

MORE ON Λ_s -SEMI- θ -CLOSED SETS

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Abstract

It is the object of this paper to study further the notion of Λ_s -semi- θ -closed sets which is defined as the intersection of a θ - Λ_s -set and a semi- θ -closed set. Moreover, we introduce some low separation axioms using the above notions. Also we present and study the notions of Λ_s -continuous functions, Λ_s -compact spaces and Λ_s -connected spaces.

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1 Introduction

The notion of θ - Λ_s -set is introduced and investigated by Caldas et al. [1] by utilizing semi- θ -open sets. These sets suggested a new class of sets which they called Λ_s -semi- θ -closed sets. They offered some properties of these sets. Among others, they proved that a topological space (X, τ) is semi- θ - T_0 if and only if every singleton of X is Λ_s -semi- θ -closed. Recall that a topological space is semi- θ - T_0 [1] if to each pair of points $x, y \in X$ and $x \neq y$, there exists a semi- θ -open set which contains one of them but not the other.

In what follows (X, τ) and (Y, σ) (or X and Y) denote topological spaces. Let A be a subset of X . We denote the interior and the closure of a set A by $int(A)$ and $cl(A)$, respectively.

2 Preliminaries

In this section we recall the definitions of $\Lambda_\theta^{\Lambda^s}$ [1] and $\Lambda_\theta^{\Lambda^s}$ -sets .

Definition 1 (see [1]) Let A be a subset of a topological space X . By $\Lambda_\theta^{\Lambda^s}(A)$ we denote the set $\cap\{O \in S\theta O(X, \tau) \mid A \subset O\}$. A subset A of a topological space (X, τ) is called a $\Lambda_\theta^{\Lambda^s}$ -set if $A = \Lambda_\theta^{\Lambda^s}(A)$.

Lemma 2.1 For subsets A and A_i ($i \in I$) of a space (X, τ) , the following hold:

- (1) $A \subset \Lambda_\theta^{\Lambda^s}(A)$.
- (2) $\Lambda_\theta^{\Lambda^s}(\Lambda_\theta^{\Lambda^s}(A)) = \Lambda_\theta^{\Lambda^s}(A)$.
- (3) If $A \subset B$, then $\Lambda_\theta^{\Lambda^s}(A) \subset \Lambda_\theta^{\Lambda^s}(B)$.
- (4) $\Lambda_\theta^{\Lambda^s}(\cap\{A_i : i \in I\}) \subset \cap\{\Lambda_\theta^{\Lambda^s}(A_i) : i \in I\}$.
- (5) $\Lambda_\theta^{\Lambda^s}(\cup\{A_i : i \in I\}) = \cup\{\Lambda_\theta^{\Lambda^s}(A_i) : i \in I\}$.
- (6) $\Lambda_\theta^{\Lambda^s}(A)$ is a $\Lambda_\theta^{\Lambda^s}$ -set.
- (7) If A is semi- θ -open, then A is a $\Lambda_\theta^{\Lambda^s}$ -set.
- (8) If A_i is $\Lambda_\theta^{\Lambda^s}$ -set for each $i \in I$, then $\cap_{i \in I} A_i$ is a $\Lambda_\theta^{\Lambda^s}$ -set.
- (9) If A_i is $\Lambda_\theta^{\Lambda^s}$ -set for each $i \in I$, then $\cup_{i \in I} A_i$ is a $\Lambda_\theta^{\Lambda^s}$ -set.

Theorem 2.2 Let X be a topological space. We set $\tau^{\Lambda_\theta^{\Lambda^s}} = \{A : A \text{ is a } \Lambda_\theta^{\Lambda^s} \text{ - set of } X\}$. The pair $(X, \tau^{\Lambda_\theta^{\Lambda^s}})$ is an Alexandroff space.

Proof. This is an immediate consequence of Lemma 2.1.

Definition 2 Let A be a subset of a topological space (X, τ) . By $\Lambda_\theta^{\Lambda^s}(A)$, we denote the set $\cup\{B \in S\theta C(X, \tau) \mid B \subset A\}$. A subset A of a topological space (X, τ) is called a $\Lambda_\theta^{\Lambda^s}$ -set if $A = \Lambda_\theta^{\Lambda^s}(A)$.

We obtain the following lemma which is similar to Lemma 2.1.

