Why Quasi-Interpolation onto Manifold has Order 4

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Abstract

We consider approximations of functions from samples where the functions take values on a submanifold of $\mathbb{R}^n$. We generalize a common quasi-interpolation scheme based on cardinal B-splines by combining it with a projection P onto the manifold. We show that for $m \geq 3$ we will have approximation order 4. We also show why higher approximation order can not be expected when the control points are constructed as projections of the filtered samples using a fixed mask.

1 Linear Theory

We start by defining cardinal B-splines.

Definition 1. *Cardinal B-splines can recursively be defined by*

$$B_0 = 1_{\left[-\frac{1}{2},\frac{1}{2}\right]} \text{ and } B_m = B_{m-1} * B_0 \text{ for all } m \geq 1$$

where $1_{\left[-\frac{1}{2},\frac{1}{2}\right]}$ *denotes the indicator function on the interval* $\left[-\frac{1}{2},\frac{1}{2}\right]$ *and* $*$ *denotes the convolution.*

Up to shift and scale cardinal B-splines are the piecewise polynomial C^{m-1}-functions with the smallest support and are therefore a popular choice for a basis of the space of piecewise polynomial C^{m-1}-functions. For a meshwidth $h > 0$ a function $f\colon [0,1] \to \mathbb{R}$ is approximated by a linear combination of shifted B-splines.

$$f_h(x) = \sum_{i\in\mathbb{Z}} c_i B_m(h^{-1}x - i) \tag{1}$$

The control points $(c_i)_{i\in\mathbb{Z}}$ can be found by applying a filter with mask $(A_i)_{i\in\mathbb{Z}}$ to the samples $(f(hi))_{i\in\mathbb{Z}}$, i.e.

$$c_i = \sum_{j\in\mathbb{Z}} A_j f(h(i+j)) \tag{2}$$

For each odd m there exists a finite sequence $(A_i)_{|i|\leq \frac{m-1}{2}}$ of length m such that

$$|f_h(x) - f(x)| \leq Ch^{m+1},$$

with a constant $C > 0$ independent of h. Careful analysis would show that C can be chosen as a multiple of $\|f^{(m+1)}\|_{L^\infty}$ This can be proven by showing polynomial reproduction, we refer Thm 3.5.4. of [2]. For small m the sequences $(A_i)_{|i|\leq \frac{m-1}{2}}$ are for example

$$\begin{aligned} m = 1 \quad &: \quad (A_0) = (1) \\ m = 3 \quad &: \quad (A_{-1}, A_0, A_1) = \left(-\frac{1}{6}, \frac{8}{6}, -\frac{1}{6}\right) \\ m = 5 \quad &: \quad (A_{-2}, A_{-1}, A_0, A_1, A_2) = \left(\frac{13}{240}, -\frac{7}{15}, \frac{73}{40}, -\frac{7}{15}, \frac{13}{240}\right). \end{aligned}$$

In [3] it is presented how these sequences can be constructed. We will consider the moments

$$a_k := \sum_{i\in\mathbb{Z}} A_i i^k, \qquad b_k := \sum_{i\in\mathbb{Z}} B_m(i) i^k \tag{3}$$

Since the sequences are symmetric, i.e. $A_{-i} = A_i$ resp. $B_m(-i) = B_m(i)$, the odd moments $a_1, a_3, \ldots$ resp. $b_1, b_3, \ldots$ are zero. The 0-th moment is always 1, i.e. $a_0 = \sum_{i\in\mathbb{Z}} A_i = 1$ and $b_0 = \sum_{i\in\mathbb{Z}} B_m(i) = 1$.

