On matrix methods of convergence of order α in (ℓ)-groups

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Abstract

We introduce a concept of convergence of order α, with $0 < \alpha \leq 1$, with respect to a summability matrix method A for sequences, taking values in (ℓ)-groups. Some main properties and differences with the classical A-convergence are investigated. A Cauchy-type criterion and a closedness result for the space of convergent sequences according our notion is proved.

1   Introduction

The notion of statistical convergence was introduced in [19, 32]. In the literature there are several extensions of this concept: indeed, statistical convergence can be viewed as a particular case of convergence with respect to a summability matrix method (see also [13, 26]) as well as ideal and filter convergence (see also [2, 21, 23, 24, 28, 29]).

In [13, 22, 26] several properties of summability matrices are investigated, while in [1] and its bibliography several applications to matrix convergence to approximation theory, in particular to Korovkin-type theorems and to approximation in statistical sense of a function by means of a sequence of linear operators, even when the limit in the classical sense does not exist (see also [3, 6, 7, 17]).

The statistical and ideal convergences were investigated in normed spaces in [30] and in topological spaces in [14, 16, 15, 25]. Recently, there are several studies and applications of ideal convergence in the setting of Riesz spaces and lattice groups. In [11] the notion of ideal convergence in (ℓ)-groups was introduced and the main properties were examined, while in

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[8, 9, 10, 11] there are some versions of basic matrix theorems and limit theorems for ideal pointwise convergent measures taking values in an (ℓ)-group R.

In [4] a natural extension of statistical convergence is presented, by replacing n with n^α, $0 < \alpha < 1$, in the definition of asymptotic density. This is motivated by the investigation of different kinds of densities, and by the problem of comparing them with the natural density.

In this paper we extend the statistical convergence of order α to convergence of order α with respect to matrix methods, and we deal with the (ℓ)-group setting. We prove a Cauchy-type criterion, some main properties and some fundamental differences of the behavior of this kind of convergence between the cases $\alpha = 1$ and $0 < \alpha < 1$. Furthermore we present a property of closedness for the space of sequences, converging according our definition.

2   Preliminaries

We begin with recalling and introducing some notions about densities and summability matrix methods, which will be useful in the sequel.

Definitions 2.1. (a) Let $\mathbb{N}$ be the set of all natural numbers and $B \subset \mathbb{N}$. If m, $n \in \mathbb{N}$, $m < n$, we denote by $B(m,n)$ the cardinality of the set $B \cap \{m, m+1, \ldots, n\}$.

(b) Let $0 < \alpha \le 1$ be a real number. The *lower* and *upper asymptotic density of order α* of the set B are defined by

$$\underline{d}^\alpha(B) = \liminf_n \frac{B(1,n)}{n^\alpha}, \quad \overline{d}^\alpha(B) = \limsup_n \frac{B(1,n)}{n^\alpha}. \tag{1}$$

If the limit $\lim_n \dfrac{B(1,n)}{n^\alpha}$ exists in $\mathbb{R}$, then the common value in (1) is said to be the *asymptotic density of the set B of order α* and is denoted by $d^\alpha(B)$.

(c) If $(x_k)_k$ is a sequence of real numbers, we say that $(x_k)_k$ *converges α-statistically to* $x_0 \in \mathbb{R}$ (shortly, $S^\alpha \lim_k x_k = x_0$) iff for each $\varepsilon > 0$ we have $d^\alpha(A(\varepsilon)) = 0$, where $A(\varepsilon) := \{k \in \mathbb{N} : |x_k - x_0| > \varepsilon\}$.

(d) Let $A := (a_{j,k})_{j,k}$ be an infinite summability matrix. For a given sequence $\underline{x} = (x_k)_k$ in $\mathbb{R}$, the *A-transform of* $\underline{x}$, denoted by $(A\underline{x})_j$, is given by

$$(A\underline{x})_j := \sum_{k=1}^\infty a_{j,k}\, x_k, \tag{2}$$

provided that the series in (2) converges for each $j \in \mathbb{N}$ (see [1]).

