An asymptotic study of nonlinear instability to Langmuir circulation in stratified shallow layers

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Abstract

The CL equations governing instability to Langmuir circulation (LC) are solved by three approximate methods, viz: a small-l asymptotic expansion where l is the spanwise wavenumber, a power series method, and a Galerkin method. Interest is focused on the CL2 instability mechanism to LC and how it is influenced by stratification throughout the layer in which LC live. Some results are provided to illustrate the CL2 instability and how it is affected by nonlinearities.

1 Introduction

Langmuir circulation (LC) is a system of counter-rotating vortices that forms below wind driven waves in the upper ocean when the wind speed exceeds 3 m/s (Leibovich, 1983) and occupies the region of fluid that is sheared by the wind. Moreover LC is made visible by its surface footprints as almost parallel streaks or windrows on the ocean surface, with spacings of up to hundreds of meters (Plueddemann *et al.*, 1996; Thorpe, 2004) and can extend for several kilometers in the direction of the wind (Thorpe, 2004). LC helps mix and form a region called the mixed layer (Langmuir, 1938) and in doing so this alters the variation with depth of density and temperature (Smith, 1992), on occasion to such a degree that the bottom of the layer is defined by a sharp change in temperature (density) known as a thermocline (pycnocline). Of interest in the present study is the role stratification plays on the evolution of LC in layers bounded by a thermocline.

The prevailing theory for LC is due to Craik & Leibovich (1976), who derived a set of evolution equations to describe them known as the CL equations. Two instability mechanisms to excite LC follow from the CL equations (Leibovich, 1980) and both rest upon the interaction between shear U' in the surface layer resulting from the wind and differential Lagrangian drift D' that results from the wave field. They are denoted CL1 and CL2. However CL2, which assumes that the drift does not vary cross stream to the wind, is considered the more likely instability to occur in Nature and is the mechanism studied in this paper.

Of course, to ensure the problem is well posed, boundary conditions must be specified at the free surface and some distance below it. Neumann conditions are an obvious choice but, when imposed on finite layers as opposed to infinite ones (in the sense of deep water waves), the linear least stable wavenumber l_c in usual circumstances is zero. This oddity was explained by Cox & Leibovich (1993), who noted that Neumann conditions ignore coupling between the perturbation flow and the extra stress it produces, implying that mixed boundary conditions that reflect that extra stress of magnitude γ, should be imposed. In doing so they found that l_c is nonzero when γ is nonzero and also that $l_c \ll 1$ when $\gamma \ll 1$. In view of that they chose to use perturbation methods to study the instability to CL2 of the simplest case $U' = D' = 1$ in the small l limit, followed by Hayes & Phillips (2016) who allowed D' and U' to be arbitrary functions of depth, while Hayes & Phillips (2017) studied the role of nonlinearities. In fact Cox & Leibovich (1993) also allowed for thermal stratification of slope $H' = 1$ and magnitude S whereas Hayes & Phillips (2016, 2017) set $S = 0$.

Our object here is to consider the role of nonlinearities when S is nonzero and D', U', H' are arbitrary functions of depth. The governing equations are stated in §2. Then in §3 and §4 we outline solution methods for the respective linearised and nonlinear problems. In §5 we discuss our results on how the parameters and nonlinearities affect the CL2 instability to LC in the small l limit when S is nonzero. This includes a revisit of Figure 3 of Cox & Leibovich (1993). In §6 we conclude this paper and discuss some possibilities for further work on LC.

2 Problem description

2.1 CL2 equations

The CL2 equations follow from perturbations to the CL equations, where the perturbation velocity $\mathbf{u} = (u, v, w)$ and perturbation temperature θ are each defined for position $\mathbf{x} = (x, y, z)$ and time t. We take the x axis to be in the direction of the imposed shear, the y axis is in the spanwise direction, and the z axis is in the vertical direction. The flow is assumed to be independent of x. In dimensionless form, the CL2 equations are then (Craik & Leibovich, 1976)

$$(\frac{\partial}{\partial t} - \nabla^2)\nabla^2\psi = RD'\frac{\partial u}{\partial y} - S\frac{\partial \theta}{\partial y} + J(\psi, \nabla^2\psi), \qquad (2.1.1)$$

$$(\frac{\partial}{\partial t} - \nabla^2)u = \frac{\partial \psi}{\partial y}U' + J(\psi, u), \qquad (2.1.2)$$

$$(\frac{\partial}{\partial t} - \tau\nabla^2)\theta = \frac{\partial \psi}{\partial y}H' + J(\psi, \theta) \qquad (2.1.3)$$

where J is the Jacobian $J(a,b) = a_y b_z - a_z b_y$. To satisfy the continuity equation the stream function ψ is defined by $v = \psi_z$ and $w = -\psi_y$. We further have that U' and H' must satisfy

$$(\frac{\partial}{\partial T} - \nabla^2)U = F, \quad (\frac{\partial}{\partial T} - \nabla^2)H = G \tag{2.1.4}$$

where T and t are disparate time scales and F, G are due to body forces and heat sources respectively. The differential drift D' results from the Stokes drift whose details depend on the wavefield. We can thus take D', U', H' to be arbitrary functions of z. Herein we allow D', U', H' to each be arbitrary polynomials of z

$$D' = \sum_{n=0}^{N} A_n z^n, \;\; U' = \sum_{n=0}^{N} B_n z^n, \;\text{ and }\; H' = \sum_{n=0}^{N} C_n z^n \tag{2.1.5}$$

where A_n, B_n and C_n are arbitrary constant coefficients. The Rayleigh number is denoted by R, the magnitude of the stratification is denoted by S, and $\tau \neq 0$ is an inverse Prandtl number. Nonlinearities are accounted for through the Jacobian J. When nonlinearities are assumed to be small we discard J to yield the linearised CL2 equations. When $S = 0$, equations (2.1.1), (2.1.2) are those used in Hayes & Phillips (2017).

2.2 Boundary conditions

We will use mixed boundary conditions on the top and bottom of the layer of fluid that are similar to those introduced by Cox & Leibovich (1993)

$$\frac{\partial^2\psi}{\partial z^2} + \gamma_1\frac{\partial\psi}{\partial z} = \frac{\partial u}{\partial z} + \gamma_2 u = \frac{\partial\theta}{\partial z} + \beta_1\theta = \psi = 0 \;\text{ on }\; z = 0, \tag{2.2.1}$$

$$\frac{\partial^2\psi}{\partial z^2} + \gamma_3\frac{\partial\psi}{\partial z} = \frac{\partial u}{\partial z} + \gamma_4 u = \frac{\partial\theta}{\partial z} + \beta_2\theta = \psi = 0 \;\text{ on }\; z = -1 \tag{2.2.2}$$

where γ_i, β_j for $i = 1, 2, 3, 4$, $j = 1, 2$ are constants. We set $z = 0$ at the top of the layer and $z = -1$ at the bottom of the layer.

3 Linear methods

3.1 Linear perturbation solution

We seek a perturbation solution to the linearised version of the CL2 equations (2.1.1), (2.1.2), (2.1.3) with boundary conditions (2.2.1) and (2.2.2) using $l \ll 1$ as a small parameter. This calculation is an extension of the work of Cox & Leibovich (1993) and Hayes & Phillips (2016). We assume

$$\psi = \mathrm{i}\sum_{k=0}^{\infty} l^{2k+1}\psi_{2k+1}\mathrm{e}^{\mathrm{i}ly}\mathrm{e}^{\sigma t}, \;\; u = \sum_{k=0}^{\infty} l^{2k}u_{2k}\mathrm{e}^{\mathrm{i}ly}\mathrm{e}^{\sigma t}, \;\; \theta = \sum_{k=0}^{\infty} l^{2k}\theta_{2k}\mathrm{e}^{\mathrm{i}ly}\mathrm{e}^{\sigma t}, \tag{3.1.1}$$

$$\sigma = \sum_{k=0}^{\infty} l^{2k+2}\sigma_{2k+2}, \;\; R = \sum_{k=0}^{\infty} l^{2k}R_{2k}, \tag{3.1.2}$$

$$\gamma_i = l^4\overline{\gamma}_i, \;\; \beta_i = \sum_{k=0}^{\infty}\beta_{i,2k}l^{2k}. \tag{3.1.3}$$

Here ψ_{2k+1}, u_{2k}, and θ_{2k} are each functions of z. To proceed we substitute the above expansions (3.1.1–3.1.3) into the linear CL2 equations and boundary conditions, equate like powers of l, and solve the resulting equations at successive orders in l. To equate like powers of l we can use the Cauchy product formula (Hardy, 1949)

$$\sum_{m=0}^{\infty} a_m x^m \sum_{n=0}^{\infty} b_n x^n = \sum_{m=0}^{\infty}\sum_{n=0}^{m} a_{m-n}b_n x^m. \tag{3.1.4}$$

In §3.1.1 we take the calculation to $O(l^4)$. In §3.1.2 an algorithm is derived so we can take the perturbation solution to $O(l^P)$ for any integer $P > 0$ within computational limits. The algorithm can then be coded into Maple. The solutions we obtain may then be used to validate more general numerical calculations. The c_i and $c_{i,j}$ appearing here are given in the Appendix. It turns out that the linear perturbation solution separates into two separate cases. We have case I: $-\beta_{1,0} + \beta_{2,0}\beta_{1,0} + \beta_{2,0} \neq 0$ and case II: $-\beta_{1,0} + \beta_{2,0}\beta_{1,0} + \beta_{2,0} = 0$. This becomes evident when applying the boundary conditions for θ_0.

3.1.1 The first few orders

At $O(l^0)$ we have

$$u_0'' = 0 \tag{3.1.5}$$

with boundary conditions

$$u_0' = 0 \;\text{ on }\; z = 0, -1. \tag{3.1.6}$$

So

$$u_0 = c_0 \tag{3.1.7}$$

where c_0 is an arbitrary constant. Without affecting σ we may let

$$u_0 = 1. \tag{3.1.8}$$

Also at $O(l^0)$ we have

$$\tau\theta_0'' = 0 \tag{3.1.9}$$

with boundary conditions

$$\theta_0' + \beta_{1,0}\theta_0 = 0 \;\text{ on }\; z = 0 \tag{3.1.10}$$

and

$$\theta_0' + \beta_{2,0}\theta_0 = 0 \;\text{ on }\; z = -1. \tag{3.1.11}$$

For case I we find

$$\theta_0 = c_3 = 0. \tag{3.1.12}$$

For case II we find

$$\theta_0 = c_3(-\beta_{1,0}z + 1) = c_3\hat{\theta}_0 \tag{3.1.13}$$

where c_3 is an arbitrary constant.
At $O(l^1)$ we have

$$\psi_1'''' = -D'R_0u_0 + S\theta_0 \tag{3.1.14}$$

with boundary conditions

$$\psi_1'' = \psi_1 = 0 \;\text{ on }\; z = 0, -1. \tag{3.1.15}$$

Solving (3.1.14), (3.1.15) gives

$$\begin{aligned}\psi_1 \;&=\; -R_0u_0\iiiint^z D'\,dz\,dz\,dz\,dz + S\iiiint^z \theta_0\,dz\,dz\,dz\,dz + c_4\frac{z^3}{6} + c_5\frac{z^2}{2} + c_6z + c_7\\ &= -R_0u_0\tilde{\psi}_1 + S\hat{\psi}_1c_3\end{aligned} \tag{3.1.16}$$

where $\tilde{\psi}_1$, $\hat{\psi}_1$ are devoid of R_0, S, and c_3. Notice here that since multiple layers of LC occurred with D' as a linear function of z in Hayes & Phillips (2016), this then means that here even with $D' = U' = H' = 1$ we can have multiple layers of LC due to θ_0 being linear in z.
At $O(l^2)$ we have

$$u_2'' = U'\psi_1 + u_0\sigma_2 + u_0 \tag{3.1.17}$$

with boundary conditions

$$u_2' = 0 \;\text{ on }\; z = 0, -1. \tag{3.1.18}$$

Solving (3.1.17), (3.1.18) gives

$$u_2 = \iint^z U'\psi_1\,dz\,dz + u_0(\sigma_2 + 1)\frac{z^2}{2} + c_8z + c_9. \tag{3.1.19}$$

The constant of integration c_9 is chosen so that there is no net flux of fluid due to the perturbation flow

$$\int_{-1}^{0} u_2\,dz = 0. \tag{3.1.20}$$

Also at $O(l^2)$ we have

$$\tau\theta_2'' = \psi_1H' + \theta_0(\tau + \sigma_2) \tag{3.1.21}$$

with boundary conditions

$$\theta_2' + \beta_{1,0}\theta_2 + \beta_{1,2}\theta_0 = 0 \;\text{ on }\; z = 0 \tag{3.1.22}$$

and

$$\theta_2' + \beta_{2,0}\theta_2 + \beta_{2,2}\theta_0 = 0 \;\text{ on }\; z = -1. \tag{3.1.23}$$

We find

$$\theta_2 = \iint^z \psi_1\frac{H'}{\tau}\,dz\,dz + \iint^z \theta_0(\frac{\sigma_2}{\tau} + 1)\,dz\,dz + c_{10}z + c_{11}. \tag{3.1.24}$$

For case II we have $\theta_2 = \tilde{\theta}_2 + c_{11}\hat{\theta}_2$ where $\tilde{\theta}_2$ and $\hat{\theta}_2$ are independent of c_{11}. The boundary conditions lead to equations for σ_2. For case I we find

$$\sigma_2 = R_0 \int_{-1}^{0} U' \tilde{\psi}_1 \, dz - 1. \tag{3.1.25}$$

For case II there are two equations involving σ_2 which can be written as the matrix equation for $\mathbf{v} = (u_0, c_3)^T$

$$M\mathbf{v} = \mathbf{0} \tag{3.1.26}$$

where the elements of the 2×2 matrix M are given in the Appendix. For a nontrivial solution the determinant of M must be zero, $M_{1,1}M_{2,2} - M_{1,2}M_{2,1} = 0$. This leads to a quadratic equation

$$a\sigma_2^2 + b\sigma_2 + c = 0 \tag{3.1.27}$$

for σ_2 where a, b, c are in the Appendix. The quadratic formula gives

$$\sigma_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.1.28}$$

The matrix equation then yields

$$c_3 = -\frac{M_{1,1}u_0}{M_{1,2}}. \tag{3.1.29}$$

At $O(l^3)$ we have

$$\psi_3'''' = (2 + \sigma_2)\psi_1'' - (R_2u_0 + R_0u_2)D' + S\theta_2 \tag{3.1.30}$$

with boundary conditions

$$\psi_3'' = \psi_3 = 0 \text{ on } z = 0, -1. \tag{3.1.31}$$

Solving (3.1.30), (3.1.31) gives

$$\begin{aligned} \psi_3 \quad = \quad & (2 + \sigma_2) \iint^z \psi_1 \, dz\, dz - \iiiint^z D'(R_2u_0 + u_2R_0) \, dz\, dz\, dz\, dz \\ & + S \iiiint^z \theta_2 \, dz\, dz\, dz\, dz + c_{12}\frac{z^3}{6} + c_{13}\frac{z^2}{2} + c_{14}z + c_{15}. \end{aligned} \tag{3.1.32}$$

For case I we have $\psi_3 = \hat{\psi}_3 - R_2\tilde{\psi}_3 + S\breve{\psi}_3$ where $\hat{\psi}_3, \tilde{\psi}_3, \breve{\psi}_3$ are each independent of R_2 and S. For case II we have $\psi_3 = \hat{\psi}_3 - R_2\tilde{\psi}_3 + c_{11}\breve{\psi}_3$ where $\hat{\psi}_3, \tilde{\psi}_3, \breve{\psi}_3$ are each independent of R_2 and c_{11}. The dependence on S appears too complicated to isolate for case II.

At $O(l^4)$ we have

$$u_4'' = u_0\sigma_4 + u_2(1 + \sigma_2) + \psi_3 U' \tag{3.1.33}$$

with boundary conditions

$$u_4' + \overline{\gamma}_2 u_0 = 0 \text{ on } z = 0 \tag{3.1.34}$$

and

$$u_4' + \overline{\gamma}_4 u_0 = 0 \text{ on } z = -1. \tag{3.1.35}$$

Solving (3.1.33), (3.1.34), (3.1.35) gives

$$u_4 = \iint^z u_2(1 + \sigma_2) \, dz\, dz + \iint^z \psi_3 U' \, dz\, dz + u_0\sigma_4\frac{z^2}{2} + c_{16}z + c_{17} \tag{3.1.36}$$

where the constant of integration c_{17} is chosen so to exclude net mass transfer as before

$$\int_{-1}^{0} u_4 \, dz = 0. \tag{3.1.37}$$

Also at $O(l^4)$ we have

$$\tau\theta_4'' = \psi_3 H' + (\tau + \sigma_2)\theta_2 + \theta_0\sigma_4 \tag{3.1.38}$$

with boundary conditions

$$\theta_4' + \beta_{1,0}\theta_4 + \beta_{1,2}\theta_2 + \beta_{1,4}\theta_0 = 0 \text{ on } z = 0 \tag{3.1.39}$$

and

$$\theta_4' + \beta_{2,0}\theta_4 + \beta_{2,2}\theta_2 + \beta_{2,4}\theta_0 = 0 \text{ on } z = -1. \tag{3.1.40}$$

We find

$$\theta_4 = \iint^z \psi_3 \frac{H'}{\tau} \, dz\, dz + \frac{\sigma_2}{\tau} \iint^z \theta_2 \, dz\, dz + \frac{\sigma_4}{\tau} \iint^z \theta_0 \, dz\, dz + \iint^z \theta_2 \, dz\, dz + c_{18}z + c_{19}. \tag{3.1.41}$$

The boundary conditions lead to equations for σ_4. For case I we find

$$\sigma_4 = -\int_{-1}^{0} \frac{U'}{u_0}(\hat{\psi}_3 - R_2\tilde{\psi}_3 + S\breve{\psi}_3)\,dz + \overline{\gamma}_4 - \overline{\gamma}_2. \tag{3.1.42}$$

For case II there are two equations for σ_4 which can be written as the matrix equation for $\mathbf{w} = (\sigma_4, c_{11})^T$

$$N\mathbf{w} = \mathbf{q} \tag{3.1.43}$$

where the elements of the 2×2 matrix N and the elements of the 2×1 vector $\mathbf{q}$ are given in the Appendix. Solving this matrix equation then yields

$$c_{11} = \frac{N_{1,1}q_2 - N_{2,1}q_1}{N_{2,2}N_{1,1} - N_{2,1}N_{1,2}} \tag{3.1.44}$$

and

$$\sigma_4 = \frac{q_1}{N_{1,1}} - \frac{N_{1,2}}{N_{1,1}}\left(\frac{N_{1,1}q_2 - N_{2,1}q_1}{N_{2,2}N_{1,1} - N_{2,1}N_{1,2}}\right). \tag{3.1.45}$$

This calculation recovers Cox & Leibovich (1993) on setting $U' = D' = H' = 1$ and recovers Hayes & Phillips (2016) on setting $S = 0$.

