On the distance between the incenter and the circumcenter of a triangle

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Abstract: In this paper, using mainly the distance in the title, we present an alternative proof of Feuerbach's theorem and some remarks.

K. W. Feuerbach proved on 1822 that the nine point circle of a triangle inscribed to the incircle and circumscribed to the excircles of a triangle. There are many proofs on Feuerbach's theorem as Pythagoras's theorem and Gauss' law of quadratic reciprocity.

We shall confine our attention to acute triangles and incircles because our theorem can be proved in the same manner about other cases. This theorem is trivial for the equilateral triangle. We remove here this triangle. Notation $\mathcal{C}(U, V)$ means the circle with the center U and radius V.

Theorem [Feuerbach's Theorem]. The nine-point circle $\mathcal{N}$ of a triangle tangents to each of the incircle and excircles of the triangle.

Proof. We set the circumcircle $\mathcal{O} := \mathcal{C}(O, R)$, incircle $\mathcal{I} := \mathcal{C}(I, r)$ and

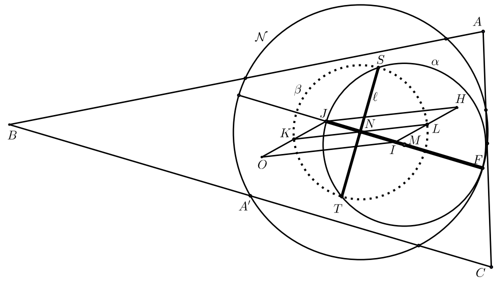

Figure (by Geogebra)

the orthocenter H of $\triangle ABC$, respectively. Moreover, let N and A' be the midpoints of OH and BC, respectively. The point J is of $JN = NI$ on the line NI and the other side from I. Points K and L are the midpoints of the segments JO and IH, respectively.

Since N is the midpoint of both lines OH and JI, the quadrilateral $JOIH$ is a parallelogram. From this and two midpoints theorem, the segments OI, KL and JH are parallel and N is on the line KL.

Let the line ℓ be perpendicular to the line JI at N and let S (over N) and T be the intersection points of ℓ and $\beta := \mathcal{C}(N, KN)$. Then $SN = NT = KN = OI/2$. The point F is the intersection point of the nine point circle $\mathcal{N} = \mathcal{C}(N, NA' = R/2)$ and the line NI on I side.

Let α be the circumcircle of $\triangle SJF$ and Let M be the intersection point of line JF and the perpendicular bisector of ST. Then the point M is of F side on the line JF by the acute angle $\angle SJN$. Since JF is the perpendicular bisector of ST at the center N of β, $\triangle SJT$ has the circumcircle $\alpha = \mathcal{C}(M, JM)$ by $JM = SM = TM$. Moreover $\triangle SJF$ is of right and three circles $\mathcal{N}, \beta, \alpha$ are at position of line JF symmetry.

α is inscribed at F to $\mathcal{N}$ by $MN = JM - JN$ from $JN = NI < NI + IF/2 = JF/2 = JM$. we obtain $IN = R/2 - r$ from the next

$$(R/2) \cdot NI = NF \cdot JN = SN^2 = KN^2 = (IO/2)^2 = (R/2)(R/2 - r). \quad \square$$

Remarks.

R0. It is essential to make the right $\triangle SJF$ with $SN^2 = R/2(R/2 - r), NF = R/2$ and points N, I on JF as the Figure by the next reason. Since $\triangle FSN$ and $\triangle SJN$ is similar, $SN/FN = JN/SN$ and hence we have $NI = R/2 - r$ by the last equation in the proof of Theorem.

R1. Since midpoints A', B', C' of three sides BC, AC, AB and midpoints A'', B'', C'' of AH, BH, CH are on the nine point circle $\mathcal{N}$, we can see that $\triangle ABC$, $\triangle A'B'C'$ and $\triangle A''B''C''$ are positions of similarity with the centers; centroid and orthocenter H with a ratio $2 : -1 : 1$, respectively.

R2. Since the pedal $\triangle A^*B^*C^*$ of $\triangle ABC$ has the incercle $\mathcal{C}(H, \rho)$ and the circumcircle $\mathcal{N} = \mathcal{C}(N, R/2)$, We have

$$OH^2 = 4 \cdot NH^2 = 4 \cdot R/2 \cdot (R/2 - 2\rho) = R^2 - 4\rho R \cdots (\sharp).$$

Thus using formula $OI^2, (\sharp), NI = R/2 - r$ and Pappus' theorem to $\triangle OIH$, we can calculate the distance

$$IH^2 = 2r^2 - 2\rho R \cdots (\flat).$$

R3. From the equations $(\sharp), (\flat)$ in R2 and $NI = R/2 - r$, we can see the following inequalities. (1) $R > 2r$, (2) $r > 2\rho$, (3) $KN > NI$, (4) $JF > OI$, (5) $OI > JI$, (6) $OH > OI$, (7) $OH > IH$.

Proof.

(1): by $NI > 0$.
(2): $r^2 > R\rho > 2r\rho$ by $(\flat)$ and (1), and hence, $r > 2\rho$.
(3): $KN^2 = (R/2)(R/2 - r) > (R/2 - r)^2 = NI^2$.
(4): $JF^2 = (R - r)^2 = R(R - 2r) + r^2 = OI^2 + r^2 > OI^2$
(5): $OI = 2KN > 2NI = JI$ by (3).
(6): $OH^2 - OI^2 = 2R(r - 2\rho) > 0$ by $(\sharp)$ and (2).
(7): $OH^2 - IH^2 = R(R-2\rho) - 2r^2 > 2r(R-2\rho-r) > 2r(r-2\rho) > 0$ by $(\sharp), (\flat)$, (1) and (2)

R4. If we give points O, I, H, then we set points N, J, K, L the line ℓ, $\beta = \mathcal{C}(N, KN)$, and points S, T as the proof of Theorem and the Figure. The point F is the intersection point of the line JI and the perpendicular line to SJ at S. We set $R := 2NF$, $r := IF$, $\mathcal{O} := \mathcal{C}(O, R)$ and $\mathcal{I} := \mathcal{C}(I, r)$. In case $r > 0$, we can draw a figure but by a size of a paper, we can not obtain a figure. Please correct the positions of O, I, H such that NI is not so small and large, where N is the midpoint of OH. From equations $OI^2 = (2KN)^2 = (2SN)^2 = 4(NI \cdot NF) = R(R - 2r)$ using $NI + r = NI + IF = NF = R/2$ and [1, p.86, 155. Theorem], we draw two tangents to the incircle $\mathcal{I}$ from any point on the circumcircle $\mathcal{O}$, we obtain triangles having the same circumcircle $\mathcal{O}$ and incircle $\mathcal{I}$.

Reference

[1] Nathan Altshiller-Court, College Geometry (An introduction to the modern geometry of the triangle and the circle), 2nd ed., Barnes & Noble,1952, p.85. 152. Euler's Theorem, p.86. 155. Theorem.