We say that A is *regular* iff $\lim_j (A\underline{x})_j = L$ whenever $\lim_k x_k = L$ with $L \in \mathbb{R}$.

The following characterization of regularity of a matrix A is known in the literature as the *Silverman-Toeplitz conditions* (see also [1, Theorem 1.6]).

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Theorem 2.2. *An infinite summability matrix $A = (a_{j,k})_{j,k}$ is regular if and only if the following conditions are satisfied:*

(i) $\sup_j \left(\sum_{k=1}^{\infty} |a_{j,k}| \right) < +\infty,$

(ii) $\lim_j a_{j,k} = 0$ *for each $k \in \mathbb{N}$,*

(iii) $\lim_j \left(\sum_{k=1}^{\infty} a_{j,k} \right) = 1.$

Using regular matrices, Freedman and Sember ([20]) extend the idea of the statistical convergence to A-statistical convergence as follows.

Definitions 2.3. (a) Let $A = (a_{j,k})_{j,k}$ be a non-negative regular summability matrix. The *A-density* of a subset $K \subset \mathbb{N}$ is defined by

$$\delta_A(K) := \lim_j \sum_{k \in K} a_{j,k}, \tag{3}$$

provided that the limit in (3) exists in $\mathbb{R}$. Similarly it is possible to define the concepts of *A-upper density* and *A-lower density* of $K \subset \mathbb{N}$ as follows:

$$\overline{\delta}_A(K) := \limsup_j \sum_{k \in K} a_{j,k}, \quad \underline{\delta}_A(K) := \liminf_j \sum_{k \in K} a_{j,k}$$

respectively.

(b) A sequence $(x_k)_k$ of real numbers is said to be *A-convergent* to $x_0 \in \mathbb{R}$ if $\delta_A(A(\varepsilon)) = 0$ or equivalently

$$\lim_j \sum_{k \in A(\varepsilon)} a_{j,k} = 0,$$

where $A(\varepsilon)$ is as in Definitions 2.1 (c). Note that, when $A := \mathcal{C}_1 = (c_{j,k})_{j,k}$ is the Cesàro matrix, defined by

$$c_{j,k} := \begin{cases} \frac{1}{j} & \text{if } 1 \leq k \leq j, \\ 0 & \text{otherwise,} \end{cases}$$

the A-density and A-convergence become the usual asymptotic density and statistical convergence respectively.

We now recall some concepts about (ℓ)-groups (see also [27, 31]).

An (ℓ)-group R is said to be *Dedekind complete* if every nonempty subset $A \subset R$, bounded from above, has a supremum in R. A Dedekind complete (ℓ)-group R is said to be *super Dedekind complete* if for any nonempty set $A \subset R$, bounded from above, there exists a countable subset $A^* \subset A$, such that $\sup A = \sup A^*$.

From now on, we always suppose that R is a Dedekind complete (ℓ)-group.

An (O)-*sequence* $(\sigma_p)_p$ in R is a monotone decreasing sequence, such that $\bigwedge_p \sigma_p = 0$.

A (D)-*sequence* or *regulator* in R is a bounded double sequence $(a_{i,j})_{i,j}$, such that for every $i \in \mathbb{N}$ the sequence $(a_{i,j})_j$ is an (O)-sequence.

An (ℓ)-group R is *weakly σ-distributive* if

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0 \tag{4}$$

for every (D)-sequence $(a_{i,j})_{i,j}$.

A sequence $(x_k)_k$ in R is (O)-*convergent to* $x \in R$ (and we write $(O)\lim_k x_k = x$) if there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there exists $\overline{k} \in \mathbb{N}$ with

$$|x_k - x| \le \sigma_p \quad \text{for all } k \ge \overline{k}.$$

A sequence $(x_k)_k$ in R is (O)-*Cauchy* if there exists an (O)-sequence $(\tau_p)_p$ with the property that for every $p \in \mathbb{N}$ there is $\overline{k} \in \mathbb{N}$ such that

$$|x_h - x_k| \le \tau_p \quad \text{whenever } h, k \ge \overline{k}.$$

An (ℓ)-group R is (O)-*complete* if every (O)-Cauchy sequence in R is (O)-convergent in R. Note that every Dedekind complete (ℓ)-group is (O)-complete (see also [12]).