Lemma 2.3 For subsets A, B and A_i ($i \in I$) of a topological space (X, τ) the following properties hold:

- (1) $\Lambda_{\theta}^{\Lambda^*}(A) \subseteq A$.
- (2) If $A \subseteq B$, then $\Lambda_{\theta}^{\Lambda^*}(A) \subseteq \Lambda_{\theta}^{\Lambda^*}(B)$.
- (3) If A is semi- θ -closed, then $\Lambda_{\theta}^{\Lambda^*}(A) = A$.
- (4) $\Lambda_{\theta}^{\Lambda^*}(\cap\{A_i : i \in I\}) = \cap\{\Lambda_{\theta}^{\Lambda^*}(A_i) : i \in I\}$.
- (5) $\cup\{\Lambda_{\theta}^{\Lambda^*}(A_i) : i \in I\} \subseteq \Lambda_{\theta}^{\Lambda^*}(\cup\{A_i : i \in I\})$.
- (6) $\Lambda_{\theta}^{\Lambda^*}(X - A) = X - \Lambda_{\theta}^{\Lambda^*}(A)$ and $\Lambda_{\theta}^{\Lambda^*}(X - A) = X - \Lambda_{\theta}^{\Lambda^*}(A)$.
- (7) $\Lambda_{\theta}^{\Lambda^*}(A)$ is a $\Lambda_{\theta}^{\Lambda^*}$ -set.
- (8) If A is a semi- θ -closed, then A is a $\Lambda_{\theta}^{\Lambda^*}$ -set.
- (9) If A_i is a $\Lambda_{\theta}^{\Lambda^*}$ -set for each $i \in I$, then $\cup\{A_i \mid i \in I\}$ and $\cap\{A_i \mid i \in I\}$ are $\Lambda_{\theta}^{\Lambda^*}$ -sets.

Observe that if X is a topological space and $\tau^{\Lambda_{\theta}^{\Lambda^*}} = \{A : A \text{ is a } \Lambda_{\theta}^{\Lambda^*} \text{ - set of } X\}$, then $(X, \tau^{\Lambda_{\theta}^{\Lambda^*}})$ is an Alexandroff space.

3 Λ_s -semi- θ -closed sets

Definition 3 A subset A of a topological space (X, τ) is called Λ_s -semi- θ -closed [1], denoted by $(\Lambda, s\theta)$ -closed, if $A = T \cap C$, where T is a $\Lambda_{\theta}^{\Lambda^*}$ -set and C is a semi- θ -closed set.

Lemma 3.1 ([1], Lemma 2.23) Let A be a subset of a space (X, τ) . Then the following conditions are equivalent:

- (1) A is $(\Lambda, s\theta)$ -closed,
- (2) $A = P \cap scl_{\theta}(A)$, where P is a $\Lambda_{\theta}^{\Lambda^*}$ -set,
- (3) If $A = \Lambda_{\theta}^{\Lambda^*}(A) \cap scl_{\theta}(A)$.

Example 3.2 Let $X = \{a, b, c, \}$ and $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$. The semi θ -closed sets of (X, τ) are $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. The set $A = \{c\}$ is $(\Lambda, s\theta)$ -closed since it is semi θ -closed but it is not closed.

Example 3.3 Let $X = \{a, b, c, \}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. The set $A = \{c\}$ is closed but it is not $(\Lambda, s\theta)$ -closed.

The Example 3.2 and Example 3.3 shown that the sets $(\Lambda, s\theta)$ -closed and closed are independent of each other.

Note that every semi θ -closed set is $(\Lambda, s\theta)$ -closed, but the converse is not true in general.

Example 3.4 Let (X, τ) as in the Example 3.2. Then $B = \{b, c\}$ is $(\Lambda, s\theta)$ -closed since it is $\Lambda_{\theta}^{\Lambda_s}$ -set, but it is not semi θ -closed.

Lemma 3.5 If A_i is $(\Lambda, s\theta)$ -closed for each $i \in I$, then $\bigcap_{i \in I} A_i$ is $(\Lambda, s\theta)$ -closed.

Proof. Suppose that A_i is $(\Lambda, s\theta)$ -closed for each $i \in I$. Then, for each $i \in I$ there exist a $\Lambda_{\theta}^{\Lambda_s}$ -set T_i and a semi- θ -closed set C_i such that $A_i = T_i \cap C_i$. Now $\bigcap_{i \in I} A_i = \bigcap_{i \in I} (T_i \cap C_i) = (\bigcap_{i \in I} T_i) \cap (\bigcap_{i \in I} C_i)$. By Lemma 2.1, $\bigcap_{i \in I} T_i$ is a $\Lambda_{\theta}^{\Lambda_s}$ -set and $\bigcap_{i \in I} C_i$ is semi- θ -closed. This shows that $\bigcap_{i \in I} A_i$ is $(\Lambda, s\theta)$ -closed. \square

Definition 4 A subset A of a space (X, τ) is said to be $(s\theta, s\theta)$ -generalized closed if $scl_{\theta}(A) \subseteq G$ holds whenever $A \subseteq G$ and $G \in S\theta O(X, \tau)$.

