RELATIVE UNIFORM CONVERGENCE OF A SEQUENCE OF FUNCTIONS AT A POINT AND KOROVKIN-TYPE APPROXIMATION THEOREMS

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Abstract. We prove a Korovkin-type approximation theorem using the relative uniform convergence of a sequence of functions at a point, which is a method stronger than the classical ones. We give some examples on this new convergence method and we study also rates of convergence.

1. Introduction

Since their discovery, the simplicity and the power of the classical theorems of Korovkin (see [33]) have impressed several mathematicians. Starting with these results, many authors have extended the Korovkin theorem to several contexts using different, new and strong convergence methods (for an overview, see e.g., [1, 2, 8] and their bibliographies).

The Korovkin-type theorems give conditions for uniform approximation of continuous functions on a compact space using sequences or nets of positive linear operators on the space of continuous functions. The classical Bohman-Korovkin theorem gives uniform convergence in the space $C([a, b])$ of all continuous real-valued functions defined on the compact subinterval $[a, b]$ of the real line, with the only hypothesis of convergence on the test functions 1, x, x^2 (see e.g., [13, 23, 32, 33]). There have been several extensions of the Korovkin theorem to abstract functional spaces, like for instance L^p spaces (see e.g., [25, 30, 37, 40]), Orlicz spaces (see e.g., [34, 38]), general modular spaces (see e.g., [6, 7, 9]). There have been also several studies about Korovkin-type theorems with respect to convergence generated by summability matrices, statistical and filter convergence (see e.g., [2, 4, 22, 26, 27, 28, 29, 41]), and "triangular A-statistical convergence", which is an extension of statistical convergence, associated with a suitable non-negative regular matrix A (see e.g., [4, 5]). In [11] it is dealt with Korovkin-type results about convergence and estimates of rates of approximation with respect to abstract convergences for nets of operators acting on an abstract modular function space and satisfying suitable axioms (see e.g., [6]), including as particular cases convergence generated by summability matrices, filter convergence and almost convergence, which is not generated by any filter (see [12]). Moreover, in [11] the general case of a net of operators, acting on an abstract modular function space, is treated, and earlier results proved in [4, 5, 6, 10, 19, 26] are extended, unifing different previous theories. Furthermore,

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these topics have several recent meaningful applications to signal processes, image reconstruction, neural networks, thermography and seismic engineering (see e.g., [17, 18, 20, 21, 39] and their bibliographies).

From now on, we assume that $I \subset \mathbb{R}$ is a compact interval.

The classical notion of uniform convergence of function sequences is formulated as follows:

Definition 1.1. The function sequence $(f_n)_n$, defined on I and with values in $\mathbb{R}$, *converges uniformly on* I to $f : I \to \mathbb{R}$ iff for every $\varepsilon > 0$ there exists an integer N such that, if $n \geq N$ and $x \in I$, then $|f_n(x) - f(x)| \leq \varepsilon$.

Observe that, in general, the notions of "uniform convergence on each closed subinterval of an open interval" and "uniform convergence on the open interval" are not equivalent. For example, the sequence $(f_n)_n$ given by $f_n(x) = x^n$ converges uniformly to 0 on any interval $[0, a]$ with $0 < a < 1$, but neither on $[0, 1]$ nor on $[0, 1)$.

Recently, the idea of uniform convergence of a sequence of functions at a point was formerly defined by J. Klippert and G. Williams (see for details [31]).

Definition 1.2. Suppose that $(f_n)_n$ is a sequence of real functions defined on I. Let $x_0 \in I$. We say that $(f_n)_n$ *converges uniformly at the point* x_0 *to* $f : I \to \mathbb{R}$ iff for every $\varepsilon > 0$ there are $\delta > 0$ and $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| \leq \varepsilon$$

whenever $n \geq N$ and $|x - x_0| \leq \delta$.

