix modelling questions, we would like to formulate the following result, that we announced long ago, in section 1 above, but had not discussed yet:

Theorem 12.7. *We have von Neumann algebras, with traces, as follows,*

$$\begin{tikzcd}
L^\infty(S) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & L^\infty(T) \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
L^\infty(U) \arrow[r, leftrightarrow] & L^\infty(K)
\end{tikzcd}$$

with $L^\infty(S) \subset L^\infty(U)$ being obtained by taking the first row algebra.

Proof. This follows indeed from the results that we already have, and notably from those in section 7 above, by using the general formalism from Theorem 12.6. □

We should mention that it is quite unclear on how to go further, in this direction. Our belief is that our quadruplets (S, T, U, K) can be axiomatized directly in terms of the associated von Neumann algebras, but we do not know so far how to do this.

In relation now with the modelling questions, we can now go ahead with our program, and discuss von Neumann algebraic extensions. We have the following result:

Theorem 12.8. *Given a matrix model $\pi : C(X) \to M_K(C(T))$, with both X, T being assumed to have integration functionals, the following are equivalent:*

(1) *π is stationary, in the sense that $\int_X = (tr \otimes \int_T)\pi$.*
(2) *π produces an inclusion $\pi' : C_{red}(X) \subset M_K(X(T))$.*
(3) *π produces an inclusion $\pi'' : L^\infty(X) \subset M_K(L^\infty(T))$.*

Moreover, in the quantum group case, these conditions imply that π is faithful.

Proof. This is standard functional analysis. Consider indeed the following diagram, with all solid arrows being the canonical maps between the algebras concerned:

$$\begin{tikzcd}
M_K(C(T)) \arrow[rr] & & M_K(L^\infty(T)) \\
C(X) \arrow[u, "\pi"] \arrow[r] & C_{red}(X) \arrow[ul, dotted, "\pi'"'] \arrow[r] & L^\infty(X) \arrow[u, dotted, "\pi''"']
\end{tikzcd}$$

With this picture in hand, the implications $(1) \iff (2) \iff (3)$ are all clear, coming from the basic properties of the GNS construction, and of the von Neumann algebras.

As for the last assertion, this is something more subtle, coming from the fact that if $L^\infty(G)$ is of type I, as required by (3), then G must be coamenable. See [82]. $\square$

The above result raises a number of interesting questions, notably in what regards the extension of the last assertion, to the case of more general homogeneous spaces.

Before going further, we would like to record as well the following key result regarding the matrix models, valid so far in the quantum group case only:

Theorem 12.9. *Consider a matrix model $\pi : C(G) \to M_K(C(T))$ for a closed subgroup $G \subset U_N^+$, with T being assumed to be a compact probability space.*

(1) *There exists a smallest subgroup $G' \subset G$, producing a factorization of type $\pi : C(G) \to C(G') \to M_K(C(T))$. The algebra $C(G')$ is called Hopf image of π.*
(2) *When π is inner faithful, in the sense that $G = G'$, we have the integration formula $\int_G = \lim_{k\to\infty} \sum_{r=1}^k \varphi^{*r}$, where $\varphi = (tr \otimes \int_T)\pi$, and $\phi * \psi = (\phi \otimes \psi)\Delta$.*

Proof. All this is well-known, but quite specialized, the idea being as follows:

(1) This follows by dividing the algebra $C(G)$ by a suitable ideal, namely the Hopf ideal generated by the kernel of the matrix model map $\pi : C(G) \to M_K(C(T))$.

(2) This follows by suitably adapting Woronowicz's proof for the existence and formula of the Haar integration functional from [99], to the matrix model situation.

For a detailed discussion of these topics, we refer to [13], [14]. $\square$

The above result is quite important, for a number of reasons. Indeed, as a main application of it, while the existence of a faithful matrix model $\pi : C(G) \subset M_K(C(T))$ forces the C^*-algebra $C(G)$ to be of type I, and so G to be coamenable, as already mentioned in the proof of Theorem 12.8 above, there is no known restriction coming from the existence of an inner faithful model $\pi : C(G) \to M_K(C(T))$. See [13], [47].

