ON CERTAIN FINITE SUMS OF INVERSE TANGENTS

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ABSTRACT. An identity is proved connecting two finite sums of inverse tangents. This identity is discretized version of Jacobi's imaginary transformation for the modular angle from the theory of elliptic functions. Some other related identities are discussed.

1. INTRODUCTION

Sums of inverse tangents have attracted a lot of attention. For example, the following sums of inverse tangents can be calculated in closed form:

$$\sum_{n=0}^{\infty} \arctan \frac{2}{(2n+1)^2} = \frac{\pi}{2}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} (-1)^{n+1} \arctan \frac{1}{F_{2n}} = \arctan \frac{\sqrt{5}-1}{2}, \tag{1.2}$$

$$\sum_{n=0}^{\infty} \arctan \frac{\sinh x}{\cosh nx} = \frac{3\pi}{4} - \arctan e^x. \tag{1.3}$$

(1.1) is a classic sum evaluated first by Glaisher in [1]. The sum (1.2), where F_n is n-th Fibonacci number, was calculated by Hoggatt and Ruegels [6]. The sum (1.3) was noted in [8]. See [2] for further references and a brief summary of research in this direction.

All summations of the type (1.1) and (1.2) seem to be based on two methods: the telescopic principle, and the method of zeroes, as was noted in [7].

Even earlier, in his studies of elliptic functions, Jacobi proved identity of which he wrote in his treatise on elliptic functions "one is obliged to rank among the most elegant formulas" [9],[10]:

$$\frac{1}{4} \arcsin k = \arctan q^{1/2} - \arctan q^{3/2} + \arctan q^{5/2} - \dots \tag{1.4}$$

Here $q = e^{-\pi K'/K}$, K is the complete elliptic integral of the first kind with modulus k

$$K = K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

$K' = K(k')$ with $k' = \sqrt{1-k^2}$ being the complementary modulus. The quantity $\arcsin k$ is called modular angle. Together with the obvious relation

$$\arcsin k + \arcsin k' = \frac{\pi}{2},$$

this implies

$$\sum_{n=1}^{\infty} \chi(n) \arctan e^{-\alpha n} + \sum_{n=1}^{\infty} \chi(n) \arctan e^{-\beta n} = \frac{\pi}{8}, \qquad \alpha\beta = \frac{\pi^2}{4}, \tag{1.5}$$

where $\chi(n) = \sin \frac{\pi n}{2}$ is Dirichlet character modulo 4 ([3], ch.14, entry 15). (1.5) is Jacobi's imaginary transformation for the modular angle.

Another arctan series related to elliptic functions was found in an unpublished manuscript by B. Cais [4]:

$$\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \arctan \frac{\sqrt{3}}{1 + 2e^{\alpha n}} + \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \arctan \frac{\sqrt{3}}{1 + 2e^{\beta n}} = \frac{\pi}{18}, \qquad \alpha\beta = \frac{4\pi^2}{9}, \tag{1.6}$$

where $\left(\frac{j}{3}\right) = \frac{2}{\sqrt{3}} \sin \frac{2\pi j}{3}$ is Legendre symbol modulo 3.

The focus of this paper will be the two reciprocal identities for finite sums of inverse tangents, Theorems 1 and 3 below. Their proofs are given in sections 2 and 3, respectively. In section 4 we mention another summation formula related to solution of Dirichlet problem on a finite rectangular grid.

Theorem 1. *Let $n, m \in \mathbb{N}_0$ and $\alpha\beta = 1$, $\alpha > 0$. Then*

$$\sum_{|j| \le n} (-1)^{n+j} \arctan\left(\sqrt{1 + \alpha^2 \cos^2 \frac{\pi j}{2n+1}} - \alpha \cos \frac{\pi j}{2n+1}\right)^{2m+1}$$

$$+ \sum_{|k| \le m} (-1)^{m+k} \arctan\left(\sqrt{1 + \beta^2 \cos^2 \frac{\pi k}{2m+1}} - \beta \cos \frac{\pi k}{2m+1}\right)^{2n+1} = \frac{\pi}{4}. \qquad (1.7)$$