Example 1.1. Define $g_n : [0, 1] \to [0, 1]$ by

$$g_n(x) = \begin{cases} x, & \text{if } n \text{ is a square} \\ 0, & \text{otherwise} \end{cases}. \tag{1.1}$$

It is readily seen that the sequence $(g_n)_n$ converges to 0 at the point 0 and does not converge at any point $x \in]0, 1]$. Now we claim that $(g_n)_n$ converges uniformly to 0 at $x_0 = 0$. Indeed, let $\varepsilon > 0$ be given, and choose $\delta = \varepsilon$ and $N = 1$. Let $n \geq N$ and $x \in [0, 1]$ with $|x| \leq \delta$. Then,

$$|g_n(x)| \leq |x| \leq \delta = \varepsilon.$$

The notion of uniform convergence of a function sequence with respect to a scale function was introduced by E. H. Moore in [36] and developed by E. W. Chittenden in [14, 15, 16]. A *scale function* is any map $\sigma : I \to \mathbb{R} \setminus \{0\}$.

Definition 1.3. A sequence $(f_n)_n$ of real-valued functions, defined on I, *converges relatively uniformly to a function* $f : I \to \mathbb{R}$ *with respect to the scale function* σ iff for every $\varepsilon > 0$ there is an integer n_ε such that for every $n \geq n_\varepsilon$ and $x \in I$ the inequality

$$|f_n(x) - f(x)| \leq \varepsilon |\sigma(x)|$$

holds.

In this paper we introduce the notion of *relative uniform convergence of a sequence of functions at a point.* We apply our new kind of convergence to prove a Korovkin-type approximation theorem. Furthermore, we study the rates of convergence, extending earlier results proved in [3, 10, 11, 19].

2. Relative uniform convergence at a point

We begin with the definition of our new convergence method.

Definition 2.1. Suppose that $(f_n)_n$ is a sequence of real-valued functions defined on I. Let $x_0 \in I$. We say that $(f_n)_n$ *converges relatively uniformly at the point* $x_0 \in I$ *to* $f : I \to \mathbb{R}$ *with respect to the scale function* σ, iff for every $\varepsilon > 0$ there are $\delta > 0$ and $N \in \mathbb{N}$ such that for every $n \geq N$, if $|x - x_0| \leq \delta$, then

$$|f_m(x) - f(x)| \leq \varepsilon |\sigma(x)| .$$

Now we give the following special cases to show the effectiveness of the new proposed method.

Remark 2.1. Observe that uniform convergence of a sequence of functions at a point is a special case of relative uniform convergence of a sequence of functions at a point, in which the scale function is a non-zero constant. If $\sigma(x)$ is bounded, then relative uniform convergence at a point implies uniform convergence at a point. However, in general, relative uniform convergence at a point does not imply uniform convergence at a point, when $\sigma(x)$ is unbounded.

Now we give the following example of a function sequence which converges relatively uniformly at $x_0 = 0$ with respect to a scale function, but does not converge uniformly at $x_0 = 0$.

Example 2.1. Define $h_n : [0, 1] \to [0, 1]$ by

$$h_n(x) = \begin{cases} \frac{nx}{1+nx}, & \text{if } n \text{ is a square} \\ 0, & \text{otherwise} \end{cases} . \tag{2.1}$$

We claim that $(h_n)_n$ converges relatively uniformly at $x_0 = 0$ to 0 with respect to the scale function

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x = 0 \end{cases} .$$

Indeed, let $\varepsilon > 0$ be given, and choose $\delta = \varepsilon$ and $N = 1$. Let $n \geq N$ and $x \in [0, 1]$ be with $x \leq \delta$. We get

$$\left| \frac{h_n(x)}{\sigma(x)} \right| \leq \frac{n\,x^2}{1+nx} \leq x \leq \delta = \varepsilon.$$

However, $(h_n)_n$ does not converge uniformly at $x_0 = 0$. Indeed, choose arbitrarily $\delta > 0$ and $N \in \mathbb{N}$, and let $n \geq N$ and $x \in [0, 1]$ be with $x \leq \delta$. For $\varepsilon = \frac{1}{2}$, $x = \frac{1}{n} \in [0, 1]$, we have $\frac{nx}{1+nx} = \frac{1}{2}$. □

3. Korovkin Type Approximation Theorems

Let $C(I)$ be the space of all continuous real-valued functions on I, and for every $x \in I$, set $e_0(x) = 1$, $e_r(x) = x^r$, $r \in \mathbb{N}$. We know that $C(I)$ is a Banach space with norm $\|f\|_{C(I)} = \sup_{x \in I} |f(x)|$. First, we give the well-known classical Korovkin approximation theorem.