3.1.2 Linear perturbation solution algorithm

At $O(l^{2j})$ for integer $j \geq 2$ we have

$$u_{2j}'' = \sum_{m=0}^{j-1} u_{2j-(2m+2)}\sigma_{2m+2} + u_{2j-2} + U'\psi_{2j-1} \tag{3.1.46}$$

with boundary conditions

$$u_{2j}' + \overline{\gamma}_2 u_{2j-4} = 0 \text{ on } z = 0 \tag{3.1.47}$$

and

$$u_{2j}' + \overline{\gamma}_4 u_{2j-4} = 0 \text{ on } z = -1. \tag{3.1.48}$$

Consistent with our progression above we choose the constant of integration so that

$$\int_{-1}^{0} u_{2j}\,dz = \delta_{0,j}u_0 \tag{3.1.49}$$

where $\delta_{i,j}$ is the Kronecker delta

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{3.1.50}$$

On solving for u_{2j} we find

$$\begin{aligned} u_{2j} \quad = \quad & \iint^{z} \sum_{m=0}^{j-1} u_{2j-(2m+2)}\sigma_{2m+2}\,dz\,dz + \iint^{z} u_{2j-2}\,dz\,dz \\ & + \iint^{z} U'\psi_{2j-1}\,dz\,dz + c_{0,j}z + c_{1,j}. \end{aligned} \tag{3.1.51}$$

Also at $O(l^{2j})$ for integer $j \geq 2$ we have

$$\theta_{2j}'' = \sum_{m=0}^{j-1} \theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau} + \theta_{2j-2} + \frac{H'}{\tau}\psi_{2j-1} \tag{3.1.52}$$

with boundary conditions

$$\theta_{2j}' + \sum_{m=0}^{j} \beta_{1,2m}\theta_{2j-2m} = 0 \text{ on } z = 0 \tag{3.1.53}$$

and

$$\theta_{2j}' + \sum_{m=0}^{j} \beta_{2,2m}\theta_{2j-2m} = 0 \text{ on } z = -1. \tag{3.1.54}$$

On solving for θ_{2j} we find

$$\begin{aligned}\theta_{2j} &= \iint^z \sum_{m=0}^{j-1} \theta_{2j-(2m+2)} \frac{\sigma_{2m+2}}{\tau}\, dz\, dz + \iint^z \theta_{2j-2}\, dz\, dz \\ &\quad + \iint^z \frac{H'}{\tau}\psi_{2j-1}\, dz\, dz + c_{2,j}z + c_{3,j}. \qquad (3.1.55)\end{aligned}$$

For case II we have $\theta_{2j} = \tilde{\theta}_{2j} + c_{3,j}\hat{\theta}_{2j}$ where $\tilde{\theta}_{2j}$ and $\hat{\theta}_{2j}$ are independent of $c_{3,j}$.
At $O(l^{2j+1})$ we have

$$\begin{aligned}\psi''''_{2j+1} &= -\sum_{m=0}^{j} D' u_{2j-2m}R_{2m} + S\theta_{2j} + \sum_{m=0}^{j-1} \psi''_{2j-2m-1}\sigma_{2m+2} \\ &\quad - \sum_{m=0}^{j-2} \psi_{2j-2m-3}\sigma_{2m+2} + 2\psi''_{2j-1} - \psi_{2j-3} \qquad (3.1.56)\end{aligned}$$

with boundary conditions

$$\psi''_{2j+1} + \overline{\gamma}_1\psi'_{2j-3} = \psi_{2j+1} = 0 \ \text{ on } \ z = 0 \qquad (3.1.57)$$

and

$$\psi''_{2j+1} + \overline{\gamma}_3\psi'_{2j-3} = \psi_{2j+1} = 0 \ \text{ on } \ z = -1. \qquad (3.1.58)$$

On solving for ψ_{2j+1} we find

$$\begin{aligned}\psi_{2j+1} &= -\iiiint^z \sum_{m=0}^{j} D' u_{2j-2m}R_{2m}\, dz\, dz\, dz\, dz + \iint^z \sum_{m=0}^{j-1} \psi_{2j-2m-1}\sigma_{2m+2}\, dz\, dz \\ &\quad - \iiiint^z \sum_{m=0}^{j-2} \psi_{2j-2m-3}\sigma_{2m+2}\, dz\, dz\, dz\, dz - \iiiint^z \psi_{2j-3}\, dz\, dz\, dz\, dz \\ &\quad +2\iint^z \psi_{2j-1}\, dz\, dz + S\iiiint^z \theta_{2j}\, dz\, dz\, dz\, dz + c_{4,j}\frac{z^3}{6} + c_{5,j}\frac{z^2}{2} + c_{6,j}z + c_{7,j}. \qquad (3.1.59)\end{aligned}$$

For case II we have $\psi_{2j+1} = \hat{\psi}_{2j+1} - R_{2j}\tilde{\psi}_{2j+1} + c_{3,j}\breve{\psi}_{2j+1}$ where $\hat{\psi}_{2j+1}$, $\tilde{\psi}_{2j+1}$, and $\breve{\psi}_{2j+1}$ are independent of R_{2j} and $c_{3,j}$. The boundary conditions lead to equations for σ_{2j}. For case I we find

$$\sigma_{2j} = -\frac{\overline{\gamma}_2}{u_0}u_{2j-4}|_{z=0} + \frac{\overline{\gamma}_4}{u_0}u_{2j-4}|_{z=-1} - \int_{-1}^{0} \frac{U'}{u_0}\psi_{2j-1}\, dz - \delta_{0,j-1}. \qquad (3.1.60)$$

For case II there are two equations for σ_{2j} which can be written as a matrix equation for $\mathbf{w} = (\sigma_{2j}, c_{3,j-1})^T$ as

$$N\mathbf{w} = \mathbf{q} \qquad (3.1.61)$$

where the elements of the 2×2 matrix N and the elements of the 2×1 vector $\mathbf{q}$ which here depend on j are given in the Appendix. Solving this matrix equation then yields

$$c_{3,j-1} = \frac{N_{1,1,j}q_{2,j} - N_{2,1,j}q_{1,j}}{N_{2,2,j}N_{1,1,j} - N_{2,1,j}N_{1,2,j}} \qquad (3.1.62)$$

and

$$\sigma_{2j} = \frac{q_{1,j}}{N_{1,1,j}} - \frac{N_{1,2,j}}{N_{1,1,j}}\left(\frac{N_{1,1,j}q_{2,j} - N_{2,1,j}q_{1,j}}{N_{2,2,j}N_{1,1,j} - N_{2,1,j}N_{1,2,j}}\right). \qquad (3.1.63)$$

The above calculation can then be coded in Maple within a loop. This calculation recovers Hayes & Phillips (2016) on setting $S = 0$.

3.2 Linear power series method

Consistent with the perturbation method above, we assume

$$\psi = \mathrm{i}A\mathrm{e}^{\mathrm{i}ly}\mathrm{e}^{\sigma t}, \ \ u = B\mathrm{e}^{\mathrm{i}ly}\mathrm{e}^{\sigma t}, \ \text{ and } \ \theta = C\mathrm{e}^{\mathrm{i}ly}\mathrm{e}^{\sigma t} \qquad (3.2.1)$$

where $A = A(z)$, $B = B(z)$, and $C = C(z)$. We substitute (3.2.1) into the linearised CL2 equations which leads to

$$A'''' - (2l^2 + \sigma)A'' + (l^4 + l^2\sigma)A + RD'Bl - SCl = 0, \tag{3.2.2}$$

$$B'' - (l^2 + \sigma)B - lAU' = 0, \tag{3.2.3}$$

and

$$\tau C'' - (\tau l^2 + \sigma)C - lAH' = 0. \tag{3.2.4}$$

Substituting (3.2.1) into the boundary conditions leads to

$$A'' + \gamma_1 A' = B' + \gamma_2 B = C' + \beta_1 C = A = 0 \;\text{ on }\; z = 0 \tag{3.2.5}$$

and

$$A'' + \gamma_3 A' = B' + \gamma_4 B = C' + \beta_2 C = A = 0 \;\text{ on }\; z = -1. \tag{3.2.6}$$

In the linear power series method we let

$$A = \sum_{m=0}^{4+M} a_m z^m, \;\; B = \sum_{m=0}^{2+M} b_m z^m, \;\text{ and }\; C = \sum_{m=0}^{2+M} c_m z^m. \tag{3.2.7}$$

We then substitute (3.2.7) into the differential equations (3.2.2), (3.2.3), (3.2.4) and boundary conditions (3.2.5), (3.2.6). Equating like powers of z in accordance with Theorem A in the Appendix then leads to a set of algebraic equations. These algebraic equations can then be solved numerically. To produce some of the results, numerical methods were combined. All of our linear power series method codes used adaptive Newton's method for systems of algebraic equations. When the minimum turning point on the neutral curve was required the Golden section algorithm was used. When finding the point where Re $\sigma = 0$ the bisection method was used. Since the numerical methods used here are iterative, rapid convergence depended upon initial guesses and the perturbation solution results shined light on appropriate initial guesses. For case II where σ has two branches, different complex valued initial guesses must be used to find both of the branches of σ. We here set $b_0 = 1$.

3.3 Linear Galerkin method

As in the linear power series method, we seek solutions of the form

$$\psi = \mathrm{i}A\mathrm{e}^{\mathrm{i}ly}\mathrm{e}^{\sigma t}, \;\; u = B\mathrm{e}^{\mathrm{i}ly}\mathrm{e}^{\sigma t}, \;\text{ and }\; \theta = C\mathrm{e}^{\mathrm{i}ly}\mathrm{e}^{\sigma t} \tag{3.3.1}$$

which leads, as above, to (3.2.2) through (3.2.6). However, we here express A, B, and C in terms of orthogonal basis functions premultiplied by coefficients

$$A = \sum_{m=0}^{M} a_m P_m, \;\; B = \sum_{m=0}^{M-2} b_m P_m, \;\text{ and }\; C = \sum_{m=0}^{M-2} c_m P_m. \tag{3.3.2}$$

Here $P_m = P_m(z)$ are shifted Legendre basis functions on $z \in [-1, 0]$ defined by

$$P_m = \frac{1}{2^m m!}\{\frac{d^m}{dx^m}[(x^2-1)^m]\}|_{x=2z+1} \tag{3.3.3}$$

and satisfy

$$\int_{-1}^{0} P_i P_j \, dz \propto \delta_{i,j}. \tag{3.3.4}$$

We substitute expansions (3.3.1) into the linearised CL2 equations to obtain equations in z whose residuals r_1, r_2, r_3 can be expanded as

$$r_1 = \sum_{i=0}^{\infty} a_i^\star P_i, \;\; r_2 = \sum_{i=0}^{\infty} b_i^\star P_i, \;\; r_3 = \sum_{i=0}^{\infty} c_i^\star P_i \tag{3.3.5}$$

where

$$a_i^\star \propto \langle r_1, P_i\rangle, \;\; b_i^\star \propto \langle r_2, P_i\rangle, \;\; c_i^\star \propto \langle r_3, P_i\rangle. \tag{3.3.6}$$

In the Galerkin method we require

$$\langle r_1, P_i\rangle = \langle r_2, P_i\rangle = \langle r_3, P_i\rangle = 0. \tag{3.3.7}$$

That is, we require

$$\int_{-1}^{0} r_1 P_j \, dz = \int_{-1}^{0} r_2 P_j \, dz = \int_{-1}^{0} r_3 P_j \, dz = 0 \tag{3.3.8}$$

for $j = 0, 1, 2, \ldots, M-4$, which yield algebraic equations for the unknown coefficients. The basis functions do not inherently satisfy the boundary conditions and extra equations are found by substituting into the boundary conditions. This technique is known as the tau-method. The resulting algebraic equations are then treated much the same as in the linear power series method. For consistency with the linear power series solutions we herein choose b_0 so that $B|_{z=0} = 1$.

4 Nonlinear methods

4.1 Nonlinear perturbation solution

We seek a nonlinear perturbation solution to the CL2 equations (2.1.1), (2.1.2), (2.1.3) with boundary conditions (2.2.1) and (2.2.2) in the small l limit. This calculation is an extension of the work of Hayes & Phillips (2017). Consistent with the linear perturbation solution we write

$$Y = ly, \quad T = l^2 t, \quad \psi(y,z,t) = l\bar{\Psi}(Y,z,T), \quad u(y,z,t) = \bar{\mathrm{u}}(Y,z,T), \quad \theta(y,z,t) = \bar{\Theta}(Y,z,T), \tag{4.1.1}$$

and

$$\gamma_i = l^4\overline{\gamma}_i, \quad \beta_i = \sum_{k=0}^{\infty}\beta_{i,2k}l^{2k}. \tag{4.1.2}$$

Equation (2.1.1) then becomes

$$\begin{aligned} & l^3\frac{\partial}{\partial T}[l^2\frac{\partial^2\bar{\Psi}}{\partial Y^2} + \frac{\partial^2\bar{\Psi}}{\partial z^2}] \\ & -l[l^4\frac{\partial^4\bar{\Psi}}{\partial Y^4} + 2l^2\frac{\partial^4\bar{\Psi}}{\partial Y^2\partial z^2} + \frac{\partial^4\bar{\Psi}}{\partial z^4}] \\ = \; & RD'\frac{\partial\bar{\mathrm{u}}}{\partial Y}l - S\frac{\partial\bar{\Theta}}{\partial Y}l + l^3\frac{\partial\bar{\Psi}}{\partial Y}\frac{\partial}{\partial z}[l^2\frac{\partial^2\bar{\Psi}}{\partial Y^2} + \frac{\partial^2\bar{\Psi}}{\partial z^2}] \\ & -l^3\frac{\partial\bar{\Psi}}{\partial z}\frac{\partial}{\partial Y}[l^2\frac{\partial^2\bar{\Psi}}{\partial Y^2} + \frac{\partial^2\bar{\Psi}}{\partial z^2}] \end{aligned} \tag{4.1.3}$$

while equation (2.1.2) becomes

$$l^2\frac{\partial\bar{\mathrm{u}}}{\partial T} - l^2\frac{\partial^2\bar{\mathrm{u}}}{\partial Y^2} - \frac{\partial^2\bar{\mathrm{u}}}{\partial z^2} = l^2\frac{\partial\bar{\Psi}}{\partial Y}U' + l^2\frac{\partial\bar{\Psi}}{\partial Y}\frac{\partial\bar{\mathrm{u}}}{\partial z} - l^2\frac{\partial\bar{\Psi}}{\partial z}\frac{\partial\bar{\mathrm{u}}}{\partial Y} \tag{4.1.4}$$

and equation (2.1.3) becomes

$$l^2\frac{\partial\bar{\Theta}}{\partial T} - \tau l^2\frac{\partial^2\bar{\Theta}}{\partial Y^2} - \tau\frac{\partial^2\bar{\Theta}}{\partial z^2} = l^2\frac{\partial\bar{\Psi}}{\partial Y}H' + l^2\frac{\partial\bar{\Psi}}{\partial Y}\frac{\partial\bar{\Theta}}{\partial z} - l^2\frac{\partial\bar{\Psi}}{\partial z}\frac{\partial\bar{\Theta}}{\partial Y}. \tag{4.1.5}$$

The boundary conditions become

$$l\frac{\partial^2\bar{\Psi}}{\partial z^2} + l^5\overline{\gamma}_1\frac{\partial\bar{\Psi}}{\partial z} = \frac{\partial\bar{\mathrm{u}}}{\partial z} + l^4\overline{\gamma}_2\bar{\mathrm{u}} = \frac{\partial\bar{\Theta}}{\partial z} + \sum_{m=0}^{\infty}\beta_{1,2m}l^{2m}\bar{\Theta} = l\bar{\Psi} = 0 \text{ on } z = 0, \tag{4.1.6}$$

$$l\frac{\partial^2\bar{\Psi}}{\partial z^2} + l^5\overline{\gamma}_3\frac{\partial\bar{\Psi}}{\partial z} = \frac{\partial\bar{\mathrm{u}}}{\partial z} + l^4\overline{\gamma}_4\bar{\mathrm{u}} = \frac{\partial\bar{\Theta}}{\partial z} + \sum_{m=0}^{\infty}\beta_{2,2m}l^{2m}\bar{\Theta} = l\bar{\Psi} = 0 \text{ on } z = -1. \tag{4.1.7}$$

We let

$$\bar{\Psi} = \sum_{k=0}^{\infty}\Psi_{2k}l^{2k}, \quad \bar{\mathrm{u}} = \sum_{k=0}^{\infty}\mathrm{u}_{2k}l^{2k}, \quad \bar{\Theta} = \sum_{k=0}^{\infty}\Theta_{2k}l^{2k}, \tag{4.1.8}$$

and

$$R = \sum_{k=0}^{\infty}R_{2k}l^{2k} \tag{4.1.9}$$

where Ψ_{2k}, u_{2k}, and Θ_{2k}, are functions of Y, z, and T. These expansions are consistent with those from the linear perturbation solution. We substitute (4.1.8) and (4.1.9) into (4.1.3 – 4.1.7) and equate like powers of l using the Cauchy product formula.
At $O(l^{2k})$ we have

$$\begin{aligned} \frac{\partial}{\partial T}\mathrm{u}_{2(k-1)} \; & - \; \frac{\partial^2}{\partial Y^2}\mathrm{u}_{2(k-1)} - \frac{\partial^2}{\partial z^2}\mathrm{u}_{2k} \\ & = \; \frac{\partial}{\partial Y}\Psi_{2(k-1)}U' + \sum_{m=0}^{k-1}\frac{\partial}{\partial Y}\Psi_{2(k-m-1)}\frac{\partial}{\partial z}\mathrm{u}_{2m} - \sum_{m=0}^{k-1}\frac{\partial}{\partial z}\Psi_{2(k-m-1)}\frac{\partial}{\partial Y}\mathrm{u}_{2m} \end{aligned} \tag{4.1.10}$$

with boundary conditions

$$\frac{\partial}{\partial z}\mathrm{u}_{2k} + \overline{\gamma}_2\mathrm{u}_{2(k-2)} = 0 \text{ on } z = 0, \tag{4.1.11}$$

$$\frac{\partial}{\partial z}\mathrm{u}_{2k} + \overline{\gamma}_4 \mathrm{u}_{2(k-2)} = 0 \ \text{ on } \ z = -1, \tag{4.1.12}$$

and

$$\begin{aligned} \frac{\partial}{\partial T}\Theta_{2(k-1)} \quad - \quad & \tau\frac{\partial^2}{\partial Y^2}\Theta_{2(k-1)} - \tau\frac{\partial^2}{\partial z^2}\Theta_{2k} \\ = \quad & \frac{\partial}{\partial Y}\Psi_{2(k-1)}H' + \sum_{m=0}^{k-1}\frac{\partial}{\partial Y}\Psi_{2(k-m-1)}\frac{\partial}{\partial z}\Theta_{2m} - \sum_{m=0}^{k-1}\frac{\partial}{\partial z}\Psi_{2(k-m-1)}\frac{\partial}{\partial Y}\Theta_{2m} \end{aligned} \tag{4.1.13}$$

with boundary conditions

$$\frac{\partial}{\partial z}\Theta_{2k} + \sum_{m=0}^{k}\beta_{1,2m}\Theta_{2k-2m} = 0 \ \text{ on } \ z = 0, \tag{4.1.14}$$

$$\frac{\partial}{\partial z}\Theta_{2k} + \sum_{m=0}^{k}\beta_{2,2m}\Theta_{2k-2m} = 0 \ \text{ on } \ z = -1. \tag{4.1.15}$$

At $O(l^{2k+1})$ we have

$$\begin{aligned} & \frac{\partial}{\partial T}\frac{\partial^2}{\partial Y^2}\Psi_{2(k-2)} + \frac{\partial}{\partial T}\frac{\partial^2}{\partial z^2}\Psi_{2(k-1)} \\ & -\frac{\partial^4}{\partial Y^4}\Psi_{2(k-2)} - 2\frac{\partial^4}{\partial Y^2\partial z^2}\Psi_{2(k-1)} - \frac{\partial^4}{\partial z^4}\Psi_{2k} \\ = \; & \sum_{m=0}^{k}R_{2(k-m)}D'\frac{\partial}{\partial Y}\mathrm{u}_{2m} - S\frac{\partial}{\partial Y}\Theta_{2k} + \sum_{m=0}^{k-2}\frac{\partial}{\partial Y}\Psi_{2(k-m-2)}\frac{\partial}{\partial z}\frac{\partial^2}{\partial Y^2}\Psi_{2m} \\ & + \sum_{m=0}^{k-1}\frac{\partial}{\partial Y}\Psi_{2(k-m-1)}\frac{\partial^3}{\partial z^3}\Psi_{2m} - \sum_{m=0}^{k-2}\frac{\partial}{\partial z}\Psi_{2(k-m-2)}\frac{\partial^3}{\partial Y^3}\Psi_{2m} \\ & - \sum_{m=0}^{k-1}\frac{\partial}{\partial z}\Psi_{2(k-m-1)}\frac{\partial}{\partial Y}\frac{\partial^2}{\partial z^2}\Psi_{2m} \end{aligned} \tag{4.1.16}$$

with boundary conditions

$$\frac{\partial^2}{\partial z^2}\Psi_{2k} + \overline{\gamma}_1\frac{\partial}{\partial z}\Psi_{2(k-2)} = \Psi_{2k} = 0 \ \text{ on } \ z = 0, \tag{4.1.17}$$

$$\frac{\partial^2}{\partial z^2}\Psi_{2k} + \overline{\gamma}_3\frac{\partial}{\partial z}\Psi_{2(k-2)} = \Psi_{2k} = 0 \ \text{ on } \ z = -1. \tag{4.1.18}$$

The equations above are to be solved for every integer $k \geqslant 0$. As in the linear perturbation solution, the nonlinear perturbation solution separates into two separate cases, that is case I: $-\beta_{1,0} + \beta_{2,0}\beta_{1,0} + \beta_{2,0} \neq 0$ and case II: $-\beta_{1,0} + \beta_{2,0}\beta_{1,0} + \beta_{2,0} = 0$.