Note that when $n = m$ and $\alpha = 1$ both sums in (1.7) are equal and we get a closed form summation:

Corollary 2. *For $n \in \mathbb{N}_0$:*

$$\sum_{|j| \le n} (-1)^{n+j} \arctan\left(\sqrt{1 + \cos^2 \frac{\pi j}{2n+1}} - \cos \frac{\pi j}{2n+1}\right)^{2n+1} = \frac{\pi}{8}. \qquad (1.8)$$

It is instructive to write (1.7) in another form by shifting the summation variable and simple rearrangement of terms

$$\sum_{j=1}^{2n} \chi_4(j) \arctan\left(\sqrt{1 + \alpha^2 \sin^2 \frac{\pi j}{4n+2}} - \alpha \sin \frac{\pi j}{4n+2}\right)^{2m+1}$$

$$+ \sum_{k=1}^{2m} \chi_4(k) \arctan\left(\sqrt{1 + \beta^2 \sin^2 \frac{\pi j}{4m+2}} - \beta \sin \frac{\pi j}{4m+2}\right)^{2n+1}$$

$$= \frac{\pi}{8} - \frac{1}{2}(-1)^n \arctan\left(\sqrt{1 + \alpha^2} - \alpha\right)^{2m+1} - \frac{1}{2}(-1)^m \arctan\left(\sqrt{1 + \beta^2} - \beta\right)^{2n+1}.$$

From this form of (1.7), it is evident that letting $n = m \to \infty$ and redefining α and β, one recovers (1.5). Thus, (1.7) is discretized version of (1.5). Our proof is completely elementary and provides an elementary proof of the modular relation (1.5).

Theorem 3. *Let n and m be positive odd numbers and $\alpha\beta = 1$. Then*

$$\sum_{j=1}^{3n/2} \left(\frac{j}{3}\right) \arctan \frac{\sqrt{3}}{1 + 2\left(\dfrac{\alpha + \tan \frac{\pi j}{3n}}{\alpha - \tan \frac{\pi j}{3n}}\right)^m} + \sum_{k=1}^{3m/2} \left(\frac{k}{3}\right) \arctan \frac{\sqrt{3}}{1 + 2\left(\dfrac{\beta + \tan \frac{\pi k}{3m}}{\beta - \tan \frac{\pi k}{3m}}\right)^n} = -\frac{\pi}{6}, \qquad (1.9)$$

where $\left(\frac{j}{3}\right)$ is Legendre symbol modulo 3.

If $n = m$ and $\alpha = 1$ the two sums in Theorem 3 are equal and we get closed form summation

Corollary 4. *For an odd positive integer n:*

$$\sum_{j=1}^{3n/2} \left(\frac{j}{3}\right) \arctan \frac{\sqrt{3}}{1 + 2\cot^n\left(\frac{\pi}{4} - \frac{\pi j}{3n}\right)} = -\frac{\pi}{12}. \qquad (1.10)$$

As an illustration of (1.10) note the case $n = 3$:

$$\arctan \frac{\sqrt{3}}{1 + 2\cot^3 \frac{\pi}{36}} - \arctan \frac{\sqrt{3}}{1 + 2\cot^3 \frac{5\pi}{36}} - \arctan \frac{\sqrt{3}}{1 - 2\cot^3 \frac{7\pi}{36}} = \frac{\pi}{12}.$$

Although (1.9) has a structure similar to (1.6) it is not clear if (1.6) can be derived from (1.9) as a limiting case. However, by combining the limiting case of Theorem 3 with (1.6) one can find the transformation formula for another infinite arctan series:

$$\sum_{n=0}^{\infty} \left(\frac{n-1}{3}\right) \arctan \frac{\sqrt{3}}{1 - 2e^{\alpha(2n+1)}} + \sum_{n=0}^{\infty} \left(\frac{n-1}{3}\right) \arctan \frac{\sqrt{3}}{1 - 2e^{\beta(2n+1)}} = \frac{2\pi}{9}, \qquad \alpha\beta = \frac{\pi^2}{9}. \qquad (1.11)$$

2. Proof of Theorem 1

We break the proof into a series of lemmas.