In the general manifold setting, talking about such things is in general not possible, unless our manifold X has some extra special structure, as for instance being an homogeneous space, in the spirit of the spaces discussed in sections 10-11 above.

However, the work on this subject is basically missing, at least so far.

Let us go back now to our basic notion of a matrix model, from Definition 12.4 above, and develop some more general theory, in that setting. We first have:

Theorem 12.10. *A 1×1 model for a manifold $X \subset S^{N-1}_{\mathbb{C},+}$ must come from a map*

$$p : T \to X_{class} \subset X$$

and π is faithful precisely when $X = X_{class}$, and when p is surjective.

Proof. According to our conventions, a 1×1 model for a manifold $X \subset S^{N-1}_{\mathbb{C},+}$ is simply a morphism of C^*-algebras $\pi : C(X) \to C(T)$. Now since the algebra $C(T)$ is commutative, this morphism must factorize through the abelianization of $C(X)$, as follows:

$$\pi : C(X) \to C(X_{class}) \to C(T)$$

Thus, our morphism π must come by transposition from a map p, as in the statement. As for the last assertion, this is clear as well, from the functoriality of $p \to \pi$. $\square$

In order to generalize the above trivial fact, we use the following definition:

Definition 12.11. *Let $X \subset S^{N-1}_{\mathbb{C},+}$. We define a closed subspace $X^{(K)} \subset X$ by*

$$C(X^{(K)}) = C(X)/J_K$$

where J_K is the common null space of matrix representations of $C(X)$, of size $L \leq K$,

$$J_K = \bigcap_{L \leq K} \bigcap_{\pi : C(X) \to M_L(\mathbb{C})} \ker(\pi)$$

and we call $X^{(K)}$ the "part of X which is realizable with $K \times K$ models".

As a basic example here, the first such space, at $K = 1$, is the classical version:

$$X^{(1)} = X_{class}$$

Observe that we have embeddings of noncommutative spaces, as follows:

$$X^{(1)} \subset X^{(2)} \subset X^{(3)} \ldots\ldots \subset X$$

As a first result now on these spaces, we have the following well-known fact:

Theorem 12.12. *The increasing union of noncommutative spaces*

$$X^{(\infty)} = \bigcup_{K \geq 1} X^{(K)}$$

equals X precisely when the algebra $C(X)$ is residually finite dimensional.

Proof. This is something well-known, coming from the general theory from [63]. We refer to [44] for a recent paper on this topic, in the context of the quantum groups. □

Getting back now to the case $K < \infty$, we first have, following [14]:

Proposition 12.13. *Consider an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$.*

1. *Given a closed subspace $Y \subset X \subset S^{N-1}_{\mathbb{C},+}$, we have $Y \subset X^{(K)}$ precisely when any irreducible representation of $C(Y)$ has dimension $\leq K$.*
2. *In particular, we have $X^{(K)} = X$ precisely when any irreducible representation of $C(X)$ has dimension $\leq K$.*

Proof. This follows from the general theory in [63], as follows:

(1) If any irreducible representation of $C(Y)$ has dimension $\leq K$, then we have $Y \subset X^{(K)}$, because the irreducible representations of a C^*-algebra separate its points [63].

Conversely, assuming $Y \subset X^{(K)}$, it is enough to show that any irreducible representation of the algebra $C(X^{(K)})$ has dimension $\leq K$. But this follows as in [63].

(2) This follows indeed from (1). □

The connection with the previous considerations comes from:

Theorem 12.14. *If $X \subset S^{N-1}_{\mathbb{C},+}$ has a faithful matrix model*

$$C(X) \to M_K(C(T))$$

then we have $X = X^{(K)}$.