4.1.1 The first few orders

At $O(l^0)$ we have

$$\frac{\partial^2 \mathrm{u}_0}{\partial z^2} = 0 \tag{4.1.19}$$

with boundary conditions

$$\frac{\partial \mathrm{u}_0}{\partial z} = 0 \ \text{ on } \ z = 0, -1. \tag{4.1.20}$$

The solution to this problem is

$$\mathrm{u}_0 = \mathrm{u}_0(Y, T) \tag{4.1.21}$$

where $\mathrm{u}_0(Y, T)$ is arbitrary.
Also at $O(l^0)$ we have

$$\tau\frac{\partial^2 \Theta_0}{\partial z^2} = 0 \tag{4.1.22}$$

with boundary conditions

$$\frac{\partial \Theta_0}{\partial z} + \beta_{1,0}\Theta_0 = 0 \ \text{ on } \ z = 0, \tag{4.1.23}$$

$$\frac{\partial \Theta_0}{\partial z} + \beta_{2,0}\Theta_0 = 0 \ \text{ on } \ z = -1. \tag{4.1.24}$$

For case I the solution to this problem is

$$\Theta_0 = c_3(Y, T) = 0. \tag{4.1.25}$$

For case II the solution to this problem is

$$\Theta_0 = c_3(Y, T)(-\beta_{1,0}z + 1) = c_3(Y, T)\hat{\Theta}_0 \tag{4.1.26}$$

where $c_3(Y, T)$ is arbitrary.
At $O(l^1)$ we have

$$\frac{\partial^4 \Psi_0}{\partial z^4} = -R_0 D' \frac{\partial \mathrm{u}_0}{\partial Y} + S \frac{\partial \Theta_0}{\partial Y} \tag{4.1.27}$$

with boundary conditions

$$\frac{\partial^2 \Psi_0}{\partial z^2} = \Psi_0 = 0 \ \text{ on } \ z = 0, -1. \tag{4.1.28}$$

The solution to this problem can be found to be

$$\begin{aligned}\Psi_0 \ &= \ -R_0 \frac{\partial \mathrm{u}_0}{\partial Y} \iiiint^z D' \, dz\, dz\, dz\, dz + S \iiiint^z \frac{\partial \Theta_0}{\partial Y} \, dz\, dz\, dz\, dz \\ &\quad + c_4(Y, T)\frac{z^3}{6} + c_5(Y, T)\frac{z^2}{2} + c_6(Y, T)z + c_7(Y, T) \\ &\quad = -R_0 \frac{\partial \mathrm{u}_0}{\partial Y} \tilde{\Psi}_0 + S \frac{\partial c_3(Y, T)}{\partial Y} \hat{\Psi}_0 \end{aligned} \tag{4.1.29}$$

where $\tilde{\Psi}_0$, $\hat{\Psi}_0$ are independent of R_0, u_0, S, and $c_3(Y, T)$. Note that $\tilde{\Psi}_0 = \tilde{\psi}_1$ and $\hat{\Psi}_0 = \hat{\psi}_1$ where $\tilde{\psi}_1$ and $\hat{\psi}_1$ are from the linear problem (3.1.16).
At $O(l^2)$ we have

$$\frac{\partial^2 \mathrm{u}_2}{\partial z^2} = \frac{\partial \mathrm{u}_0}{\partial T} - \frac{\partial^2 \mathrm{u}_0}{\partial Y^2} - \frac{\partial \Psi_0}{\partial Y} U' + \frac{\partial \Psi_0}{\partial z}\frac{\partial \mathrm{u}_0}{\partial Y} - \frac{\partial \Psi_0}{\partial Y}\frac{\partial \mathrm{u}_0}{\partial z} \tag{4.1.30}$$

with boundary conditions

$$\frac{\partial \mathrm{u}_2}{\partial z} = 0 \ \text{ on } \ z = 0, -1. \tag{4.1.31}$$

We find

$$\begin{aligned}\mathrm{u}_2 &= -\iint^z \frac{\partial \Psi_0}{\partial Y} U' \, dz\, dz + \iint^z \frac{\partial \Psi_0}{\partial z}\frac{\partial \mathrm{u}_0}{\partial Y} \, dz\, dz \\ &+ \left(\frac{\partial \mathrm{u}_0}{\partial T} - \frac{\partial^2 \mathrm{u}_0}{\partial Y^2}\right)\frac{z^2}{2} + c_8(Y, T)z + c_9(Y, T) = \tilde{\mathrm{u}}_2 + \hat{\mathrm{u}}_2 c_9(Y, T)\end{aligned} \tag{4.1.32}$$

where $c_9(Y, T)$ is an arbitrary function of Y and T. Here $\tilde{\mathrm{u}}_2$ and $\hat{\mathrm{u}}_2$ are independent of $c_9(Y, T)$.
Also at $O(l^2)$ we have

$$\tau \frac{\partial^2 \Theta_2}{\partial z^2} = \frac{\partial \Theta_0}{\partial T} - \tau \frac{\partial^2 \Theta_0}{\partial Y^2} - \frac{\partial \Psi_0}{\partial Y} H' + \frac{\partial \Psi_0}{\partial z}\frac{\partial \Theta_0}{\partial Y} - \frac{\partial \Psi_0}{\partial Y}\frac{\partial \Theta_0}{\partial z} \tag{4.1.33}$$

with boundary conditions

$$\frac{\partial \Theta_2}{\partial z} + \beta_{1,0}\Theta_2 + \beta_{1,2}\Theta_0 = 0 \ \text{ on } \ z = 0, \tag{4.1.34}$$

$$\frac{\partial \Theta_2}{\partial z} + \beta_{2,0}\Theta_2 + \beta_{2,2}\Theta_0 = 0 \ \text{ on } \ z = -1. \tag{4.1.35}$$

We find

$$\begin{aligned}\Theta_2 \ &= \ \iint^z \frac{\partial \Theta_0}{\partial T}\frac{1}{\tau} \, dz\, dz - \iint^z \frac{\partial^2 \Theta_0}{\partial Y^2} \, dz\, dz - \iint^z \frac{\partial \Psi_0}{\partial Y}\frac{H'}{\tau} \, dz\, dz - \iint^z \frac{1}{\tau}\frac{\partial \Psi_0}{\partial Y}\frac{\partial \Theta_0}{\partial z} \, dz\, dz \\ &+ \iint^z \frac{1}{\tau}\frac{\partial \Psi_0}{\partial z}\frac{\partial \Theta_0}{\partial Y} \, dz\, dz + c_{10}(Y, T)z + c_{11}(Y, T).\end{aligned} \tag{4.1.36}$$

For case II we have $\Theta_2 = \hat{\Theta}_2 c_{11}(Y, T) + \tilde{\Theta}_2$ where $\hat{\Theta}_2$ and $\tilde{\Theta}_2$ are independent of $c_{11}(Y, T)$. The boundary conditions lead to further equations which differ for the separate cases. For case I we find

$$\frac{\partial \mathrm{u}_0}{\partial T} - \frac{\partial^2 \mathrm{u}_0}{\partial Y^2} = \int_{-1}^{0} \frac{\partial \Psi_0}{\partial Y} U' \, dz. \tag{4.1.37}$$

If we now use (4.1.29) for Ψ_0 we obtain an equation for $\mathfrak{u}_0$

$$\frac{\partial \mathfrak{u}_0}{\partial T} - \frac{\partial^2 \mathfrak{u}_0}{\partial Y^2}\left(1 - R_0 \int_{-1}^{0} \tilde{\psi}_1 U' \, dz\right) = 0 \tag{4.1.38}$$

which on making use of equation (3.1.25) becomes

$$\frac{\partial \mathfrak{u}_0}{\partial T} + \sigma_2 \frac{\partial^2 \mathfrak{u}_0}{\partial Y^2} = 0. \tag{4.1.39}$$

A periodic in y Fourier cosine solution to equation (4.1.39) is (Hayes & Phillips, 2017)

$$\mathfrak{u}_0(Y,T) = \sum_{p=0}^{\infty} h_p \mathrm{e}^{\sigma_2 p^2 T} \cos pY \tag{4.1.40}$$

where h_p are constant coefficients. For case II, we have two coupled nonlinear partial differential equations for $\mathfrak{u}_0$ and $c_3(Y,T)$ as (4.1.37) and a further lengthy equation in the Appendix. In special cases such as $\beta_{i,2m} = 0$ for $m \neq 1$ these partial differential equations are then linear and exact solutions can be found. Moreover when $\beta_{i,2m} = 0$ for $m \neq 1$ there are two equations in terms of $\mathfrak{u}_0$ and $c_3(Y,T)$ as

$$\frac{\partial \mathfrak{u}_0}{\partial T} - \frac{\partial^2 \mathfrak{u}_0}{\partial Y^2} = \int_{-1}^{0} (-R_0 \frac{\partial^2 \mathfrak{u}_0}{\partial Y^2} \tilde{\Psi}_0 + S \frac{\partial^2 c_3}{\partial Y^2} \hat{\Psi}_0) U' \, dz, \tag{4.1.41}$$

$$\frac{1}{\tau}\frac{\partial c_3}{\partial T} - \frac{\partial^2 c_3}{\partial Y^2} = \int_{-1}^{0} (-R_0 \frac{\partial^2 \mathfrak{u}_0}{\partial Y^2} \tilde{\Psi}_0 + S \frac{\partial^2 c_3}{\partial Y^2} \hat{\Psi}_0) \frac{H'}{\tau} \, dz - \beta_{1,2} c_3 + \beta_{2,2} c_3. \tag{4.1.42}$$

We assume

$$\mathfrak{u}_0 = \sum_{p=0}^{\infty} f_p(T) \cos pY, \quad c_3 = \sum_{p=0}^{\infty} g_p(T) \cos pY. \tag{4.1.43}$$

Substituting into the two coupled partial differential equations for $\mathfrak{u}_0$ and c_3 and equating like harmonics yields

$$\dot{f}_p(T) + a_p f_p(T) + b_p g_p(T) = 0, \tag{4.1.44}$$

$$\dot{g}_p(T) + c_p f_p(T) + d_p g_p(T) = 0 \tag{4.1.45}$$

where the constants a_p, b_p, c_p, d_p are given in the Appendix. If $b_p \neq 0$ we find

$$g_p(T) = \frac{-\dot{f}_p(T) - a_p f_p(T)}{b_p}, \tag{4.1.46}$$

$$\ddot{f}_p(T) + (a_p + d_p)\dot{f}_p(T) + (d_p a_p - c_p b_p) f_p(T) = 0. \tag{4.1.47}$$

The latter is a simple second order differential equation. We will omit the expressions for $f_p(T)$, $g_p(T)$. Note for this case that $(a_p + d_p)/\tau = b$ and $(d_p a_p - c_p b_p)/\tau = c$ when $p = 1$ where b and c appear in (3.1.28).
At $O(l^3)$ we have

$$\begin{aligned}\frac{\partial^4 \Psi_2}{\partial z^4} &= \frac{\partial^3 \Psi_0}{\partial T \partial z^2} - 2\frac{\partial^4 \Psi_0}{\partial Y^2 \partial z^2} - R_2 D' \frac{\partial \mathfrak{u}_0}{\partial Y} - R_0 D' \frac{\partial \mathfrak{u}_2}{\partial Y} + S \frac{\partial \Theta_2}{\partial Y} \\ &\quad - \frac{\partial \Psi_0}{\partial Y}\frac{\partial^3 \Psi_0}{\partial z^3} + \frac{\partial \Psi_0}{\partial z}\frac{\partial^3 \Psi_0}{\partial Y \partial z^2}\end{aligned} \tag{4.1.48}$$

with boundary conditions

$$\frac{\partial^2 \Psi_2}{\partial z^2} = \Psi_2 = 0 \;\; \text{on} \;\; z = 0, -1. \tag{4.1.49}$$

The solution to this problem can be found to be

$$\begin{aligned}\Psi_2 &= \iint^z \frac{\partial \Psi_0}{\partial T} \, dz\, dz - \iint^z 2\frac{\partial^2 \Psi_0}{\partial Y^2} \, dz\, dz - \iiiint^z D' R_2 \frac{\partial \mathfrak{u}_0}{\partial Y} \, dz\, dz\, dz\, dz \\ &\quad - \iiiint^z D' R_0 \frac{\partial \mathfrak{u}_2}{\partial Y} \, dz\, dz\, dz\, dz + S \iiiint^z \frac{\partial \Theta_2}{\partial Y} \, dz\, dz\, dz\, dz \\ &\quad + \iiiint^z \frac{\partial \Psi_0}{\partial z}\frac{\partial^3 \Psi_0}{\partial Y \partial z^2} \, dz\, dz\, dz\, dz - \iiiint^z \frac{\partial \Psi_0}{\partial Y}\frac{\partial^3 \Psi_0}{\partial z^3} \, dz\, dz\, dz\, dz \\ &\quad + c_{12}(Y,T)\frac{z^3}{6} + c_{13}(Y,T)\frac{z^2}{2} + c_{14}(Y,T) z + c_{15}(Y,T).\end{aligned} \tag{4.1.50}$$

For case I we have $\Psi_2 = \hat{\Psi}_2 - R_2\tilde{\Psi}_2 + S\breve{\Psi}_2$ where $\hat{\Psi}_2$, $\tilde{\Psi}_2$, and $\breve{\Psi}_2$ are each independent of R_2 and S. For case II we have $\Psi_2 = \breve{\Psi}_2 + \tilde{\Psi}_2\frac{\partial c_9(Y,T)}{\partial Y} + \hat{\Psi}_2\frac{\partial c_{11}(Y,T)}{\partial Y}$ where $\hat{\Psi}_2$, $\tilde{\Psi}_2$, and $\breve{\Psi}_2$ are each independent of $c_9(Y,T)$ and $c_{11}(Y,T)$.
At $O(l^4)$ we have

$$\begin{aligned}\frac{\partial^2 \mathrm{u}_4}{\partial z^2} &= \frac{\partial \mathrm{u}_2}{\partial T} - \frac{\partial^2 \mathrm{u}_2}{\partial Y^2} - \frac{\partial \Psi_2}{\partial Y}U' \\ &\quad -\frac{\partial \Psi_2}{\partial Y}\frac{\partial \mathrm{u}_0}{\partial z} - \frac{\partial \Psi_0}{\partial Y}\frac{\partial \mathrm{u}_2}{\partial z} + \frac{\partial \Psi_2}{\partial z}\frac{\partial \mathrm{u}_0}{\partial Y} + \frac{\partial \Psi_0}{\partial z}\frac{\partial \mathrm{u}_2}{\partial Y} \end{aligned} \tag{4.1.51}$$

with boundary conditions as

$$\frac{\partial \mathrm{u}_4}{\partial z} + \overline{\gamma}_2 \mathrm{u}_0 = 0 \text{ on } z = 0, \tag{4.1.52}$$

$$\frac{\partial \mathrm{u}_4}{\partial z} + \overline{\gamma}_4 \mathrm{u}_0 = 0 \text{ on } z = -1. \tag{4.1.53}$$

We find

$$\begin{aligned}\mathrm{u}_4 &= \iint^z \frac{\partial \mathrm{u}_2}{\partial T}\,dz\,dz - \iint^z \frac{\partial^2 \mathrm{u}_2}{\partial Y^2}\,dz\,dz - \iint^z \frac{\partial \Psi_2}{\partial Y}U'\,dz\,dz \\ &\quad - \iint^z \frac{\partial \Psi_0}{\partial Y}\frac{\partial \mathrm{u}_2}{\partial z}\,dz\,dz + \iint^z \frac{\partial \Psi_2}{\partial z}\frac{\partial \mathrm{u}_0}{\partial Y}\,dz\,dz + \iint^z \frac{\partial \Psi_0}{\partial z}\frac{\partial \mathrm{u}_2}{\partial Y}\,dz\,dz \\ &\quad + c_{16}(Y,T)z + c_{17}(Y,T)\end{aligned} \tag{4.1.54}$$

where $c_{17}(Y,T)$ is arbitrary.
Also at $O(l^4)$ we have

$$\begin{aligned}\tau\frac{\partial^2 \Theta_4}{\partial z^2} &= \frac{\partial \Theta_2}{\partial T} - \tau\frac{\partial^2 \Theta_2}{\partial Y^2} - \frac{\partial \Psi_2}{\partial Y}H' \\ &\quad -\frac{\partial \Psi_2}{\partial Y}\frac{\partial \Theta_0}{\partial z} - \frac{\partial \Psi_0}{\partial Y}\frac{\partial \Theta_2}{\partial z} + \frac{\partial \Psi_2}{\partial z}\frac{\partial \Theta_0}{\partial Y} + \frac{\partial \Psi_0}{\partial z}\frac{\partial \Theta_2}{\partial Y}\end{aligned} \tag{4.1.55}$$

with boundary conditions as

$$\frac{\partial \Theta_4}{\partial z} + \beta_{1,0}\Theta_4 + \beta_{1,2}\Theta_2 + \beta_{1,4}\Theta_0 = 0 \text{ on } z = 0, \tag{4.1.56}$$

$$\frac{\partial \Theta_4}{\partial z} + \beta_{2,0}\Theta_4 + \beta_{2,2}\Theta_2 + \beta_{2,4}\Theta_0 = 0 \text{ on } z = -1. \tag{4.1.57}$$

We find

$$\begin{aligned}\Theta_4 &= \iint^z \frac{1}{\tau}\frac{\partial \Theta_2}{\partial T}\,dz\,dz - \iint^z \frac{\partial^2 \Theta_2}{\partial Y^2}\,dz\,dz - \iint^z \frac{\partial \Psi_2}{\partial Y}\frac{H'}{\tau}\,dz\,dz \\ &\quad - \iint^z \frac{1}{\tau}\frac{\partial \Psi_2}{\partial Y}\frac{\partial \Theta_0}{\partial z}\,dz\,dz - \iint^z \frac{1}{\tau}\frac{\partial \Psi_0}{\partial Y}\frac{\partial \Theta_2}{\partial z}\,dz\,dz + \iint^z \frac{1}{\tau}\frac{\partial \Psi_2}{\partial z}\frac{\partial \Theta_0}{\partial Y}\,dz\,dz \\ &\quad + \iint^z \frac{1}{\tau}\frac{\partial \Psi_0}{\partial z}\frac{\partial \Theta_2}{\partial Y}\,dz\,dz + c_{18}(Y,T)z + c_{19}(Y,T).\end{aligned} \tag{4.1.58}$$

The boundary conditions lead to equations for $c_9(Y,T)$ and $c_{11}(Y,T)$. For case I we have a single partial differential equation for $c_9(Y,T)$ appearing as

$$\begin{aligned}& -\int_{-1}^{0} \frac{\partial \mathrm{u}_2}{\partial T}\,dz + \int_{-1}^{0} \frac{\partial^2 \mathrm{u}_2}{\partial Y^2}\,dz + \int_{-1}^{0} \frac{\partial \Psi_2}{\partial Y}U'\,dz \\ & + \int_{-1}^{0} \frac{\partial \Psi_0}{\partial Y}\frac{\partial \mathrm{u}_2}{\partial z}\,dz - \int_{-1}^{0} \frac{\partial \Psi_0}{\partial z}\frac{\partial \mathrm{u}_2}{\partial Y}\,dz - \overline{\gamma}_2\mathrm{u}_0 + \overline{\gamma}_4\mathrm{u}_0 = 0.\end{aligned} \tag{4.1.59}$$

For case II we have two coupled partial differential equations in terms of $c_9(Y,T)$ and $c_{11}(Y,T)$ as (4.1.59) and a further very lengthy equation in the Appendix. At higher orders the complexity of the calculation becomes unwieldy. This calculation recovers Hayes & Phillips (2017) on setting $S = 0$. For time varying solutions it may be more convenient to use numerical methods such as those in §4.2, §4.3.