We now fix an exponent $0 < \alpha \le 1$, and introduce the concepts of A-density and A-convergence of order α for a summability matrix $A = (a_{j,k})_{j,k}$.

Definition 2.4. The A-*density of order* α of a subset $K \subset \mathbb{N}$ is defined by

$$\delta_A^{\alpha}(K) := \lim_j \frac{1}{j^{\alpha-1}} \sum_{k \in K} a_{j,k}, \tag{5}$$

provided that the limit in (5) exists in $\mathbb{R}$. Analogously we define the *upper A-density of order α* and *lower A-density of order α* as follows:

$$\overline{\delta}_A^{\alpha}(K) := \limsup_j \frac{1}{j^{\alpha-1}} \sum_{k \in K} a_{j,k}, \quad \underline{\delta}_A^{\alpha}(K) := \liminf_j \frac{1}{j^{\alpha-1}} \sum_{k \in K} a_{j,k},$$

respectively. Note that for $A = \mathcal{C}_1$, A-density of order α reduces to density of order α (see Definition 2.1 (b)).

Remark 2.5. Observe that, if $0 < \alpha < 1$ and $E \subset \mathbb{N}$ is such that $\delta_A^{\alpha}(E) = 0$, then $\delta_A^{\alpha}(\mathbb{N} \setminus E) = +\infty$. Indeed we get

$$\lim_j \frac{1}{j^{\alpha-1}} \sum_{k=1}^{\infty} a_{j,k} = \lim_j \frac{1}{j^{\alpha-1}} \sum_{k \in E} a_{j,k} + \tag{6}$$

$$+ \ \lim_j \frac{1}{j^{\alpha-1}} \sum_{k \in \mathbb{N} \setminus E} a_{j,k} = \delta_A^{\alpha}(E) + \delta_A^{\alpha}(\mathbb{N} \setminus E),$$

provided that the limits involved exist in $[0, +\infty]$. Since $\delta_A^\alpha(E) = 0$, from (6) and regularity of A we obtain

$$\delta_A^\alpha(\mathbb{N} \setminus E) = \lim_j \frac{1}{j^{\alpha-1}} \sum_{k=1}^\infty a_{j,k} = +\infty. \tag{7}$$

Definitions 2.6. (a) A sequence $(x_k)_k$ in an (ℓ)-group R is *A-convergent of order α or A^α-convergent* to $x_0 \in R$ (shortly, $A^\alpha \lim_k x_k = x_0$) if there exists an (O)-sequence $(\sigma_p)_p$ such that, for every $p \in \mathbb{N}$, $\delta_A^\alpha(B_p) = 0$, where

$$B_p := \{k \in \mathbb{N} : |x_k - x_0| \not\leq \sigma_p\}, \tag{8}$$

or equivalently

$$\lim_j \frac{1}{j^{\alpha-1}} \sum_{k \in B_p} a_{j,k} = 0.$$

In this case we write $A^\alpha \lim_n x_n = x_0$.

(b) A sequence $(x_k)_k$ in R is A^α-*Cauchy* if there is an (O)-sequence $(\tau_p)_p$ such that for every $p \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with

$$\delta_A^\alpha(\{k \in \mathbb{N} : |x_k - x_n| \not\leq \tau_p\}) = 0.$$

Remark 2.7. For $A = \mathcal{C}_1$, the Cesàro matrix, A^α-convergence reduces to statistical convergence of order α (see [4]). Furthermore, if $(x_k)_k$ is A-convergent of order α when $\alpha = 1$, we see simply that $(x_k)_k$ is *A-convergent*. The collection of all sequences in an (ℓ)-group R which are A-convergent and A^α-convergent are denoted by Am_0 and Am_0^α respectively.

