FUNCTIONS WITH STRONGLY SEMI-θ-CLOSED GRAPHS *

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Abstract

In this note, we study some other properties of functions with strongly semi-θ-closed graphs by utilizing semi-θ-open sets and the semi-θ-closure operator.

1 Introduction

The notion of minimal structures was introduced by Popa and Noiri in [19] and [20]. A subfamily m_x of the power set $P(X)$ of a nonempty set X is called a minimal structure on X if X and $\emptyset$ set are in m_x. Each member of m_x is said to be m_x-open and the complement of an m_x-open set is said to be m_x-closed. Now it is the common viewpoint of many topologists that generalized open sets are important ingredients in General Topology and they are now the research topics of many topologists worldwide of which lots of important and interesting results emerged. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc by utilizing generalized open sets. One of the most well-known notions and also an inspiration source is the notion of semi-open sets introduced by N. Levine [13] in 1963. In 1987, Di Maio and Noiri[8] used this notion and the semi-closure [6] of a set to introduce the concepts of semi-θ-open and semi-θ-closed sets which provide a formulation of the semi-θ-closure of a set in a topological space. Mukherjee and Basu [15]

*2000 Mathematics Subject Classification: 54B05, 54C08.
Key words and phrases: semi-θ-open set, s-θ-continuity, semi-θ-T_2 space, strongly semi-θ-closed graph.

continued the work of Di Maio and Noiri and defined the concepts of semi-θ-connectedness, semi-θ-components and semi-θ-quasi-components. Also Park and Park [18] have used these sets to define the notion of weaker forms of irresolute functions. Dontchev and Noiri [11] obtained, among others, that a topological space is semi-Hausdorff if and only if each singleton is semi-θ-closed. Recently the authors [1, 2, 3, 5, 4] have also obtained several new and important results and notions related to these sets. In this direction we shall study some other properties of functions with strongly semi-θ-closed graphs by utilizing semi-θ-open sets and the semi-θ-closure operator. The notion of functions with strongly semi-θ-closed graphs is a particular case of functions with strongly m-closed graphs introduced by Popa and Noiri in [19, 21].

2 Preliminaries

Since we shall require the following known definitions, notations and some properties, we recall them in this section.

Let (X, τ) be a topological space and S a subset of X. We denote the closure and the interior of S by $Cl(S)$ and $Int(S)$, respectively. A subset S is said to be semi-open [13] if there exists an open set U such that $U \subset S \subset Cl(U)$, or equivalently if $S \subset Cl(Int(S))$. The complement of a semi-open set is said to be semi-closed [6]. The intersection of all semi-closed sets containing S is called the semi-closure [6] of S and is denoted by $sCl(S)$. A subset S is said to be semi-regular [8] if it is both semi-open and semi-closed. The family of all semi-open sets (resp. semi-regular sets, open sets) of (X, τ) is denoted by $SO(X, \tau)$ (resp. $SR(X, \tau)$, $O(X, \tau)$). The semi-θ-closure of S [8], denoted by $sCl_\theta(S)$, is defined to be the set of all $x \in X$ such that $sCl(O) \cap S \neq \emptyset$ for every $O \in SO(X, \tau)$ with $x \in O$. A subset S is said to be semi-θ-closed [9] if $S = sCl_\theta(S)$. The complement of a semi-θ-closed set is said to be semi-θ-open. The family of all semi-θ-open (resp. semi-θ-closed) subsets of X is denoted by $S\theta O(X, \tau)$ (resp. $S\theta C(X, \tau)$). We set $S\theta O(X, x) = \{U : x \in U \in S\theta O(X, \tau)\}$ and $S\theta C(X, x) = \{U : x \in U \in S\theta C(X, \tau)\}$. The family $S\theta O(X, \tau)$ of semi-θ-open sets is a minimal structure on X.

We recall the following results which are obtained by Di Maio and Noiri [8, 10], Mukherjee and Basu [15] and Caldas et al. [5].