4.1.2 Nonlinear perturbation solution algorithm

In light of the nonlinear perturbation solution above we let

$$\bar{\Psi} = \sum_{m=0}^{L}\sum_{k=0}^{\infty} \Psi_{2k,m}\sin(mY)l^{2k}, \quad \bar{\mathrm{u}} = \sum_{m=0}^{L}\sum_{k=0}^{\infty} \mathrm{u}_{2k,m}\cos(mY)l^{2k}, \quad \bar{\Theta} = \sum_{m=0}^{L}\sum_{k=0}^{\infty} \Theta_{2k,m}\cos(mY)l^{2k} \tag{4.1.60}$$

with

$$R = \sum_{k=0}^{\infty} R_{2k}l^{2k}, \quad \gamma_i = l^4\overline{\gamma}_i, \quad \beta_i = \sum_{k=0}^{\infty}\beta_{i,2k}l^{2k} \tag{4.1.61}$$

where $\Psi_{2k,m}$, $\mathrm{u}_{2k,m}$, and $\Theta_{2k,m}$ are functions of z and T. We substitute (4.1.60), (4.1.61) into equations (4.1.3) to (4.1.7) and discard harmonics in Y in the residuals that are of higher order than in the expansion of the solution in accordance with Theorem B in the Appendix. We then equate like harmonics in Y and like powers of l and then need to solve the resulting equations for $\Psi_{2k,m}$, $\mathrm{u}_{2k,m}$, and $\Theta_{2k,m}$ at each order in l. With the nonlinear perturbation solution we are particularly interested in the nonlinear steady states, for which we set $\partial/\partial T = 0$. In this case, arbitrary constants of integration will appear in the nonlinear perturbation solution. We choose them so that $\mathrm{u}_{i,j}|_{z=0} = \delta_{i,0}\delta_{j,1}$. Note that while this choice is dissimilar to that in the linear perturbation solutions it is similar to that in the linear numerical solutions. We found that the nonlinear steady states appear to require restrictions on the boundary conditions at $O(l^6)$ such as $\gamma_3 = \gamma_4 = 0$. This may be related to observations where LC tend to curl up near the bottom of the mixed layer. This calculation recovers Hayes & Phillips (2017) on setting $S = 0$.

4.2 Nonlinear power series method

Here we look for solutions of the form

$$\psi = \sum_{k=0}^{L}\sum_{m=0}^{4+M} a_{m,k}\sin(kly)z^m, \quad u = \sum_{k=0}^{L}\sum_{m=0}^{2+M} b_{m,k}\cos(kly)z^m, \quad \theta = \sum_{k=0}^{L}\sum_{m=0}^{2+M} c_{m,k}\cos(kly)z^m \tag{4.2.1}$$

where the coefficients $a_{m,k}$, $b_{m,k}$, and $c_{m,k}$ are unknown functions of t. Here ψ is a Fourier sine series in y while u and θ are both Fourier cosine series in y; each are Maclaurin series in z. Substituting into the governing equations and equating the appropriate like coefficients in accordance with Theorem A and Theorem B leads to a system of nonlinear ordinary differential equations for $a_{m,k}$, $b_{m,k}$, and $c_{m,k}$ which can be numerically solved for by using methods such as the Runge–Kutta method. In the case for which $da_{m,k}/dt = db_{m,k}/dt = dc_{m,k}/dt = 0$ this leads to a system of algebraic equations. These algebraic equations are treated much the same as in the linear power series method. We here set $b_{0,1} = 1$ for consistency with the nonlinear perturbation solutions.

4.3 Nonlinear Galerkin method

In this method we look for solutions of the form

$$\psi = \sum_{k=0}^{L}\sum_{m=0}^{M} a_{m,k}P_m\sin(kly), \quad u = \sum_{k=0}^{L}\sum_{m=0}^{M-2} b_{m,k}P_m\cos(kly), \quad \theta = \sum_{k=0}^{L}\sum_{m=0}^{M-2} c_{m,k}P_m\cos(kly) \tag{4.3.1}$$

where $a_{m,k}$, $b_{m,k}$, and $c_{m,k}$ are unknown functions of t. Here ψ is a Fourier sine series in y while u and θ are Fourier cosine series' in y. Different are the basis functions. $P_m(z)$ are shifted Legendre basis functions on $z \in [-1, 0]$. We substitute these expansions into the CL2 equations, discard the higher order harmonics, and collect like trigonometrical terms in accordance with Theorem B to obtain a set of equations in z and t whose residuals we call $r_{1,i}(z,t)$, $r_{2,i}(z,t)$, and $r_{3,i}(z,t)$. In the Galerkin method we require

$$\int_{-1}^{0} r_{1,i}P_j\,dz = \int_{-1}^{0} r_{2,i}P_j\,dz = \int_{-1}^{0} r_{3,i}P_j\,dz = 0 \tag{4.3.2}$$

for $i = 0, 1, \ldots, L$, and $j = 0, 1, \ldots, M-4$. We obtain the further equations required to close the system by substitution of (4.3.1) into the boundary conditions. This results in a system of nonlinear ordinary differential equations which can be solved numerically by using methods such as the Runge–Kutta method. For the case of nonlinear steady states they reduce to a set of algebraic equations. Once again, these algebraic equations are treated much the same as in the linear power series method. Herein we choose $b_{0,1}$ so that the coefficient of $\cos(ly)$ in $u|_{z=0}$ is unity for consistency with the nonlinear perturbation solutions and nonlinear power series solutions.

5 Results

In this section we are interested in how the parameters and nonlinearities affect the CL2 instability to LC over a restricted parameter range. For case I we let $\beta_{i,2m} = 0$ for $m \neq 0$ and for case II we let $\beta_{i,2m} = 0$ for $m \neq 1$. Herein $\epsilon = 0$ is for the linear case and $\epsilon = 1$ is for its nonlinear counterpart. We here choose $L = 1$ and $15 \leqslant M \leqslant 20$.

5.1 Growth rate

We consider first the growth rate σ. For case I we find that σ is real valued and for case II σ is complex valued where we see that in accord with (3.1.28) there are two solutions. When Re $\sigma < 0$ the motion is stable and when Re $\sigma > 0$ the motion is unstable. When $\sigma = 0$ there is neutral instability. The instability is oscillatory when Im $\sigma \neq 0$. The growth rate σ from the linear perturbation solution to $O(l^4)$ for case I where D', U', H' are constants is

$$\sigma = \frac{1}{79833600}(-530R^2D'^2U'^2 - 67320RD'U' - \frac{691((\beta_2 - \frac{2077}{691})\beta_1 + \frac{2077}{691}\beta_2 - \frac{5544}{691})RD'U'SH'}{((\beta_2-1)\beta_1+\beta_2)\tau})l^4$$
$$+(-1+\frac{RD'U'}{120})l^2 + \gamma_4 - \gamma_2 \tag{5.1.1}$$

and the growth rate σ from the linear perturbation solution to $O(l^2)$ for case II where D', U', H' are constants is

$$\sigma = \frac{1}{2}(\frac{RD'U'}{120} - (\beta_{1,2} - \beta_{2,2} + 1)\tau - \frac{SH'}{120} - 1 \tag{5.1.2}$$
$$\pm\sqrt{(\frac{RD'U'}{120})^2 - 2\frac{RD'U'}{120}(\frac{SH'}{120} - \tau(\beta_{1,2} - \beta_{2,2} + 1) + 1) + (\frac{SH'}{120} + \tau(\beta_{1,2} - \beta_{2,2} + 1) - 1)^2})l^2.$$

There can be uncertainty in deciding when case I or case II is appropriate. What happens is either the case I result or the case II result will converge or both case I and case II result will converge each for separate parts of the domain of discourse, and the appropriate case is that which converges. This is the competition between case I and case II as mentioned in Cox & Leibovich (1993). A good indication of whether the instability is case I or case II is whenever $\sigma_4 > \sigma_2$ for case I then σ for case I is likely to diverge and so the appropriate instability is then case II. Plots of σ which do illustrate this competition are shown in Figures 1, 2. Plots of Re σ vs R and Im σ vs R for case II are shown

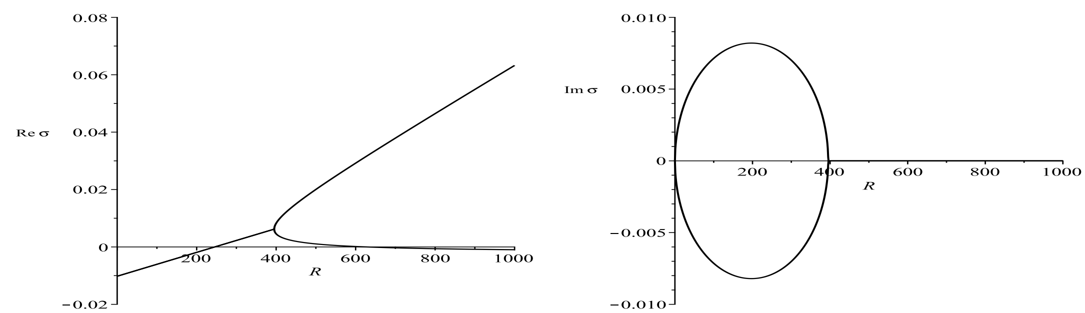

Figure 1: Plots of linear growth rate (left) Re σ vs R and (right) Im σ vs R for $\beta_1 = 1/100$. Here $D' = U' = H' = 1$, $S = 100$, $l = 1/10$, $\gamma_1 = 1/20000$, $\gamma_2 = 1/10000$, $\gamma_3 = \gamma_4 = 0$, $\beta_2 = 0$, and $\tau = 1/10$.

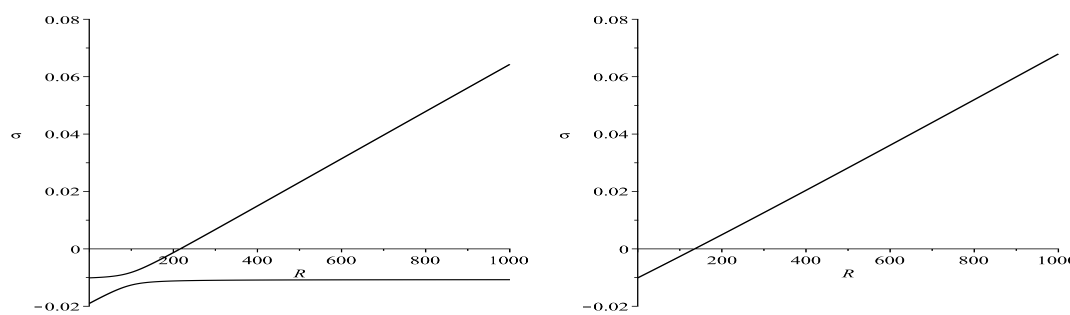

Figure 2: Plots of linear growth rate σ vs R for (left) $\beta_1 = 1/10$ and (right) $\beta_1 = 1$. Here $D' = U' = H' = 1$, $S = 100$, $l = 1/10$, $\gamma_1 = 1/20000$, $\gamma_2 = 1/10000$, $\gamma_3 = \gamma_4 = 0$, $\beta_2 = 0$, and $\tau = 1/10$.

in Figure 1 and plots of σ vs R for case I and case II are shown in Figure 2. In these plots we see that the instability changes from case II to case I as β_1 increases. We also see that the fluid motion switches from stable to unstable as R increases and so here increasing R is destabilising. It is then quite obvious from (5.1.1) and (5.1.2) how the parameters would affect σ in the small l limit where the expressions are valid. For example, increasing D' or U' is destabilising whenever increasing R is destabilising, and increasing H' is stabilising whenever increasing S is stabilising. For the boundary conditions of Cox & Leibovich (1993) we see in case I that increasing R or τ is destabilising and increasing S is stabilising. We also see in case I that increasing $\gamma_2 - \gamma_4$ is stabilising. For case II with the boundary conditions of Cox & Leibovich (1993) and on assuming σ remains complex we see that increasing R, D', or U' is destabilising and increasing S, H', or τ is stabilising.

5.2 Neutral instability

For case II, we see from (3.1.28) that neutral instability for which $\sigma = 0$ is seldom possible. Linear neutral curves and nonlinear steady states do exist for case I. From the case I linear and nonlinear perturbation solution for neutral instability at $O(l^2)$ we have

$$R_0 = \frac{1}{\int_{-1}^{0} \tilde{\psi}_1 U' \, dz}. \tag{5.2.1}$$

From the case I linear perturbation solution for neutral instability at $O(l^4)$ we have

$$R_2 = \frac{\int_{-1}^{0} (\hat{\psi}_3 + S\check{\psi}_3) U' \, dz + u_0(\overline{\gamma}_2 - \overline{\gamma}_4)}{\int_{-1}^{0} \tilde{\psi}_3 U' \, dz} = \tilde{R}_2 + \hat{R}_2(\overline{\gamma}_2 - \overline{\gamma}_4) + \check{R}_2 S. \tag{5.2.2}$$

From the case I nonlinear perturbation solution for neutral instability at $O(l^4)$ we have

$$\begin{aligned} R_2 \;=\; & (S\int_{-1}^{0} U' \frac{\partial \check{\Psi}_2}{\partial Y} dz - \overline{\gamma}_2 \mathfrak{u}_0 + \overline{\gamma}_4 \mathfrak{u}_0 + \int_{-1}^{0} \frac{\partial^2 \mathfrak{u}_2}{\partial Y^2} dz + \int_{-1}^{0} \frac{\partial \hat{\Psi}_2}{\partial Y} U' \, dz \\ & + \int_{-1}^{0} \frac{\partial \Psi_0}{\partial Y} \frac{\partial \mathfrak{u}_2}{\partial z} dz - \int_{-1}^{0} \frac{\partial \Psi_0}{\partial z} \frac{\partial \mathfrak{u}_2}{\partial Y} dz) / \int_{-1}^{0} U' \frac{\partial \tilde{\Psi}_2}{\partial Y} dz = \tilde{R}_2 + \hat{R}_2(\overline{\gamma}_2 - \overline{\gamma}_4) + \check{R}_2 S. \end{aligned} \tag{5.2.3}$$

In these equations $\tilde{R}_2$, $\hat{R}_2$, and $\check{R}_2$ are each independent of $\overline{\gamma}_i$ and S. Also note that nonlinear R_2 is here projected onto a mode in Y. We here choose $L = 1$ in the nonlinear expansions. For both the case I linear and nonlinear problems the expression for R appears as

$$R = R_0 + (\tilde{R}_2 + \check{R}_2 S) l^2 + \frac{\hat{R}_2}{l^2}(\gamma_2 - \gamma_4) + \dots. \tag{5.2.4}$$

The neutral curve from the perturbation solution to $O(l^4)$ for case I where D', U', H' are constants is

$$R = \frac{5455}{231}\frac{l^2}{D'U'} + \frac{691}{5544}\frac{((\beta_2 - \frac{2077}{691})\beta_1 + \frac{2077}{691}\beta_2 - \frac{5544}{691}) S H' l^2}{\tau D'U'((\beta_2 - 1)\beta_1 + \beta_2)} + \frac{1550}{21}\frac{\epsilon l^2}{D'U'^3} + \frac{120}{D'U'} + \frac{120(\gamma_2 - \gamma_4)}{D'U'l^2}. \tag{5.2.5}$$

Here it is evident that

$$\tilde{R}_2 = \frac{5455}{231}\frac{1}{D'U'} + \frac{1550}{21}\frac{\epsilon}{D'U'^3}, \tag{5.2.6}$$

$$\check{R}_2 = \frac{691}{5544}\frac{((\beta_2 - \frac{2077}{691})\beta_1 + \frac{2077}{691}\beta_2 - \frac{5544}{691}) H'}{\tau D'U'((\beta_2 - 1)\beta_1 + \beta_2)}, \tag{5.2.7}$$

$$\hat{R}_2 = \frac{120}{D'U'} = R_0. \tag{5.2.8}$$

We plot linear neutral curves and nonlinear steady states as R vs l in the small l limit in Figure 3 (left). For this case,

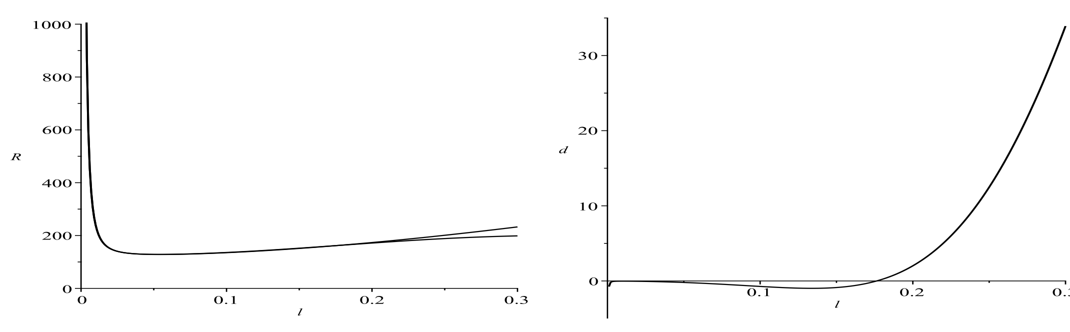

Figure 3: (left) Plots of neutral curve R vs l, linear (top) and nonlinear (bottom). (right) Plot of $d = R_L - R_{NL}$ vs l. Here $D' = U' = H' = 1$, $S = 100$, $\gamma_1 = 1/20000$, $\gamma_2 = 1/10000$, $\gamma_3 = \gamma_4 = 0$, $\beta_1 = 1$, $\beta_2 = 0$, and $\tau = 1/10$.

since the fluid motion switches from stable to unstable as σ passes through zero with increasing R, any point above the neutral curve is unstable, while any point below the neutral curve is stable. In Figure 3 (right) we see that nonlinearities are small when $l \ll 1$ similarly to as shown in Hayes & Phillips (2017) for the case $S = 0$. In the small l limit we see that nonlinearities have a stabilising effect. It is also quite obvious from (5.2.5) how the parameters and nonlinearities would affect neutral instability in the small l limit where this expression is valid. From (5.2.5) we see for the boundary conditions of Cox & Leibovich (1993) that increasing S is stabilising. This is consistent with Langmuir (1938) in that temperature is thought to be secondary to the formation of LC. Increasing H' or decreasing τ has a similar effect as increasing S. Also we see from (5.2.5) that nonlinearities are stabilising. When using the boundary conditions of Cox & Leibovich (1993) we find for the nonlinear problem that increasing D' or U' is destabilising and increasing U' is more effective in destabilising the flow than increasing D'. The effect of increasing $\gamma_2 - \gamma_4$ is stabilising.

