CHARACTERIZATIONS OF PRE-R_0 AND PRE-R_1 TOPOLOGICAL SPACES

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Abstract

In this paper we introduce two new classes of topological spaces called pre-R_0 and pre-R_1 spaces in terms of the concept of preopen sets and investigate some of their fundamental properties.

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1 Introduction

The notion of R_0 topological space is introduced by N. A. Shanin [20] in 1943. Later, A. S. Davis [3] rediscovered it and studied some properties of this weak separation axiom. Several topologists (e. g. [6], [9], [10], [19]) further investigated properties of R_0 topological spaces and many interesting results have been obtained in various contexts. In the same paper, A. S. Davis also introduced the notion of R_1 topological space which are independent of both T_0 and T_1 but strictly weaker than T_2. M. G. Murdeshwar and S. A. Naimpally [18] studied some of the fundamental properties of the class of R_1 topological spaces. In 1963, N. Levine [14] offered a new notion to the field of general topology by introducing semi-open sets. He defined this notion by utilizing the known notion of closure of an open set, i.e., a subset of a topological space is semi-open if it is contained in the closure of its interior. Since the advent of this notion, several new notions are defined in terms of semi-open sets of which two are semi-R_0 and semi-R_1 introduced by S. N. Maheshwari and R. Prasad [15] and C. Dorsett [5], respectively. These two notions are defined as natural generalizations of the separation axioms R_0

and R_1 by replacing the closure operator with the semiclosure operator and openness with semi-openness. In 1982, A. S. Mashhour et al. [17] introduced the notion of preopen set which is also known under the name of locally dense set [2] in the literature. Since then, this notion received wide usage in general topology. In this paper, we continue the study of the above mentioned classes of topological spaces satisfying these axioms by introducing two more notions in terms of preopen sets called pre-R_0 and pre-R_1. It turns out that pre-R_0 and pre-R_1 are equivalent with pre-T_1 and pre-T_2, respectively.

Throughout the paper (X, τ) (or simply X) will always denote a topological space. For a subset A of X, the closure, interior and complement of A in X are denoted by $Cl(A), Int(A)$ and $X - A$ respectively. By $PO(X, \tau)$ and $PC(X, \tau)$ we denote the collection of all preopen sets and the collection of all preclosed sets of (X, τ), respectively.

2 Preliminaries

Since we shall require the following known definitions, notations and some properties, we recall them in this section.

Definition 1 *Let A be a subset of a topological space (X, τ). Then :*
(1) A is preopen [17], if $A \subset Int(Cl(A))$.
(2) A is preclosed [17], if $X - A$ is preopen or equivalently if $Cl(Int(A)) \subset A$.
(3) The intersection of all preclosed sets containing A is called the preclosure of A [7] and is denoted by $pCl(A)$.
(4) (X, τ) is pre-T_1 [13], if to each pair of distinct points x and y of X, there exists a pair of preopen sets one containing x but not y and the other containing y but not x.
(5) (X, τ) is pre-T_2 [13], if to each pair of distinct points x and y of X, there exists a pair of disjoint preopen sets, one containing x and the other containing y.

Lemma 2.1 *(El-Deeb et al. [8]). Let (X, τ) be a topological space and A, B subsets of X. Then the following hold:*
(1) $x \in pCl(A)$ if and only if $A \cap V \neq \emptyset$ for every $V \in PO(X, \tau), x \in V$.
(2) A is preclosed in (X, τ) if and only if $A = pCl(A)$.
(3) $pCl(A) \subset pCl(B)$ if $A \subset B$.
(4) $pCl(pCl(A)) = pCl(A)$.

Recall, that a subset B_x of a topological space X is said to be a pre-neighbourhood of a point $x \in X$ [13] if there exists a preopen set U such that $x \in U \subset B_x$.

Lemma 2.2 *A subset of a topological space X is preopen in X if and only if it is a pre-neighbourhood of each of its points.*

3 Pre-R_0 spaces

Definition 2 *Let (X, τ) be a topological space and $A \subset X$. Then the pre-kernel of A [11], denoted by $pKer(A)$ is defined to be the set $pKer(A) = \cap\{G \in PO(X, \tau) \mid A \subset G\}$.*

Lemma 3.1 *Let (X, τ) be a topological space and $x \in X$. Then, $y \in pKer(\{x\})$ if and only if $x \in pCl(\{y\})$.*

Proof. Suppose that $y \notin pKer(\{x\})$. Then there exists a preopen set V containing x such that $y \notin V$. Therefore, we have $x \notin pCl(\{y\})$. The converse is similarly shown.

