Division by Zero Calculus For Differentiable Functions in Multiply Dimensions

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Abstract: Based on the preprint survey paper ([25]), we will give a fundamental relation among the basic concepts of division by zero calculus and derivatives as a direct extension of the preprints ([29, 30]) which gave the generalization of the division by zero calculus to differentiable functions. Here, we will consider the case of multiply dimensions. In particular, we will find a new viewpoint and applications to the gradient and nabla.

Key Words: Division by zero, division by zero calculus, differentiable, analysis, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0, [(z^n)/n]_{n=0} = \log z$, $[e^{(1/z)}]_{z=0} = 1$, gradient, vector analysis, nabla, Green's function, Gâteaux differentiable, Frêchet differentiable.

AMS Mathematics Subject Classifications: 00A05, 00A09, 42B20, 30E20.

1 Division by zero calculus

In order to state the new results in a self-contained way, we will recall the simple background on the division by zero calculus for differentiable functions based on ([29, 30]). For the basic references on the division by zero and the division by zero calculus, see the references cited in the reference.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{1.1}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as** $1/0 = 0$ **and** $0/0 = 0$**, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense. However, we gave several definitions for $1/0 = 0$ and $0/0 = 0$. See, for example, [26].

In addition, when the function $f(x)$ is not differentiable, by many meanings of zero, we **should define** as

$$\frac{f(x)}{x-a}|_{x=a} = 0,$$

for example, since 0 represents impossibility. In particular, the value of the function $y = |x|/x$ at $x = 0$ is zero.

We will note its naturality of the definition.

Indeed, we consider the function $F(x) = f(x) - f(a)$ and by the definition, we have

$$\frac{F(x)}{x-a}|_{x=a} = F'(a) = f'(a).$$

Meanwhile, by the definition, we have

$$\lim_{x\to a} \frac{F(x)}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a). \tag{1.2}$$

For many applications, see the references cited in the reference.

The identity (1.1) may be regarded as an interpretation of the differential coefficient $f'(a)$ by the concept of the division by zero. Here, we do not use the concept of limitings. This means that NOT

$$\lim_{x\to a}\frac{f(x)}{x-a}$$

BUT

$$\frac{f(x)}{x-a}|_{x=a}.$$

Note that $f'(a)$ represents the principal variation of order $x-a$ of the function $f(x)$ at $x=a$ which is defined independently of $f(a)$ in (1.2). This is a basic meaning of the division by zero calculus $\frac{f(x)}{x-a}|_{x=a}$.

Following this idea, we can accept the formula, naturally, for also $n=0$ for the general formula; that is,

$$\frac{f(x)}{(x-a)^0}|_{x=a} = \frac{f^{(0)}(a)}{0!} = f(a).$$

In the expression (1.1), the value $f'(a)$ in the right hand side is represented by the point a, meanwhile the expression

$$\frac{f(x)}{x-a}|_{x=a} \tag{1.3}$$

in the left hand side, is represented by the dummy variable $x-a$ that represents the property of the function around the point $x=a$ with the sense of the division

$$\frac{f(x)}{x-a}.$$

For $x \neq a$, it represents the usual division.

Of course, by our definition

$$\frac{f(x)}{x-a}|_{x=a} = \frac{f(x)-f(a)}{x-a}|_{x=a}, \tag{1.4}$$

however, here $f(a)$ may be replaced by any constant. This fact looks like showing that the function

$$\frac{1}{x-a}$$

is zero at $x = a$ in a sense. Of course, this result is derived immediately from the definition of the division by zero calculus.

When we apply the relation (1.1) to the elementary formulas for differentiable functions, we can imagine some deep results. For example, in the simple formula

$$(u + v)' = u' + v',$$

we have the result

$$\frac{u(x) + v(x)}{x - a}|_{x=a} = \frac{u(x)}{x - a}|_{x=a} + \frac{v(x)}{x - a}|_{x=a},$$

that is not trivial in our definition. This is a result from the property of derivatives.