Lemma 2.1 *Let A be a subset of a topological space (X, τ).*

(i) If $A \in SO(X,\tau)$*, then* $sCl(A) \in SR(X)$*.*
(ii) $A \in SR(X)$ *if and only if* $A \in S\theta O(X) \cap S\theta C(X)$*.*
(iii) If $A \in \tau$*, then* $sCl(A) = Int(Cl(A))$*.*

Lemma 2.2 *For the semi-θ-closure of a subset* A *of a topological space* (X,τ)*, the following properties hold:*
(i) $A \subset sCl(A) \subset sCl_\theta(A)$ *and* $sCl(A) = sCl_\theta(A)$ *if* $A \in SO(X)$*.*
(ii) If $A \subset B$*, then* $sCl_\theta(A) \subset sCl_\theta(B)$*.*
(iii) If $A_\alpha \in S\theta C(X)$ *for each* $\alpha \in A$*, then* $\bigcap\{A_\alpha \mid \alpha \in A\} \in S\theta C(X)$*.*
(iv) $sCl_\theta(sCl_\theta(A)) = sCl_\theta(A)$ *and* $sCl_\theta(A) \in S\theta C(X)$*.*

Lemma 2.3 *For subsets of a topological space* (X,τ)*, the following properties hold:*
(i) A *is a semi-θ-open set if and only if* A *is the union of semi-regular sets.*
(ii) If $A_\alpha \in S\theta O(X)$ *for each* $\alpha \in A$*, then* $\bigcup\{A_\alpha \mid \alpha \in A\} \in S\theta O(X)$*.*

3 s-θ-continuous functions

Definition 1 *A function* $f : (X,\tau) \to (Y,\sigma)$ *is said to be*
(i) irresolute [7] (resp. semi-continuous [13]) if $f^{-1}(V) \in SO(X)$ *for every* $V \in SO(Y)$ *(resp.* $V \in \sigma$*) ;*
(ii) quasi-irresolute [10](resp. semi weakly-continuous [17]) if for each $x \in X$ *and each* $V \in SO(Y)$ *(resp.* $V \in \sigma$*) containing* $f(x)$*, there exists* $U \in SO(X)$ *containing* x *such that* $f(U) \subset sCl(V)$*;*
(iii) s-θ-continuous [15](resp. almost s-θ-continuous, weakly s-θ-continuous) if for each $x \in X$ *and each* $V \in \sigma$ *containing* $f(x)$*, there exists* $U \in S\theta O(X,x)$ *such that* $f(U) \subset V$ *(resp.* $f(U) \subset sCl(V)$*,* $f(U) \subset Cl(V)$*).*

First, we recall the following results concerning quasi-irresolute functions.

Lemma 3.1 *(Di Maio and Noiri [10]). For a function* $f : (X,\tau) \to (Y,\sigma)$*, the following properties are equivalent:*
(1) f *is quasi-irresolute;*
(2) For each $x \in X$ *and each* $V \in SO(Y, f(x))$*, there exists* $U \in SO(X,x)$ *such that* $f(sCl(U)) \subset sCl(V)$*;*
(3) $f^{-1}(K) \in SR(X)$ *for each* $K \in SR(Y)$*;*
(4) $f^{-1}(V) \in S\theta O(X)$ *for each* $V \in S\theta O(Y)$*;*
(5) f *is s-θ-irresolute ([15]).*

Proof. This is shown in Propositions 3.3 and 3.4 of [10]. Mukherjee and Basu [15] called a function f to be s-θ-irresolute if it satisfies the statement (4).

Proposition 3.2 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*
(1) f is quasi-irresolute;
(2) For each $x \in X$ and each $V \in SR(X, f(x))$, there exists $U \in SR(X, x)$ such that $f(U) \subset V$;
(3) For each $x \in X$ and each $V \in S\theta O(Y, f(x))$, there exists $U \in S\theta O(X, x)$ such that $f(U) \subset V$;
(4) For each $x \in X$ and each $V \in S\theta O(Y, f(x))$, there exists $U \in S\theta O(X, x)$ such that $f(U) \subset sCl_\theta(V)$.