3   The main results

We begin with a Cauchy-type criterion, which extends [11, Proposition 2.13] to A^α-convergence in the (ℓ)-group setting.

Proposition 3.1. *A sequence $(x_k)_k$ in R is A^α-convergent if and only if it is A^α-Cauchy.*

Proof. We begin with the sufficient part. Let $(\sigma_p)_p$ be an (O)-sequence, related with the Cauchy condition. There is a sequence $(n_p)_p$ in $\mathbb{N}$ with $\delta_A^\alpha(\mathbb{N} \setminus F_p) = 0$ for all $p \in \mathbb{N}$, where

$$F_p := \{k \in \mathbb{N} : |x_k - x_{n_p}| \leq \sigma_p\}. \tag{9}$$

We now claim that $F_p \cap F_q \neq \emptyset$ whenever $p \neq q$. Otherwise we have $F_p \subset \mathbb{N} \setminus F_q$. But $\delta_A^\alpha(\mathbb{N} \setminus F_q) = 0$, while $\delta_A^\alpha(F_p) = +\infty$ or 1 according as $0 < \alpha < 1$ or $\alpha = 1$ (see also (7)). This is a contradiction, which proves the claim.

Let now $p \neq q$. There exists $k_{p,q} \in \mathbb{N}$ with $|x_{k_{p,q}} - x_{n_p}| \leq \sigma_p$ and $|x_{k_{p,q}} - x_{n_q}| \leq \sigma_q$. So we get $|x_{n_p} - x_{n_q}| \leq \sigma_p + \sigma_q$, and hence $(x_{n_p})_p$ is an (O)-Cauchy sequence. As every Dedekind complete (ℓ)-group is (O)-complete (see [12]), there is $y \in R$ with $(O)\lim_p x_{n_p} = y$. So for each $p \in \mathbb{N}$ we get

$$\{k \in \mathbb{N} : |x_k - y| \not\leq 2\sigma_p\} \subset \{k \in \mathbb{N} : |x_k - x_{n_p}| + |x_{n_p} - y| \not\leq 2\sigma_p\}$$
$$\subset \{k \in \mathbb{N} : |x_{n_p} - y| \not\leq \sigma_p\} \cup \{k \in \mathbb{N} : |x_k - x_{n_p}| \not\leq \sigma_p\},$$

and hence

$$\delta_A^\alpha(\{k \in \mathbb{N} : |x_k - y| \not\leq 2\sigma_p\}) = 0.$$

So, $A^\alpha \lim_k x_k = y$, which proves the sufficient part.

We now turn to the necessary part. We know that there exist an (O)-sequence $(\sigma_p)_p$ and $y \in R$ with $d_A^\alpha(B_p) = 0$ for every $p \in \mathbb{N}$, where

$$B_p := \{k \in \mathbb{N} : |x_k - y| \not\leq \sigma_p\}.$$

Observe that $\mathbb{N} \setminus B_p \neq \emptyset$ for all $p \in \mathbb{N}$, because $d_A^\alpha(B_p) = +\infty$ or 1 according as $0 < \alpha < 1$ or $\alpha = 1$ (see also (7)). If $k, n \in \mathbb{N} \setminus B_p$, then

$$|x_k - x_n| \leq |x_k - y| + |x_n - y| \leq 2\sigma_p. \tag{10}$$

Let $V_p := \{k \in \mathbb{N} : |x_k - x_n| \not\leq 2\sigma_p\}$, $p \in \mathbb{N}$. From (10), for any $p \in \mathbb{N}$ we get $V_p \subset B_p$, and thus $d_A^\alpha(V_p) = 0$, since $d_A^\alpha(B_p) = 0$. Thus the assertion of the necessary part follows. This concludes the proof. $\qquad \square$

We now prove uniqueness of the A^α-limit in the (ℓ)-group setting.