2 Nonlinear theory

Assume now that $f\colon [0,1] \to M \subset \mathbb{R}^n$, where $M \subset \mathbb{R}^d$ is a smooth Riemannian submanifold of $\mathbb{R}^d$. We consider again the linear combination (2). In general $c_i \notin M$. We will apply a projection $P\colon U \subset \mathbb{R}^n \to M$ to c_i. Usually this is the shortest point projection, i.e. $P(q) := \operatorname{argmin}_{p\in M} |p - q|$. However since we will only require P to be a projection onto M (i.e. a map whose image is M and whose restriction to M is the identity on M) and to be sufficiently smooth we could take any other sufficiently smooth projection onto the manifold. For small h the projection of c_i is possible as then c_i is sufficiently close to the manifold such that the projection is well-defined. Projecting will reduce the degrees of freedom for a control point c_i from that of the ambient space to the dimension of the manifold which can be quite a large reduction. Then we apply the linear combination (1). Finally, we apply the projection P which makes the approximation M-valued. Our approximation therefore is

$$f_h(x) = P\left(\sum_{i\in\mathbb{Z}} P(c_i) B_m(h^{-1}x - i)\right)$$

This method is not new, it has been described in [1], Section 3.5 of [2] and probably earlier.

3 Proof

We show that we have an order 4 approximation.

Theorem 1. *Let $m \geq 3$ be odd, $f \in C^4([0,1], M)$ with $M \subset \mathbb{R}^n$ such that the projection P is well-defined for h small enough and C^4. Define f_h as above. Then we have*

$$|f_h(x) - f(x)| \leq Ch^4$$

with a constant $C > 0$ independent of h.

Proof. The idea is to use Taylor expansion at x for f and at $f(x)$ for P. We have

$$\begin{aligned}
c_i &= \sum_j A_j f(h(i+j)) && (4)\\
&= \sum_j A_j \sum_{k=0}^{m} \frac{f^{(k)}(x)}{k!}(hi + hj - x)^k + \mathcal{O}(h^{m+1}) && (5)\\
&= \sum_{k=0}^{m} \sum_j A_j (hi + hj - x)^k \frac{f^{(k)}(x)}{k!} + \mathcal{O}(h^{m+1}) && (6)\\
&= \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{k}{j} h^j a_j (hi - x)^{k-j} \frac{f^{(k)}(x)}{k!} + \mathcal{O}(h^{m+1}) && (7)
\end{aligned}$$

Since $a_0 = 1$ and $a_1 = 0$ we have

$$c_i = f(x) + (hi - x)f'(x) + \sum_{k=2}^{m} \frac{f^{(k)}(x)}{k!} \sum_{j=0}^{k} \binom{k}{j} h^j a_j (hi - x)^{k-j} + \mathcal{O}(h^{m+1})$$

Now using Taylor expansion of P at $f(x)$ yields

$$\begin{aligned}
&P(c_i) && (8)\\
&= f(x) && (9)\\
&+P'(f(x)) \left[(hi - x)f'(x) + \sum_{k=2}^{m} \sum_{j=0}^{k} \binom{k}{j} h^j a_j (hi - x)^{k-j} \frac{f^{(k)}(x)}{k!} \right] && (10)\\
&+\frac{1}{2} P''(f(x))[f'(x), f'(x)](hi - x)^2 && (11)\\
&+\frac{1}{2} P''(f(x))[f'(x), f''(x)](hi - x) \frac{1}{2} \sum_{j=0}^{2} \binom{2}{j} h^j a_j (hi - x)^{2-j} && (12)\\
&+\frac{1}{6} P'''(f(x))[f'(x), f'(x), f'(x)](hi - x)^3 && (13)\\
&+\mathcal{O}(h^4) && (14)
\end{aligned}$$

It follows that

$$\sum_{i\in\mathbb{Z}} P(c_i)B_m(h^{-1}x-i) \tag{15}$$

$$= f(x) \tag{16}$$

$$+P'(f(x))\left[\sum_{i\in\mathbb{Z}}\sum_{k=1}^{m}\sum_{j=0}^{k}\binom{k}{j}h^j a_j(hi-x)^{k-j}B_m(h^{-1}x-i)\right] \tag{17}$$

$$+\frac{1}{2}P''(f(x))[f'(x),f'(x)]\sum_{i\in\mathbb{Z}}(hi-x)^2B_m(h^{-1}x-i) \tag{18}$$

$$+\frac{1}{4}P''(f(x))[f'(x),f''(x)]\sum_{i\in\mathbb{Z}}\sum_{j=0}^{2}\binom{2}{j}h^j a_j(hi-x)^{3-j}B_m\left(\frac{x}{h}-i\right) \tag{19}$$