Lemma 3.6 A subset A of a space (X, τ) is $(s\theta, s\theta)$ -generalized closed if and only if $scl_{\theta}(A) \subseteq \Lambda_{\theta}^{\Lambda_s}(A)$.

Proof. Necessity: Suppose that there is a point $x \in X$ such that $x \notin \Lambda_\theta^{\Lambda_s}(A)$. Then exists a subset $O \in \mathcal{S}\theta O(X, \tau)$ such that $A \subseteq O$ and $x \notin O$. This implies that $scl_\theta(A) \subseteq O$. Hence $x \notin scl_\theta(A)$ since A is $(s\theta, s\theta)$ -generalized closed. Sufficiency: Obvious.

Theorem 3.7 *A subset A of a space (X, τ) is semi- θ -closed if and only if A is $(s\theta, s\theta)$ -generalized closed and $(\Lambda, s\theta)$ -closed.*

Proof. Necessity: Every semi θ -closed set is both $(s\theta, s\theta)$ -generalized closed and $(\Lambda, s\theta)$ -closed..

Sufficiency: Since A is $(s\theta, s\theta)$ -generalized closed, then by Lemma 3.3, $scl_\theta(A) \subseteq \Lambda_\theta^{\Lambda_s}(A)$. By assumption and Lemma 3.1 $A = \Lambda_\theta^{\Lambda_s}(A) \cap scl_\theta(A) = scl_\theta(A)$. i.e., A is semi θ -closed.

Definition 5 *A subset A of a topological space (X, τ) is called $(\Lambda, s\theta)$ -open if $X \setminus A$ is $(\Lambda, s\theta)$ -closed.*

Theorem 3.8 *The union of any family of $(\Lambda, s\theta)$ -open sets is a $(\Lambda, s\theta)$ -open set.*

Proof. The proof of this theorem follows by the fact that the intersection of a family of $(\Lambda, s\theta)$ -closed sets is $(\Lambda, s\theta)$ -closed. \square

Lemma 3.9 *The following statements are equivalent for a subset A of a topological space X :*

- (1) A is $(\Lambda, s\theta)$ -open
- (2) $A = T \cup C$, where T is a $\Lambda_\theta^{\Lambda_s^*}$ -set and C is a semi- θ -open set.

Proof. The proof of this lemma is clear. \square

Lemma 3.10 Every $\Lambda_\theta^{\Lambda^*}$ -set is $(\Lambda, s\theta)$ -open.

Proof. Take $A = A \cup \emptyset$, where A is a $\Lambda_\theta^{\Lambda^*}$ -set, X is semi- θ -closed and $\emptyset = X \setminus X$. \square

Definition 6 A subset A of a topological space X is called a $\Lambda_\theta^{\Lambda^s}$ - D set if there are two $(\Lambda, s\theta)$ -open sets U and V in X such that $U \neq X$ and $A = U - V$.

It is true that every $(\Lambda, s\theta)$ -open set U different from X is a $\Lambda_\theta^{\Lambda^s}$ - D set if $A = U$ and $V = \emptyset$.

Definition 7 A topological space (X, τ) is called:

- (i) $\Lambda_\theta^{\Lambda^s}$ - D_0 if for any distinct pair of points x and y of X there exists a $\Lambda_\theta^{\Lambda^s}$ - D set of X containing x but not y or a $\Lambda_\theta^{\Lambda^s}$ - D set of X containing y but not x .
- (ii) $\Lambda_\theta^{\Lambda^s}$ - D_1 if for any distinct pair of points x and y of X there exist a $\Lambda_\theta^{\Lambda^s}$ - D set of X containing x but not y and a $\Lambda_\theta^{\Lambda^s}$ - D set of X containing y but not x .
- (iii) $\Lambda_\theta^{\Lambda^s}$ - D_2 if for any distinct pair of points x and y of X there exist disjoint $\Lambda_\theta^{\Lambda^s}$ - D sets G and E of X containing x and y , respectively.

A topological space (X, τ) satisfies the $(\Lambda, s\theta)$ -property if for any distinct pair of points in X , there is a $(\Lambda, s\theta)$ -open set containing one of the points but not the other.