Theorem 3.1. (see also [32, 33]) Suppose that $(L_n)_n$ is a sequence of positive linear operators acting from $C(I)$ into itself, satisfying the following conditions:

$$\lim_{n\to\infty} \|L_n(e_r) - e_r\|_{C(I)} = 0, \quad r = 0, 1, 2.$$

Then, for all $f \in C(I)$,

$$\lim_{n\to\infty} \|L_n(f) - f\|_{C(I)} = 0.$$

Now we present our following main theorem.

Theorem 3.2. *Let $(L_n)_n$ be a sequence of positive linear operators acting from $C(I)$ into itself. Then $(L_n(e_r))_n$, $r = 0, 1, 2$, converges relatively uniformly at x_0 to e_r with respect to the (possibly unbounded) scale function σ_r if and only if for all $f \in C(I)$, $(L_n(f))_n$ converges relatively uniformly at x_0 to f with respect to the scale function σ defined by*

$$\sigma(x) = \max\{|\sigma_r(x)| : r = 0, 1, 2\}. \tag{3.1}$$

Proof. Let $I = [a, b]$, with $a < b \in \mathbb{R}$, and let $x_0 \in I$ be fixed. Since each $e_r \in C(I)$, the sufficient condition is obvious. Now, let $f \in C(I)$ and $x \in I$ be fixed. Let $Q = \max\{-a, b\}$, $R = \max\{Q, Q^2\}$. Of course, $|x| \le R$ and $x^2 \le R$ for every $x \in I$. By the continuity of f on I, there is a positive real number S with $|f(x)| \le S$ for every $x \in I$. Therefore, we get

$$|f(t) - f(x)| \le |f(t)| + |f(x)| \le 2S.$$

Moreover, since f is uniformly continuous on I, for every $\varepsilon > 0$ there exists $\eta > 0$ with $|f(t) - f(x)| \le \varepsilon/4$ for all $t \in I$ satisfying $|t - x| \le \eta$. Hence, for each x, $t \in I$ we have

$$|f(t) - f(x)| \le \frac{\varepsilon}{4} + \frac{2S}{\eta^2}(t - x)^2,$$

that is

$$-\frac{\varepsilon}{4} - \frac{2S}{\eta^2}(t - x)^2 \le f(t) - f(x) \le \frac{\varepsilon}{4} + \frac{2S}{\eta^2}(t - x)^2.$$

Without loss of generality, ε can be chosen such that $0 < \varepsilon \le 1$, so that $\varepsilon^2 \le \varepsilon$. By hypothesis, in correspondence with $\min\left\{\frac{\varepsilon}{4}, \frac{\varepsilon}{4\,S}, \frac{\varepsilon\,\eta^2}{32\,R\,S}\right\}$ and $r = 0, 1, 2$ there are $\delta_r > 0$ and $N_r \in \mathbb{N}$ with

$$|L_n(e_r; x) - e_r(x)| \le \min\left\{\frac{\varepsilon}{4}, \frac{\varepsilon}{4\,S}, \frac{\varepsilon\,\eta^2}{32\,R\,S}\right\} |\sigma_r(x)| \tag{3.2}$$

whenever $n \ge N_r$ and $|x - x_0| \le \delta_r$. From (3.2) we get

$$|L_n(e_r; x) - e_r(x)| \le \min\left\{\frac{\varepsilon}{4}, \frac{\varepsilon}{4\,S}, \frac{\varepsilon\,\eta^2}{32\,R\,S}\right\} \sigma(x) \tag{3.3}$$

for every $n \geq N$ and $x \in I$ with $|x - x_0| \leq \delta$, where $\delta = \min\{\delta_r : r = 0, 1, 2\}$ and $N = \max\{N_r : r = 0, 1, 2\}$. We have

$$\begin{aligned}
& L_n((\cdot - x)^2; x) \\
= \; & |L_n(e_2 - 2\,x\,e_1 + x^2; x) - x^2 + 2\,x^2 - x^2| \\
\leq \; & |L_n(e_2; x) - x^2| + 2\,|x|\,|L_n(e_1; x) - x| + x^2\,|L_n(e_0; x) - 1| \\
= \; & |L_n(e_2; x) - x^2| + 2\,|x|\,|L_n(e_1; x) - e_1(x)| + x^2\,|L_n(e_0; x) - e_0(x)| \\
\leq \; & \frac{\varepsilon\,\eta^2}{8\,S}\,\sigma(x)
\end{aligned}$$