Theorem 3.2. *Let $(x_k)_k$ be an A^α-convergent sequence. Then, its A^α-limit is unique.*

Proof. Let $(x_k)_k$ be a sequence in R, with $A^\alpha \lim_k x_k = x_0$ and $A^\alpha \lim_k x_k = y_0$. Then there are two (O)-sequences $(\sigma_p)_p$, $(\tau_p)_p$ such that

$$\delta_A^\alpha(\{k \in \mathbb{N} : |x_k - x_0| \not\leq \sigma_p\}) = \delta_A^\alpha(\{k \in \mathbb{N} : |x_k - y_0| \not\leq \tau_p\}) = 0 \text{ for all } p \in \mathbb{N}.$$

Fix now $p \in \mathbb{N}$, and let

$$D_1 := \{k \in \mathbb{N} : |x_k - x_0| \leq \sigma_p\}, \quad D_2 := \{k \in \mathbb{N} : |x_k - x_0| \leq \tau_p\}.$$

If $D_1 \cap D_2 = \emptyset$, then $D_1 \subset \mathbb{N} \setminus D_2$, and hence $\overline{\delta}_A^\alpha(D_1) \leq \overline{\delta}_A^\alpha(\mathbb{N} \setminus D_2)$, but $\overline{\delta}_A^\alpha(\mathbb{N} \setminus D_2) = 0$, while $\overline{\delta}_A^\alpha(D_1) = +\infty$ or 1 according as $0 < \alpha < 1$ or $\alpha = 1$ (see also (7)), a contradiction. Hence, $D_1 \cap D_2 \neq \emptyset$. Let $k \in D_1 \cap D_2$, then

$$|x_0 - y_0| \leq |x_0 - x_k| + |x_k - y_0| \leq \sigma_p + \tau_p.$$

Since $(\sigma_p)_p$, $(\tau_p)_p$ are (O)-sequences, by arbitrariness of p we get $x_0 = y_0$. $\qquad \square$

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The next result is a fundamental property of A^α-convergence in the case $0 < \alpha < 1$, and extends [4, Theorem 1] to the (ℓ)-group context.

Theorem 3.3. *Let A be a regular and lower triangular summability matrix. If a sequence $(x_k)_k$ in R is A^α-convergent to x_0 (with $0 < \alpha < 1$), then there is a set $C := \{k_1 < k_2 < \ldots < k_n < \ldots\} \subset \mathbb{N}$ with $\overline{\delta}_A^\alpha(C) = +\infty$ and $(O) \lim_n x_{k_n} = x_0$.*

Proof. Suppose that $(x_n)_n$ is A^α-convergent to x_0. Then there exists an (O)-sequence $(\sigma_p)_p$ with $\delta_A^\alpha(B_p) = 0$ for each $p \in \mathbb{N}$, where B_p is as in (8). Set $C_p = \mathbb{N} \setminus B_p$. Since $\delta_A^\alpha(B_p) = 0$, then $\delta_A^\alpha(C_p) = +\infty$ (see also 7)). Moreover, it is easy to see that the sequence $(C_p)_p$ is decreasing.

Let now $(G_i)_i$ be a strictly increasing sequence of positive real numbers. Choose arbitrarily $v_1 \in C_1$. Since $\delta_A^\alpha(C_2) = +\infty$, there is $v_2 \in C_2$, $v_2 > v_1$, with

$$\sum_{k \in C_2} a_{j,k} > G_2 \cdot j^{\alpha - 1} \quad \text{for each } j \geq v_2$$

In particular, we get

$$\sum_{k \in C_2} a_{v_2,k} > G_2 \cdot v_2{}^{\alpha - 1}.$$

At the next step, since $\delta_A^\alpha(C_3) = +\infty$, we can find an element $v_3 \in C_3$, $v_3 > v_2$, with

$$\sum_{k \in C_3} a_{v_3,k} > G_3 \cdot v_3{}^{\alpha - 1}.$$

Proceeding by induction, we construct a strictly increasing sequence $(v_j)_j$ of positive integers, with $v_j \in C_j$ and

$$\sum_{k \in C_j} a_{v_j,k} > G_j \cdot v_j{}^{\alpha - 1} \quad \text{for all } j \in \mathbb{N}.$$