Remark 3.11 (i) If (X, τ) satisfies the $(\Lambda, s\theta)$ -property, then it is $\Lambda_\theta^{\Lambda^s}$ - D_0 .
(ii) If (X, τ) is $\Lambda_\theta^{\Lambda^s}$ - D_i , then it is $\Lambda_\theta^{\Lambda^s}$ - D_{i-1} , where $i = 1, 2$.

Theorem 3.12 For a topological space (X, τ) , the following statements are true:

- (1) (X, τ) is $\Lambda_\theta^{\Lambda^s}$ - D_0 if and only if it satisfies the $(\Lambda, s\theta)$ -property.
- (2) (X, τ) is $\Lambda_\theta^{\Lambda^s}$ - D_1 if and only if it is $\Lambda_\theta^{\Lambda^s}$ - D_2 .

Proof. The sufficiency for (1) and (2) follows from the above Remark 3.5.

Necessity condition for (1). Let (X, τ) be $\Lambda_{\theta}^{\Lambda^s}$ - D_0 so that for any distinct pair of points x and y of X at least one belongs to a $\Lambda_{\theta}^{\Lambda^s}$ - D set O . Therefore we choose $x \in O$ and $y \notin O$. Suppose $O = U - V$ for which $U \neq X$ and U and V are $(\Lambda, s\theta)$ -open sets in X . This implies that $x \in U$. For the case that $y \notin O$ we have (i) $y \notin U$, (ii) $y \in U$ and $y \in V$. For (i), the space X satisfies the $(\Lambda, s\theta)$ -property since $x \in U$ and $y \notin U$. For (ii), the space X also satisfies the $(\Lambda, s\theta)$ -property since $y \in V$ but $x \notin V$.

Necessity condition for (2). Suppose that X is $\Lambda_{\theta}^{\Lambda^s}$ - D_1 . It follows from the definition that for any distinct points x and y in X there exist $\Lambda_{\theta}^{\Lambda^s}$ - D sets G and E such that G containing x but not y and E containing y but not x . Let $G = U - V$ and $E = W - D$, where U, V, W and D are $(\Lambda, s\theta)$ -open sets in X . By the fact that $x \notin E$, we have two cases, i.e. either $x \notin W$ or both W and D contain x . If $x \notin W$, then from $y \notin G$ either (i) $y \notin U$ or (ii) $y \in U$ and $y \in V$. If (i) is the case, then it follows from $x \in U - V$ that $x \in U - (V \cup W)$, and also it follows from $y \in W - D$ that $y \in W - (U \cup D)$. Thus we have $U - (V \cup W)$ and $W - (U \cup D)$ which are disjoint. If (ii) is the case, it follows that $x \in U - V$, $y \in V$ and $(U - V) \cap V = \emptyset$. If $x \in W$ and $x \in D$, we have $y \in W - D$, $x \in D$ and $(W - D) \cap D = \emptyset$. This shows that X is $\Lambda_{\theta}^{\Lambda^s}$ - D_2 . \square

Definition 8 Let (X, τ) be a topological space. A point $x \in X$ which has only X as the $(\Lambda, s\theta)$ -neighborhood is called a $\Lambda_{\theta}^{\Lambda^s}$ -neat point.

Theorem 3.13 For a topological space (X, τ) that satisfies the $(\Lambda, s\theta)$ -property the following are equivalent:

- (1) (X, τ) is $\Lambda_{\theta}^{\Lambda^s}$ - D_1 ;
- (2) (X, τ) has no $\Lambda_{\theta}^{\Lambda^s}$ -neat point.

Proof. (1) \rightarrow (2). Since (X, τ) is $\Lambda_\theta^{\Lambda^s}$ - D_1 , so each point x of X is contained in a $\Lambda_\theta^{\Lambda^s}$ - D set $O = U - V$ and thus in U . By definition $U \neq X$. This implies that x is not a $\Lambda_\theta^{\Lambda^s}$ -neat point.