Lemma 3.2 *Let (X, τ) be a topological space and A a subset of X. Then, $pKer(A) = \{x \in X \mid pCl(\{x\}) \cap A \neq \emptyset\}$.*

Proof. Let $x \in pKer(A)$ and suppose $pCl(\{x\}) \cap A = \emptyset$. Hence, $x \notin X - pCl(\{x\})$ which is a preopen set containing A.This is impossible, since $x \in pKer(A)$. Consequently, $pCl(\{x\}) \cap A \neq \emptyset$. Next, let $x \in X$ such that $pCl(\{x\}) \cap A \neq \emptyset$ and suppose that $x \notin pKer(A)$. Then, there exists a preopen set U containing A and $x \notin U$. Let $y \in pCl(\{x\}) \cap A$. Hence, U is a pre-neigbourhood of y which does not contain x. By this contradiction $x \in pKer(A)$ and the claim.

Definition 3 *A topological space (X, τ) is said to be a pre-R_0 space if every preopen set contains the preclosure of each of its singletons.*

Lemma 3.3 *A topological space (X, τ) is pre-R_0 if and only if for each $U \in PO(X, \tau), x \in U$ implies $Cl(Int(\{x\}) \subset U$.*

Proposition 3.4 *For a topological space (X, τ), the following properties are equivalent :*
(1) (X, τ) is pre-R_0 space;
(2) For any $F \in PC(X, \tau), x \notin F$ implies $F \subset U$ and $x \notin U$ for some $U \in PO(X, \tau)$;
(3) For any $F \in PC(X, \tau), x \notin F$ implies $F \cap pCl(\{x\}) = \emptyset$;
(4) For any distinct points x and y of X, either $pCl(\{x\}) = pCl(\{y\})$ or $pCl(\{x\}) \cap pCl(\{y\}) = \emptyset$.

Proof. (1) $\rightarrow$ (2) : Let $F \in PC(X, \tau)$ and $x \notin F$. Then by (1) $pCl(\{x\}) \subset X - F$. Set $U = X - pCl(\{x\})$, then $U \in PO(X, \tau), F \subset U$ and $x \notin U$.
(2) $\rightarrow$ (3) : Let $F \in PC(X, \tau)$ and $x \notin F$. There exists $U \in PO(X, \tau)$ such that $F \subset U$ and $x \notin U$. Since $U \in PO(X, \tau), U \cap pCl(\{x\}) = \emptyset$ and $F \cap pCl(\{x\}) = \emptyset$.
(3) $\rightarrow$ (4) :Suppose that $pCl(\{x\}) \neq pCl(\{y\})$ for distinct points $x, y \in X$. There exists $z \in pCl(\{x\})$ such that $z \notin pCl(\{y\})$ (or $z \in pCl(\{y\})$ such that $z \notin pCl(\{x\})$). There exists $V \in PO(X, \tau)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin pCl(\{y\})$. By (3), we obtain $pCl(\{x\}) \cap pCl(\{y\}) = \emptyset$. The proof for otherwise is similar.
(4) $\rightarrow$ (1) : Let $V \in PO(X, \tau)$ and $x \in V$. For each $y \notin V, x \neq y$ and $x \notin pCl(\{y\})$. This shows that $pCl(\{x\}) \neq pCl(\{y\})$. By (4), $pCl(\{x\}) \cap pCl(\{y\}) = \emptyset$ for each $y \in X - V$ and hence $pCl(\{x\}) \cap (\bigcup_{y \in X - V} pCl(\{y\})) = \emptyset$. On the other hand, since $V \in PO(X, \tau)$ and $y \in X - V$, we have $pCl(\{y\}) \subset X - V$ and hence $X - V = \bigcup_{y \in X - V} pCl(\{y\})$. Therefore, we obtain $(X - V) \cap pCl(\{x\}) = \emptyset$ and $pCl(\{x\}) \subset V$. This shows that (X, τ) is a pre-R_0 space.