Put now

$$C := [1, v_1] \cup \left(\bigcup_{j=2}^{\infty} ([v_{j-1}, v_j] \cap C_j) \right).$$

Since A is lower triangular, we get

$$a_{v_j,k} = 0 \quad \text{for all } j \in \mathbb{N} \text{ and } k > v_j. \tag{11}$$

As $C \cap [1, v_j] \supset C_j \cap [1, v_j]$ for every $j \in \mathbb{N}$, from (11) we get

$$\sum_{k \in C} a_{v_j,k} \geq \sum_{k \in C_j} a_{v_j,k} > G_j \cdot v_j{}^{\alpha - 1},$$

that is

$$\frac{1}{v_j{}^{\alpha - 1}} \sum_{k \in C} a_{v_j,k} > G_j, \quad \text{for all } j \in \mathbb{N}. \tag{12}$$

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From (12) it follows that
$$\limsup_j \frac{1}{v_j^{\alpha-1}} \sum_{k \in C} a_{v_j,k} = +\infty,$$
namely $\overline{\delta}_A^\alpha(C) = +\infty$.

Now, to show that the sequence $(x_k)_{k \in C}$ (O)-converges to x_0, it is sufficient to observe that for all $j \in \mathbb{N}$ and $k \geq v_j$ we have $|x_k - x_0| \leq \sigma_j$, where $(\sigma_j)_j$ is an (O)-sequence. This completes the proof. $\qquad\square$

Remarks 3.4. (a) Some examples of lower triangular non-negative regular matrices, whose $\mathcal{C}_1$ is a particular case, are the matrices which generate the Erdős-Ulam ideals (see [18, Example 1.2.3 (d)]).

(b) Observe that, for $\alpha = 1$, A^α-convergence reduces to A-convergence, that is the convergence generated by the ideal of those subsets of $\mathbb{N}$ having A-density zero. From this (see also [23, Lemma 4] and [11, Proposition 2.8]) it follows that in a Dedekind complete (ℓ)-group R, a sequence $(x_k)_k$ is A-convergent to x_0 if and only if there is a subset $C \subset \mathbb{N}$, $C = \{k_1 < \ldots < k_n < \ldots\}$, with $\delta_A(C) = 1$ and $(O) \lim_n x_{k_n} = x_0$.

(c) For $0 < \alpha < 1$, the converse of Theorem 3.3 is in general not true, as can be seen by taking
$$x_k := \begin{cases} (4,4), & \text{if } k = n^2, n \in \mathbb{N}. \\ (0,0), & \text{otherwise.} \end{cases} \tag{13}$$

Indeed, for $\alpha = \frac{1}{4}$, $\overline{\delta}_A^\alpha(C) = +\infty$, where $C = \{n^2 : n \in \mathbb{N}\}$, but it is not true that $A^\alpha \lim_k x_k = (4,4)$.

From now on we do not require lower triangularity of the summability matrix involved, but only non-negativity and regularity.

We now check the following inclusion, extending [4, Theorem 2].

Theorem 3.5. *Let $0 < \alpha \leq \beta \leq 1$. Then $Am_0^\alpha \subset Am_0^\beta$.*

Proof. Let $(x_k)_k \in Am_0^\alpha$ and with $A^\alpha \lim_k x_k = x_0$. Then there is an (O)-sequence $(\sigma_p)_p$ such that for all $p \in \mathbb{N}$ we get $\delta_A^\alpha(B_p) = 0$, namely
$$\lim_j \frac{1}{j^{\alpha-1}} \sum_{k \in B_p} a_{j,k},$$
where $B_p := \{k \in \mathbb{N} : |x_k - x_0| \not\leq \sigma_p\}$. Since
$$0 \leq \limsup_j \frac{1}{j^{\beta-1}} \sum_{k \in B_p} a_{j,k} \leq \limsup_j \frac{1}{j^{\alpha-1}} \sum_{k \in B_p} a_{j,k} = 0,$$
then we get $\delta_A^\beta(B_p) = 0$, which shows that $(x_k)_k \in Am_0^\beta$. $\qquad\square$

If we take $\beta = 1$ in Theorem 3.5, then we obtain the following result.