(2) \rightarrow (1). Since X satisfies the $(\Lambda, ps\theta)$ -property, then for each distinct pair of points $x, y \in X$, at least one of them, choose x for example has a $(\Lambda, s\theta)$ -neighborhood U containing x and not y . Thus U which is different from X is a $\Lambda_\theta^{\Lambda^s}$ - D set. If X has no $\Lambda_\theta^{\Lambda^s}$ -neat point, then y is not a $\Lambda_\theta^{\Lambda^s}$ -neat point. This means that there exists a $(\Lambda, s\theta)$ -neighborhood V of y such that $V \neq X$. Thus $y \in (V - U)$ but not x and $V - U$ is a $\Lambda_\theta^{\Lambda^s}$ - D set. Hence X is $\Lambda_\theta^{\Lambda^s}$ - D_1 . \square

Remark 3.14 *It is clear that a topological space (X, τ) that satisfies the $(\Lambda, s\theta)$ -property is not $\Lambda_\theta^{\Lambda^s}$ - D_1 if and only if there is a unique $\Lambda_\theta^{\Lambda^s}$ -neat point in X . It is unique because if x and y are both $\Lambda_\theta^{\Lambda^s}$ -neat point in X , then at least one of them say x has a $(\Lambda, s\theta)$ -neighborhood U containing x but not y . But this is a contradiction since $U \neq X$.*

Definition 9 *Let (X, τ) be a topological spaces and $A \subseteq X$. A point $x \in X$ is called $(\Lambda, s\theta)$ -cluster point of A if for every $(\Lambda, s\theta)$ -open set U of X containing x we have $A \cap U \neq \emptyset$. The set of all $(\Lambda, s\theta)$ -cluster points is called the $(\Lambda, s\theta)$ -closure of A and is denoted by $A^{(\Lambda, s\theta)}$.*

Lemma 3.15 *Let A and B be subsets of a topological space (X, τ) . For the $(\Lambda, s\theta)$ -closure, the following properties hold.*

- (1) $A \subset A^{(\Lambda, s\theta)}$.
- (2) $A^{(\Lambda, s\theta)} = \cap \{F \mid A \subset F \text{ and } F \text{ is } (\Lambda, s\theta)\text{-closed}\}$.
- (3) If $A \subset B$, then $A^{(\Lambda, s\theta)} \subset B^{(\Lambda, s\theta)}$.
- (4) A is $(\Lambda, s\theta)$ -closed if and only if $A = A^{(\Lambda, s\theta)}$.
- (5) $A^{(\Lambda, s\theta)}$ is $(\Lambda, s\theta)$ -closed.

Proof. Straightforward.

Definition 10 A topological space (X, τ) is called a $(\Lambda, s\theta)$ -symmetric if for x and y in X , $x \in y^{(\Lambda, s\theta)}$ implies $y \in x^{(\Lambda, s\theta)}$.

Theorem 3.16 A topological space (X, τ) is $(\Lambda, s\theta)$ -symmetric if and only if for $x \in X$, $x^{(\Lambda, s\theta)} \subseteq E$ whenever $x \in E$ and E is $(\Lambda, s\theta)$ -open in (X, τ) .

Proof. Assume that $x \in y^{(\Lambda, s\theta)}$ but $y \notin x^{(\Lambda, s\theta)}$. This means that the complement of $x^{(\Lambda, s\theta)}$ contains y . Therefore the set $\{y\}$ is a subset of the complement of $x^{(\Lambda, s\theta)}$. This implies that $y^{(\Lambda, s\theta)}$ is a subset of the complement of $x^{(\Lambda, s\theta)}$. Now the complement of $x^{(\Lambda, s\theta)}$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subset E$ and E is $(\Lambda, s\theta)$ -open in (X, τ) but $x^{(\Lambda, s\theta)}$ is not a subset of E . This means that $x^{(\Lambda, s\theta)}$ and the complement of E are not disjoint. Let y belongs to their intersection. Now we have $x \in y^{(\Lambda, s\theta)}$ which is a subset of the complement of E and $x \notin E$. But this is a contradiction. \square

Theorem 3.17 For a $(\Lambda, s\theta)$ -symmetric topological space (X, τ) , the following are equivalent:

- (1) (X, τ) satisfies the $(\Lambda, s\theta)$ -property;
- (2) (X, τ) is $\Lambda_\theta^{\Lambda_s}$ - D_0 ;
- (3) (X, τ) is $\Lambda_\theta^{\Lambda_s}$ - D_1 .

Proof. (1) \leftrightarrow (2) : Lemma 3.10.

(3) \rightarrow (2) : Remark 3.11.