$$+\frac{1}{6}P'''(f(x))[f'(x),f'(x),f'(x)]\sum_{i\in\mathbb{Z}}(hi-x)^3B_m(h^{-1}x-i) \tag{20}$$

$$+\mathcal{O}(h^4) \tag{21}$$

By the linear theory Term (17) is zero. By Lemma 2 the constant is equal to $h^2b_2 > 0$, hence Term (18) does not vanish. For Term (19) we have by Lemma 2 and the fact that $a_i = b_i = 0$ for odd i.

$$\sum_{i\in\mathbb{Z}}\sum_{j=0}^{2}\binom{2}{j}h^j a_j(hi-x)^{3-j} \tag{22}$$

$$\underbrace{b_3}_{0}a_0 + 2b_2\underbrace{a_1}_{0} + \underbrace{b_1}_{0}a_2 \tag{23}$$

$$= 0. \tag{24}$$

By Lemma 2, Term (20) is zero as well. Hence (18) is the only term left and we have

$$f_h(x) = P\left(f(x) + P''(f(x))[f'(x),f'(x)]\frac{b_2h^2}{2}\right) \tag{25}$$

$$+\mathcal{O}(h^4) \tag{26}$$

$$= f(x) \tag{27}$$

$$+P'(f(x))\left[P''(f(x))[f'(x),f'(x)]\frac{b_2h^2}{2}\right] \tag{28}$$

$$+\mathcal{O}(h^4). \tag{29}$$

Term (28) is zero by Lemma 4. □

In numerical experiments one can observe that, unlike in the linear case, the approximation order does not exceed 4. This has been observed in [4]. If we try to generalize the previous proof beyond 4 we end up with the following order 4 terms for $\sum_{i\in\mathbb{Z}} P(c_i)B_m(h^{-1}x-i)$:

$$\frac{1}{24}P''''(f(x))[f'(x),f'(x),f'(x),f'(x)]b_4h^4 \tag{30}$$
$$\frac{3}{6}P'''(f(x))[f'(x),f'(x),f''(x)]\frac{1}{2}(b_4+b_2a_2)h^4 \tag{31}$$
$$\frac{2}{2}P''(f(x))[f'(x),f'''(x)]\frac{1}{6}(b_4+3b_2a_2)h^4 \tag{32}$$
$$\frac{1}{2}P''(f(x))[f''(x),f''(x)]\frac{1}{4}(b_4+2b_2a_2+b_0a_2^2)h^4 \tag{33}$$

By taking four derivatives of $P(f(x)) = f(x)$ we get

$$\begin{aligned}
& P''''(f(x))[f'(x),f'(x),f'(x),f'(x)] && (34)\\
& +6P'''(f(x))[f'(x),f'(x),f''(x)] && (35)\\
& +4P''(f(x))[f'(x),f'''(x)] && (36)\\
& +3P''(f(x))[f''(x),f''(x)] && (37)\\
= \;& (Id - P'(f(x)))[f''''(x)]. && (38)
\end{aligned}$$

The RHS and therefore also the LHS yield zero when applied to $P'(f(x))$. By comparison one can see that in order for the terms (30)-(33) to be a multiple of (34)-(38) one would for example need $b_2a_2 = 0$. However $b_2 > 0$ and in order to be exact for polynomials of degree 2 one needs $a_2 = -b_2$ and hence we have $b_2a_2 = -b_2^2 \neq 0$. Hence in general there does not exist a linear sequence $(A_i)_{i\in\mathbb{Z}}$ such that we have optimal approximation order for any manifold. An alternative way to find control points with optimal approximation order is described in Section 3.5.3 of [2].

The analysis above also shows that the constant $C > 0$ in Theorem 1 depends not only on $f^{(4)}(x) = f''''(x)$ but also on lower order derivatives as well as on the projection P. Additionally, for f_h we will also have the 4-th order term

$$\frac{1}{2}P''(f(x))\left[P''(f(x))[f'(x),f'(x)],P''(f(x))[f'(x),f'(x)]\right]\left(\frac{b_2h^2}{2}\right)^2.$$

4 Appendix

The appendix consists of a part regarding linear combinations of B-splines and a part regarding the projection P onto the manifold.