Proof. (1) $\Rightarrow$ (2): For each $V \in SR(Y, f(x))$, by Lemma 3.1 $f^{-1}(V) \in SR(X)$. Put $U = f^{-1}(V)$, then $U \in SR(X, x)$ and $f(U) \subset V$.
(2) $\Rightarrow$ (3): For each $x \in X$ and each $V \in S\theta O(Y, f(x))$, by Lemma 2.3 there exists $W \in SR(Y)$ such that $f(x) \in W \subset V$. By (2), there exists $U \in SR(X, x)$ such that $f(U) \subset V$. However, by Lemma 2.1 $U \in SR(X, x) \subset S\theta O(X, x)$ and (3) holds.
(3) $\Rightarrow$ (4):This is obvious.
(4) $\Rightarrow$ (1): For each $x \in X$ and each $V \in SO(Y, f(x))$, by Lemma 2.1 $sCl(V) \in SR(Y, f(x)) \subset S\theta O(Y, f(x)) \cap S\theta C(Y, f(x))$ and by (4) there exists $U \in S\theta O(X, x)$ such that $f(U) \subset sCl_\theta(sCl(V))$; hence $f(U) \subset sCl(V)$. This shows that f is quasi-irresolute.

Proposition 3.3 *If $f : (X, \tau) \to (Y, \sigma)$ is quasi-irresolute, then f is almost s-θ-continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. By Lemma 2.1, $sCl(V) \in SR(Y)$ and by Lemma 3.1 $f^{-1}(sCl(V)) \in SR(X)$. Since $x \in f^{-1}(V) \subset f^{-1}(sCl(V)) \in S\theta O(X)$, put $U = f^{-1}(sCl(V))$. Then $U \in S\theta O(X, x)$ and $f(U) \subset sCl(V)$. Therefore, f is almost s-θ-continuous.

Remark 3.4 *It is obvious that (1) irresoluteness implies quasi-irresoluteness and (2) s-θ-continuity implies almost s-θ-continuity which implies weak s-θ-continuity.*

By Definition 1, Proposition 3.3 and Remark 3.4, we obtain the following relations.

DIAGRAM

$$\begin{array}{ccccc}
\textit{irresolutness} & & \textit{s-}\theta\textit{-continuity} & \rightarrow & \textit{semi-continuity} \\
\downarrow & & \downarrow & & \downarrow \\
\textit{quasi-irresolutness} & \rightarrow & \textit{almost s-}\theta\textit{-continuity} & \rightarrow & \textit{semi weak continuity} \\
 & & \downarrow & & \\
 & & \textit{weak s-}\theta\textit{-continuity} & &
\end{array}$$

Remark 3.5 *Irresoluteness and s-θ-continuity are independent (by Example 3.6). Moreover, s-θ-continuity does not imply quasi-irresoluteness (by Example 3.7) and quasi-irresoluteness does not imply semi-continuity (by Example 3.8). Also semi-continuity does not imply almost s-θ-continuity (by Example 3.9). Observe that none of the implications in DIAGRAM is reversible.*

Example 3.6 *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b\}, \{b, c\}\}$. Then the identity function $f : (X, \tau) \rightarrow (X, \tau)$ is irresolute and not s-θ-continuous by Example 4.3 of [15].*

Example 3.7 *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $f : (X, \tau) \rightarrow (X, \tau)$ be the function defined by setting $f(a) = f(b) = a$ and $f(c) = c$. Then, by Example 7.2 of [10], f is not quasi-irresolute and hence not irresolute. However, f is s-θ-continuous.*

Example 3.8 *(Di Maio and Noiri [10]). Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{b, c\}, X\}$. Then the identity function $f : (X, \tau) \rightarrow (Y, \sigma)$ is quasi-irresolute. However, f is not semi-continuous and hence not irresolute.*

Example 3.9 *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then the identity function $f : (X, \tau) \rightarrow (Y, \sigma)$ is semi-continuous and weakly s-θ-continuous. However, f is not almost s-θ-continuous.*