Lemma 5. *The following identity holds for $\alpha > 0$, $n, m \in \mathbb{N}_0$ and $j \in \mathbb{Z}$*

$$2 \arctan\left(\sqrt{1 + \alpha^2 \cos^2 \frac{\pi j}{2n+1}} - \alpha \cos \frac{\pi j}{2n+1}\right)^{2m+1} = \frac{\pi}{2} - \arctan\left(\sinh(2m+1)\alpha_j\right)$$

where α_j is the positive solution of $\sinh \alpha_j = \alpha \cos \frac{\pi j}{2n+1}$.

Proof. By denoting $s = 2m + 1$ for brevity we obtain

$$
\begin{aligned}
2 \arctan\left(\sqrt{1 + \alpha^2 \cos^2 \frac{\pi j}{2n+1}} - \alpha \cos \frac{\pi j}{2n+1}\right)^{2m+1} &= 2 \arctan\left(\cosh \alpha_j - \sinh \alpha_j\right)^s \\
&= 2 \arctan e^{-s\alpha_j} \\
&= \frac{\pi}{2} - \left(\arctan e^{s\alpha_j} - \arctan e^{-s\alpha_j}\right) \\
&= \frac{\pi}{2} - \arctan \frac{e^{s\alpha_j} - e^{-s\alpha_j}}{2}.
\end{aligned}
$$

Since $\dfrac{e^x - e^{-x}}{2} = \sinh x$ the proof is complete. $\qquad\square$

Lemma 6. *For $n, m \in \mathbb{N}_0$, $j \in \mathbb{Z}$, and α_j as was defined in the previous lemma, one has*

$$\frac{\pi}{2} - \arctan\left(\sinh(2m+1)\alpha_j\right) = (-1)^m \sum_{|k| \leq m} \arctan \frac{\cos \frac{2\pi k}{2m+1}}{\alpha \cos \frac{\pi j}{2n+1}},$$

Proof. Using properties of complex numbers we write

$$
\begin{aligned}
\frac{\pi}{2} - \arctan\left(\sinh(2m+1)\alpha_j\right) &= \arg(i) + \arg\left(1 - i\sinh(2m+1)\alpha_j\right) \\
&= \arg\left(\sinh(2m+1)\alpha_j + i\right) \\
&= (-1)^m \arg\left(\sinh(2m+1)\alpha_j + \sinh \frac{\pi i(2m+1)}{2}\right).
\end{aligned}
$$

This expression can be factorised according to the formula

$$\sinh(2m+1)a + \sinh(2m+1)b = 2^{2m} \prod_{|k| \leq m} \left(\sinh a + \sinh\left(b + \frac{2\pi i k}{2m+1}\right)\right).$$

Its validity is easy to check by standard methods: both sides are polynomials in $\sinh a$ with leading coefficient 2^{2m} and zeroes $-\sinh\left(b + \frac{2\pi i k}{2m+1}\right)$, $|k| \leq m$.

Thus

$$
\begin{aligned}
\frac{\pi}{2} - \arctan\left(\sinh(2m+1)\alpha_j\right) &= (-1)^m \arg\left(2^{2m} \prod_{|k| \leq m} \left(\sinh \alpha_j + \sinh\left(\frac{\pi i}{2} + \frac{2\pi i k}{2m+1}\right)\right)\right) \\
&= (-1)^m \arg\left(\prod_{|k| \leq m} \left(\alpha \cos \frac{\pi j}{2n+1} + i \cos \frac{2\pi k}{2m+1}\right)\right) \\
&= (-1)^m \sum_{|k| \leq m} \arctan \frac{\cos \frac{2\pi k}{2m+1}}{\alpha \cos \frac{\pi j}{2n+1}},
\end{aligned}
$$

as required. $\qquad\square$

Lemma 7. *For $n, m \in \mathbb{N}_0$, $j \in \mathbb{Z}$, one has*

$$\sum_{|k| \leq m} \arctan \frac{\cos \frac{2\pi k}{2m+1}}{\alpha \cos \frac{\pi j}{2n+1}} = \sum_{|k| \leq m} (-1)^k \arctan \frac{\cos \frac{\pi k}{2m+1}}{\alpha \cos \frac{\pi j}{2n+1}}.$$