Corollary 3.5 *A topological space (X, τ) is a pre-R_0 space if and only if for any x and y in X, $pCl(\{x\}) \neq pCl(\{y\})$ implies $pCl(\{x\}) \cap pCl(\{y\}) = \emptyset$.*

Proof. This is an immediate consequence of Proposition 3.4.

Lemma 3.6 *The following statements are equivalent for any points x and y in a topological space (X, τ) :*
(1) $pKer(\{x\}) \neq pKer(\{y\})$;
(2) $pCl(\{x\}) \neq pCl(\{y\})$.

Proof. (1) $\rightarrow$ (2) : Suppose that $pKer(\{x\}) \neq pKer(\{y\})$, then there exists a point z in X such that $z \in pKer(\{x\})$ and $z \notin pKer(\{y\})$. From $z \in pKer(\{x\})$ it follows that $\{x\} \cap pCl(\{z\}) \neq \emptyset$ which implies $x \in pCl(\{z\})$. By $z \notin pKer(\{y\})$, we have $\{y\} \cap pCl(\{z\}) = \emptyset$. Since $x \in pCl(\{z\})$, $pCl(\{x\}) \subset pCl(\{z\})$ and $\{y\} \cap pCl(\{x\}) = \emptyset$. Therefore it follows that $pCl(\{x\}) \neq pCl(\{y\})$. Now $pKer(\{x\}) \neq pKer(\{y\})$ implies that $pCl(\{x\}) \neq pCl(\{y\})$.
(2) $\rightarrow$ (1) : Suppose that $pCl(\{x\}) \neq pCl(\{y\})$. Then there exists a point z in X such that $z \in pCl(\{x\})$ and $z \notin pCl(\{y\})$. It follows that there exists a preopen set containing z and therefore x but not y, namely, $y \notin pKer(\{x\})$ and thus $pKer(\{x\}) \neq pKer(\{y\})$.

Theorem 3.7 *A topological space (X, τ) is a pre-R_0 space if and only if for any points x and y in X , $pKer(\{x\}) \neq pKer(\{y\})$ implies $pKer(\{x\}) \cap pKer(\{y\}) = \emptyset$.*

Proof. Suppose that (X, τ) is a pre-R_0 space. Thus by Lemma 3.6, for any points x and y in X if $pKer(\{x\}) \neq pKer(\{y\})$ then $pCl(\{x\}) \neq pCl(\{y\})$. Now we prove that $pKer(\{x\}) \cap pKer(\{y\}) = \emptyset$. Assume that $z \in pKer(\{x\}) \cap pKer(\{y\})$. By $z \in pKer(\{x\})$, it follows that $x \in pCl(\{z\})$. Since $x \in pCl(\{x\})$, by Corollary 3.5 $pCl(\{x\}) = pCl(\{z\})$. Similarly, we have $pCl(\{y\}) = pCl(\{z\}) = pCl(\{x\})$. This is a contradiction. Therefore, we have $pKer(\{x\}) \cap pKer(\{y\}) = \emptyset$.
Conversely , let (X, τ) be a topological space such that for any points x and y in X, $pKer(\{x\}) \neq pKer(\{y\})$ implies $pKer(\{x\}) \cap pKer(\{y\}) = \emptyset$. If $pCl(\{x\}) \neq pCl(\{y\})$, then by Lemma 3.1, $pKer(\{x\}) \neq pKer(\{y\})$. Therefore $pKer(\{x\}) \cap pKer(\{y\}) = \emptyset$ which implies $pCl(\{x\}) \cap pCl(\{y\}) = \emptyset$. Because $z \in pCl(\{x\})$ implies that $x \in pKer(\{z\})$ and therefore $pKer(\{x\}) \cap pKer(\{z\}) \neq \emptyset$. By hypothesis, we therefore have $pKer(\{x\}) = pKer(\{z\})$. Then $z \in pCl(\{x\}) \cap pCl(\{y\})$ implies that $pKer(\{x\}) = pKer(\{z\}) = pKer(\{y\})$. This is a contradiction. Therefore, $pCl(\{x\}) \cap pCl(\{y\}) = \emptyset$ and by Corollary 3.5 (X, τ) is a pre-R_0 space.