4 Semi-θ-T_2 spaces

Definition 2 *A topological space* (X, τ) *is said to be semi-*θ*-*T_2 *[5] (resp. semi-*T_2 *[14]) if for every pair of distinct points* x *and* y*, there exist two semi-*θ*-open (resp. semi-open) sets* U *and* V *such that* $x \in U$*,* $y \in V$ *and* $U \cap V = \emptyset$*.*

Theorem 4.1 *For a topological space* (X, τ)*, the following properties are equivalent:*
(1) For every pair of distinct points $x, y \in X$*, there exist* $U \in S\theta O(X, x)$ *and* $V \in S\theta O(X, y)$ *such that* $sCl_\theta(U) \cap sCl_\theta(V) = \emptyset$*;*
(2) (X, τ) *is semi-*θ*-*T_2*;*
(3) (X, τ) *is semi-*T_2*;*
(4) For every pair of distinct points $x, y \in X$*, there exist* $U, V \in SO(X)$ *such that* $x \in U$*,* $y \in V$ *and* $sCl(U) \cap sCl(V) = \emptyset$*;*
(5) For every pair of distinct points $x, y \in X$*, there exist* $U, V \in SR(X)$ *such that* $x \in U$*,* $y \in V$ *and* $U \cap V = \emptyset$*.*

Proof. (1) $\Rightarrow$ (2): This is obvious.
(2) $\Rightarrow$ (3): Since $S\theta O(X) \subset SO(X)$, the proof is obvious.
(3) $\Rightarrow$ (4): This follows from Proposition 4.2 of [10].
(4) $\Rightarrow$ (5): By Lemma 2.1, $sCl(U) \in SR(X)$ for every $U \in SO(X)$ and the proof immediately follows.
(5) $\Rightarrow$ (1): By Lemma 2.1, every semi-regular set is semi-θ-open and semi-θ-closed. Hence the proof is obvious.

Definition 3 *A topological space* X *is called a* $s\theta$*-space if the union of any two semi-*θ*-closed sets is a semi-*θ*-closed set.*

Theorem 4.2 *Let* $f, g : X \to Y$ *be functions,* Y *a semi-*θ*-*T_2 *and* X *a* $s\theta$*-space. If* f *and* g *are s-*θ*-irresolute, then* $A = \{x \in X \mid f(x) = g(x)\}$ *is semi-*θ*-closed in* X*.*

Proof. If $x \in X \setminus A$, then $f(x) \neq g(x)$. Since Y is semi-θ-T_2 , there exist $V_1 \in S\theta O(Y, f(x))$ and $V_2 \in S\theta O(Y, g(x))$ such that $V_1 \cap V_2 = \emptyset$. By the fact that f and g are s-θ-irresolute, $f^{-1}(V_1)$ and $g^{-1}(V_2)$ are semi-θ-open. Since X is a $s\theta$-space, $x \in f^{-1}(V_1) \cap g^{-1}(V_2) \in S\theta O(X, \tau)$. Put $U = f^{-1}(V_1) \cap g^{-1}(V_2)$, then $U \in S\theta O(X, \tau)$ and $U \cap A = \emptyset$. This shows that A is semi-θ-closed in X.

We say that a function $f : X \to Y$ is semi-θ-open if $f(A) \in S\theta O(Y)$ for all $A \in S\theta O(X)$.

Lemma 4.3 *Let a bijection $f : X \to Y$ be semi-θ-open. Then for any $B \in S\theta C(X)$, $f(B) \in S\theta C(Y)$.*

Semi-θ-T_2 spaces remain invariant under bijective semi-θ-open functions as shown in the next theorem.

Theorem 4.4 *If a bijection $f : X \to Y$ is semi-θ-open and X is semi-θ-T_2, then Y is semi-θ-T_2.*

Proof. Let $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Since f is bijective, $f^{-1}(y_1)$, $f^{-1}(y_2) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$. The semi-$\theta$-$T_2$ property of X gives the existence of sets $U \in S\theta O(X, f^{-1}(y_1))$ and $V \in S\theta O(X, f^{-1}(y_2))$ such that $U \cap V = \emptyset$. Thus semi-θ-openness of f gives the existence of two sets $f(U) \in S\theta O(Y, y_1)$ and $f(V) \in S\theta O(Y, y_2)$ such that $f(U) \cap f(V) = \emptyset$ which assures that Y is semi-θ-T_2.