Proof. This follows from the above and from standard representation theory from [63]. For full details on all this, we refer to [14]. □

We now discuss the universal $K \times K$-matrix model, a C^*-algebra analogue of the character varieties for discrete groups or finite dimensional algebras [73]:

Theorem 12.15. *Given $X\subset S^{N-1}_{\mathbb C,+}$ algebraic, the category of its $K\times K$ matrix models, with $K\geq1$ being fixed, has a universal object as follows:*

$$\pi_K:C(X)\to M_K(C(T_K))$$

That is, if $\rho:C(X)\to M_K(C(T))$ is a matrix model, we have a diagram of type

$$\begin{tikzcd}
C(X) \arrow[rr, "\pi"] \arrow[dr, "\rho"'] & & M_K(C(T_K)) \arrow[dl] \\
& M_K(C(T)) &
\end{tikzcd}$$

where the map on the right is unique and arises from a continuous map $T\to T_K$.

Proof. Consider the universal commutative C^*-algebra generated by elements $x_{ij}(a)$, with $1\leq i,j\leq K$, $a\in\mathcal O(X)$, subject to the relations ($a,b\in\mathcal O(X)$, $\lambda\in\mathbb C$, $1\leq i,j\leq K$):

$$x_{ij}(a+\lambda b)=x_{ij}(a)+\lambda x_{ij}(b)$$

$$x_{ij}(ab)=\sum_kx_{ik}(a)x_{kj}(b)$$

$$x_{ij}(1)=\delta_{ij}$$

$$x_{ij}(a)^*=x_{ji}(a^*)$$

This is indeed well-defined because of the following relations:

$$\sum_l\sum_kx_{ik}(z_l^*)x_{ki}(z_l)=1$$

Let T_K be the spectrum of this C^*-algebra. Since X is algebraic, we have:

$$\pi:C(X)\to M_K(C(T_K)),\ \pi(z_k)=(x_{ij}(z_k))$$

By construction of T_K and π, we have the universal matrix model. See [14]. $\square$

Getting now to the case of the algebraic manifolds, we first have here:

Proposition 12.16. *Let $X\subset S^{N-1}_{\mathbb C,+}$ with X algebraic and $X_{class}\neq\emptyset$, and let*

$$\pi:C(X)\to M_K(C(T_K))$$

be the universal matrix model. Then we have

$$C(X^{(K)})=C(X)/Ker(\pi)$$

and hence $X=X^{(K)}$ if and only if X has a faithful $K\times K$-matrix model.

Proof. We have to show that $Ker(\pi) = J_K$, the latter ideal being the intersection of the kernels of all matrix representations $C(X) \to M_L(\mathbb{C})$, for any $L \leq K$.

For $a \notin Ker(\pi)$, we see that $a \notin J_K$ by evaluating at an appropriate element of T_K.

Conversely, assume that we are given $a \in Ker(\pi)$. Let $\rho : C(X) \to M_L(\mathbb{C})$ be a representation with $L \leq K$, and let $\varepsilon : C(X) \to \mathbb{C}$ be a representation. We can extend ρ to a representation $\rho' : C(X) \to M_K(\mathbb{C})$ by letting, for any $b \in C(X)$:

$$\rho'(b) = \begin{pmatrix} \rho(b) & 0 \\ 0 & \varepsilon(b) I_{K-L} \end{pmatrix}$$

The universal property of the universal matrix model yields that $\rho'(a) = 0$, since $\pi(a) = 0$. Hence $\rho(a) = 0$. We thus have $a \in J_K$, and $Ker(\pi) \subset J_K$, and the first statement is proved. The last statement follows from the first one. See [14]. □

Next, we have the following result, also from [14]:

Proposition 12.17. *Let $X \subset S^{N-1}_{\mathbb{C},+}$ be algebraic, and satisfying $X_{class} \neq \emptyset$. Then $X^{(K)}$ is algebraic as well.*

Proof. We keep the notations above, and consider the following map:

$$\pi_0 : \mathcal{O}(X) \to M_K(C(T_K)) \quad , \quad z_l \to (x_{ij}(z_l))$$

This induces a $*$-algebra map, as follows:

$$\tilde{\pi_0} : C^*(\mathcal{O}(X)/Ker(\pi_0)) \to M_K(C(T_K))$$

We need to show that $\tilde{\pi_0}$ is injective. For this purpose, observe that the universal model factorizes as follows, where p is canonical surjection:

$$\pi : C(X) \xrightarrow{p} C^*(\mathcal{O}(X)/Ker(\pi_0)) \xrightarrow{\tilde{\pi_0}} M_K(C(T_K))$$