4.1 B-spline sums

Lemma 1. *For $0 \le k \le m$ we let $G\colon \mathbb{R}\to\mathbb{R}$ be defined by*

$$G(x) = \sum_{i\in\mathbb{Z}} B_m(x-i)i^k$$

for all $x \in \mathbb{R}$. Then G is a polynomial of degree k with leading term x^k.

Proof. By definition of B-splines we have $B'_m(x-i) = B_{m-1}(x-i+1/2) - B_{m-1}(x-i-1/2)$. Hence we have

$$\begin{aligned} G'(x) &= \sum_{i\in\mathbb{Z}} B'_m(x-i)i^k && (39)\\ &= \sum_{i\in\mathbb{Z}} \left(B_{m-1}(x-i+1/2) - B_{m-1}(x-i-1/2)\right) i^k && (40)\\ &= \sum_{i\in\mathbb{Z}} B_{m-1}(x-i+1/2)\left(i^k-(i-1)^k\right) && (41) \end{aligned}$$

When repeatedly applying this rule the polynomial degree of the term on the right hand side reduces by 1 every time. Hence by applying k times we get

$$G^{(k)}(x) = \sum_{i\in\mathbb{Z}} B_{m-k}(x-i+k/2)k! = k!.$$

Since the k-th derivative of G is therefore constant to $k!$ the claim follow. ☐

Lemma 2. *For $0 \le k \le m$ we have for all $x \in \mathbb{R}$*

$$\sum_{i\in\mathbb{Z}} B_m(x-i)(x-i)^k = b_k,$$

where b_k is defined in (3)*.*

In particular for odd k the sum is zero by the symmetry of the B-splines.

Proof. By Lemma 1 the function

$$\begin{aligned} F(x) &:= \sum_{i\in\mathbb{Z}} B_m(x-i)(x-i)^k && (42)\\ &= \sum_{j=0}^{k}(-1)^j\binom{k}{j}x^{k-j}\sum_{i\in\mathbb{Z}} B_m(x-i)i^j && (43) \end{aligned}$$

is a polynomial. On the other hand we have $F(x+1) = F(x)$, i.e. it is periodic. Hence it follows that F is constant and that $F(x) = F(0) = b_k$ for all $x \in \mathbb{R}$. ☐

4.2 Properties of a Projection onto a manifold

Lemma 3. *Let $P\colon U \subset \mathbb{R}^n \to M$ be a projection onto a manifold M. Then for each $p \in M$ the map $P'(p)\colon \mathbb{R}^n \to T_pM \subset \mathbb{R}^n$ is a projection as well, i.e. we have $P'(p) \circ P'(p) = P'(p)$.*

Proof. Let $p \in M, v \in \mathbb{R}^n$ and $g\colon \mathbb{R} \to \mathbb{R}^n$ be defined by $g(t) = p + tv$. The function $t \mapsto P(g(t))$ is well-defined for $|t|$ sufficiently small. As P is a projection we have $P \circ P = P$ and hence also $P(P(g)) = P(g)$. Taking the derivative and using the chain-rule we get

$$P'(P(g(0))) \circ P'(g(0))g'(0) = P'(g(0))g'(0) \Rightarrow P'(p) \circ P'(p)v = P'(p)v$$

Since this is true for all $v \in \mathbb{R}^n$ we get $P'(p) \circ P'(p) = P'(p)$. ☐

Lemma 4. *Let* $f\colon [0,1] \to M$ *and* P *be a projection onto the manifold* M*. Then we have*

$$P'(f(x))\,[P''(f(x))[f''(x), f''(x)]] = 0$$

for all $x \in [0,1]$

Proof. Taking two derivative of $P(f(x)) = f(x)$ yields

$$P''(f(x))[f'(x), f'(x)] + P'(f(x))[f''(x)] = f''(x).$$

Applying $P'(f(x))$ on both sides and using Lemma 3 yields the claim. ☐

References

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