Corollary 3.6. *If a sequence $(x_k)_k$ in R is A^α-convergent to x_0 for some $0 < \alpha \leq 1$, then it is A-convergent to x_0, that is $Am_0^\alpha \subset Am_0$.*

We now prove a closedness property for the space of A^α-convergent sequences, extending [4, Theorem 3] to the (ℓ)-group setting.

Theorem 3.7. *Let $\alpha \in (0, 1]$, $(\underline{x}^{(k)})_k$ be a sequence in Am_0^α, where $\underline{x}^{(k)} = (x_j^{(k)})_j$ and for every $k \in \mathbb{N}$ $(x_j^{(k)})_j$ is A^α-convergent with respect to a common (O)-sequence $(\sigma_p)_p$, independent of k. If $\underline{x} = (x_j)_j$ is such that*

$$(O) \lim_k \left(\bigvee_j |x_j^{(k)} - x_j| \right) = 0, \tag{14}$$

then $\underline{x} \in Am_0^\alpha$.

Proof. Let $(\underline{x}^{(k)})_k$ satisfy (14), where $\underline{x}^{(k)} \in Am_0$ for all $k \in \mathbb{N}$. Suppose that

$$A^\alpha \lim_j x_j^{(k)} = y_k \quad \text{for all } k \in \mathbb{N}. \tag{15}$$

By (15) there is an (O)-sequence $(\sigma_p)_p$ with

$$\delta_A^\alpha(\{n \in \mathbb{N} : |x_n^{(k)} - y_k| \not\leq \sigma_p\}) = 0 \quad \text{for all } k, p \in \mathbb{N}.$$

Fix now $p \in \mathbb{N}$, and put

$$E_k := \{n \in \mathbb{N} : |x_n^{(k)} - y_k| \leq \sigma_p\} \tag{16}$$

for all $k \in \mathbb{N}$. If $E_k \cap E_r = \emptyset$, then $E_k \subset \mathbb{N} \setminus E_r$ and hence $\overline{\delta}_A^\alpha(E_k) \leq \overline{\delta}_A^\alpha(\mathbb{N} \setminus E_r) = 0$. But $\overline{\delta}_A^\alpha(E_k) = +\infty$ or 1 according as $0 < \alpha < 1$ or $\alpha = 1$ (see also (7)), and so we obtain a contradiction. Thus for every $k, r \in \mathbb{N}$ there exists $\overline{n} \in E_k \cap E_r$.

By (14) there is an (O)-sequence $(\tau_p)_p$ such that for all $p \in \mathbb{N}$ there is $n_0 = n_0(p) \in \mathbb{N}$ with

$$\bigvee_{j=1}^\infty |x_j^{(k)} - x_j| \leq \tau_p \quad \text{for all } k \geq n_0.$$

Then

$$|x_j^{(k)} - x_j^{(r)}| \leq |x_j^{(k)} - x_j| + |x_j^{(r)} - x_j|$$
$$\leq \bigvee_{j=1}^\infty |x_j^{(k)} - x_j| + \bigvee_{j=1}^\infty |x_j^{(r)} - x_j| \leq 2\,\tau_p$$

for every $j \in \mathbb{N}$ and $k, r \geq n_0$. Thus we get

$$|y_k - y_r| \leq |y_k - x_{\overline{n}}^{(k)}| + |x_{\overline{n}}^{(k)} - x_{\overline{n}}^{(r)}| + |y_k - x_{\overline{n}}^{(k)}|$$

$$\leq 2\sigma_p + \bigvee_{j=1}^{\infty} |x_j^{(k)} - x_j| + \bigvee_{j=1}^{\infty} |x_j^{(r)} - x_j| \leq 2\sigma_p + 2\tau_p$$