5.3 Figure 3 of Cox & Leibovich (1993) revisited

We are interested in the nonlinear counterpart to Figure 3 of Cox & Leibovich (1993). Figure 4 (left) is Figure 3 of Cox & Leibovich (1993). The flat parts of Figure 4 (left) represent the stability margin for oscillatory convection and the parabolic parts of Figure 4 (left) represent the stability margin to steady convection (Cox & Leibovich, 1993). The nonlinear version of Figure 3 of Cox & Leibovich (1993) is plotted within Figure 4 (right) for the corresponding nonlinear steady states only. In Figure 4, l_c increases with increasing β_1. To obtain the flat parts of Figure 4 (left)

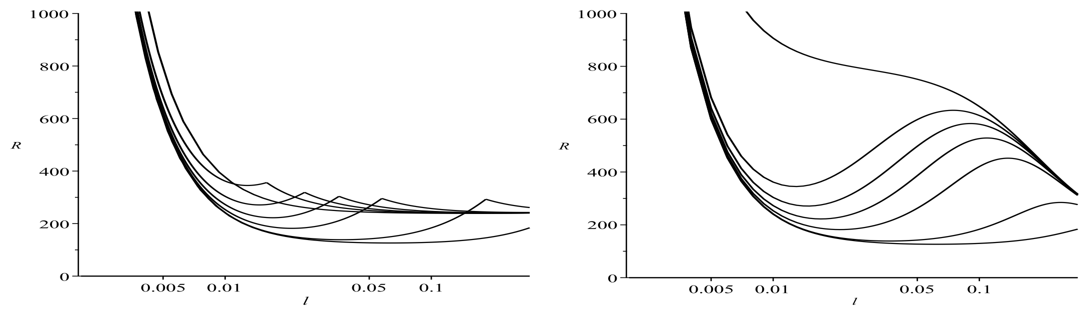

Figure 4: (left) Figure 3 of Cox & Leibovich (1993). (right) nonlinear steady states R vs l. Here $D' = U' = H' = 1$, $S = 100$, $\gamma_1 = 1/40000$, $\gamma_2 = -\gamma_3 = -\gamma_4 = 1/20000$, $\beta_1 = -\beta_2 \in \{1/20000, 1/2000, 1/1000, 1/500, 1/200, 1/20, 1/2\}$, and $\tau = 10/67$.

where $\mathrm{Re}\,\sigma = 0$ with the case II linear perturbation solution, we solve $b = 0$ for R_0 and then $\mathrm{Re}\,\sigma_2 = 0$ providing $c \geqslant 0$ where b and c appear in (3.1.28). Then $\mathrm{Re}\,\sigma_{2j} = 0$ can be solved for R_{2j-2} for $j > 1$ on assuming $c \geqslant 0$. Figure 4 (right) was obtained by using the Galerkin method as outlined in §4. The Galerkin method used for finding the steady states assumes $\partial/\partial t = 0$ and so Figure 4 (right) represents the nonlinear counterpart of the stability margin to steady convection in Figure 4 (left) only up to a certain l value for each curve. Moreover, if we plot $\mathrm{Re}\,\sigma$ vs R and $\mathrm{Im}\,\sigma$ vs R for values consistent with Figure 4, then for l before the bifurcation point in Figure 4 (left) the growth rate is case I like that of Figure 2 (right) and for l after the bifurcation point in Figure 4 (left) the growth rate is case II like that of Figure 1. If we solve for $\sigma = 0$ in case II our Galerkin method will find where only one of the branches of σ is zero, but both branches must be included for the solution to be real valued. A similar idea must also apply to the nonlinear case because in the nonlinear perturbation solution it is found that the solution is linear up until $O(l^2)$. Finding a nonlinear counterpart to σ at higher orders appears to be quite difficult and is omitted. In Figure 4 (right) nonlinearities are small and stabilising in the small l limit. When Figure 4 (left) and (right) are overlayed the neutral curves for the linear case appear indistinguishable to the corresponding nonlinear steady states about their minimum turning points.

5.4 Onset

Onset occurs at the minimum point on the neutral curve $R = R(l)$, which we denote by (l_c, R_c) where l_c and R_c are called the critical wavenumber and critical Rayleigh number respectively. From the perturbation solutions onset is found by solving $\frac{dR}{dl} = 0$. We find from the $O(l^4)$ perturbation solutions for case I that

$$l_c = \left(\frac{(\gamma_2 - \gamma_4)\hat{R}_2}{\tilde{R}_2 + \check{R}_2 S}\right)^{\frac{1}{4}} \tag{5.4.1}$$

and thus that

$$R_c = R_0 + 2((\gamma_2 - \gamma_4)\hat{R}_2)^{1/2}(\tilde{R}_2 + \check{R}_2 S)^{1/2} = R_0 + 2l_c^2(\tilde{R}_2 + \check{R}_2 S). \tag{5.4.2}$$

The critical wavenumber l_c from the perturbation solution to $O(l^4)$ for case I where D', U', H' are constants is

$$l_c = \left(\frac{(\gamma_2 - \gamma_4)\frac{120}{D'U'}}{\frac{5455}{231}\frac{1}{D'U'} + \frac{1550}{21}\frac{\epsilon}{D'U'^3} + \frac{691}{5544}\frac{((\beta_2 - \frac{2077}{691})\beta_1 + \frac{2077}{691}\beta_2 - \frac{5544}{691})H'}{\tau D'U'((\beta_2 - 1)\beta_1 + \beta_2)}S}\right)^{\frac{1}{4}} \tag{5.4.3}$$

and the corresponding critical Rayleigh number R_c from the perturbation solution to $O(l^4)$ for case I where D', U', H' are constants is

$$R_c = \frac{120}{D'U'} + 2((\gamma_2 - \gamma_4)\frac{120}{D'U'})^{1/2}(\frac{5455}{231}\frac{1}{D'U'} + \frac{1550}{21}\frac{\epsilon}{D'U'^3} + \frac{691}{5544}\frac{((\beta_2 - \frac{2077}{691})\beta_1 + \frac{2077}{691}\beta_2 - \frac{5544}{691})H'}{\tau D'U'((\beta_2 - 1)\beta_1 + \beta_2)}S)^{1/2}. \tag{5.4.4}$$

It is here quite obvious from (5.4.3) and (5.4.4) how the parameters and nonlinearities would affect l_c and R_c in the small l limit where these expression are valid. From (5.4.3) and (5.4.4) we see for the boundary conditions of Cox & Leibovich (1993) that increasing S reduces l_c and increases R_c. Here we also see that increasing H' or decreasing τ has a similar effect as increasing S. Also we see that nonlinearities reduce the value of l_c and increase R_c. Increasing

D' has no effect on l_c. Increasing U' only has an effect on l_c in the presence of nonlinearities. When using the boundary conditions of Cox & Leibovich (1993) we find for the nonlinear problem that increasing U' increases l_c, and increasing D' or U' decreases R_c where increasing U' is more effective in decreasing R_c than increasing D'. The effect of increasing $\gamma_2 - \gamma_4$ is to increase the value of l_c and increase the value of R_c, and we see that $\gamma_2 = \gamma_4 = 0$ leads to unphysical results. Another peculiarity is that depending on the choice of β_1, β_2, and different from Cox & Leibovich (1993), we see that there can be a singularity of l_c when S increases. In Figure 5 are plots of l_c vs S and R_c vs S for the linear and nonlinear problems. In Figure 5 the effect of increasing S is to lower the value of l_c and increase the

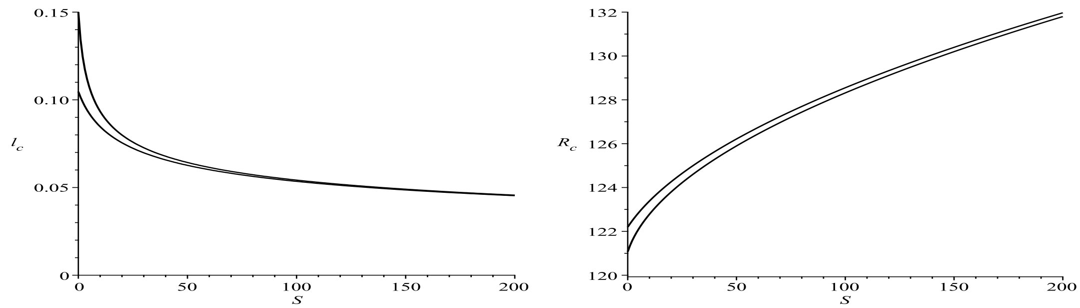

Figure 5: (left) Plots of l_c vs S, linear (top) and nonlinear (bottom). (right) Plots of R_c vs S, linear (bottom) and nonlinear (top). Here $D' = U' = H' = 1$, $\gamma_1 = 1/20000$, $\gamma_2 = 1/10000$, $\gamma_3 = \gamma_4 = 0$, $\beta_1 = 1$, $\beta_2 = 0$, and $\tau = 1/10$.

value of R_c. In Figure 6 are plots of the ratio of linear to nonlinear critical wavenumber $\kappa = l_{c,\text{linear}}/l_{c,\text{nonlinear}}$ vs S and plots of the ratio of linear to nonlinear critical Rayleigh number $\rho = R_{c,\text{linear}}/R_{c,\text{nonlinear}}$ vs S for parameters consistent with Figure 5. In Figure 6 we see that the nonlinearities appear to diminish as S increases. Figure 7 shows how l_c and

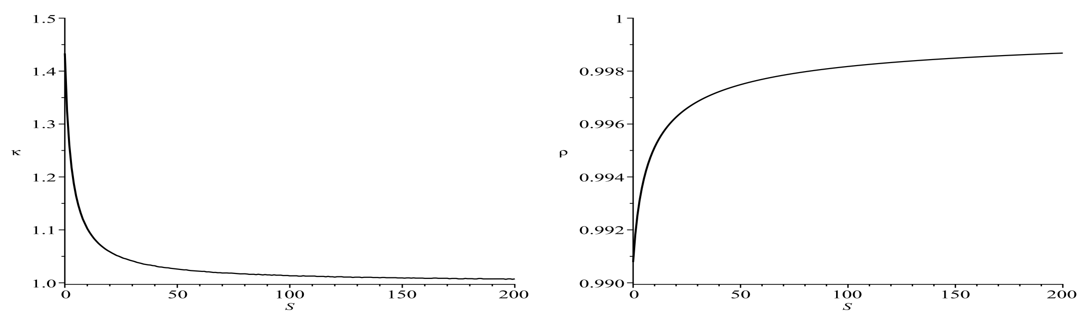

Figure 6: (left) Plots of κ vs S. (right) Plots of ρ vs S. Here $D' = U' = H' = 1$, $\gamma_1 = 1/20000$, $\gamma_2 = 1/10000$, $\gamma_3 = \gamma_4 = 0$, $\beta_1 = 1$, $\beta_2 = 0$, and $\tau = 1/10$.

R_c varies with β_1 for both the linear and nonlinear cases with $S = 100$ and other parameters consistent with Figure 5. In Figure 7 we see that l_c increases with increasing β_1 and R_c decreases with increasing β_1. In Figure 8 are plots

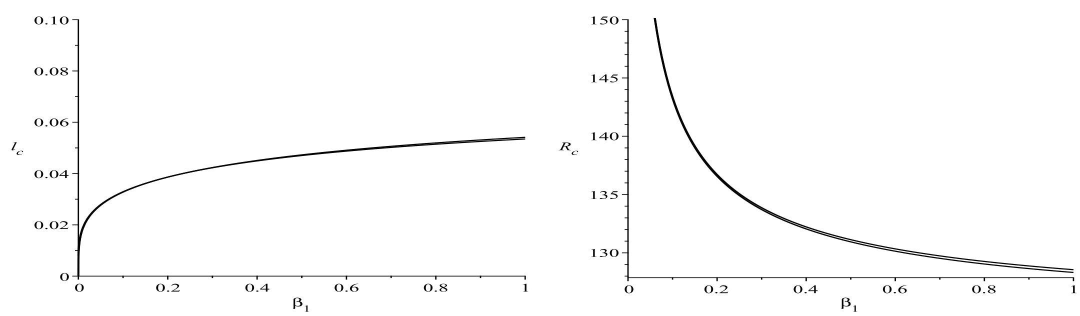

Figure 7: (left) Plots of l_c vs β_1, linear (top) and nonlinear (bottom). (right) Plots of R_c vs β_1, linear (bottom) and nonlinear (top). Here $D' = U' = H' = 1$, $\gamma_1 = 1/20000$, $\gamma_2 = 1/10000$, $\gamma_3 = \gamma_4 = 0$, $S = 100$, $\beta_2 = 0$, and $\tau = 1/10$.

of the ratio of linear to nonlinear critical wavenumber $\kappa = l_{c,\text{linear}}/l_{c,\text{nonlinear}}$ vs β_1 and plots of the ratio of linear to nonlinear critical Rayleigh number $\rho = R_{c,\text{linear}}/R_{c,\text{nonlinear}}$ vs β_1 for $S \in \{0, 100, 200\}$ and other parameters consistent with Figure 5. In Figure 8 the κ curves decrease for increasing S and the ρ curves increase for increasing S. We see for $S = 0$ that κ and ρ are independent of β_1 as expected. For $S = 100$ and $S = 200$ we see that κ increases with increasing β_1 and ρ decreases with increasing β_1. Also, the nonlinearities appear to be small for small β_1. For $S = 0$ we find that

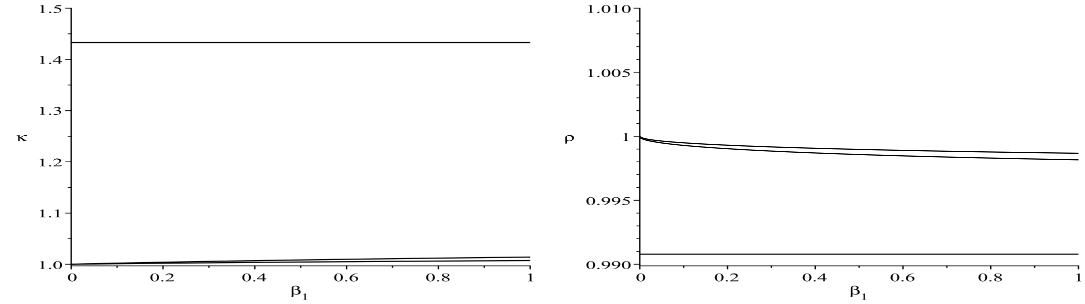

Figure 8: (left) Plots of κ vs β_1. (right) Plots of ρ vs β_1. Here $D' = U' = H' = 1$, $\gamma_1 = 1/20000$, $\gamma_2 = 1/10000$, $\gamma_3 = \gamma_4 = 0$, $S \in \{0, 100, 200\}$, $\beta_2 = 0$, and $\tau = 1/10$.

$\kappa \approx 1.425$ at $O(l^4)$ which is consistent with the value reported in Hayes & Phillips (2017). As shown in Figure 8 this is $\kappa \approx 1.433$ at $O(l^6)$. In Figures 5 to 8 we see that nonlinearities reduce the value of l_c and increase R_c.

6 Discussion

The methods used in this paper are very useful for the LC problem. The perturbation method is particularly useful in that the effect of altering parameters and of nonlinearities is evident in the small l limit by inspecting the simple expressions found from the perturbation solutions. Note that when $\gamma_i = O(1)$ our perturbation solutions would require a more direct perturbation expansion. For example, in the linear perturbation solution σ would then need to have an $O(1)$ term σ_0. This then leads to a messy calculation, especially for its nonlinear counterpart, with many separate subcases. The preferable strategy may then be to use the numerical methods such as the nonlinear power series and nonlinear Galerkin methods presented in this paper. I have also constructed animations of LC varying with time. In the nonlinear realm there is flexibility for animations of LC to show LC spacing changing with time due to the fact that the number of modes in y can increase as time increases. This is to be explored in further work on LC.

7 Appendix

7.1 Linear perturbation solution details

From ψ_1

$$c_4 = -R_0 u_0 \iint^z D'\,dz\,dz|_{z=-1} + S \iint^z \theta_0\,dz\,dz|_{z=-1} + c_5,$$

$$c_5 = R_0 u_0 \iint^z D'\,dz\,dz|_{z=0} - S \iint^z \theta_0\,dz\,dz|_{z=0},$$

$$c_6 = -R_0 u_0 \iiiint^z D'\,dz\,dz\,dz\,dz|_{z=-1} + S \iiiint^z \theta_0\,dz\,dz\,dz\,dz|_{z=-1} - \frac{1}{6}c_4 + \frac{1}{2}c_5 + c_7,$$

$$c_7 = R_0 u_0 \iiiint^z D'\,dz\,dz\,dz\,dz|_{z=0} - S \iiiint^z \theta_0\,dz\,dz\,dz\,dz|_{z=0}.$$

From u_2

$$c_8 = -\int^z U'\psi_1\,dz|_{z=0},$$

$$c_9 = -\int_{-1}^0 \iint^z U'\psi_1\,dz\,dz\,dz - \frac{u_0}{6}(\sigma_2 + 1) + \frac{1}{2}c_8.$$

From θ_2 for case I

$$c_{10} = -\beta_{1,0}\Big(\iint^z \frac{H'}{\tau}\psi_1\,dz\,dz|_{z=0} + c_{11}\Big) - \int^z \frac{H'}{\tau}\psi_1\,dz|_{z=0},$$

$$\begin{aligned} c_{11} \quad = \quad & \Big(\int_{-1}^0 \frac{H'}{\tau}\psi_1\,dz + (\beta_{1,0} - \beta_{1,0}\beta_{2,0}) \iint^z \frac{H'}{\tau}\psi_1\,dz\,dz|_{z=0} \\ & -\beta_{2,0} \iint^z \frac{H'}{\tau}\psi_1\,dz\,dz|_{z=-1} - \beta_{2,0} \int^z \frac{H'}{\tau}\psi_1\,dz|_{z=0}\Big)/(-\beta_{1,0} + \beta_{1,0}\beta_{2,0} + \beta_{2,0}). \end{aligned}$$

From θ_2 for case II

$$
\begin{aligned}
c_{10} \quad = \quad & -\beta_{1,0}(\iint^z \frac{\theta_0\sigma_2}{\tau}\,dz\,dz|_{z=0} + \iint^z \theta_0\,dz\,dz|_{z=0} + \iint^z \frac{H'}{\tau}\psi_1\,dz\,dz|_{z=0} + c_{11}) \\
& -\int^z \frac{\theta_0\sigma_2}{\tau}\,dz|_{z=0} - \int^z \theta_0\,dz|_{z=0} - \int^z \frac{H'}{\tau}\psi_1\,dz|_{z=0} - \beta_{1,2}\theta_0|_{z=0}.
\end{aligned}
$$