Theorem 3.8 *For a topological space (X, τ), the following properties are equivalent :*
(1) (X, τ) is a pre-R_0 space;
(2) For any nonempty set A and $G \in PO(X, \tau)$ such that $A \cap G \neq \emptyset$, there exists $F \in PC(X, \tau)$ such that $A \cap F \neq \emptyset$ and $F \subset G$;

(3) Any $G \in PO(X, \tau)$ *,* $G = \cup\{F \in PC(X, \tau) \mid F \subset G\}$*;*
(4) Any $F \in PC(X, \tau)$*,* $F = \cap\{G \in PO(X, \tau) \mid F \subset G\}$*;*
(5) For any $x \in X, pCl(\{x\}) \subset pKer(\{x\})$*.*

Proof. (1) $\rightarrow$ (2) : Let A be a nonempty set of X and $G \in PO(X, \tau)$ such that $A \cap G \neq \emptyset$. There exists $x \in A \cap G$. Since $x \in G \in PO(X, \tau), pCl(\{x\}) \subset G$. Set $F = pCl(\{x\})$, then $F \in PC(X, \tau), F \subset G$ and $A \cap F \neq \emptyset$.
(2) $\rightarrow$ (3) : Let $G \in PO(X, \tau)$, then $G \supset \cup\{F \in PC(X, \tau) \mid F \subset G\}$. Let x be any point of G. There exists $F \in PC(X, \tau)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \cup\{F \in PC(X, \tau) \mid F \subset G\}$ and hence $G = \cup\{F \in PC(X, \tau) \mid F \subset G\}$.
(3) $\rightarrow$ (4) : This is obvious.
(4) $\rightarrow$ (5) : Let x be any point of X and $y \notin pKer(\{x\})$. There exists $V \in PO(X, \tau)$ such that $x \in V$ and $y \notin V$; hence $pCl(\{y\}) \cap V = \emptyset$. By (4) $(\cap\{G \in PO(X, \tau) \mid pCl(\{y\}) \subset G\}) \cap V = \emptyset$ and there exists $G \in PO(X, \tau)$ such that $x \notin G$ and $pCl(\{y\}) \subset G$. Therefore, $pCl(\{x\}) \cap G = \emptyset$ and $y \notin pCl(\{x\})$. Consequently, we obtain $pCl(\{x\}) \subset pKer(\{x\})$.
(5) $\rightarrow$ (1) : Let $G \in PO(X, \tau)$ and $x \in G$. Let $y \in pKer(\{x\})$, then $x \in pCl(\{y\})$ and $y \in G$. This implies that $pKer(\{x\}) \subset G$. Therefore, we obtain $x \in pCl(\{x\}) \subset pKer(\{x\}) \subset G$. This shows that (X, τ) is a pre-R_0 space.

Corollary 3.9 *For a topological space* (X, τ)*, the following properties are equivalent :*
(1) (X, τ) *is a pre-*R_0 *space;*
(2) $pCl(\{x\}) = pKer(\{x\})$ *for all* $x \in X$*.*

Proof. (1) $\rightarrow$ (2) : Suppose that (X, τ) is a pre-R_0 space. By Theorem 3.8, $pCl(\{x\}) \subset pKer(\{x\})$ for each $x \in X$. Let $y \in pKer(\{x\})$, then $x \in pCl(\{y\})$ and by Corollary 3.5 $pCl(\{x\}) = pCl(\{y\})$. Therefore, $y \in pCl(\{x\})$ and hence $pKer(\{x\}) \subset pCl(\{x\})$. This shows that $pCl(\{x\}) = pKer(\{x\})$.
(2) $\rightarrow$ (1) : This is obvious by Theorem 3.8.

The following lemma due to Maki et al. [16] is very useful and important.