for each $n \geq N$ and $x \in I$ with $|x - x_0| \leq \delta$. As the operators L_n are linear and positive, taking into acount (3.3), we have

$$\begin{aligned}
& |L_n(f; x) - f(x)| \\
\leq \; & |L_n(f; x) - f(x)L_n(e_0; x)| + |f(x)L_n(e_0; x) - f(x)| \\
\leq \; & \frac{\varepsilon}{4}\,L_n(e_0; x) + \frac{2S}{\eta^2}\,L_n((\cdot - x)^2; x) \\
& + S\;|L_n(e_0; x) - e_0(x)| \\
\leq \; & \frac{\varepsilon}{4}\;|L_n(e_0; x) - e_0(x)| + \frac{\varepsilon}{4}\,e_0(x) + \frac{2S}{\eta^2}\,L_n((\cdot - x)^2; x) \\
& + S\;|L_n(e_0; x) - e_0(x)| \\
\leq \; & \frac{\varepsilon^2}{4}\,\sigma(x) + \frac{\varepsilon}{4}\,\sigma(x) + \frac{\varepsilon}{4}\,\sigma(x) + \frac{\varepsilon}{4}\,\sigma(x) \leq \varepsilon\,\sigma(x).
\end{aligned}$$

whenever $n \geq N$ and $|x - x_0| \leq \delta$. This ends the proof. □

When the involved scale functions are non-zero constants, the next result follows immediately from our main Korovkin-type approximation theorem.

Corollary 3.3. *Let $(L_n)_n$ be a sequence of positive linear operators acting from $C(I)$ into itself. Then $(L_n(e_r))_n$, $r = 0, 1, 2$, converges uniformly at x_0 to e_r if and only if for all $f \in C(I)$, $(L_n(f))_n$ converges uniformly at x_0 to f.*

In the next example we will show that our main Korovkin-type approximation theorem is stronger.

Example 3.1. Let $I = [0, 1]$ and consider the following Meyer-König and Zeller polynomials introduced by W. Meyer-König and K. Zeller in [35]:

$$M_n(f; x) = (1 - x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} x^k, \quad f \in C[0, 1].$$

It is well-known that $M_n(1; x) = 1$, $M_n(t; x) = x$ and

$$M_n\left(t^2; x\right) = x^2 + \eta_n(x) \leq x^2 + \frac{x(1 - x)}{n + 1},$$

where

$$\eta_n(x) = x(1 - x)^{n+1} \sum_{k=0}^{\infty} \binom{n + k - 1}{k} \frac{x^k}{n + k + 1}.$$

Using these polynomials, we define the following positive linear operators on $C[0, 1]$:

$$T_n(f; x) = (1 + h_n(x))M_n(f; x)\,, \tag{3.4}$$

where h_n is given by (2.1), and we choose $\sigma_r(x) = \sigma(x)$, $r = 0, 1, 2$, where

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x = 0 \end{cases}.$$

Now we claim that $(T_n(e_r))_n$ $r = 0, 1, 2$, converges uniformly at $x_0 = 0$ to e_r with respect to the scale function σ_r. Let $\varepsilon > 0$ be given. Choose $\delta_0 = \varepsilon$ and $N_0 = 1$. Let $n \geq N_0$ and $x \in [0,1]$ with $|x| \leq \delta_0$. Then,

$$\left| \frac{T_n(1;x) - 1}{\sigma_0(x)} \right| = \left| \frac{h_n(x)}{\sigma(x)} \right| \leq |x| \leq \delta_0 = \varepsilon.$$

Also, choose $\delta_1 = \sqrt{\varepsilon}$ and $N_1 = 1$. Let $n \geq N_1$ and $x \in [0,1]$ with $|x| \leq \delta_1$. Then,

$$\begin{aligned} \left| \frac{T_n(t;x) - x}{\sigma_1(x)} \right| &= \left| \frac{x\, h_n(x)}{\sigma(x)} \right| \\ &= |x| \left| \frac{h_n(x)}{\sigma(x)} \right| \leq |x|\,|x| \leq \delta_1^2 = \varepsilon. \end{aligned}$$