whenever $k, r \geq n_0$. This shows that $(y_k)_k$ is an (O)-Cauchy sequence in R. Since R is Dedekind complete, then R is (O)-complete too (see also [12]), and so the sequence $(y_k)_k$ is (O)-convergent to some element $y \in R$. Hence there is an (O)-net $(\eta_p)_p$ such that for each $p \in \mathbb{N}$ there exists $n^* = n^*(p)$ with

$$|y_k - y| \leq \eta_p \quad \text{for all } k \geq n^*.$$

Choose $k \geq \max\{n_0, n^*\}$. Then for all $j \in \mathbb{N}$ we have

$$\begin{aligned}
|x_j - y| &\leq |x_j - x_j^{(k)}| + |x_j^{(k)} - y_k| + |y_k - y| \\
&\leq \tau_p + \eta_p + |x_j^{(k)} - y_k|.
\end{aligned} \tag{17}$$

Observe that, if E_k is as in (16) and $j \in E_k$, then $|x_j - y| \leq w_p$, where $w_p = \sigma_p + \tau_p + \eta_p$. So $(w_p)_p$ is an (O)-sequence, and

$$\overline{E} := \{j \in \mathbb{N} : |x_j - y| \not\leq w_p\} \subset \mathbb{N} \setminus E_k.$$

Since by hypothesis $\delta_A^\alpha(\mathbb{N} \setminus E_k) = 0$, we get also $\delta_A^\alpha(\overline{E}) = 0$, which completes the proof. $\qquad \square$

We now recall a condition, under which it is possible to replace a countable family of (O)-sequences with one (O)-sequence ([5, Lemma 2.8]).

Lemma 3.8. *Let R be a super Dedekind complete and weakly σ-distributive (ℓ)-group. If $(\sigma_p^{(k)})_p$ is an equibounded family of (O)-sequences, then there is an (O)-sequence $(b_r)_r$ with the property that for every $k, r \in \mathbb{N}$ there exists $\overline{p} = \overline{p}(k, r)$, with $\sigma_{\overline{p}}^{(k)} \leq b_r$.*

Remark 3.9. Observe that, if R is a super Dedekind complete and weakly σ-distributive (ℓ)-group and $(x_j^{(k)})_j$, $k \in \mathbb{N}$, are as in Theorem 3.7, A^α-convergent for every k and order equibounded in R (namely with a positive element $u \in R$ with $|x_j^{(k)}| \leq u$ for all $j, k \in \mathbb{N}$), then the $(x_j^{(k)})_j$'s are A^α-convergent with respect to a same (O)-sequence $(b_r)_r$, independent of k. Indeed, let $(y_k)_k$ be a sequence in R such that for all $k \in \mathbb{N}$ there exists an (O)-sequence $(\sigma_p^{(k)})_p$ with $\delta_A^\alpha(B_p^{(k)}) = 0$ for all $k, p \in \mathbb{N}$, where

$$B_p^{(k)} := \{j \in \mathbb{N} : |x_j^{(k)} - y_k| \not\leq \sigma_p^{(k)}\}. \tag{18}$$

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Note that order equiboundedness of the double sequence $(x_j^{(k)} - y_k)_{j,k}$ is a consequence of order equiboundedness of $(x_j^{(k)})_{j,k}$. So, without loss of generality, we can assume that $\sigma_p^{(k)} \leq 2\,u$ for each k, $p \in \mathbb{N}$.

By Lemma 3.8 there exists an (O)-sequence $(b_r)_r$ such that for every k, $r \in \mathbb{N}$ there exists $\overline{p} \in \mathbb{N}$ with $\sigma_{\overline{p}}^{(k)} \leq b_r$. From this and (18) it follows that $D_r^{(k)} \subset B_{\overline{p}}^{(k)}$, where $D_r^{(k)} := \{j \in \mathbb{N} : |x_j^{(k)} - y_k| \not\leq b_r\}$. Hence we get $\delta_A^\alpha(D_r^{(k)}) = 0$, since $\delta_A^\alpha(B_{\overline{p}}^{(k)}) = 0$. This proves the claim, by virtue of arbitrariness of r.

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