We therefore obtain $Ker(\pi) = Ker(p)$, and hence, according to the previous proposition, we conclude that:

$$C(X^{(K)}) = C(X)/Ker(p) = C^*(\mathcal{O}(X)/Ker(\pi_0))$$

Thus $X^{(K)}$ is indeed algebraic. Since $\mathcal{O}(X)/Ker(\pi_0)$ is isomorphic to a $*$-subalgebra of $M_K(C(T_K))$, it satisfies the standard Amitsur-Levitski polynomial identity:

$$S_{2K}(x_1, \ldots, x_{2K}) = 0$$

Byy density, so does $C^*(\mathcal{O}(X)/Ker(\pi_0))$. Hence any irreducible representation of $C^*(\mathcal{O}(X)/Ker(\pi_0))$ has dimension $\leq K$, by [63]. Thus if $a \in C^*(\mathcal{O}(X)/Ker(\pi_0))$ is a nonzero element, we can, by the same reasoning as in the proof of the previous proposition, find a representation $\rho : C^*(\mathcal{O}(X)/Ker(\pi_0)) \to M_K(\mathbb{C})$ such that $\rho(a) \neq 0$ (because a given algebra map $\varepsilon : C(X) \to \mathbb{C}$ induces an algebra map $C(T_K) \to \mathbb{C}$, $x_{ij}(a) \mapsto \delta_{ij}\varepsilon(a)$, which enables us to extend representations similarly as before). By construction the universal model space yields an algebra map $M_K(C(T_K)) \to M_K(\mathbb{C})$ whose composition with $\tilde{\pi_0}p = \pi$ is ρp, so $\tilde{\pi_0}(a) \neq 0$, and $\tilde{\pi_0}$ is injective. □

Summarizing, we have proved the following result:

Theorem 12.18. *Let $X\subset S^{N-1}_{\mathbb{C},+}$ be algebraic, satisfying $X_{class}\neq\emptyset$. Then we have an increasing sequence of algebraic submanifolds*

$$X_{class}=X^{(1)}\subset X^{(2)}\subset X^{(3)}\subset\ldots\ldots\subset X$$

where $C(X^{(K)})\subset M_K(C(T_K))$ is obtained by factorizing the universal matrix model.

Proof. This follows indeed from the above results. See [14]. □

Let us discuss how the half-liberation operation, which is connected to $X^{(2)}$. We restrict the attention to the real case. The half-classical version is constructed as follows:

Definition 12.19. *The half-classical version of a noncommutative real compact algebraic manifold X is the intermediate noncommutative manifold $X^\times\subset X^*\subset X$ given by:*

$$C(X^*)=C(X)\Big/\left\langle abc=cba\Big|\forall a,b,c\in\{x_i\}\right\rangle$$

We say that X is half-classical when $X=X^*$.

In order to understand the structure of X^*, we use an old matrix model method, which goes back to [37], and then to [36]. This is based on the following observation:

Proposition 12.20. *For any $z\in\mathbb{C}^N$, the matrices*

$$X_i=\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}$$

are self-adjoint, and half-commute.

Proof. The matrices X_i are indeed self-adjoint, and their products are given by:

$$X_iX_j=\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}\begin{pmatrix}0&z_j\\ \bar{z}_j&0\end{pmatrix}=\begin{pmatrix}z_i\bar{z}_j&0\\ 0&\bar{z}_iz_j\end{pmatrix}$$

$$X_iX_jX_k=\begin{pmatrix}z_i\bar{z}_j&0\\ 0&\bar{z}_iz_j\end{pmatrix}\begin{pmatrix}0&z_k\\ \bar{z}_k&0\end{pmatrix}=\begin{pmatrix}0&z_i\bar{z}_jz_k\\ \bar{z}_iz_j\bar{z}_k&0\end{pmatrix}$$