Finally, choose $\delta_2 = \dfrac{2\varepsilon}{7}$ and $N_2 = 1$. Let $n \geq N_2$ and $x \in [0,1]$ with $|x| \leq \delta_2$. Then, we obtain

$$\begin{aligned} \left| \frac{T_n(t^2;x) - x^2}{\sigma_2(x)} \right| &\leq \left| \frac{(1 + h_n(x))\left(x^2 + \frac{x(1-x)}{n+1}\right) - x^2}{\sigma(x)} \right| \\ &\leq \left| \frac{x(1-x)}{(n+1)\sigma(x)} \right| + \left| \frac{h_n(x)}{\sigma(x)} \right| \left| x^2 + \frac{x(1-x)}{n+1} \right| \leq \frac{7}{2}|x| \leq \frac{7}{2}\delta = \varepsilon. \end{aligned}$$

Hence, by Theorem 3.2, for $\varepsilon > 0$ there are $\delta = \min\left\{\varepsilon, \sqrt{\varepsilon}, \dfrac{2\varepsilon}{7}\right\}$ and $N = 1$ such that for every $n \geq N$,

$$\left| \frac{T_n(f;x) - f(x)}{\sigma(x)} \right| \leq \varepsilon$$

holds for all $x \in I = [0,1]$ satisfying $|x| \leq \delta$. However, since $|T_n(1;x) - 1| = |(1 + h_n(x)) - 1| = \begin{cases} \frac{nx}{1+nx}, & n \text{ is square} \\ 0, & \text{otherwise} \end{cases}$, the sequence $(T_n(e_0))$ is not uniformly convergent to $e_0(x) = 1$ and also, $(T_n(e_0))$ is not converges uniformly at $x_0 = 0$ to e_0. Hence, we can say that Theorem 3.1 (classical Korovkin type theorem) and Corollary 3.3 do not work for our operators defined by (3.4). $\square$

Example 3.2. Let $I = [0,1]$, and consider the classical Bernstein polynomials

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

on $C[0,1]$. It is well-known that $B_n(1;x) = 1$, $B_n(t;x) = x$ and

$$B_n(t^2;x) = x^2 + \frac{x(1-x)}{n}.$$

Using these polynomials, we define the following positive linear operators on $C[0,1]$:

$$T_n^*(f;x) = (1 + h_n(x)) B_n(f;x), \tag{3.5}$$

where h_n is given by (2.1), and we choose $\sigma_r(x) = \sigma(x)$, $r = 0, 1, 2$, where

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x = 0 \end{cases}$$

Then it is not difficult to see that the sequence of the operators defined in (3.5) converges relatively uniformly at $x_0 = 0$ with respect to the scale function σ, but does not converge uniformly at $x_0 = 0$.

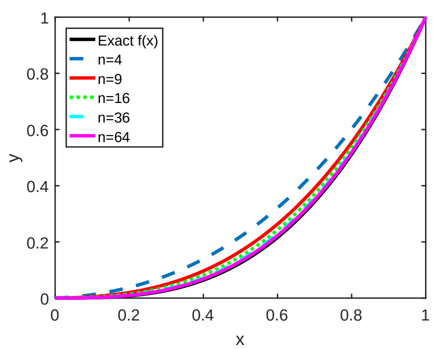

FIGURE 1. The operators $B_n(f;x)$ and the function $f(x) = x^3$.

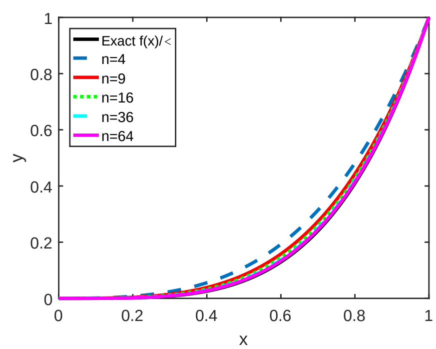

FIGURE 2. The operators $\frac{B_n(f;x)}{\sigma(x)}$ and the function $\frac{f(x)}{\sigma(x)}$.

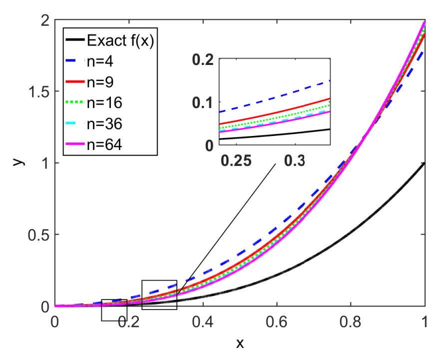

FIGURE 3. The operators $(1 + h_n(x)) B_n(f;x)$ and the function $f(x)$.