Lemma 3.10 *In every topological space, each singleton is preopen or preclosed.*

Theorem 3.11 *A topological space* (X, τ) *is pre-*R_0 *if and only if it is pre-*T_1*.*

Proof. Necessity. Suppose that (X, τ) is pre-R_0. For each point $x \in X$, by Lemma 3.10 $\{x\}$ is preopen or preclosed in X. If $\{x\}$ is preopen, then we have $pCl(\{x\}) \subset \{x\}$ and hence $\{x\}$ is preclosed by Lemma 2.1. This shows that (X, τ) is pre-T_1.
Sufficiency. Let U be any preopen set of X and $x \in U$. Since $\{x\}$ is preclosed, $pCl(\{x\}) = \{x\} \subset U$. Therefore (X, τ) is pre-R_0.

Theorem 3.12 *For a topological space (X, τ), the following properties are equivalent :*
(1) (X, τ) is a pre-R_0 space;
(2) $x \in pCl(\{y\})$ if and only if $y \in pCl(\{x\})$.

Proof. (1) $\rightarrow$ (2) : This is obvious from Theorem 3.11.
(2) $\rightarrow$ (1) : Let U be a preopen set and $x \in U$. If $y \notin U$, then $x \notin pCl(\{y\})$ and hence $y \notin pCl(\{x\})$. This implies that $pCl(\{x\}) \subset U$. Hence (X, τ) is pre-R_0.

Theorem 3.13 *For a topological space (X, τ), the following properties are equivalent :*
(1) (X, τ) is a pre-R_0 space;
(2) If F is preclosed, then $F = pKer(F)$;
(3) If F is preclosed and $x \in F$, then $pKer(\{x\}) \subset F$;
(4) If $x \in X$, then $pKer(\{x\}) \subset pCl(\{x\})$.

Proof. (1) $\rightarrow$ (2) : Let F be preclosed and $x \notin F$. Thus $X - F$ is preopen and contains x. Since (X, τ) is pre-R_0, $pCl(\{x\}) \subset X - F$. Thus $pCl(\{x\}) \cap F = \emptyset$ and by Lemma 3.2 $x \notin pKer(F)$. Therefore $pKer(F) = F$.
(2) $\rightarrow$ (3) : In general, $A \subset B$ implies $pKer(A) \subset pKer(B)$. Therefore, it follows from (2) that $pKer(\{x\}) \subset pKer(F) = F$.
(3) $\longleftrightarrow$ (4) : Since $x \in pCl(\{x\})$ and $pCl(\{x\})$ is preclosed, by (3) $pKer(\{x\}) \subset pCl(\{x\})$.
(4) $\longleftrightarrow$ (1) : We show the implication by using Theorem 3.12. Let $x \in pCl(\{y\})$. Then by Lemma 3.1 $y \in pKer(\{x\})$. Since $x \in pCl(\{x\})$ and $pCl(\{x\})$ is preclosed, by (4) we obtain $y \in pKer(\{x\}) \subset pCl(\{x\})$. Therefore $x \in pCl(\{y\})$ implies $y \in pCl(\{x\})$. The converse is obvious and (X, τ) is pre-R_0.

Definition 4 *A filterbase F is called p-convergent to a point x in X [11], if for any preopen set U of X containing x, there exists B in F such that B is a subset of U.*

Lemma 3.14 *Let (X,τ) be a topological space and let x and y be any two points in X such that every net in X p-converging to y p-converges to x. Then $x \in pCl(\{y\})$.*

Proof. Suppose that $x_n = y$ for each $n \in \mathbf{N}$. Then $\{x_n\}_{n\in\mathbf{N}}$ is a net in $pCl(\{y\})$. By the fact that $\{x_n\}_{n\in\mathbf{N}}$ p-converges to y , then $\{x_n\}_{n\in\mathbf{N}}$ p-converges to x and this means that $x \in pCl(\{y\})$.

Theorem 3.15 *For a topological space (X,τ), the following statements are equivalent :*
(1) (X,τ) is a pre-R_0 space;
(2) If $x, y \in X$, then $y \in pCl(\{x\})$ if and only if every net in X p-converging to y p-converges to x.