From the matrix M for case II

$$
M_{1,1} = \sigma_2 + 1 - R_0\int_{-1}^{0}\tilde{\psi}_1 U'\,dz,
$$

$$
M_{1,2} = S\int_{-1}^{0}\hat{\psi}_1 U'\,dz,
$$

$$
\begin{aligned}
M_{2,1} \quad = \quad & R_0\int^z \tilde{\psi}_1\frac{H'}{\tau}\,dz|_{z=-1} - \beta_{1,0}R_0\iint^z \tilde{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=0} - R_0\int^z \tilde{\psi}_1\frac{H'}{\tau}\,dz|_{z=0} \\
& +\beta_{2,0}R_0\iint^z \tilde{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=-1} + \beta_{2,0}\beta_{1,0}R_0\iint^z \tilde{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=0} + \beta_{2,0}R_0\int^z \tilde{\psi}_1\frac{H'}{\tau}\,dz|_{z=0},
\end{aligned}
$$

$$
\begin{aligned}
M_{2,2} \quad = \quad & -\sigma_2(\int^z \frac{\hat{\theta}_0}{\tau}\,dz|_{z=-1} + \beta_{2,0}\iint^z \frac{\hat{\theta}_0}{\tau}\,dz\,dz|_{z=-1}) - \int^z \hat{\theta}_0\,dz|_{z=-1} \\
& -S\int^z \hat{\psi}_1\frac{H'}{\tau}\,dz|_{z=-1} + \beta_{1,0}S\iint^z \hat{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=0} + S\int^z \hat{\psi}_1\frac{H'}{\tau}\,dz|_{z=0} \\
& -\beta_{2,0}\iint^z \hat{\theta}_0\,dz\,dz|_{z=-1} - \beta_{2,0}S\iint^z \hat{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=-1} \\
& -\beta_{2,0}\beta_{1,0}S\iint^z \hat{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=0} - \beta_{2,0}S\int^z \hat{\psi}_1\frac{H'}{\tau}\,dz|_{z=0} \\
& +\beta_{1,2}(1-\beta_{2,0})\hat{\theta}_0|_{z=0} - \beta_{2,2}\hat{\theta}_0|_{z=-1}.
\end{aligned}
$$

From the quadratic equation for σ_2 for case II

$$
a = -\int^z \frac{\hat{\theta}_0}{\tau}\,dz|_{z=-1} - \beta_{2,0}\iint^z \frac{\hat{\theta}_0}{\tau}\,dz\,dz|_{z=-1},
$$

$$
\begin{aligned}
b \quad = \quad & -\int^z \hat{\theta}_0\,dz|_{z=-1} - S\int^z \hat{\psi}_1\frac{H'}{\tau}\,dz|_{z=-1} + \beta_{1,0}S\iint^z \hat{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=0} + S\int^z \hat{\psi}_1\frac{H'}{\tau}\,dz|_{z=0} \\
& -\beta_{2,0}\iint^z \hat{\theta}_0\,dz\,dz|_{z=-1} - \beta_{2,0}S\iint^z \hat{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=-1} - \beta_{1,0}\beta_{2,0}S\iint^z \hat{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=0} \\
& -\beta_{2,0}S\int^z \hat{\psi}_1\frac{H'}{\tau}\,dz|_{z=0} + \beta_{1,2}(1-\beta_{2,0})\hat{\theta}_0|_{z=0} - \beta_{2,2}\hat{\theta}_0|_{z=-1} \\
& -(1 - R_0\int_{-1}^{0}\tilde{\psi}_1 U'\,dz)(\int^z \frac{\hat{\theta}_0}{\tau}\,dz|_{z=-1} + \beta_{2,0}\iint^z \frac{\hat{\theta}_0}{\tau}\,dz\,dz|_{z=-1}),
\end{aligned}
$$

$$
\begin{aligned}
c \quad = \quad & (1 - R_0\int_{-1}^{0}\tilde{\psi}_1 U'\,dz)(-\int^z \hat{\theta}_0\,dz|_{z=-1} - S\int^z \hat{\psi}_1\frac{H'}{\tau}\,dz|_{z=-1} + \beta_{1,0}S\iint^z \hat{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=0} \\
& +S\int^z \hat{\psi}_1\frac{H'}{\tau}\,dz|_{z=0} - \beta_{2,0}\iint^z \hat{\theta}_0\,dz\,dz|_{z=-1} - \beta_{2,0}S\iint^z \hat{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=-1} \\
& -\beta_{2,0}\beta_{1,0}S\iint^z \hat{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=0} - \beta_{2,0}S\int^z \hat{\psi}_1\frac{H'}{\tau}\,dz|_{z=0} + \beta_{1,2}(1-\beta_{2,0})\hat{\theta}_0|_{z=0} - \beta_{2,2}\hat{\theta}_0|_{z=-1}), \\
& -S\int_{-1}^{0}\hat{\psi}_1 U'\,dz(R_0\int^z \tilde{\psi}_1\frac{H'}{\tau}\,dz|_{z=-1} - \beta_{1,0}R_0\iint^z \tilde{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=0} - R_0\int^z \tilde{\psi}_1\frac{H'}{\tau}\,dz|_{z=0} \\
& +\beta_{2,0}R_0\iint^z \tilde{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=-1} + \beta_{2,0}\beta_{1,0}R_0\iint^z \tilde{\psi}_1\frac{H'}{\tau}\,dz\,dz|_{z=0} + \beta_{2,0}R_0\int^z \tilde{\psi}_1\frac{H'}{\tau}\,dz|_{z=0}).
\end{aligned}
$$

From ψ_3

$$
c_{12} = -\iint^z D'(R_2u_0 + u_2R_0)\,dz\,dz|_{z=-1} + S\iint^z \theta_2\,dz\,dz|_{z=-1} + c_{13},
$$

$$c_{13} = \iint^z D'(R_2u_0 + u_2R_0)\,dz\,dz|_{z=0} - S\iint^z \theta_2\,dz\,dz|_{z=0},$$

$$\begin{aligned} c_{14} &= \iint^z \psi_1(2+\sigma_2)\,dz\,dz|_{z=-1} - \iiiint^z D'(R_2u_0 + u_2R_0)\,dz\,dz\,dz\,dz|_{z=-1} \\ &+S\iiiint^z \theta_2\,dz\,dz\,dz\,dz|_{z=-1} - \frac{1}{6}c_{12} + \frac{1}{2}c_{13} + c_{15}, \end{aligned}$$

$$\begin{aligned} c_{15} &= \iiiint^z D'(R_2u_0 + u_2R_0)\,dz\,dz\,dz\,dz|_{z=0} \\ &- \iint^z \psi_1(2+\sigma_2)\,dz\,dz|_{z=0} - S\iiiint^z \theta_2\,dz\,dz\,dz\,dz|_{z=0}. \end{aligned}$$

From u_4

$$c_{16} = -\overline{\gamma}_2 u_0 - \int^z u_2(1+\sigma_2)\,dz|_{z=0} - \int^z U'\psi_3\,dz|_{z=0},$$

$$\begin{aligned} c_{17} &= -\int_{-1}^0 \iint^z u_2(1+\sigma_2)\,dz\,dz\,dz \\ &+\frac{1}{2}c_{16} - \frac{1}{6}u_0\sigma_4 - \int_{-1}^0 \iint^z U'\psi_3\,dz\,dz\,dz. \end{aligned}$$

From θ_4 for case I

$$\begin{aligned} c_{18} &= -\beta_{1,0}\Big(\iint^z \frac{H'}{\tau}\psi_3\,dz\,dz|_{z=0} + (\frac{\sigma_2}{\tau}+1)\iint^z \theta_2\,dz\,dz|_{z=0} + c_{19}\Big) \\ &- \int^z \frac{H'}{\tau}\psi_3\,dz|_{z=0} - (\frac{\sigma_2}{\tau}+1)\int^z \theta_2\,dz|_{z=0} - \beta_{1,2}\theta_2|_{z=0}, \end{aligned}$$

$$\begin{aligned} c_{19} &= \Big(-\int^z \frac{H'}{\tau}\psi_3\,dz|_{z=-1} - (\frac{\sigma_2}{\tau}+1)\int^z \theta_2\,dz|_{z=-1} \\ &+\Big(-\beta_{1,0}\Big(\iint^z \frac{H'}{\tau}\psi_3\,dz\,dz|_{z=0} + (\frac{\sigma_2}{\tau}+1)\iint^z \theta_2\,dz\,dz|_{z=0}\Big) \\ &- \int^z \frac{H'}{\tau}\psi_3\,dz|_{z=0} - (\frac{\sigma_2}{\tau}+1)\int^z \theta_2\,dz|_{z=0} - \beta_{1,2}\theta_2|_{z=0}\Big)(\beta_{2,0}-1) \\ &-\beta_{2,0}\Big(\iint^z \frac{H'}{\tau}\psi_3\,dz\,dz|_{z=-1} + (\frac{\sigma_2}{\tau}+1)\iint^z \theta_2\,dz\,dz|_{z=-1}\Big) - \beta_{2,2}\theta_2|_{z=-1}\Big)/(-\beta_{1,0} + \beta_{1,0}\beta_{2,0} + \beta_{2,0}). \end{aligned}$$

From θ_4 for case II

$$\begin{aligned} c_{18} &= -\beta_{1,0}\Big(\iint^z \frac{H'}{\tau}\psi_3\,dz\,dz|_{z=0} + (\frac{\sigma_2}{\tau}+1)\iint^z \theta_2\,dz\,dz|_{z=0} + \iint^z \theta_0\frac{\sigma_4}{\tau}\,dz\,dz|_{z=0} + c_{19}\Big) \\ &- \int^z \frac{H'}{\tau}\psi_3\,dz|_{z=0} - (\frac{\sigma_2}{\tau}+1)\int^z \theta_2\,dz|_{z=0} - \int^z \theta_0\frac{\sigma_4}{\tau}\,dz|_{z=0} - \beta_{1,2}\theta_2|_{z=0} - \beta_{1,4}\theta_0|_{z=0}. \end{aligned}$$

From the matrix N for case II

$$N_{1,1} = u_0,$$

$$N_{1,2} = \int_{-1}^0 \check{\psi}_3 U'\,dz,$$

$$\begin{aligned} N_{2,1} &= \int^z \frac{\theta_0}{\tau}\,dz|_{z=-1} - \beta_{1,0}\iint^z \frac{\theta_0}{\tau}\,dz\,dz|_{z=0} - \int^z \frac{\theta_0}{\tau}\,dz|_{z=0} \\ &+\beta_{2,0}\iint^z \frac{\theta_0}{\tau}\,dz\,dz|_{z=-1} + \beta_{2,0}\beta_{1,0}\iint^z \frac{\theta_0}{\tau}\,dz\,dz|_{z=0} + \beta_{2,0}\int^z \frac{\theta_0}{\tau}\,dz|_{z=0}, \end{aligned}$$

$$
\begin{aligned}
N_{2,2} \;=\;& \int^z \hat{\theta}_2 \frac{\sigma_2}{\tau}\,dz|_{z=-1} + \int^z \hat{\theta}_2\,dz|_{z=-1} + \int^z \breve{\psi}_3 \frac{H'}{\tau}\,dz|_{z=-1} - \beta_{1,0} \iint^z \hat{\theta}_2 \frac{\sigma_2}{\tau}\,dz\,dz|_{z=0} \\
& -\beta_{1,0} \iint^z \hat{\theta}_2\,dz\,dz|_{z=0} - \beta_{1,0} \iint^z \breve{\psi}_3 \frac{H'}{\tau}\,dz\,dz|_{z=0} - \int^z \hat{\theta}_2 \frac{\sigma_2}{\tau}\,dz|_{z=0} - \int^z \hat{\theta}_2\,dz|_{z=0} - \int^z \breve{\psi}_3 \frac{H'}{\tau}\,dz|_{z=0} \\
& +\beta_{2,0} \iint^z \hat{\theta}_2 \frac{\sigma_2}{\tau}\,dz\,dz|_{z=-1} + \beta_{2,0} \iint^z \hat{\theta}_2\,dz\,dz|_{z=-1} + \beta_{2,0} \iint^z \breve{\psi}_3 \frac{H'}{\tau}\,dz\,dz|_{z=-1} \\
& +\beta_{1,0}\beta_{2,0} \iint^z \hat{\theta}_2 \frac{\sigma_2}{\tau}\,dz\,dz|_{z=0} + \beta_{1,0}\beta_{2,0} \iint^z \hat{\theta}_2\,dz\,dz|_{z=0} + \beta_{1,0}\beta_{2,0} \iint^z \breve{\psi}_3 \frac{H'}{\tau}\,dz\,dz|_{z=0} \\
& +\beta_{2,0} \int^z \hat{\theta}_2 \frac{\sigma_2}{\tau}\,dz|_{z=0} + \beta_{2,0} \int^z \hat{\theta}_2\,dz|_{z=0} + \beta_{2,0} \int^z \breve{\psi}_3 \frac{H'}{\tau}\,dz|_{z=0} \\
& -\beta_{1,2}(1-\beta_{2,0})\hat{\theta_2}|_{z=0} + \beta_{2,2}\hat{\theta_2}|_{z=-1}.
\end{aligned}
$$

From the vector $\mathbf{q}$ for case II

$$
q_1 = -\int_{-1}^{0} \hat{\psi}_3 U'\,dz + R_2 \int_{-1}^{0} \tilde{\psi}_3 U'\,dz + \overline{\gamma}_4 u_0 - \overline{\gamma}_2 u_0,
$$

$$
\begin{aligned}
q_2 \;=\;& -\int^z \tilde{\theta}_2 \frac{\sigma_2}{\tau}\,dz|_{z=-1} - \int^z \tilde{\theta}_2\,dz|_{z=-1} - \int^z \hat{\psi}_3 \frac{H'}{\tau}\,dz|_{z=-1} + R_2 \int^z \tilde{\psi}_3 \frac{H'}{\tau}\,dz|_{z=-1} \\
& +\beta_{1,0}\Big(\iint^z \tilde{\theta}_2 \frac{\sigma_2}{\tau}\,dz\,dz|_{z=0} + \iint^z \tilde{\theta}_2\,dz\,dz|_{z=0} + \iint^z \hat{\psi}_3 \frac{H'}{\tau}\,dz\,dz|_{z=0} - R_2 \iint^z \tilde{\psi}_3 \frac{H'}{\tau}\,dz\,dz|_{z=0}\Big) \\
& +\int^z \tilde{\theta}_2 \frac{\sigma_2}{\tau}\,dz|_{z=0} + \int^z \tilde{\theta}_2\,dz|_{z=0} + \int^z \hat{\psi}_3 \frac{H'}{\tau}\,dz|_{z=0} - R_2 \int^z \tilde{\psi}_3 \frac{H'}{\tau}\,dz|_{z=0} \\
& -\beta_{2,0}\Big(\iint^z \tilde{\theta}_2 \frac{\sigma_2}{\tau}\,dz\,dz|_{z=-1} + \iint^z \tilde{\theta}_2\,dz\,dz|_{z=-1} + \iint^z \hat{\psi}_3 \frac{H'}{\tau}\,dz\,dz|_{z=-1} - R_2 \iint^z \tilde{\psi}_3 \frac{H'}{\tau}\,dz\,dz|_{z=-1} \\
& -\Big(-\beta_{1,0}\Big(\iint^z \tilde{\theta}_2 \frac{\sigma_2}{\tau}\,dz\,dz|_{z=0} + \iint^z \tilde{\theta}_2\,dz\,dz|_{z=0} + \iint^z \hat{\psi}_3 \frac{H'}{\tau}\,dz\,dz|_{z=0} - R_2 \iint^z \tilde{\psi}_3 \frac{H'}{\tau}\,dz\,dz|_{z=0}\Big) \\
& -\int^z \tilde{\theta}_2 \frac{\sigma_2}{\tau}\,dz|_{z=0} - \int^z \tilde{\theta}_2\,dz|_{z=0} - \int^z \hat{\psi}_3 \frac{H'}{\tau}\,dz|_{z=0} + R_2 \int^z \tilde{\psi}_3 \frac{H'}{\tau}\,dz|_{z=0}\Big)\Big) \\
& +\beta_{1,2}(1-\beta_{2,0})\tilde{\theta_2}|_{z=0} + \beta_{1,4}(1-\beta_{2,0})\theta_0|_{z=0} - \beta_{2,2}\tilde{\theta_2}|_{z=-1} - \beta_{2,4}\theta_0|_{z=-1}.
\end{aligned}
$$

7.2 Linear perturbation solution algorithm details

From u_{2j} in the linear perturbation solution algorithm

$$
c_{0,j} = -\overline{\gamma}_2 u_{2j-4}|_{z=0} - \int_{-1}^{0} U'\psi_{2j-1}\,dz|_{z=0} - \int^z u_{2j-2}\,dz|_{z=0} - \int^z \sum_{m=0}^{j-1} u_{2j-(2m+2)}\sigma_{2m+2}\,dz|_{z=0},
$$

$$
c_{1,j} = -\int_{-1}^{0}\Big[\iint^z \sum_{m=0}^{j-1} u_{2j-(2m+2)}\sigma_{2m+2}\,dz\,dz + \iint^z u_{2j-2}\,dz\,dz + \iint^z U'\psi_{2j-1}\,dz\,dz\Big]\,dz + \delta_{0,j}u_0 + \frac{c_{0,j}}{2}.
$$

From θ_{2j} in the linear perturbation solution algorithm for case I

$$
\begin{aligned}
c_{2,j} \;=\;& -\beta_{1,0}\Big(\iint^z \theta_{2j-2}\,dz\,dz|_{z=0} + \iint^z \frac{H'}{\tau}\psi_{2j-1}\,dz\,dz|_{z=0} \\
& + \iint^z \sum_{m=0}^{j-1} \theta_{2j-(2m+2)} \frac{\sigma_{2m+2}}{\tau}\,dz\,dz|_{z=0} + c_{3,j}\Big) - \int^z \theta_{2j-2}\,dz|_{z=0} - \int^z \frac{H'}{\tau}\psi_{2j-1}\,dz|_{z=0} \\
& - \int^z \sum_{m=0}^{j-1} \theta_{2j-(2m+2)} \frac{\sigma_{2m+2}}{\tau}\,dz|_{z=0} - \sum_{m=1}^{j} \beta_{1,2m}\theta_{2j-2m}|_{z=0},
\end{aligned}
$$

$$\begin{aligned}
c_{3,j} \;=\; & (\int^z \theta_{2j-2}\,dz|_{z=-1} + \int^z \frac{H'}{\tau}\psi_{2j-1}\,dz|_{z=-1} + \int^z \sum_{m=0}^{j-1}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz|_{z=-1} \\
& -\beta_{1,0}(\iint^z \theta_{2j-2}\,dz\,dz|_{z=0} + \iint^z \frac{H'}{\tau}\psi_{2j-1}\,dz\,dz|_{z=0} + \iint^z \sum_{m=0}^{j-1}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz\,dz|_{z=0}) \\
& -\int^z \theta_{2j-2}\,dz|_{z=0} - \int^z \frac{H'}{\tau}\psi_{2j-1}\,dz|_{z=0} - \int^z \sum_{m=0}^{j-1}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz|_{z=0} - \sum_{m=1}^{j}\beta_{1,2m}\theta_{2j-2m}|_{z=0} \\
& +\beta_{2,0}(\iint^z \theta_{2j-2}\,dz\,dz|_{z=-1} + \iint^z \frac{H'}{\tau}\psi_{2j-1}\,dz\,dz|_{z=-1} + \iint^z \sum_{m=0}^{j-1}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz\,dz|_{z=-1} \\
& +\beta_{1,0}(\iint^z \theta_{2j-2}\,dz\,dz|_{z=0} + \iint^z \frac{H'}{\tau}\psi_{2j-1}\,dz\,dz|_{z=0} + \iint^z \sum_{m=0}^{j-1}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz\,dz|_{z=0}) \\
& +\int^z \theta_{2j-2}\,dz|_{z=0} + \int^z \frac{H'}{\tau}\psi_{2j-1}\,dz|_{z=0} + \int^z \sum_{m=0}^{j-1}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz|_{z=0} + \sum_{m=1}^{j}\beta_{1,2m}\theta_{2j-2m}|_{z=0}) \\
& +\sum_{m=1}^{j}\beta_{2,2m}\theta_{2j-2m}|_{z=-1})/(\beta_{1,0} - \beta_{1,0}\beta_{2,0} - \beta_{2,0}).
\end{aligned}$$