The latter quantity being symmetric in i,k, we have $X_iX_jX_k=X_kX_jX_i$, as desired. □

In order to connect the algebra of the classical coordinates z_i to that of the noncommutative coordinates X_i, we will need an abstract definition, as follows:

Definition 12.21. *Given a noncommutative polynomial $f\in\mathbb{R}<x_1,\ldots,x_N>$, we define a usual polynomial $f^\circ\in\mathbb{R}[z_1,\ldots,z_N,\bar{z}_1,\ldots,\bar{z}_N]$, by setting*

$$f=x_{i_1}x_{i_2}x_{i_3}x_{i_4}\ldots\implies f^\circ=z_{i_1}\bar{z}_{i_2}z_{i_3}\bar{z}_{i_4}\ldots$$

in the monomial case, and then by extending this correspondence, by linearity.

As a basic example here, the polynomial defining the free real sphere $S^{N-1}_{\mathbb{R},+}$ produces in this way the polynomial defining the complex sphere $S^{N-1}_{\mathbb{C}}$:

$$f = x_1^2 + \ldots + x_N^2 \implies f^\circ = |z_1|^2 + \ldots + |z_N|^2$$

Also, given a polynomial $f \in \mathbb{R} < x_1, \ldots, x_N >$, we can decompose it into its even and odd parts, $f = g + h$, by putting into g/h the monomials of even/odd length. Observe that with $z = (z_1, \ldots, z_N)$, these odd and even parts are given by:

$$g(z) = \frac{f(z) + f(-z)}{2} \quad , \quad h(z) = \frac{f(z) - f(-z)}{2}$$

With these conventions, we have the following result:

Proposition 12.22. *Given a noncommutative real compact algebraic manifold X, coming from a family of noncommutative polynomials $\{f_\alpha\} \subset \mathbb{R} < x_1, \ldots, x_N >$, we have a morphism of unital C^*-algebras as follows,*

$$\pi : C(X) \to M_2(\mathbb{C}) \quad , \quad \pi(x_i) = \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

precisely when $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ belongs to the real algebraic manifold

$$Y = \left\{ z \in \mathbb{C}^N \Big| g_\alpha^\circ(z_1, \ldots, z_N) = h_\alpha^\circ(z_1, \ldots, z_N) = 0, \forall \alpha \right\}$$

where $f_\alpha = g_\alpha + h_\alpha$ is the even/odd decomposition of f_α.

Proof. Let X_i be the matrices in the statement. In order for $x_i \to X_i$ to define a morphism of algebras, these matrices must satisfy the equations defining X. Thus, the model space Z in the statement consists of those points $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ satisfying:

$$f_\alpha(X_1, \ldots, X_N) = 0 \quad , \quad \forall \alpha$$

We have already seen, in the proof of Proposition 12.20, the formulae of the products X_iX_j and $X_iX_jX_k$. In general, the matrices X_i multiply as follows:

$$X_{i_1}X_{j_1} \ldots X_{i_k}X_{j_k} = \begin{pmatrix} z_{i_1}\bar{z}_{j_1} \ldots z_{i_k}\bar{z}_{j_k} & 0 \\ 0 & \bar{z}_{i_1}z_{j_1} \ldots \bar{z}_{i_k}z_{j_k} \end{pmatrix}$$

$$X_{i_1}X_{j_1} \ldots X_{i_k}X_{j_k}X_{i_{k+1}} = \begin{pmatrix} 0 & z_{i_1}\bar{z}_{j_1} \ldots z_{i_k}\bar{z}_{j_k}z_{i_{k+1}} \\ \bar{z}_{i_1}z_{j_1} \ldots \bar{z}_{i_k}z_{j_k}\bar{z}_{i_{k+1}} & 0 \end{pmatrix}$$

We therefore obtain, in terms of the even/odd decomposition $f_\alpha = g_\alpha + h_\alpha$:

$$f_\alpha(X_1, \ldots, X_N) = \begin{pmatrix} g_\alpha^\circ(z_1, \ldots, z_N) & h_\alpha^\circ(z_1, \ldots, z_N) \\ \overline{h_\alpha^\circ(z_1, \ldots, z_N)} & \overline{g_\alpha^\circ(z_1, \ldots, z_N)} \end{pmatrix}$$

Thus, we obtain the equations for Y from the statement. ∎

As a first consequence, of theoretical interest, a necessary condition for X to exist is that the manifold $Y\subset\mathbb C^N$ constructed above must be compact.