Proof. Let f be an odd function. Then

$$\sum_{|k|\leq m}(-1)^k f\left(\cos\frac{\pi k}{2m+1}\right) = \sum_{|k|\leq m} f\left(\cos\left(\frac{\pi k}{2m+1}-\pi k\right)\right)$$

$$= \sum_{|k|\leq m} f\left(\cos\frac{2\pi km}{2m+1}\right)$$

$$= \sum_{|k|\leq m} f\left(\cos\frac{2\pi k}{2m+1}\right).$$

The last equality is explained as follows. First, note that cos has period 2π. The sum $\sum_{|k|\leq m}$ is over residue class mod $2m+1$. When $m>0$, the numbers m and $2m+1$ are coprime. Hence, when k runs over residue class mod $2m+1$, the set of numbers km runs over residue class mod $2m+1$.

To complete the proof of the lemma set $f(x)=\arctan\dfrac{x}{\alpha\cos\frac{\pi j}{2n+1}}$. $\qquad\square$

Lemma 8.

$$\sum_{|j|\leq n}(-1)^j = (-1)^n, \qquad n\in\mathbb{N}_0.$$

Proof. The sum is trivial when $n=0$. Let's assume that $n>0$. Then

$$\sum_{|j|\leq n}(-1)^j = (-1)^n\frac{1-(-1)^{2n+1}}{1-(-1)} = (-1)^n. \qquad\square$$

Now, we are in a position to prove Theorem 1. According to lemmas 5-8 we have that the LHS of equation (1.7) equals

$$\sum_{|j|\leq n}(-1)^{n+j}\frac{1}{2}\sum_{|k|\leq m}(-1)^{m+k}\arctan\frac{\cos\frac{\pi k}{2m+1}}{\alpha\cos\frac{\pi j}{2n+1}} + \sum_{|k|\leq m}(-1)^{m+k}\frac{1}{2}\sum_{|j|\leq n}(-1)^{n+j}\arctan\frac{\cos\frac{\pi j}{2n+1}}{\beta\cos\frac{\pi k}{2m+1}}$$

$$= \frac{1}{2}(-1)^{n+m}\sum_{|j|\leq n}\sum_{|k|\leq m}(-1)^{j+k}\frac{\pi}{2}\text{sign}\left(\alpha\cos\frac{\pi j}{2n+1}\cos\frac{\pi k}{2m+1}\right)$$

$$= \frac{\pi}{4}(-1)^{n+m}\sum_{|j|\leq n}(-1)^j\sum_{|k|\leq m}(-1)^k = \frac{\pi}{4}.$$

3. PROOF OF THEOREM 3

Again, as we did in the previous section, it is convenient to break the proof into several parts.

Lemma 9. *We have the partial fractions expansion for arbitrary positive integer m:*

$$\frac{\sinh\left(m\tanh^{-1}z\right)}{\sinh\left(3m\tanh^{-1}z\right)}\frac{1}{1-z^2} = \frac{1}{m\sqrt{3}}\sum_{k=1}^{3m/2}\left(\frac{k}{3}\right)\frac{\tan\frac{\pi k}{3m}}{z^2+\tan^2\frac{\pi k}{3m}}. \tag{3.1}$$

Proof. Since $\frac{\sinh t}{\sinh 3t} = \frac{1}{2\cosh 2t+1}$, $\tanh^{-1}z = \frac{1}{2}\ln\frac{1+z}{1-z}$, and

$$2\cosh\left(2m\tanh^{-1}z\right) = \left(\frac{1+z}{1-z}\right)^m + \left(\frac{1-z}{1+z}\right)^m,$$

the LHS of (3.1) is a rational function of z of the form $f(z)=\frac{(1-z^2)^{m-1}}{P_{2m}(z)}$, where $P_{2m}(z)$ is polynomial of degree exactly $2m$. This rational function has poles at $z_k = i\tan\frac{\pi k}{3m}$, $k=3r-1$ or $3r-2$ with $r=1,2,3,...,m$. Rezidues of $f(z)$ at z_k are

$$\frac{(-1)^k}{3im}\sin\frac{\pi k}{3} = \frac{1}{2im\sqrt{3}}\left(\frac{k}{3}\right).$$