From θ_{2j} in the linear perturbation solution algorithm for case II

$$\begin{aligned}
c_{2,j} \;=\; & -\beta_{1,0}(\iint^z \theta_{2j-2}\,dz\,dz|_{z=0} + \iint^z \frac{H'}{\tau}\psi_{2j-1}\,dz\,dz|_{z=0} \\
& +\iint^z \sum_{m=0}^{j-1}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz\,dz|_{z=0} + c_{3,j}) - \sum_{m=1}^{j}\beta_{1,2m}\theta_{2j-2m}|_{z=0} - \int^z \theta_{2j-2}\,dz|_{z=0} - \int^z \frac{H'}{\tau}\psi_{2j-1}\,dz|_{z=0} \\
& -\int^z \sum_{m=0}^{j-1}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz|_{z=0}.
\end{aligned}$$

From ψ_{2j+1} in the linear perturbation solution algorithm

$$\begin{aligned}
c_{4,j} \;=\; & -\iint^z \sum_{m=0}^{j} D'u_{2j-2m}R_{2m}\,dz\,dz|_{z=-1} + S\iint^z \theta_{2j}\,dz\,dz|_{z=-1} \\
& -\iint^z \sum_{m=0}^{j-2}\psi_{2j-2m-3}\sigma_{2m+2}\,dz\,dz|_{z=-1} - \iint^z \psi_{2j-3}\,dz\,dz|_{z=-1} + c_{5,j} + \overline{\gamma}_3\psi'_{2j-3}|_{z=-1},
\end{aligned}$$

$$\begin{aligned}
c_{5,j} \;=\; & \iint^z \sum_{m=0}^{j} D'u_{2j-2m}R_{2m}\,dz\,dz|_{z=0} - S\iint^z \theta_{2j}\,dz\,dz|_{z=0} \\
& +\iint^z \sum_{m=0}^{j-2}\psi_{2j-2m-3}\sigma_{2m+2}\,dz\,dz|_{z=0} + \iint^z \psi_{2j-3}\,dz\,dz|_{z=0} - \overline{\gamma}_1\psi'_{2j-3}|_{z=0},
\end{aligned}$$

$$\begin{aligned}
c_{6,j} \;=\; & -\iiiint^z \sum_{m=0}^{j} D'u_{2j-2m}R_{2m}\,dz\,dz\,dz\,dz|_{z=-1} + S\iiiint^z \theta_{2j}\,dz\,dz\,dz\,dz|_{z=-1} \\
& +\iint^z \sum_{m=0}^{j-1}\psi_{2j-2m-1}\sigma_{2m+2}\,dz\,dz|_{z=-1} - \iiiint^z \sum_{m=0}^{j-2}\psi_{2j-2m-3}\sigma_{2m+2}\,dz\,dz\,dz\,dz|_{z=-1} \\
& -\iiiint^z \psi_{2j-3}\,dz\,dz\,dz\,dz|_{z=-1} + 2\iint^z \psi_{2j-1}\,dz\,dz|_{z=-1} - \frac{1}{6}c_{4,j} + \frac{1}{2}c_{5,j} + c_{7,j},
\end{aligned}$$

$$
\begin{aligned}
c_{7,j} \;=\;& \iiiint^z \sum_{m=0}^{j} D' u_{2j-2m} R_{2m}\, dz\, dz\, dz\, dz|_{z=0} - S \iiiint^z \theta_{2j}\, dz\, dz\, dz\, dz|_{z=0} \\
& - \iint^z \sum_{m=0}^{j-1} \psi_{2j-2m-1} \sigma_{2m+2}\, dz\, dz|_{z=0} + \iiiint^z \sum_{m=0}^{j-2} \psi_{2j-2m-3} \sigma_{2m+2}\, dz\, dz\, dz\, dz|_{z=0} \\
& + \iiiint^z \psi_{2j-3}\, dz\, dz\, dz\, dz|_{z=0} - 2 \iint^z \psi_{2j-1}\, dz\, dz|_{z=0}.
\end{aligned}
$$

From the matrix N in the linear perturbation solution algorithm for case II

$$
N_{1,1,j} = u_0,
$$

$$
N_{1,2,j} = \int_{-1}^{0} \breve{\psi}_{2j-1} U'\, dz,
$$

$$
\begin{aligned}
N_{2,1,j} \;=\;& \int^z \frac{\theta_0}{\tau}\, dz|_{z=-1} - \beta_{1,0} \iint^z \frac{\theta_0}{\tau}\, dz\, dz|_{z=0} - \int^z \frac{\theta_0}{\tau}\, dz|_{z=0} \\
& + \beta_{2,0} \iint^z \frac{\theta_0}{\tau}\, dz\, dz|_{z=-1} + \beta_{2,0}\beta_{1,0} \iint^z \frac{\theta_0}{\tau}\, dz\, dz|_{z=0} + \beta_{2,0} \int^z \frac{\theta_0}{\tau}\, dz|_{z=0},
\end{aligned}
$$

$$
\begin{aligned}
N_{2,2,j} \;=\;& \int^z \hat{\theta}_{2j-2}\, dz|_{z=-1} + \int^z \breve{\psi}_{2j-1} \frac{H'}{\tau}\, dz|_{z=-1} + \int^z \hat{\theta}_{2j-2} \frac{\sigma_2}{\tau}\, dz|_{z=-1} \\
& - \beta_{1,0}\Big(\iint^z \hat{\theta}_{2j-2}\, dz\, dz|_{z=0} + \iint^z \breve{\psi}_{2j-1} \frac{H'}{\tau}\, dz\, dz|_{z=0} + \iint^z \hat{\theta}_{2j-2} \frac{\sigma_2}{\tau}\, dz\, dz|_{z=0}\Big) \\
& - \int^z \hat{\theta}_{2j-2}\, dz|_{z=0} - \int^z \breve{\psi}_{2j-1} \frac{H'}{\tau}\, dz|_{z=0} - \int^z \hat{\theta}_{2j-2} \frac{\sigma_2}{\tau}\, dz|_{z=0} \\
& + \beta_{2,0}\Big(\iint^z \hat{\theta}_{2j-2}\, dz\, dz|_{z=-1} + \iint^z \breve{\psi}_{2j-1} \frac{H'}{\tau}\, dz\, dz|_{z=-1} + \iint^z \hat{\theta}_{2j-2} \frac{\sigma_2}{\tau}\, dz\, dz|_{z=-1} \\
& + \beta_{1,0}\Big(\iint^z \hat{\theta}_{2j-2}\, dz\, dz|_{z=0} + \iint^z \breve{\psi}_{2j-1} \frac{H'}{\tau}\, dz\, dz|_{z=0} + \iint^z \hat{\theta}_{2j-2} \frac{\sigma_2}{\tau}\, dz\, dz|_{z=0}\Big) \\
& + \int^z \hat{\theta}_{2j-2}\, dz|_{z=0} + \int^z \breve{\psi}_{2j-1} \frac{H'}{\tau}\, dz|_{z=0} + \int^z \hat{\theta}_{2j-2} \frac{\sigma_2}{\tau}\, dz|_{z=0}\Big) \\
& - \beta_{1,2}(1 - \beta_{2,0}) \hat{\theta}_{2j-2}|_{z=0} + \beta_{2,2} \hat{\theta}_{2j-2}|_{z=-1}.
\end{aligned}
$$

From the vector $\mathbf{q}$ in the linear perturbation solution algorithm for case II

$$
q_{1,j} = -\delta_{0,j-1} u_0 - \int_{-1}^{0} \hat{\psi}_{2j-1} U'\, dz + R_{2j-2} \int_{-1}^{0} \tilde{\psi}_{2j-1} U'\, dz + \overline{\gamma}_4 u_{2j-4}|_{z=-1} - \overline{\gamma}_2 u_{2j-4}|_{z=0},
$$

$$
\begin{aligned}
q_{2,j} \;=\;& -\int^z \tilde{\theta}_{2j-2}\,dz|_{z=-1} - \int^z (\hat{\psi}_{2j-1} - R_{2j-2}\tilde{\psi}_{2j-1})\frac{H'}{\tau}\,dz|_{z=-1} - \int^z \sum_{m=1}^{j-2}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz|_{z=-1}\\
& - \int^z \tilde{\theta}_{2j-2}\frac{\sigma_2}{\tau}\,dz|_{z=-1} + \beta_{1,0}\Big(\iint^z \tilde{\theta}_{2j-2}\,dz\,dz|_{z=0} + \iint^z (\hat{\psi}_{2j-1} - R_{2j-2}\tilde{\psi}_{2j-1})\frac{H'}{\tau}\,dz\,dz|_{z=0}\\
& + \iint^z \sum_{m=1}^{j-2}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz\,dz|_{z=0} + \iint^z \tilde{\theta}_{2j-2}\frac{\sigma_2}{\tau}\,dz\,dz|_{z=0}\Big) + \int^z \tilde{\theta}_{2j-2}\,dz|_{z=0}\\
& + \int^z (\hat{\psi}_{2j-1} - R_{2j-2}\tilde{\psi}_{2j-1})\frac{H'}{\tau}\,dz|_{z=0} + \int^z \sum_{m=1}^{j-2}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz|_{z=0} + \int^z \tilde{\theta}_{2j-2}\frac{\sigma_2}{\tau}\,dz|_{z=0}\\
& -\beta_{2,0}\Big(\iint^z \tilde{\theta}_{2j-2}\,dz\,dz|_{z=-1} + \iint^z (\hat{\psi}_{2j-1} - R_{2j-2}\tilde{\psi}_{2j-1})\frac{H'}{\tau}\,dz\,dz|_{z=-1} + \iint^z \sum_{m=1}^{j-2}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz\,dz|_{z=-1}\\
& + \iint^z \tilde{\theta}_{2j-2}\frac{\sigma_2}{\tau}\,dz\,dz|_{z=-1} - \Big(-\beta_{1,0}\Big(\iint^z \tilde{\theta}_{2j-2}\,dz\,dz|_{z=0} + \iint^z (\hat{\psi}_{2j-1} - R_{2j-2}\tilde{\psi}_{2j-1})\frac{H'}{\tau}\,dz\,dz|_{z=0}\\
& + \iint^z \sum_{m=1}^{j-2}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz\,dz|_{z=0} + \iint^z \tilde{\theta}_{2j-2}\frac{\sigma_2}{\tau}\,dz\,dz|_{z=0}\Big) - \int^z \tilde{\theta}_{2j-2}\,dz|_{z=0}\\
& - \int^z (\hat{\psi}_{2j-1} - R_{2j-2}\tilde{\psi}_{2j-1})\frac{H'}{\tau}\,dz|_{z=0} - \int^z \sum_{m=1}^{j-2}\theta_{2j-(2m+2)}\frac{\sigma_{2m+2}}{\tau}\,dz|_{z=0} - \int^z \tilde{\theta}_{2j-2}\frac{\sigma_2}{\tau}\,dz|_{z=0}\Big)\Big)\\
& +\Big(\sum_{m=2}^{j}\beta_{1,2m}\theta_{2j-2m}|_{z=0} + \beta_{1,2}\tilde{\theta}_{2j-2}|_{z=0}\Big)(1-\beta_{2,0}) - \sum_{m=2}^{j}\beta_{2,2m}\theta_{2j-2m}|_{z=-1} - \beta_{2,2}\tilde{\theta}_{2j-2}|_{z=-1}.
\end{aligned}
$$

7.3 Nonlinear perturbation solution details

From Ψ_0

$$
c_4(Y,T) = -\iint^z D'R_0\frac{\partial \mathfrak{u}_0}{\partial Y}\,dz\,dz|_{z=-1} + S\iint^z \frac{\partial \Theta_0}{\partial Y}\,dz\,dz|_{z=-1} + c_5(Y,T),
$$

$$
c_5(Y,T) = \iint^z D'R_0\frac{\partial \mathfrak{u}_0}{\partial Y}\,dz\,dz|_{z=0} - S\iint^z \frac{\partial \Theta_0}{\partial Y}\,dz\,dz|_{z=0},
$$

$$
\begin{aligned}
c_6(Y,T) \;=\;& -\iiiint^z D'R_0\frac{\partial \mathfrak{u}_0}{\partial Y}\,dz\,dz\,dz\,dz|_{z=-1} + S\iiiint^z \frac{\partial \Theta_0}{\partial Y}\,dz\,dz\,dz\,dz|_{z=-1}\\
& -\frac{1}{6}c_4(Y,T) + \frac{1}{2}c_5(Y,T) + c_7(Y,T),
\end{aligned}
$$

$$
c_7(Y,T) = \iiiint^z D'R_0\frac{\partial \mathfrak{u}_0}{\partial Y}\,dz\,dz\,dz\,dz|_{z=0} - S\iiiint^z \frac{\partial \Theta_0}{\partial Y}\,dz\,dz\,dz\,dz|_{z=0}.
$$

From $\mathfrak{u}_2$

$$
c_8(Y,T) = \int^z \frac{\partial \Psi_0}{\partial Y}U'\,dz|_{z=0}.
$$

From Θ_2 for case I

$$
c_{10}(Y,T) = \int^z \frac{\partial \Psi_0}{\partial Y}\frac{H'}{\tau}\,dz|_{z=0} - \beta_{1,0}\Big(-\iint^z \frac{\partial \Psi_0}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=0} + c_{11}(Y,T)\Big),
$$

$$
\begin{aligned}
c_{11}(Y,T) \;=\;& \Big(\int^z \frac{\partial \Psi_0}{\partial Y}\frac{H'}{\tau}\,dz|_{z=-1} - \int^z \frac{\partial \Psi_0}{\partial Y}\frac{H'}{\tau}\,dz|_{z=0} - \beta_{1,0}\iint^z \frac{\partial \Psi_0}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=0}\\
& +\beta_{2,0}\iint^z \frac{\partial \Psi_0}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=-1} + \beta_{2,0}\int^z \frac{\partial \Psi_0}{\partial Y}\frac{H'}{\tau}\,dz|_{z=0}\\
& +\beta_{2,0}\beta_{1,0}\iint^z \frac{\partial \Psi_0}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=0}\Big)/(-\beta_{1,0} + \beta_{2,0}\beta_{1,0} + \beta_{2,0}).
\end{aligned}
$$

From Θ_2 for case II

$$\begin{aligned}
c_{10}(Y,T) &= -\int^z \frac{1}{\tau}\frac{\partial\Theta_0}{\partial T}\,dz|_{z=0} + \int^z \frac{\partial^2\Theta_0}{\partial Y^2}\,dz|_{z=0} + \int^z \frac{\partial\Psi_0}{\partial Y}\frac{H'}{\tau}\,dz|_{z=0} \\
&\quad + \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz|_{z=0} - \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz|_{z=0} \\
&\quad -\beta_{1,0}(\iint^z \frac{1}{\tau}\frac{\partial\Theta_0}{\partial T}\,dz\,dz|_{z=0} - \iint^z \frac{\partial^2\Theta_0}{\partial Y^2}\,dz\,dz|_{z=0} - \iint^z \frac{\partial\Psi_0}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=0} \\
&\quad - \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz\,dz|_{z=0} + \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz\,dz|_{z=0} + c_{11}(Y,T)) - \beta_{1,2}\Theta_0|_{z=0}.
\end{aligned}$$

For case II, the second coupled nonlinear partial differential equation for $\mathfrak{u}_0$ and $c_3(Y,T)$ at $O(l^2)$ is

$$\begin{aligned}
&\int^z \frac{1}{\tau}\frac{\partial\Theta_0}{\partial T}\,dz|_{z=-1} - \int^z \frac{\partial^2\Theta_0}{\partial Y^2}\,dz|_{z=-1} - \int^z \frac{\partial\Psi_0}{\partial Y}\frac{H'}{\tau}\,dz|_{z=-1} - \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz|_{z=-1} \\
&+ \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz|_{z=-1} - \beta_{1,0}(\iint^z \frac{1}{\tau}\frac{\partial\Theta_0}{\partial T}\,dz\,dz|_{z=0} - \iint^z \frac{\partial^2\Theta_0}{\partial Y^2}\,dz\,dz|_{z=0} - \iint^z \frac{\partial\Psi_0}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=0} \\
&- \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz\,dz|_{z=0} + \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz\,dz|_{z=0}) - \int^z \frac{1}{\tau}\frac{\partial\Theta_0}{\partial T}\,dz|_{z=0} + \int^z \frac{\partial^2\Theta_0}{\partial Y^2}\,dz|_{z=0} \\
&+ \int^z \frac{\partial\Psi_0}{\partial Y}\frac{H'}{\tau}\,dz|_{z=0} + \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz|_{z=0} - \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz|_{z=0} + \beta_{2,0}(\iint^z \frac{1}{\tau}\frac{\partial\Theta_0}{\partial T}\,dz\,dz|_{z=-1} \\
&- \iint^z \frac{\partial^2\Theta_0}{\partial Y^2}\,dz\,dz|_{z=-1} - \iint^z \frac{\partial\Psi_0}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=-1} - \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz\,dz|_{z=-1} \\
&+ \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz\,dz|_{z=-1} + \beta_{1,0}(\iint^z \frac{1}{\tau}\frac{\partial\Theta_0}{\partial T}\,dz\,dz|_{z=0} - \iint^z \frac{\partial^2\Theta_0}{\partial Y^2}\,dz\,dz|_{z=0} \\
&- \iint^z \frac{\partial\Psi_0}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=0} - \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz\,dz|_{z=0} + \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz\,dz|_{z=0}) \\
&+ \int^z \frac{1}{\tau}\frac{\partial\Theta_0}{\partial T}\,dz|_{z=0} - \int^z \frac{\partial^2\Theta_0}{\partial Y^2}\,dz|_{z=0} - \int^z \frac{\partial\Psi_0}{\partial Y}\frac{H'}{\tau}\,dz|_{z=0} - \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz|_{z=0} \\
&+ \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz|_{z=0}) - \beta_{1,2}(1-\beta_{2,0})\Theta_0|_{z=0} + \beta_{2,2}\Theta_0|_{z=-1} = 0.
\end{aligned}$$

The constants appearing in the differential equations (4.1.44), (4.1.45) are

$$a_p = p^2 - R_0 p^2 \int_{-1}^{0} \tilde{\Psi}_0 U'\,dz,$$

$$b_p = S\,p^2 \int_{-1}^{0} \hat{\Psi}_0 U'\,dz,$$

$$c_p = -R_0 p^2 \int_{-1}^{0} \tilde{\Psi}_0 H'\,dz,$$

$$d_p = S\,p^2 \int_{-1}^{0} \hat{\Psi}_0 H'\,dz - \tau(\beta_{2,2} - \beta_{1,2} - p^2).$$