In order to discuss modelling questions, we will need as well:

Definition 12.23. *Assuming that we are given a noncommutative complex compact algebraic manifold Z, appearing as follows, for certain polynomials $f_\alpha\in\mathbb C<z_1,\ldots,z_N>$,*

$$C(Z)=C^*\left(z_1,\ldots,z_N\Big|f_\alpha(z_1,\ldots,z_N)=0\right)$$

we define the projective version of Z to be the quotient space $Z\to PZ$ corresponding to the subalgebra $C(PZ)\subset C(Z)$ generated by the variables $x_{ij}=z_iz_j^*$.

The relation with the half-classical manifolds comes from the fact that the projective version of a half-classical manifold is classical. Indeed, from $abc=cba$ we obtain:

$$ab\cdot cd=(abc)d=(cba)d=c(bad)=c(dab)=cd\cdot ab$$

Finally, let us call "matrix model" any morphism of unital C^*-algebras $f:A\to B$, with target algebra $B=M_K(C(Y))$, with $K\in\mathbb N$, and Y being a compact space.

Following [36], we have the following result:

Proposition 12.24. *Given a half-classical manifold X which is symmetric, in the sense that all its defining polynomials f_α are even, its universal 2×2 antidiagonal model,*

$$\pi:C(X)\to M_2(C(Y))$$

where Y is the manifold constructed in Proposition 12.22, is faithful. In addition, the construction $X\to Y$ is such that X exists precisely when Y is compact.

Proof. We can proceed as in [36]. Indeed, the universal model π in the statement induces, at the level of projective versions, a certain representation $C(PX)\to M_2(C(PY))$.

By using the multiplication formulae from the proof of Proposition 12.22, the image of this representation consists of diagonal matrices, and the upper left components of these matrices are the standard coordinates of PY. Thus, we have an isomorphism $PX\simeq PY$, and we can conclude as in [36], by using a grading trick. See [36]. □

As a first observation, this result shows that when X is symmetric, we have $X^*\subset X^{(2)}$. Of course, going beyond this observation is an interesting problem.

In what follows, we will rather need a more detailed version of the above result. For this purpose, we can use the following definition:

Definition 12.25. *Associated to any compact manifold $Y\subset\mathbb C^N$ is the real compact half-classical manifold $[Y]$, having as coordinates the following variables,*

$$X_i=\begin{pmatrix}0&z_i\\ \bar z_i&0\end{pmatrix}$$

where $z_1,\ldots,z_N$ are the standard coordinates on Y. In other words, $[Y]$ is given by the fact that $C([Y])\subset M_2(C(Y))$ is the algebra generated by these matrices.

Here the fact that $[Y]$ is indeed half-classical follows from the results above. As for the fact that $[Y]$ is indeed algebraic, this follows from Proposition 12.24.

We can now reformulate the result in Proposition 12.24, as follows:

Theorem 12.26. *The symmetric half-classical manifolds X appear as follows:*

(1) *We have $X = [Y]$, for a certain conjugation-invariant subspace $Y \subset \mathbb{C}^N$.*
(2) *$PX = P[Y]$, and X is maximal with this property.*
(3) *In addition, we have an embedding $C([X]) \subset C(X) \rtimes \mathbb{Z}_2$.*

Proof. This follows from Proposition 12.24, with the embedding in (3) being constructed as in [36], by $x_i = z_i \otimes \tau$, where τ is the standard generator of $\mathbb{Z}_2$. See [36]. □

As a conclusion to all this, the half-classical geometry can be developed in a quite efficient way, at a technical level which is close to that of the classical one, by using 2×2 matrix models. There are of course higher analogues of all this developed, using $K \times K$ matrix models. We refer to [14], [36], [37] for more on these topics.