Hence, taking into account that $\left(\frac{k}{3}\right)=0$ when $k\equiv 0\pmod 3$

$$\frac{\sinh\left(m\tanh^{-1}z\right)}{\sinh\left(3m\tanh^{-1}z\right)}\frac{1}{1-z^2} = \frac{1}{2im\sqrt{3}}\sum_{k=1}^{3m}\left(\frac{k}{3}\right)\frac{1}{z-i\tan\frac{\pi k}{3m}}.$$

Due to $\left(\frac{3m-k}{3}\right) = -\left(\frac{k}{3}\right)$ and $\tan\frac{\pi(3m-k)}{3m} = -\tan\frac{\pi k}{3m}$ this is equivalent to (3.1). $\qquad\square$

Lemma 10. *For arbitrary positive integers n and m we have the transformation formula*

$$\sum_{j=1}^{3n/2} \left(\frac{j}{3}\right) \frac{\sinh\left(m\tanh^{-1}\left(z\tan\frac{\pi j}{3n}\right)\right)}{\sinh\left(3m\tanh^{-1}\left(z\tan\frac{\pi j}{3n}\right)\right)} \frac{\tan\frac{\pi j}{3n}}{1 - z^2\tan^2\frac{\pi j}{3n}}$$

$$-\frac{n}{mz^2}\sum_{k=1}^{3m/2}\left(\frac{k}{3}\right)\frac{\sinh\left(n\tanh^{-1}\left(z^{-1}\tan\frac{\pi k}{3m}\right)\right)}{\sinh\left(3n\tanh^{-1}\left(z^{-1}\tan\frac{\pi k}{3m}\right)\right)}\frac{\tan\frac{\pi k}{3m}}{1 - z^{-2}\tan^2\frac{\pi k}{3m}} = 0. \qquad (3.2)$$

Proof. In the previous lemma, replace z with $z\tan\frac{\pi j}{3n}$, then multiply the resulting identity with

$$\frac{z}{n}\left(\frac{j}{3}\right)\tan\frac{\pi j}{3n},$$

and sum wrt j from 1 to $3n/2$. It is easy to see the symmetry of the resulting double sum under the transformation $n \to m$, $m \to n$, $z \to 1/z$, from which the identity in the lemma follows. $\qquad\square$

Lemma 11.

$$\sqrt{3}\int_s^\infty \frac{\sinh t}{\sinh 3t}\,dt = \frac{\pi}{6} - \arctan\frac{\tanh s}{\sqrt{3}} = \arctan\frac{\sqrt{3}}{1 + 2e^{2s}}.$$

Proof. The proof of this lemma is given in [4], but we reproduce it here for the sake of completeness. Since

$$\frac{\sinh t}{\sinh 3t} = \frac{1}{2\cosh 2t + 1} = \frac{1}{\cosh^2 t}\frac{1}{3 + \tanh^2 t},$$

the integral can be easily calculated. The second equality follows from the elementary formula

$$\arctan x - \arctan y = \arctan\frac{x - y}{1 + xy}$$

with $x = \frac{1}{\sqrt{3}}$, $y = \frac{\tanh s}{\sqrt{3}}$ and the identity

$$\frac{\frac{1}{\sqrt{3}} - \frac{\tanh s}{\sqrt{3}}}{1 + \frac{\tanh s}{3}} = \frac{\sqrt{3}}{1 + 2e^s}. \qquad\square$$

Lemma 12. *For an odd positive integer n:*

$$\sum_{j=1}^{3n/2}\left(\frac{j}{3}\right) = 1.$$

Proof. This is obvious for $n = 1$. For arbitrary odd n its validity follows from the fact that the sum of Legendre symbols mod 3 for three consecutive integers is 0. $\qquad\square$

The formula in Theorem 3 now follows easily from these lemmas. We integrate equation (3.2) wrt z from $1/\alpha$ to ∞ using lemma 11. Then assuming that n is odd we complete the proof using lemma 12.