From Ψ_2

$$\begin{aligned}
c_{12}(Y,T) &= \frac{\partial\Psi_0}{\partial T}|_{z=-1} - 2\frac{\partial^2\Psi_0}{\partial Y^2}|_{z=-1} - \iint^z D' R_2 \frac{\partial\mathfrak{u}_0}{\partial Y}\,dz\,dz|_{z=-1} \\
&\quad - \iint^z D' R_0 \frac{\partial\mathfrak{u}_2}{\partial Y}\,dz\,dz|_{z=-1} + S\iint^z \frac{\partial\Theta_2}{\partial Y}\,dz\,dz|_{z=-1} \\
&\quad + \iint^z \frac{\partial\Psi_0}{\partial z}\frac{\partial^3\Psi_0}{\partial Y\partial z^2}\,dz\,dz|_{z=-1} - \iint^z \frac{\partial\Psi_0}{\partial Y}\frac{\partial^3\Psi_0}{\partial z^3}\,dz\,dz|_{z=-1} + c_{13}(Y,T),
\end{aligned}$$

$$\begin{aligned}
c_{13}(Y,T) &= -\frac{\partial\Psi_0}{\partial T}|_{z=0} + 2\frac{\partial^2\Psi_0}{\partial Y^2}|_{z=0} + \iint^z D' R_2 \frac{\partial\mathfrak{u}_0}{\partial Y}\,dz\,dz|_{z=0} \\
&\quad + \iint^z D' R_0 \frac{\partial\mathfrak{u}_2}{\partial Y}\,dz\,dz|_{z=0} - S\iint^z \frac{\partial\Theta_2}{\partial Y}\,dz\,dz|_{z=0} \\
&\quad - \iint^z \frac{\partial\Psi_0}{\partial z}\frac{\partial^3\Psi_0}{\partial Y\partial z^2}\,dz\,dz|_{z=0} + \iint^z \frac{\partial\Psi_0}{\partial Y}\frac{\partial^3\Psi_0}{\partial z^3}\,dz\,dz|_{z=0},
\end{aligned}$$

$$
\begin{aligned}
c_{14}(Y,T) \quad = \quad & \iint^{z} \frac{\partial \Psi_0}{\partial T}\, dz\, dz|_{z=-1} - \iint^{z} 2\frac{\partial^2 \Psi_0}{\partial Y^2}\, dz\, dz|_{z=-1} - \iiiint^{z} D' R_2 \frac{\partial \mathfrak{u}_0}{\partial Y}\, dz\, dz\, dz\, dz|_{z=-1} \\
& - \iiiint^{z} D' R_0 \frac{\partial \mathfrak{u}_2}{\partial Y}\, dz\, dz\, dz\, dz|_{z=-1} + S \iiiint^{z} \frac{\partial \Theta_2}{\partial Y}\, dz\, dz\, dz\, dz|_{z=-1} \\
& + \iiiint^{z} \frac{\partial \Psi_0}{\partial z} \frac{\partial^3 \Psi_0}{\partial Y \partial z^2}\, dz\, dz\, dz\, dz|_{z=-1} - \iiiint^{z} \frac{\partial \Psi_0}{\partial Y} \frac{\partial^3 \Psi_0}{\partial z^3}\, dz\, dz\, dz\, dz|_{z=-1} \\
& - \frac{1}{6} c_{12}(Y,T) + \frac{1}{2} c_{13}(Y,T) + c_{15}(Y,T),
\end{aligned}
$$

$$
\begin{aligned}
c_{15}(Y,T) \quad = \quad & -\iint^{z} \frac{\partial \Psi_0}{\partial T}\, dz\, dz|_{z=0} + \iint^{z} 2\frac{\partial^2 \Psi_0}{\partial Y^2}\, dz\, dz|_{z=0} + \iiiint^{z} D' R_2 \frac{\partial \mathfrak{u}_0}{\partial Y}\, dz\, dz\, dz\, dz|_{z=0} \\
& + \iiiint^{z} D' R_0 \frac{\partial \mathfrak{u}_2}{\partial Y}\, dz\, dz\, dz\, dz|_{z=0} - S \iiiint^{z} \frac{\partial \Theta_2}{\partial Y}\, dz\, dz\, dz\, dz|_{z=0} \\
& - \iiiint^{z} \frac{\partial \Psi_0}{\partial z} \frac{\partial^3 \Psi_0}{\partial Y \partial z^2}\, dz\, dz\, dz\, dz|_{z=0} + \iiiint^{z} \frac{\partial \Psi_0}{\partial Y} \frac{\partial^3 \Psi_0}{\partial z^3}\, dz\, dz\, dz\, dz|_{z=0}.
\end{aligned}
$$

From $\mathfrak{u}_4$

$$
\begin{aligned}
c_{16}(Y,T) \quad = \quad & -\int^{z} \frac{\partial \mathfrak{u}_2}{\partial T}\, dz|_{z=0} + \int^{z} \frac{\partial^2 \mathfrak{u}_2}{\partial Y^2}\, dz|_{z=0} + \int^{z} \frac{\partial \Psi_2}{\partial Y} U'\, dz|_{z=0} \\
& + \int^{z} \frac{\partial \Psi_0}{\partial Y} \frac{\partial \mathfrak{u}_2}{\partial z}\, dz|_{z=0} - \int^{z} \frac{\partial \Psi_2}{\partial z} \frac{\partial \mathfrak{u}_0}{\partial Y}\, dz|_{z=0} - \int^{z} \frac{\partial \Psi_0}{\partial z} \frac{\partial \mathfrak{u}_2}{\partial Y}\, dz|_{z=0} - \overline{\gamma}_2 \mathfrak{u}_0.
\end{aligned}
$$

From Θ_4 for case I

$$
\begin{aligned}
c_{18}(Y,T) \quad = \quad & -\int^{z} \frac{1}{\tau} \frac{\partial \Theta_2}{\partial T}\, dz|_{z=0} + \int^{z} \frac{\partial^2 \Theta_2}{\partial Y^2}\, dz|_{z=0} + \int^{z} \frac{\partial \Psi_2}{\partial Y} \frac{H'}{\tau}\, dz|_{z=0} \\
& + \int^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial Y} \frac{\partial \Theta_2}{\partial z}\, dz|_{z=0} - \int^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial z} \frac{\partial \Theta_2}{\partial Y}\, dz|_{z=0} \\
& - \beta_{1,0}(\iint^{z} \frac{1}{\tau} \frac{\partial \Theta_2}{\partial T}\, dz\, dz|_{z=0} - \iint^{z} \frac{\partial^2 \Theta_2}{\partial Y^2}\, dz\, dz|_{z=0} - \iint^{z} \frac{\partial \Psi_2}{\partial Y} \frac{H'}{\tau}\, dz\, dz|_{z=0} \\
& - \iint^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial Y} \frac{\partial \Theta_2}{\partial z}\, dz\, dz|_{z=0} + \iint^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial z} \frac{\partial \Theta_2}{\partial Y}\, dz\, dz|_{z=0} + c_{19}(Y,T)) - \beta_{1,2} \Theta_2|_{z=0},
\end{aligned}
$$

$$
\begin{aligned}
c_{19}(Y,T) \quad = \quad & (-\int^{z} \frac{1}{\tau} \frac{\partial \Theta_2}{\partial T}\, dz|_{z=-1} + \int^{z} \frac{\partial^2 \Theta_2}{\partial Y^2}\, dz|_{z=-1} + \int^{z} \frac{\partial \Psi_2}{\partial Y} \frac{H'}{\tau}\, dz|_{z=-1} + \int^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial Y} \frac{\partial \Theta_2}{\partial z}\, dz|_{z=-1} \\
& - \int^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial z} \frac{\partial \Theta_2}{\partial Y}\, dz|_{z=-1} - (-\int^{z} \frac{1}{\tau} \frac{\partial \Theta_2}{\partial T}\, dz|_{z=0} + \int^{z} \frac{\partial^2 \Theta_2}{\partial Y^2}\, dz|_{z=0} + \int^{z} \frac{\partial \Psi_2}{\partial Y} \frac{H'}{\tau}\, dz|_{z=0} \\
& + \int^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial Y} \frac{\partial \Theta_2}{\partial z}\, dz|_{z=0} - \int^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial z} \frac{\partial \Theta_2}{\partial Y}\, dz|_{z=0} \\
& - \beta_{1,0}(\iint^{z} \frac{1}{\tau} \frac{\partial \Theta_2}{\partial T}\, dz\, dz|_{z=0} - \iint^{z} \frac{\partial^2 \Theta_2}{\partial Y^2}\, dz\, dz|_{z=0} - \iint^{z} \frac{\partial \Psi_2}{\partial Y} \frac{H'}{\tau}\, dz\, dz|_{z=0} \\
& - \iint^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial Y} \frac{\partial \Theta_2}{\partial z}\, dz\, dz|_{z=0} + \iint^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial z} \frac{\partial \Theta_2}{\partial Y}\, dz\, dz|_{z=0}) - \beta_{1,2} \Theta_2|_{z=0})(1 - \beta_{2,0}) \\
& - \beta_{2,0}(\iint^{z} \frac{1}{\tau} \frac{\partial \Theta_2}{\partial T}\, dz\, dz|_{z=-1} - \iint^{z} \frac{\partial^2 \Theta_2}{\partial Y^2}\, dz\, dz|_{z=-1} - \iint^{z} \frac{\partial \Psi_2}{\partial Y} \frac{H'}{\tau}\, dz\, dz|_{z=-1} \\
& - \iint^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial Y} \frac{\partial \Theta_2}{\partial z}\, dz\, dz|_{z=-1} + \iint^{z} \frac{1}{\tau} \frac{\partial \Psi_0}{\partial z} \frac{\partial \Theta_2}{\partial Y}\, dz\, dz|_{z=-1}) \\
& - \beta_{2,2} \Theta_2|_{z=-1})/(-\beta_{1,0} + \beta_{2,0}\beta_{1,0} + \beta_{2,0}).
\end{aligned}
$$

From Θ_4 for case II

$$\begin{aligned}
c_{18}(Y,T) \quad = \quad & -\int^z \frac{1}{\tau}\frac{\partial\Theta_2}{\partial T}\,dz|_{z=0} + \int^z \frac{\partial^2\Theta_2}{\partial Y^2}\,dz|_{z=0} + \int^z \frac{\partial\Psi_2}{\partial Y}\frac{H'}{\tau}\,dz|_{z=0} + \int^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz|_{z=0} \\
& + \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_2}{\partial z}\,dz|_{z=0} - \int^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz|_{z=0} - \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_2}{\partial Y}\,dz|_{z=0} - \beta_{1,0}(\iint^z \frac{1}{\tau}\frac{\partial\Theta_2}{\partial T}\,dz\,dz|_{z=0} \\
& - \iint^z \frac{\partial^2\Theta_2}{\partial Y^2}\,dz\,dz|_{z=0} - \iint^z \frac{\partial\Psi_2}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=0} - \iint^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz\,dz|_{z=0} \\
& - \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_2}{\partial z}\,dz\,dz|_{z=0} + \iint^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz\,dz|_{z=0} + \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_2}{\partial Y}\,dz\,dz|_{z=0} + c_{19}(Y,T)) \\
& -\beta_{1,2}\Theta_2|_{z=0} - \beta_{1,4}\Theta_0|_{z=0}.
\end{aligned}$$

For case II, the second coupled partial differential equation in terms of $c_9(Y,T)$ and $c_{11}(Y,T)$ at $O(l^4)$ is

$$\begin{aligned}
& \int^z \frac{1}{\tau}\frac{\partial\Theta_2}{\partial T}\,dz|_{z=-1} - \int^z \frac{\partial^2\Theta_2}{\partial Y^2}\,dz|_{z=-1} - \int^z \frac{\partial\Psi_2}{\partial Y}\frac{H'}{\tau}\,dz|_{z=-1} - \int^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz|_{z=-1} \\
& - \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_2}{\partial z}\,dz|_{z=-1} + \int^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz|_{z=-1} + \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_2}{\partial Y}\,dz|_{z=-1} \\
& - \int^z \frac{1}{\tau}\frac{\partial\Theta_2}{\partial T}\,dz|_{z=0} + \int^z \frac{\partial^2\Theta_2}{\partial Y^2}\,dz|_{z=0} + \int^z \frac{\partial\Psi_2}{\partial Y}\frac{H'}{\tau}\,dz|_{z=0} + \int^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz|_{z=0} \\
& + \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_2}{\partial z}\,dz|_{z=0} - \int^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz|_{z=0} - \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_2}{\partial Y}\,dz|_{z=0} \\
& -\beta_{1,0}(\iint^z \frac{1}{\tau}\frac{\partial\Theta_2}{\partial T}\,dz\,dz|_{z=0} - \iint^z \frac{\partial^2\Theta_2}{\partial Y^2}\,dz\,dz|_{z=0} - \iint^z \frac{\partial\Psi_2}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=0} - \iint^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz\,dz|_{z=0} \\
& - \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_2}{\partial z}\,dz\,dz|_{z=0} + \iint^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz\,dz|_{z=0} + \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_2}{\partial Y}\,dz\,dz|_{z=0}) \\
& +\beta_{2,0}(\iint^z \frac{1}{\tau}\frac{\partial\Theta_2}{\partial T}\,dz\,dz|_{z=-1} - \iint^z \frac{\partial^2\Theta_2}{\partial Y^2}\,dz\,dz|_{z=-1} - \iint^z \frac{\partial\Psi_2}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=-1} - \iint^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz\,dz|_{z=-1} \\
& - \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_2}{\partial z}\,dz\,dz|_{z=-1} + \iint^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz\,dz|_{z=-1} + \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_2}{\partial Y}\,dz\,dz|_{z=-1} \\
& + \int^z \frac{1}{\tau}\frac{\partial\Theta_2}{\partial T}\,dz|_{z=0} - \int^z \frac{\partial^2\Theta_2}{\partial Y^2}\,dz|_{z=0} - \int^z \frac{\partial\Psi_2}{\partial Y}\frac{H'}{\tau}\,dz|_{z=0} - \int^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz|_{z=0} \\
& - \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_2}{\partial z}\,dz|_{z=0} + \int^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz|_{z=0} + \int^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_2}{\partial Y}\,dz|_{z=0} \\
& +\beta_{1,0}(\iint^z \frac{1}{\tau}\frac{\partial\Theta_2}{\partial T}\,dz\,dz|_{z=0} - \iint^z \frac{\partial^2\Theta_2}{\partial Y^2}\,dz\,dz|_{z=0} - \iint^z \frac{\partial\Psi_2}{\partial Y}\frac{H'}{\tau}\,dz\,dz|_{z=0} - \iint^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial Y}\frac{\partial\Theta_0}{\partial z}\,dz\,dz|_{z=0} \\
& - \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial Y}\frac{\partial\Theta_2}{\partial z}\,dz\,dz|_{z=0} + \iint^z \frac{1}{\tau}\frac{\partial\Psi_2}{\partial z}\frac{\partial\Theta_0}{\partial Y}\,dz\,dz|_{z=0} + \iint^z \frac{1}{\tau}\frac{\partial\Psi_0}{\partial z}\frac{\partial\Theta_2}{\partial Y}\,dz\,dz|_{z=0})) \\
& +(-\beta_{1,2}\Theta_2|_{z=0} - \beta_{1,4}\Theta_0|_{z=0})(1-\beta_{2,0}) + \beta_{2,2}\Theta_2|_{z=-1} + \beta_{2,4}\Theta_0|_{z=-1} = 0.
\end{aligned}$$

7.4 A theorem for a class of nonlinear differential equations

The following Theorem A formalises a procedure outlined in Hildebrand (1956):

Theorem A

Provided that the $L+1$ term Maclaurin series of the exact general solution,

$$A = \sum_{l=0}^{L} \frac{d^l A}{dx^l}|_{x=0}\frac{x^l}{l!} \tag{7.4.1}$$

to an M^{th} order ordinary differential equation

$$\frac{d^M A}{dx^M} = \xi \tag{7.4.2}$$

exists and all the derivatives and integrals of A are defined at $x=0$, it only solves the coefficients of x^l, $l \in \{0, 1, \ldots, L-M\}$ in the residual of (7.4.2) provided ξ is expandable in a Maclaurin series as

$$\xi = \sum_{l=0}^{\infty} \frac{d^l \xi}{dx^l}|_{x=0}\frac{x^l}{l!}, \tag{7.4.3}$$

where all the derivatives and integrals of ξ are defined at $x = 0$ and the right hand side of (7.4.2) does not contain $\frac{d^M A}{dx^M}$.

Proof of Theorem A

Since the Maclaurin series of A and ξ exist and all their derivatives and integrals are defined at $x = 0$, we can integrate (7.4.2) M times and substitute the result into (7.4.1) to find

$$A = \sum_{l=0}^{L} \frac{d^{(l-M)}\xi}{dx^{(l-M)}}|_{x=0} \frac{x^l}{l!}. \tag{7.4.4}$$

Substituting (7.4.4) into the residual r of (7.4.2) then gives

$$\mathrm{r} = \sum_{l=0}^{L} \frac{d^{(l-M)}\xi}{dx^{(l-M)}}|_{x=0} \frac{x^{l-M}}{(l-M)!} - \sum_{l=0}^{\infty} \frac{d^l\xi}{dx^l}|_{x=0} \frac{x^l}{l!}, \tag{7.4.5}$$

provided ξ is expandable in a Maclaurin series as in (7.4.3). Equating like powers of x in (7.4.5) then yields

$$\mathrm{r} = -\sum_{l=L-M+1}^{\infty} \frac{d^l\xi}{dx^l}|_{x=0} \frac{x^l}{l!}, \tag{7.4.6}$$

which shows that Theorem A is true. □

7.5 Another theorem for a class of nonlinear differential equations

The following Theorem B is of the essence of that given in various texts (see for example Muscalu & Schlag, 2013):

Theorem B

Provided that the $2L + 1$ term complex Fourier series of the exact general solution

$$A = \sum_{n=-L}^{L} P(A, \mathrm{e}^{\mathrm{i}nlx})\mathrm{e}^{\mathrm{i}nlx}, \quad 0 < l < \infty, \tag{7.5.1}$$

to an M^{th} order ordinary differential equation

$$\frac{d^M A}{dx^M} = \xi, \tag{7.5.2}$$

exists, it only solves the coefficients of $\mathrm{e}^{\mathrm{i}nlx}$ for $n \in [-L, L]$ in the residual of (7.5.2) if ξ is expandable as a complex Fourier series as

$$\xi = \sum_{n=-\infty}^{\infty} P(\xi, \mathrm{e}^{\mathrm{i}nlx})\mathrm{e}^{\mathrm{i}nlx}, \quad 0 < l < \infty. \tag{7.5.3}$$

Here A and ξ are periodic with period $\frac{2\pi}{l}$ and all of their derivatives and integrals are continuous for all x. Moreover the right hand side of (7.5.2) must not contain $\frac{d^M A}{dx^M}$ and $P(a, \mathrm{e}^{\mathrm{i}nlx})$ denotes the projection of a onto $\mathrm{e}^{\mathrm{i}nlx}$.

Proof of Theorem B

Since the complex Fourier series of A and ξ exist and because A and ξ are periodic with period $\frac{2\pi}{l}$ and all their derivatives and integrals are continuous for all x, we can integrate (7.5.2) M times and substitute the result into (7.5.1) to find

$$A = \sum_{n=-L}^{L} P(\frac{d^{(-M)}\xi}{dx^{(-M)}}, \mathrm{e}^{\mathrm{i}nlx})\mathrm{e}^{\mathrm{i}nlx}, \tag{7.5.4}$$

where the notation $\frac{d^{(-M)}\xi}{dx^{(-M)}}$ denotes the M^{th} integral of ξ with respect to x. Substituting (7.5.4) into the residual r of (7.5.2) then gives

$$\mathrm{r} = \frac{d^M}{dx^M} \sum_{n=-L}^{L} P(\frac{d^{(-M)}\xi}{dx^{(-M)}}, \mathrm{e}^{\mathrm{i}nlx})\mathrm{e}^{\mathrm{i}nlx} - \sum_{n=-\infty}^{\infty} P(\xi, \mathrm{e}^{\mathrm{i}nlx})\mathrm{e}^{\mathrm{i}nlx}, \tag{7.5.5}$$

provided ξ is expandable in complex Fourier series as in (7.5.3). Then equation (7.5.5) can be written as

$$\mathrm{r} = -\sum_{n \notin [-L,L]} P(\xi, \mathrm{e}^{\mathrm{i}nlx})\mathrm{e}^{\mathrm{i}nlx}, \tag{7.5.6}$$

which shows that Theorem B is true. □

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