As an overall conclusion now, we have seen that the compact quantum groups lead to new classes of noncommutative algebraic manifolds, which look quite fundamental.

Importantly, these manifolds have some Riemannian features, including a bit of differential geometry, notably in the form of a Laplacian, and also, especially, due to the existence of an integration functional, which is explicitely computable.

All this is not very far from Connes' noncommutative geometry [52], and as explained above, our belief is that there should be some common unification ground for this.

In regards now with applications, the potential connections with quantum physics coming via the work of Jones [70], [71], [72] and Voiculescu [87], [88], [89], involving statistical mechanics, quantum field theories, and random matrices, must be understood.

All this is not done yet, but we have reasons to be quite optimistic here. Indeed, the whole modern theory of compact quantum groups was developed with the mathematical work of Jones and Voiculescu at its foundations, and by keeping a constant eye on what is going on, in subfactor theory and free probability. And all this remains valid, somehow by definition, for the noncommutative geometry theory developed here.

Next in line, a massive number of occurrences of operator algebras and related noncommutative manifolds in the context of modern mathematical physics come from the work of Connes and his collaborators [52], [53], [54], [55], [56], [57], [58], [59]. All this material is waiting to be studied, from our present point of view, mixing compact quantum groups, subfactors and free probability, and noncommutative algebraic manifolds.

The work of Connes and Kreimer on Feynman diagrams [58] is probably a good starting point here. Indeed, the Hopf algebra constructed by them is something of "Lie algebra" type, a bit as those of Drinfeld and Jimbo [64], [69], and looking for some kind of underlying compact quantum group in relation with all this makes sense.

This is something which is not done yet, but there are no reasons of thinking that this is not possible. After all, the compact quantum groups are well-connected to the planar algebra theory of Jones [72], which regards the Feynman diagrams from a quite advanced point of view, 3-dimensional, coming from TQFT and related theories [97].

To a certain extent, the material in the other above-mentioned papers, which is quite mathematical, might have some quantum group connections too. However, this is for the moment pure speculation, and nothing concrete is known so far.

Getting now to the real thing, all this is of course about the Standard Model. Thus, as a third step of the program, the papers of Chamseddine, Connes, Lott and Marcolli [41], [42], [43], [60] are waiting to be studied, from a quantum group perspective.

An interesting observation here, from the paper of Bhowmick, D'Andrea and Dabrowski [33], and their subsequent paper with Das [34], is that the gauge group of the Standard Model, when regarded à la Chamseddine-Connes, naturally extends into a "free gauge group", with the various U_N components replaced by their free versions U_N^+.

All this is quite old, from about 10 years ago, and getting beyond this observation is a key open problem, with no further advances on it, at least so far.

A natural idea here would be that of restricting the attention to the quark part, and so to U_3^+, and then using the fact, coming from [18], that PU_3^+ appears as a twist of the quantum permutation group S_9^+. Indeed, we would have here a connection between quarks and quantum permutations, which would be very interesting.

Mathematically speaking, the quantum permutation groups are indeed those compact quantum groups, and by far, whose theory is the most advanced, and which can potentially help, in connection with difficult questions in physics. In fact, while the study in the continuous case has been subject to some ups and lows, in what regards quantum permutations the theory has been relentlessly developed since Wang's 1998 paper [93], and is now at a decent technical level, not far from what the professional physicists are doing. From the latest wave of papers here, let us mention [38], [74], [80].

From a particle physics perspective, all this would be extremely interesting. Indeed, assuming that at least a bit of what has been said above makes sense, this would suggest that QCD, and perhaps the whole Standard Model, might be actually "twisted" in their present known form, and so perhaps waiting to be untwisted. Needless to say, all this is wild speculation, which remains to be investigated. By the way the problem is not new, the paper [18], potentially complementing [33], [34], being 10 years old as well.

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