In the following well-known formulas, we have some **deep meanings** on the division by zero calculus.

$$(uv)' = u'v + uv',$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

and the famous laws

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

Note also the logarithm derivative, for $u, v > 0$

$$(\log(uv))' = \frac{u'}{u} + \frac{v'}{v}$$

and for $u > 0$

$$(u^v)' = u^v \left(v' \log u + v\frac{u'}{u}\right).$$

For the second order differentials, we have the familiar formulas:

$$(uv)'' = u''v + 2u'v' + uv'',$$

$$\frac{d^2 f(g(t))}{dt^2} = f''(g(t))g'(t) + f'(g(t))g''(t),$$

$$\left(\frac{1}{f}\right)'' = \frac{2(f')^2 - ff''}{f^3}$$

and

$$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3}.$$

The representation of the higher order differential coefficients $f^{(n)}(a)$ is very simple and, for example, for the Taylor expansion we have the beautiful representation

$$f(a) = \sum_{n=0}^{\infty} \frac{f(x)}{(x-a)^n}|_{x=a} \cdot (x-a)^n.$$

Further note that

$$\frac{f(x)}{(x-a)^2}|_{x=a} = \frac{f''(a)}{2}$$

$$= \lim_{x \to 0} \frac{f(a+x) + f(a-x) - 2f(a)}{2x^2}.$$

We note the basic relation for analytic functions $f(z)$ for the analytic extension of $f(x)$ to complex variable z

$$\frac{f(x)}{(x-a)^n}|_{x=a} = \frac{f^{(n)}(a)}{n!} = Res._{\zeta=a}\left\{\frac{f(\zeta)}{(\zeta-a)^{n+1}}\right\}.$$

We therefore see the basic identities among the division by zero calculus, differential coefficients and residues in the case of analytic functions. Among these basic concepts, the differential coefficients are studied deeply and so, from the results of the differential coefficient properties, we can derive another results for the division by zero calculus and residures. See [30].

2 Definition of division by zero calculus for multiply dimensions

We would like to consider some general situation in Section 1, and we can consider in the following on some more abstract way on the line of Gâteaux differentiable and Frêchet differentiable functions. However, as the first step, we would like to consider the prototype case of the Taylor expansion in the three dimensional case as an essential and typical case.

We first recall the Taylor expansion

$$f(x,y,z) = f(a,b,c) + \left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} + (z-c)\frac{\partial}{\partial z}\right) f(a,b,c) +$$

$$+\frac{1}{2!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} + (z-c)\frac{\partial}{\partial z}\right)^2 f(a,b,c) + \cdots$$

$$+\frac{1}{n!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} + (z-c)\frac{\partial}{\partial z}\right)^n f(a,b,c) + \cdots.$$

Then, in particular, note that as in the one dimensional way, for $R = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$

$$f(x,y,z) = f(a,b,c) + \left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} + (z-c)\frac{\partial}{\partial z}\right) f(a,b,c) + o(R).$$

When we consider the case $R = 0$, we should consider its direction as in

$$\lim_{R\to 0}\left(\frac{x-a}{R}, \frac{y-b}{R}, \frac{y-b}{R}\right) = (\ell, m, n).$$

We will denote the unit vector (ℓ, m, n) at (a,b,c) by $\mathbf{u}(a,b,c)$. Then, we **define** the division by zero calculus

$$\frac{f(x,y,z)}{R}|_{R=0} \tag{2.1}$$

by

$$(\nabla f)(a,b,c) \cdot \mathbf{u}(a,b,c);$$

that is,

$$\frac{f(x,y,z)}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}|_{(a,b,c)} = (\nabla f)(a,b,c) \cdot \mathbf{u}(a,b,c).$$

We can define a general order division by zero calculus

$$\frac{f(x,y,z)}{R^n}|_{R=0}$$

similarly.

As in one dimensional case, if a functions is not differentiable in the definition, then we shall define it as zero.

As in the one dimensional case, we can apply formulas for ∇ to the division by zero calculus. For example, in the formulas

$$\nabla(f+g) = \nabla f + \nabla g,$$

$$\nabla(fg) = g\nabla f + f\nabla g,$$

$$\nabla(f(g)) = \frac{df}{dg}\nabla g,$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B}),$$

and

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2},$$

for example, we obtain

$$\frac{f(x,y,z)g(x,y,z)}{R}|_{R=0} = g(a,b,c)\frac{f(x,y,z)}{R}|_{R=0} + f(a,b,c)\frac{g(x,y,z)}{R}|_{R=0}.$$

Anyhow, with our definition, we can consider the division by zero calculus

$$\frac{f(x,y,z)}{R^n}|_{R=0}$$

that appears in many formulas.

3 Examples

We shall examine examples.

1. The value of the function $y = x/|x|$ at $x = 0$ is ± 1 in our sense, here;

$$y = \frac{x}{|x|}|_{x=0} = \pm 1,$$

that depends on the unit vector $\mathbf{u}$.

However, since the division by zero calculus is not always almighty, we should consider the value 0 also;

$$y = \frac{x}{|x|}|_{x=0} = \frac{0}{0} = 0,$$

in some practical sense. Note that this function is an odd function and 0 is the mean value around the origin.