Proof. (1) $\rightarrow$ (2) : Let $x, y \in X$ such that $y \in pCl(\{x\})$. Let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a net in X such that $\{x_\alpha\}_{\alpha\in\Lambda}$ p-converges to y. Since $y \in pCl(\{x\})$, by Corollary 3.5 we have $pCl(\{x\}) = pCl(\{y\})$. Therefore $x \in pCl(\{y\})$. This means that $\{x_\alpha\}_{\alpha\in\Lambda}$ p-converges to x. Conversely, let $x, y \in X$ such that every net in X p-converging to y p-converges to x. Then $x \in pCl(\{y\})$ by Lemma 3.2. By Corollary 3.5, we have $pCl(\{x\}) = pCl(\{y\})$. Therefore $y \in pCl(\{x\})$.
(2) $\rightarrow$ (1) : Assume that x and y are any two points of X such that $pCl(\{x\}) \cap pCl(\{y\}) \neq \emptyset$. Let $z \in pCl(\{x\}) \cap pCl(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $pCl(\{x\})$ such that $\{x_\alpha\}_{\alpha\in\Lambda}$ p-converges to z. Since $z \in pCl(\{y\})$, then $\{x_\alpha\}_{\alpha\in\Lambda}$ p-converges to y. It follows that $y \in pCl(\{x\})$. By the same token we obtain $x \in pCl(\{y\})$. Therefore $pCl(\{x\}) = pCl(\{y\})$ and by Corollary 3.5 (X,τ) is pre-R_0.

4 Pre-R_1 spaces

Definition 5 *A topological space (X,τ) is said to be pre-R_1 if for x, y in X with $pCl(\{x\}) \neq pCl(\{y\})$, there exist disjoint preopen sets U and V such that $pCl(\{x\})$ is a subset of U and $pCl(\{y\})$ is a subset of V.*

Proposition 4.1 *If (X,τ) is pre-R_1, then (X,τ) is pre-R_0.*

Proof. Let U be preopen and $x \in U$. If $y \notin U$, then since $x \notin pCl(\{y\})$, $pCl(\{x\}) \neq pCl(\{y\})$. Hence, there exists a preopen V_y such that $pCl(\{y\}) \subset V_y$ and $x \notin V_y$, which implies $y \notin pCl(\{x\})$. Thus $pCl(\{x\}) \subset U$. Therefore (X,τ) is pre-R_0.

Theorem 4.2 *For a topological space (X, τ), the following statements are equivalent :*
(1) (X, τ) is pre-T_2 ;
(2) (X, τ) is pre-R_1.

Proof. (1) $\rightarrow$ (2) : Since X is pre-T_2, then X is pre-T_1. If $x, y \in X$ such that $pCl(\{x\}) \neq pCl(\{y\})$, then $x \neq y$. There exists disjoint preopen sets U and V such that $x \in U$ and $y \in V$; hence $pCl(\{x\}) = \{x\} \subset U$ and $pCl(\{y\}) = \{y\} \subset V$. Hence X is pre-R_1.
(2) $\rightarrow$ (1) : Suppose that (X, τ) is pre-R_1. By Proposition 4.1, (X, τ) is pre-R_0 and hence it is pre-T_1 by Theorem 3.11. Let $x, y \in X$ such that $x \neq y$. Since $pCl(\{x\}) = \{x\} \neq \{y\} = pCl(\{y\})$, there exist disjoint preopen sets U and V such that $x \in U$ and $y \in V$. Hence X is pre-T_2.

Theorem 4.3 *For a topological space (X, τ), the following statements are equivalent :*
(1) (X, τ) is pre-R_1 ;
(2) If $x, y \in X$ such that $pCl(\{x\}) \neq pCl(\{y\})$, then there exists preclosed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

Proof. (1) $\rightarrow$ (2) : Let $x, y \in X$ such that $pCl(\{x\}) \neq pCl(\{y\})$. By Theorem 4.2, X is pre-T_2 and hence $x \neq y$. Therefore, there exists disjoint preopen sets U_1 and U_2 such that $x \in U_1$ and $y \in U_2$. Then $F_1 = X - U_2$ and $F_2 = X - U_1$ are preclosed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.
(2) $\rightarrow$ (1) : Suppose that x and y are distinct points of X. By Lemma 3.10, there are three cases as follows :
(i) $\{x\}, \{y\} \in PO(X, \tau)$. Then $pKer(\{x\}) = \{x\} \neq \{y\} = pKer(\{y\})$. By Lemma 3.6, we have $pCl(\{x\}) \neq pCl(\{y\})$.
(ii) $\{x\} \in PO(X, \tau), \{y\} \in PC(X, \tau)$. Then $pCl(\{y\}) = \{y\}$ and hence $pCl(\{x\}) \neq pCl(\{y\})$.
(iii) $\{x\}, \{y\} \in PC(X, \tau)$. Then $pCl(\{x\}) = \{x\} \neq \{y\} = pCl(\{y\})$ and $pCl(\{x\}) \neq pCl(\{y\})$.