(1) \rightarrow (3) : Let $x \neq y$ and by (1), we may assume that $x \in E \subset \{y\}^c$ for some E $(\Lambda, s\theta)$ -open in (X, τ) . Then $x \notin y^{(\Lambda, s\theta)}$ and hence $y \notin x^{(\Lambda, s\theta)}$. Hence there exists a $(\Lambda, s\theta)$ -open set F such that $y \in F \subset \{x\}^c$. Since every $(\Lambda, s\theta)$ -open set is a $\Lambda_\theta^{\Lambda_s}$ - D set, we have that (X, τ) is a $\Lambda_\theta^{\Lambda_s}$ - D_1 space. \square

4 $(\Lambda, s\theta)$ -continuous functions

Definition 11 Let (X, τ) and (Y, σ) two topological spaces. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $(\Lambda, s\theta)$ -continuous at a point $x \in X$ if for every (Λ, p) -open set V of Y such that $f(x) \in V$ there exists a $(\Lambda, s\theta)$ -open set U of X such that $x \in U$ and $f(U) \subseteq V$.

The function f is called $(\Lambda, s\theta)$ -continuous if f is $(\Lambda, s\theta)$ -continuous at every point $x \in X$.

Definition 12 Let (X, τ) be a topological space, $x \in X$ and $\{x_s, s \in S\}$ be a net of X . We say that the net $\{x_s, s \in S\}$ $(\Lambda, s\theta)$ -converges to x if for every $(\Lambda, s\theta)$ -open set U containing x there exists an element $s_0 \in S$ such that $s \geq s_0$ implies $x_s \in U$.

Theorem 4.1 Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in A^{(\Lambda, s\theta)}$ if and only if there exists a net $\{x_s, s \in S\}$ of A which $(\Lambda, s\theta)$ -converges to x .

Proof. The existence of such a net clearly implies that $x \in A^{(\Lambda, s\theta)}$. Suppose $x \in A^{(\Lambda, s\theta)}$ and let us denote by \mathcal{U} the set of all $(\Lambda, s\theta)$ -open subsets U of X such that $x \in U$ directed by the relation \subseteq , i.e., let us define that $U_1 \leq U_2$ if $U_2 \subseteq U_1$. The net $\{x_U, U \in \mathcal{U}\}$, where x_U is an arbitrary point of $A \cap U$, $(\Lambda, s\theta)$ -converges to x . \square

Theorem 4.2 For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following are equivalent:

- (1) f is $(\Lambda, s\theta)$ -continuous;
- (2) $f^{-1}(V)$ is $(\Lambda, s\theta)$ -open in (X, τ) for every $(\Lambda, s\theta)$ -open set V of (Y, σ) ;
- (3) $f^{-1}(F)$ is $(\Lambda, s\theta)$ -closed in (X, τ) for every $(\Lambda, s\theta)$ -closed set F of (Y, σ) ;

- (4) $f(A^{(\Lambda, s\theta)}) \subset [f(A)]^{(\Lambda, s\theta)}$ for each subset A of X ;
(5) $[f^{-1}(B)]^{(\Lambda, s\theta)} \subset f^{-1}(B^{(\Lambda, s\theta)})$ for each subset B of Y ;
(6) For every $x \in X$ and every net $\{x_s, s \in S\}$ of X which $(\Lambda, s\theta)$ -converges to x in X , the net $\{f(x_s), s \in S\}$ $(\Lambda, s\theta)$ -converges to $f(x)$ in Y .

Proof. (1) \rightarrow (2): Let V be any $(\Lambda, s\theta)$ -open set of (Y, σ) and $x \in f^{-1}(V)$. Since f is $(\Lambda, s\theta)$ -continuous, there exists a $(\Lambda, s\theta)$ -open set U_x containing x such that $f(U_x) \subset V$. Therefore, we have $x \in U_x \subset f^{-1}(V)$ and hence $f^{-1}(V) = \cup\{U_x \mid x \in f^{-1}(V)\}$. By Theorem 3.8, $f^{-1}(V)$ is $(\Lambda, s\theta)$ -open in (X, τ) .

(2) \rightarrow (1): This is obvious.

(2) \leftrightarrow (3): This is obvious from Definition 5.

(3) \rightarrow (4): Let A be any subset of X . Since $A \subset f^{-1}([f(A)]^{(\Lambda, s\theta)})$, by Lemma 3.15 we have $A^{(\Lambda, s\theta)} \subset f^{-1}([f(A)]^{(\Lambda, s\theta)})$ and hence $f(A^{(\Lambda, s\theta)}) \subset [f(A)]^{(\Lambda, s\theta)}$.

(4) \rightarrow (5): Let B be any subset of Y . By (4) we have $f([f^{-1}(B)]^{(\Lambda, s\theta)}) \subset [f(f^{-1}(B))]^{(\Lambda, s\theta)} \subset B^{(\Lambda, s\theta)}$ and hence $[f^{-1}(B)]^{(\Lambda, s\theta)} \subset f^{-1}(B^{(\Lambda, s\theta)})$.