2.
$$\frac{x\exp(xy)}{\sqrt{(x-a)^2+(y-b)^2}}|_{(a,b,c)} = (\ell(1+ab)+ma^2)e^{ab}.$$

3.
$$\frac{\log\sqrt{x^2+y^2+z^2}}{R}|_{R=0} = \frac{\ell a+mb+nc}{a^2+b^2+c^2}.$$

4.
$$\frac{(x^2+y^2+z^2)^{n/2}}{R}|_{R=0} = n(\ell a{+}mb{+}nc)\left(a^2+b^2+c^2\right)^{(n-2)/2)}, n = -1, 1, 2, ...$$

5.
$$\frac{f(\sqrt{x^2+y^2+z^2})}{R}|_{R=0} = \frac{\ell a+mb+nc}{\sqrt{a^2+b^2+c^2}}f'(\sqrt{a^2+b^2+c^2}).$$

6.
$$\frac{(x^2+y^2+z^2)\exp(-\sqrt{x^2+y^2+z^2})}{R}|_{R=0}$$
$$= (\ell a+mb+nc)\left(2-\sqrt{a^2+b^2+c^2}\right)\exp(-\sqrt{a^2+b^2+c^2}).$$

4 Green's functions

For the fundamental solution of the Laplace equation

$$\Delta G(R) = -\delta(\mathbf{r}-\mathbf{r}'),$$

we have, for $R = |\mathbf{r}-\mathbf{r}'|$,

$$G_1(R) = -\frac{1}{2}R,$$

$$G_2(R) = -\frac{1}{2\pi}\log R,$$

and

$$G_3(R) = -\frac{1}{4\pi R},$$

depending on the dimensions. We know that at the singular point $R = 0$, they are all zero.

For the fundamental solutions of the Helmholtz type equation

$$(\Delta + k^2)G(R) = -\delta(\mathbf{r} - \mathbf{r}'),$$

we have

$$G_1^{\pm}(R) = \frac{\pm i}{2k}\exp(\pm ikR)$$

and

$$G_3^{\pm}(R) = \frac{1}{4\pi R}\exp(\pm ikR).$$

depending on the dimensions and selections of branches. Then we have:

$$G_1^{\pm}(0) = \frac{\pm i}{2k}$$

and

$$G_3^{\pm}(R)|_{R=0} = \frac{\pm ik}{4\pi}.$$

For the fundamental solutions of the Klein-Gordon equation

$$(\Delta - \mu^2)G(R) = -\delta(\mathbf{r} - \mathbf{r}'),$$

we have

$$G_1(R) = \frac{1}{2\mu}\exp(-\mu R)$$

and

$$G_3(R) = \frac{1}{4\pi R}\exp(-\mu R).$$

depending on the dimensions. Then we have:

$$G_1(0) = \frac{1}{2\mu}$$

and

$$G_3(R)|_{R=0} = \frac{-\mu}{4\pi}.$$

In addition, we have:

$$\frac{\cos kR}{4\pi R}|_{R=0} = 0$$

and

$$\frac{\sin kR}{4\pi R}|_{R=0} = \frac{k}{4\pi}.$$

5 One dimensional case and multiply dimensional case

In Section 2, we considered the division by zero calculus in the natural way for multiply dimensional cases. However, we see that its basic idea is similar with the one dimensional case, indeed, when we consider it on some line as in $x - a = \ell R, y - b = mR, z - c = nR$, we can see that the result is the same as the definition of the division by zero calculus in one dimensional case.

At this moment, we have another idea for (2.1); indeed, it looks like that the definition (2.1) is independent of the vector $\mathbf{u}(a, b, c)$. Following this idea, we can consider it as follows

$$\frac{f(x, y, z)}{R}|_{R=0} = (\nabla f)(a, b, c).$$

Whether this definition is good or bad will depend on the global properties of the division by zero calculus.

We shall consider one example in the multiply dimensional case.

On the complex plane, we shall consider the point η such that the two lines through (β and γ) and (α and η) are orthogonal and η is on the line through (β and γ).

Then, we obtain the formula

$$\eta = \frac{\overline{\alpha}(\beta - \gamma) + \alpha(\overline{\beta} - \overline{\gamma}) + \beta\overline{\gamma} - \overline{\beta}\gamma}{2(\beta - \gamma)}. \tag{5.1}$$

We will consider the case $\beta = \gamma$. Then, we obtain the formula, by the division by zero calculus for complex analysis

$$\eta = \frac{1}{2}(\overline{\alpha} + \overline{\gamma}).$$

Meanwhile, when we consider β on some line through the point γ, then we obtain the natural result that η is on the line and the line through (α and η) are orthogonal with the line.

We considered the division by zero calculus for the several complex variable case in [27] that shows some more delicate properties.

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