5 Strongly semi-θ-closed graphs

If $f : (X, \tau) \to (Y, \sigma)$ is any function, then the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f (see: [12]).

Definition 4 *A function $f : X \to Y$ is said to have a strongly semi-θ-closed graph if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in S\theta O(X, x)$ and $V \in S\theta O(Y, y)$ such that $f(U) \cap sCl_\theta(V) = \emptyset$.*

Definition 5 *A function $f : X \to Y$ is said to be strongly semi-θ-closed [10] (resp. sθ-C-semiclosed [5]) if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in SO(X, x)$ and $V \in SO(Y, y)$ (resp. $V \in S\theta O(Y, y)$) such that $(sCl(U) \times sCl(V)) \cap G(f) = \emptyset$ (resp. $f(sCl(U)) \cap V = \emptyset$).*

Theorem 5.1 *For a graph $G(f)$ of a function $f : X \to Y$, the following properties are equivalent:*
(1) $G(f)$ is strongly semi-θ-closed;
(2) For each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in S\theta O(X, x)$ and $V \in$

$S\theta O(Y,y)$ such that $f(U) \cap V = \emptyset$;
(3) $G(f)$ is $s\theta$-C-semiclosed;
(4) For each $(x,y) \in (X \times Y)\backslash G(f)$, there exist $U \in SR(X,x)$ and $V \in SR(Y,y)$ such that $f(U) \cap V = \emptyset$;
(5) $G(f)$ is strongly semi-θ-closed in the sense of Di Maio and Noiri [10].

Proof. (1) $\Rightarrow$ (2): The proof is obvious.
(2) $\Rightarrow$ (3): For $U \in S\theta O(X,x)$, there exists $U_0 \in SO(X,x)$ such that $x \in U_0 \subset sCl(U_0) \subset U$.
(3) $\Rightarrow$ (4): For each $(x,y) \in (X \times Y)\backslash G(f)$, by (3) there exist $U \in SO(X,x)$ and $V_0 \in S\theta O(Y,y)$ such that $f(sCl(U)) \cap V_0 = \emptyset$. By Lemma 2.1, $sCl(U) \in SR(X)$ and there exist $V \in SR(Y)$ such that $y \in V \subset V_0$; hence $f(sCl(U)) \cap V = \emptyset$.
(4) $\Rightarrow$ (5): The proof is obvious.
(5) $\Rightarrow$ (1): For each $(x,y) \notin G(f)$, there exist $U \in SO(X,x)$ and $V \in SO(Y,y)$ such that $f(sCl(U)) \cap sCl(V) = \emptyset$. Since $x \in U \subset sCl(U) \in SR(X,x) \subset S\theta O(X,x)$ and $y \in V \subset sCl(V) \in SR(Y,y) \subset S\theta O(Y,y) \cap S\theta C(Y,y)$, $f(sCl(U)) \cap sCl_\theta(sCl(V)) = f(sCl(U)) \cap sCl(V) = \emptyset$.

It is shown in Proposition 4.4. of [10] that $f : X \to Y$ is quasi-irresolute and Y is semi-T_2, then $G(f)$ is strongly semi-θ-closed. Similarly, we have the following property.

Theorem 5.2 *If $f : X \to Y$ is weakly s-θ-continuous and Y is Hausdorff, then $G(f)$ is strongly semi-θ-closed.*

Proof. Let $(x,y) \notin G(f)$. Since $y \neq f(x)$ and Y is Hausdorff, there exist open sets V, W in Y such that $f(x) \in W$, $y \in V$ and $W \cap V = \emptyset$; hence $Cl(W) \cap Int(Cl(V)) = \emptyset$. Since f is weakly s-θ-continuous, there exists $U \in S\theta O(X,x)$ such that $f(U) \subset Cl(W)$. Therefore, we have $f(U) \cap Int(Cl(V)) = \emptyset$. Since $Int(Cl(V)) \in SR(Y, f(x)) \subset S\theta O(Y, f(x))$, by Theorem 5.1 $G(f)$ is strongly semi-θ-closed.