(5) \rightarrow (3): Let F be any $(\Lambda, s\theta)$ -closed set in (Y, σ) . By Lemma 3.15, $[f^{-1}(F)]^{(\Lambda, s\theta)} \subset f^{-1}(F^{(\Lambda, s\theta)}) = f^{-1}(F)$ and $[f^{-1}(F)]^{(\Lambda, s\theta)} \subset f^{-1}(F)$. Therefore, we obtain $[f^{-1}(F)]^{(\Lambda, s\theta)} = f^{-1}(F)$. This shows that $f^{-1}(F)$ is $(\Lambda, s\theta)$ -closed in (X, τ) .

(1) \rightarrow (6): Let $x \in X$ and $\{x_s \mid s \in S\}$ be a net $(\Lambda, s\theta)$ -converging to x . For any $(\Lambda, s\theta)$ -open set of (Y, σ) containing $f(x)$, by (1) there exists a $(\Lambda, s\theta)$ -open set U of X containing x such that $f(U) \subset V$. Since $\{x_s \mid s \in S\}$ converges to x , there exists $s_0 \in S$ such that $s \geq s_0$ implies $x_s \in U$. Therefore, $f(x_s) \in V$ for any $s \geq s_0$ and the net $\{f(x_s) \mid s \in S\}$ $(\Lambda, s\theta)$ -converges to $f(x)$.

(6) \rightarrow (1): Let us suppose that there exists a point $x \in X$ and a $(\Lambda, s\theta)$ -open neighbourhood V of $f(x)$ such that for every $(\Lambda, s\theta)$ -open set U of X containing x such that $f(U) \not\subseteq V$. Then for every $(\Lambda, s\theta)$ -open set U of X such that $x \in U$, we choose an element $x_U \in U$ such that $f(x_U) \notin V$. Let \mathcal{U} be the set of all $(\Lambda, s\theta)$ -open sets U of X containing x and is directed by the relation \subseteq i.e., let us define that $U_1 \leq U_2$ if $U_2 \subseteq U_1$. Easily, the net $\{x_U, U \in \mathcal{U}\}$ $(\Lambda, s\theta)$ -converges to x but the net $\{f(x_U), U \in \mathcal{U}\}$ does not $(\Lambda, s\theta)$ -converge to $f(x)$ which is a contradiction. Thus there exists a $(\Lambda, s\theta)$ -open set U of X such that $x \in U$ and $f(U) \subseteq V$. \square

We recall that a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be quasi irresolute [3] if $f^{-1}(V)$ is semi- θ -open in (X, τ) for each semi- θ -open set V of (Y, σ) .

Clearly, if a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is quasi irresolute, then $f : (X, \tau^{\Lambda_\theta^*}) \rightarrow (Y, \sigma^{\Lambda_\theta^*})$ is continuous.

Indeed let V be any Λ_θ^* -set of (Y, σ) . Then $V = \Lambda_\theta^*(V) = \cup\{W \mid V \supset W \in S\theta C(Y, \sigma)\}$. Since f is quasi irresolute, we have $f^{-1}(V) = \cup\{f^{-1}(W) \mid f^{-1}(V) \supset f^{-1}(W) \in S\theta C(X, \tau)\} \subset \cup\{U \mid f^{-1}(V) \supset U \in S\theta C(X, \tau)\} = \Lambda_\theta^*(f^{-1}(V))$. By Lemma 2.3, we have $f^{-1}(V) \supset \Lambda_\theta^*(f^{-1}(V))$ and hence $f^{-1}(V)$ is a Λ_θ^* -set of (X, τ) . \square

Observe that if a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is quasi irresolute, then $f : (X, \tau^{\Lambda_\theta^{As}}) \rightarrow (Y, \sigma^{\Lambda_\theta^{As}})$ is continuous.

Theorem 4.3 *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a quasi irresolute function, then it is $(\Lambda, s\theta)$ -continuous.*

Proof. Let F be a $(\Lambda, s\theta)$ -closed set of (Y, σ) . Then there exist a Λ_θ^{As} -set T and a semi- θ -closed set C such that $F = T \cap C$. Since f is quasi irresolute $f^{-1}(T)$ is a Λ_θ^{As} -set of (X, τ) and $f^{-1}(C)$ is semi- θ -closed. Therefore,

$f^{-1}(F) = f^{-1}(T) \cap f^{-1}(C)$ is $(\Lambda, s\theta)$ -closed in (X, τ) . By Theorem 4.2, f is $(\Lambda, s\theta)$ -continuous. \square

Example 4.4 Let $X = \{a, b, c, \}$ and $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. The semi θ -closed sets of (X, τ) are $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$, the $(\Lambda, s\theta)$ -closed sets of (X, τ) are $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and the semi θ -closed sets of (X, σ) are $\{\emptyset, X, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then f is $(\Lambda, s\theta)$ -continuous but it is not quasi-irresolute since $f^{-1}(\{b, c\})$ is not semi θ -closed in (X, τ) .