4. Other reciprocal relations

In our previous paper [13], we have found many relations of the form $P(n,m) = P(m,n)$ for finite products of trigonometric functions. However, the identity in Theorem 1 is of the type $S(n,m)+S(m,n) = C$, where C is independent of n and m. There is simple method to find other relations of this type. It is based on the solution of Dirichlet problem on a finite rectangular grid [11]. For example

$$m\sum_{j=1}^{n}(-1)^j\cot\frac{\pi j}{2n}\frac{\sinh y\alpha_j}{\sinh m\alpha_j}\sin\frac{\pi jx}{n} + n\sum_{k=1}^{m}(-1)^k\cot\frac{\pi k}{2m}\frac{\sinh x\beta_k}{\sinh n\beta_k}\sin\frac{\pi ky}{m} = -xy, \qquad (4.1)$$

where $1 \le x \le n$, $1 \le y \le m$ are integers and

$$\cos\frac{\pi j}{n} + \cosh\alpha_j = 2, \quad \cos\frac{\pi k}{m} + \cosh\beta_k = 2 \qquad (1 \le j \le n,\ 1 \le k \le m). \qquad (4.2)$$

In particular, when $x = y$, $n = m$ this gives the closed form summation

$$\sum_{j=1}^{n}(-1)^j\cot\frac{\pi j}{2n}\frac{\sinh x\alpha_j}{\sinh n\alpha_j}\sin\frac{\pi jx}{n} = -\frac{x^2}{n}, \qquad \sinh\frac{\alpha_j}{2} = \sin\frac{\pi j}{2n}. \qquad (4.3)$$

Laplace operator on a finite rectangular grid is defined as

$$\Delta f(x, y, k) = f(x - 1, y) + f(x + 1, y) + f(x, y - 1) + f(x, y + 1) - 4f(x, y).$$

One can see that the RHS of (4.1) satisfies the discrete Laplace equation

$$\Delta f(x, y) = 0, \quad (0 <\leq x \leq n, 0 \leq y \leq m)$$

on a rectangular grid of size $n \times m$. Also $-xy = f_1(x, y) + f_2(x, y)$, where $f_1(x, y)$ and $f_2(x, y)$ are solutions of the Laplace equation with boundary conditions

$$\begin{cases} f_1(0, y) = f_1(n, y) = 0, \ 0 \leq y \leq m, \\ f_1(x, 0) = 0, f_1(x, m) = xm, \ 0 \leq x \leq n, \end{cases} \tag{4.4}$$

$$\begin{cases} f_2(x, 0) = f_2(x, m) = 0, \ 0 \leq x \leq n, \\ f_2(0, y) = 0, f_2(n, y) = ny, \ 0 \leq y \leq m. \end{cases} \tag{4.5}$$

Partial solutions of Laplace equation corresponding to boundary conditions (4.4) and (4.5) are given by, respectively

$$u_j^{(1)}(x, y) = \sin \frac{\pi j x}{n} \sinh y\alpha_j, \quad (1 \leq j \leq n).$$

$$u_k^{(2)}(x, y) = \sin \frac{\pi k y}{m} \sinh x\beta_k, \quad (1 \leq k \leq m).$$

In fact this method is quite well known and there are many examples in electrodynamics and heat conduction problems in physics (e.g., [12]).

One could generalize (4.1) to include one continuous parameter α by requiring that α_j and β_k be defined by

$$\sinh \frac{\alpha_j}{2} = \alpha \sin \frac{\pi j}{2n}, \ \sinh \frac{\beta_k}{2} = \frac{1}{\alpha} \sin \frac{\pi k}{2m}, \qquad (1 \leq j \leq n, 1 \leq k \leq m)$$

instead of (4.2). However to obtain a closed form summation we would need $\alpha = 1$, so this does not generalize (4.3).

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