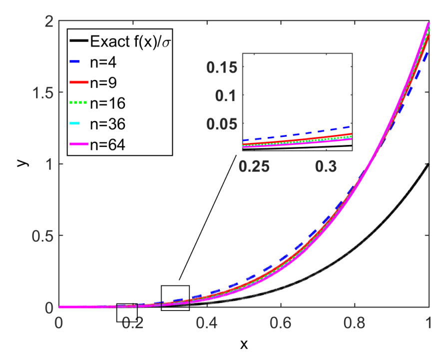

FIGURE 4. The operators $\frac{(1+h_n(x))B_n(f;x)}{\sigma(x)}$ and the function $\frac{f(x)}{\sigma(x)}$.

Figure 1 : We can see the Bernstein operators, which converge uniformly, and also converge uniformly at the point $x_0 = 0$.

Figure 2 : We can see the Bernstein operators, divided by the scale function, that converge uniformly with respect to the scale function σ, and also converge uniformly at the point $x_0 = 0$ with respect to the scale function σ.

Figure 3 : We can see the new operators given via the Bernstein operators with the function sequence $(h_n)_n$, which do not converge uniformly at the point $x_0 = 0$.

Figure 4 : We can see the new operators, divided by the scale function, given via the Bernstein operators with the function sequence $(h_n)_n$, that converge uniformly at the point $x_0 = 0$ with respect to the scale function σ.

4. Rates of Convergence

In this section we study the rate of convergence with the aid of the modulus of continuity, which is defined by

$$\omega(f,\delta) = \sup_{t,x\in I, |t-x|\leq \delta} |f(t)-f(x)|, \quad f\in C(I), \quad \delta>0.$$

It is readily seen that, for any $\lambda > 0$ and $f \in C(I)$,

$$\omega(f,\lambda\,\delta) \leq (1+[\lambda])\,\omega(f,\delta),$$

where $[\lambda]$ denotes the greatest integer less than or equal to λ.

Theorem 4.1. *Let $(L_n)_n$ be a sequence of positive linear operators acting from $C(I)$ into itself. Assume that the following conditions hold:*

(*i*) *$(L_n(e_0))_n$ converges relatively uniformly at x_0 to e_0 with respect to the scale function σ_0;*

(*ii*) *$\lim_{n\to\infty} \dfrac{\omega(f,\delta_n)}{|\sigma_1(x)|} = 0$ for each $x \in I$, where*

$$\delta_n = \sqrt{L_n\left((\cdot - x)^2; x\right)}, \quad n\in\mathbb{N}. \tag{4.1}$$

Then, for every $f \in C(I)$, $(L_n(f))_n$ converges relatively uniformly at x_0 to f with respect to the scale function σ, where

$$\sigma(x) = \max\{|\sigma_r(x)| : r=0,1\}.$$

Proof. Let $x \in I$ and $f \in C(I)$ be fixed. Since the operators L_n are linear and positive, then for every $n\in\mathbb{N}$ and $\delta > 0$ we have

$$\begin{aligned} & |L_n(f;x) - f(x)| \\ \leq\ & L_n(|f(\cdot) - f(x)|; x) + |f(x)|\, L_n(1;x) \\ \leq\ & L_n\left(\left(1 + \frac{(\cdot - x)^2}{\delta^2}\right)\omega(f,\delta); x\right) + |f(x)|\, L_n(1;x) \\ =\ & \omega(f,\delta)\ L_n(1;x) \\ & + \frac{\omega(f,\delta)}{\delta^2}\left[L_n\left((\cdot - x)^2; x\right)\right] + |f(x)|\, L_n(1;x). \end{aligned}$$

Now, let $\delta = \delta_n$ be as in (4.1). We get

$$\begin{aligned} \frac{|L_n(f;x) - f(x))|}{\sigma(x)} \leq\ & [\omega(f,\delta_n) + |f(x)|]\, \frac{L_n(1;x)}{|\sigma_0(x)|} \\ & + 2\,\frac{\omega(f,\delta_n)}{|\sigma_1(x)|}\, [L_n(1;x) + 1]. \end{aligned}$$

The assertion follows by using (*i*) and (*ii*). □

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