Theorem 5.3 *If $f : X \to Y$ is surjective and has a strongly semi-θ-closed graph $G(f)$, then Y is semi-θ-T_2.*

Proof. Let y_1, y_2 $(y_1 \neq y_2) \in Y$. The surjectivity of f gives a $x_1 \in X$ such that $f(x_1) = y_1$. Now $(x_1, y_2) \in (X \times Y)\backslash G(f)$. The strong semi-$\theta$-closedness of $G(f)$ provides $U \in S\theta O(X, x_1)$ and $V \in S\theta O(Y, y_2)$ such that $f(U) \cap sCl_\theta(V) = \emptyset$, whence one infers that $y_1 \notin sCl_\theta(V)$. This means that there exists $W \in SO(Y, y_1)$ such that $sCl(W) \cap V = \emptyset$. So, Y is semi-θ-T_2.

Theorem 5.4 *A space X is semi-θ-T_2 if and only if the identity function $id : X \to X$ has a strongly semi-θ-closed graph $G(id)$.*

Proof. *Necessity.* Let X be a semi-θ-T_2 space. Since the identity function $id : X \to X$ is quasi irresolute by Lemma 3.1, it follows from Proposition 4.4 of [10] that $G(id)$ is strongly semi-θ-closed.
Sufficiency. Let $G(id)$ be a strongly semi-θ-closed graph. Then the surjectivity of id and strong semi-θ-closedness of $G(id)$ together imply, by Theorem 5.3, that X is semi-θ-T_2.

Theorem 5.5 *If $f : X \to Y$ is an injection and $G(f)$ is strongly semi-θ-closed, then X is semi-θ-T_2.*

Proof. Since f is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. Then $(x_1, f(x_2)) \in (X \times Y) \backslash G(f)$. Since $G(f)$ is strongly semi-θ-closed, by Theorem 5.1 there exist $U \in S\theta O(X, x_1)$ and $V \in S\theta O(Y, f(x_2))$ such that $f(U) \cap V = \emptyset$. Therefore $x_2 \notin U$. By Lemma 2.3, there exists $R \in SR(X, x)$ such that $x_1 \in R \subset U$. Therefore, $x_2 \in X \setminus U \subset X \setminus R \in SR(X)$ and by Theorem 4.1, X is semi-θ-T_2.

Theorem 5.6 *If $f : X \to Y$ is a bijective function with a strongly semi-θ-closed graph, then both X and Y are semi-θ-T_2.*

Proof. The proof is an immediate consequence of Theorems 5.3 and 5.5.

Theorem 5.7 *If a bijection $f : X \to Y$ is semi-θ-open and X is semi-θ-T_2, then $G(f)$ is strongly semi-θ-closed.*

Proof. Let $(x, y) \in (X \times Y) \backslash G(f)$. Then $y \neq f(x)$. Since f is bijective, $x \neq f^{-1}(y)$. Since X is semi-θ-T_2, there exist U_x, $U_y \in S\theta O(X)$ such that $x \in U_x$, $f^{-1}(y) \in U_y$ and $U_x \cap U_y = \emptyset$. Moreover, since f is semi-θ-open and bijective, $f(x) \in f(U_x) \in S\theta O(Y)$, $y \in f(U_y) \in S\theta O(Y)$ and $f(U_x) \cap f(U_y) = \emptyset$. This shows that $G(f)$ is strongly semi-θ-closed.

Theorem 5.8 *If $f : X \to Y$ is a function with a strongly semi-θ-closed graph, then for each $x \in X$, $f(x) = \cap\{sCl_\theta(f(U)) : U \in S\theta O(X, x)\}$.*

Proof. Suppose that there exists a point $y \in Y$ such that $y \neq f(x)$ and $y \in \cap\{sCl_\theta(f(U)) : U \in S\theta O(X, x)\}$. This implies that $y \in sCl_\theta(f(U))$ for every $U \in S\theta O(X, x)$. So $sCl(V) \cap f(U) \neq \emptyset$ for every $V \in SO(Y, y)$. This indicates that $sCl_\theta(V) \cap f(U) \supset sCl(V) \cap f(U) \neq \emptyset$ which contradicts the hypothesis that f is a function with a strongly semi-θ-closed graph. Hence the theorem holds.