For each case we have $pCl(\{x\}) \neq pCl(\{y\})$ and there exist preclosed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$. Now, we set $U_1 = X - F_2$ and $U_2 = X - F_1$, then we obtain that $x \in U_1, y \in U_2, U_1 \cap U_2 = \emptyset$ and U_1, U_2 are preopen. This shows that (X, τ) is pre-T_2. It follows from Theorem 4.2 that (X, τ) is pre-R_1.

Theorem 4.4 *A topological space (X, τ) is pre-R_1 if and only if for $x, y \in X, pKer(\{x\}) \neq pKer(\{y\})$, there exist disjoint preopen sets U and V such that $pCl(\{x\}) \subset U$ and $pCl(\{y\}) \subset V$.*

Proof. It follows from Lemma 3.6.

The following notions are due to Dontchev et al. [4]:

A point x of a topological space (X, τ) is a pre-θ-accumulation point of a subset $A \subset X$, if for each preopen U of X containing x, $pCl(U) \cap A \neq \emptyset$. The set $pCl_\theta(A)$ of all pre-θ-accumulation points of A is called the pre-θ-closure of A. The set A is said to be pre-θ-closed if $pCl_\theta(A) = A$. Complement of a pre-θ-closed set is said to be pre-θ-open.

Lemma 4.5 *For any subset A of a topological space (X, τ), $pCl(A) \subset pCl_\theta(A)$.*

Lemma 4.6 *Let x and y are points in a topological space (X, τ). Then $y \in pCl_\theta(\{x\})$ if and only if $x \in pCl_\theta(\{y\})$.*

Theorem 4.7 *A topological space (X, τ) is pre-R_1 if and only if for each $x \in X$, $pCl(\{x\}) = pCl_\theta(\{x\})$.*

Proof. Necessity. Assume that X is pre-R_1 and $y \in pCl_\theta(\{x\}) - pCl(\{x\})$. Then There exists a preopen set U containing y such that $pCl(U) \cap \{x\} \neq \emptyset$ but $U \cap \{x\} = \emptyset$. Thus $pCl(\{y\}) \subset U$, $pCl(\{x\}) \cap U = \emptyset$. Hence $pCl(\{x\}) \neq pCl(\{y\})$. Since X is pre-R_1, there exist disjoint preopen sets U_1 and U_2 such that $pCl(\{x\}) \subset U_1$ and $pCl(\{y\}) \subset U_2$. Therefore $X - U_1$ is a preclosed pre-neigbourhood at y which does not contain x. Thus $y \notin pCl_\theta(\{x\})$. This is a contradiction.

Sufficiency. Suppose that $pCl(\{x\}) = pCl_\theta(\{x\})$ for each $x \in X$. We first prove that X is pre-R_0. Let x belong to the preopen set U and $y \notin U$. Since $pCl_\theta(\{y\}) = pCl(\{y\}) \subset X - U$, we have $x \notin pCl_\theta(\{y\})$ and by Lemma 4.6 $y \notin pCl_\theta(\{x\}) = pCl(\{x\})$. It follows that $pCl(\{x\}) \subset U$. Therefore (X, τ) is pre-R_0. Now, let $a, b \in X$ with $pCl(\{a\}) \neq pCl(\{b\})$. By Theorem 3.11, (X, τ) is pre-T_1 and $a \neq b$. Since $pCl(\{a\}) = pCl_\theta(\{a\})$ for each $a \in X$, $b \notin pCl_\theta(\{a\})$ and hence there exists a preopen set U containing b such that $a \notin pCl(U)$. Therefore, we obtain $b \in U$, $a \in X - pCl(U)$ and $U \cap (X - pCl(U)) = \emptyset$. This shows that (X, τ) is pre-T_2. It follows from Theorem 4.2 that (X, τ) is pre-R_1.

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