5 $(\Lambda, s\theta)$ -compactness and $(\Lambda, s\theta)$ -connectedness

Definition 13 A topological space (X, τ) is called $(\Lambda, s\theta)$ -compact (resp. semi- θ -compact) if every cover of $(\Lambda, s\theta)$ -open (resp. semi- θ -open) sets has a finite subcover.

Theorem 5.1 A topological space (X, τ) is $(\Lambda, s\theta)$ -compact (resp. semi- θ -compact) if and only if for every family $\{A_i : i \in I\}$ of $(\Lambda, s\theta)$ -closed (resp. semi- θ -closed) sets in X satisfying $\bigcap \{A_i : i \in I\} = \emptyset$, there is a finite subfamily A_{i_1}, \dots, A_{i_n} with $\bigcap \{A_{i_k} : k = 1, \dots, n\} = \emptyset$.

Proof. Straightforward. \square

Theorem 5.2 For a topological space (X, τ) , the following hold:

- (1) If $(X, \tau^{\Lambda_{\theta}^{s*}})$ is compact, then (X, τ) is semi- θ -compact.
- (2) If (X, τ) is $(\Lambda, s\theta)$ -compact, then (X, τ) is semi- θ -compact.
- (3) If (X, τ) is $(\Lambda, s\theta)$ -compact, then $(X, \tau^{\Lambda_{\theta}^{s*}})$ is compact.

Proof. (1) This follows from Lemma 2.1.

(2) This follows from Theorem 5.1 and of the fact that every semi- θ -closed set is $(\Lambda, s\theta)$ -closed.

(3) This follows from Lemma 3.10. \square

Theorem 5.3 *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $(\Lambda, s\theta)$ -continuous surjection and (X, τ) is a $(\Lambda, s\theta)$ -compact space, then (Y, σ) is $(\Lambda, s\theta)$ -compact.*

Proof. Let $\{V_i \mid i \in I\}$ be any cover of Y by $(\Lambda, s\theta)$ -open sets of (Y, σ) . Since f is $(\Lambda, s\theta)$ -continuous, by Theorem 4.2 $\{f^{-1}(V_i \mid i \in I)\}$ is a cover of X by $(\Lambda, s\theta)$ -open sets of (X, τ) . By $(\Lambda, s\theta)$ -compactness of (X, τ) , there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_i \mid i \in I_0)\}$. Since f is surjective, we obtain $Y = f(X) = \cup_{i \in I_0} V_i$. This shows that (Y, σ) is $(\Lambda, s\theta)$ -compact. \square

Corollary 5.4 *The $(\Lambda, s\theta)$ -compactness is preserved by quasi irresolute surjections.*

Proof. This is an immediate consequence of Theorem 5.3 and Theorem 4.3. \square

Definition 14 *A topological space (X, τ) is called $(\Lambda, s\theta)$ -connected if X cannot be written as a disjoint union of two non-empty $(\Lambda, s\theta)$ -open sets.*

Theorem 5.5 *For a topological space (X, τ) , the following statements are equivalent:*

- (1) *The space X is $(\Lambda, s\theta)$ -connected;*
- (2) *The only subsets of X , which are both $(\Lambda, s\theta)$ -open and $(\Lambda, s\theta)$ -closed are the empty set \emptyset and X .*

Proof. Straightforward. \square

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References

- [1] M. Caldas and S. Jafari, On θ -semigeneralized closed sets in topology, Kyungpook Math. J. 43(2003), 135-148.
- [2] G. Di Maio and T. Noiri, On s -closed spaces, Indian J. Pure Appl. Math. 18(3)(1987), 226-233.
- [3] G. Di Maio and T. Noiri, Weak and strong forms of irresolute functions, Rend. Circ. Mat. Palermo (2) Suppl. 18(1988), 255-273.
- [4] S. Ganguly and C. K. Basu Further characterizations of s -closed spaces, Indian J. Pure Appl. Math. 23(9)(1992), 635-641.
- [5] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo 19(2)(1970), 89-96.

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