Definition 6 *A subset A of a topological space X is said to be s-closed relative to X [8] if for every cover $\{V_\alpha \mid \alpha \in V\}$ of A by semi-open sets of X, there exists a finite subset $\bigtriangledown_0$ of $\bigtriangledown$ such that $A \subset \bigcup\{sCl(V_\alpha) \mid \alpha \in \bigtriangledown_0\}$. If $A = X$, the space X is said to be s-closed [8].*

Noiri [16] showed that if $G(f)$ is strongly closed then f has the following property:
(P) For every set B quasi H-closed relative to Y, $f^{-1}(B)$ is a closed set of X.

Analogously, we have the following result.

Theorem 5.9 *Let (X, τ) be a $s\theta$-space. If $f : X \to Y$ has a strongly semi-θ-closed graph $G(f)$, then it has the following property:*

(P^*) For every set F which is s-closed relative to Y, $f^{-1}(F)$ is semi-θ-closed in X.

Proof. Suppose that $x \in X \setminus f^{-1}(F)$ and $y \in F$. Then $(x, y) \in (X \times Y)\backslash G(f)$. By Theorem 5.1, strong semi-θ-closedness of $G(f)$ gives the existence of $U_y(x) \in SR(X, x)$ and $V_y \in SR(Y, y)$ such that $f(U_y(x)) \cap V_y = \emptyset$....(*).
Clearly $\{V_y : y \in F\}$ is a cover of F by semi-regular sets in Y. Since F is s-closed relative to Y, by Proposition 4.1 of [8] there exist finite semi-regular sets $V_{y_1}, V_{y_2}, ..., V_{y_n}$ in Y such that $F \subset \bigcup\limits_{i=1}^{n} V_{y_i}$.
Let $U = \bigcap\limits_{i=1}^{n} U_{y_i}(x)$, where $U_{y_i}(x)$ are the semi-θ-open sets in X satisfying (*). Also $U \in S\theta O(X, x)$. Now $f(U) \cap F \subset f[\bigcap\limits_{i=1}^{n} U_{y_i}(x)] \cap (\bigcup\limits_{i=1}^{n} V_{y_i}) \subset \bigcup\limits_{i=1}^{n} (f[U_{y_i}(x)] \cap V_{y_i}) = \emptyset$. Therefore $U \cap f^{-1}(F) = \emptyset$. This shows that $f^{-1}(F)$ is semi-θ-closed in X.

Lemma 5.10 *If K is a semi-θ-closed set of a s-closed space (X, τ), then K is s-closed relative to X.*

Proof. This follows from Proposition 4.2 of [8].

Theorem 5.11 *Let (X, τ) be a $s\theta$-space. If Y is s-closed, then a function $f : X \to Y$ with a strongly semi-θ-closed graph is quasi-irresolute.*

Proof. Let K be any semi-θ-closed set of Y. Since Y is s-closed, by Lemma 5.10 K is s-closed relative to Y and it follows from Theorem 5.9 that $f^{-1}(K)$ is semi-θ-closed in X. By Lemma 3.1, f is quasi-irresolute.

Corollary 5.12 *Let (X, τ) be a $s\theta$-space and (Y, σ) a s-closed semi-T_2 space. Then for a function $f : (X, \tau) \to (Y, \sigma)$ the following properties are equivalent:*
(1) $G(f)$ is strongly semi-θ-closed;
(2) $f^{-1}(K)$ is semi-θ-closed in X for every subset K which is s-closed relative to Y;
(3) f is quasi-irresolute.

Proof. (1) $\Rightarrow$ (2): This is an immediate consequence of Theorem 5.9.
(2) $\Rightarrow$ (3): This follows from Lemmas 5.10 and 3.1.
(3) $\Rightarrow$ (1): This follows from Proposition 4.4 of [10] and Theorem 5.1.

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