Solid Angle of a Rectangular Plate

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The solid angle covered by a rectangular plate of length a and width b at a distance d to the observer is calculated. [vixra:2001.0603]

I. ON-AXIS GEOMETRY

We consider a rectangular plate of size $a \times b$ (area ab) at a distance d to an observer. We consider only the on-axis alignment where the vector normal of the plate surface starting at the plate center points to the observer, which means the plate surface is perpendicular to the line of sight.

In a Cartesian coordinate system with the center of coordinates at the observer, the plate surface coordinates are confined to

$$-a/2 \le x \le a/2; \quad -b/2 \le y \le b/2; \quad z = d. \tag{1}$$

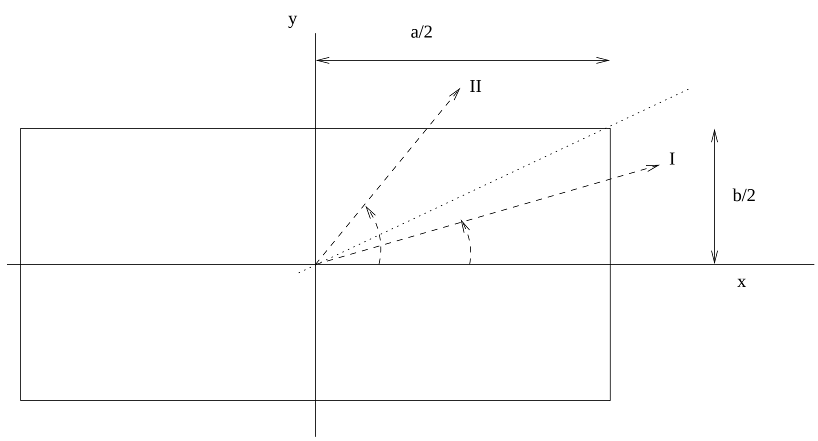

FIG. 1. The plate seen by the observer looking into the direction $+z$.

The 9 Cartesian coordinates of three corners $\vec{A}$, $\vec{B}$, and $\vec{C}$ of the quadrangle—with the observer at the origin—are reduced to the lengths a, b and distance d by fundamental vector algebra: the lengths a and b are obtained by the Pythagorean formula for the length of the pair-wise differences of adjacent corners. The value of d, the distance of their common plane to the origin, is given by decomposing $\vec{A}$ into components parallel and orthogonal to the plane:

$$\vec{A} = x(\vec{B} - \vec{A}) + y(\vec{C} - \vec{A}) + d\vec{T}. \tag{2}$$

Compute $\vec{T}$ by the cross product $\vec{T} = (\vec{B} - \vec{A}) \times (\vec{C} - \vec{A})$, and obtain d by the ratio of dot products: $d = \vec{A} \cdot \vec{T}/(\vec{T} \cdot \vec{T})$. Because only d^2 will be used below, the sign that emerges from this formula does not matter.

II. COORDINATE LIMITS

In a spherical coordinate system centered at the observer, with polar angle θ ($0 \le \theta \le \pi$) and azimuth angle φ ($0 \le \varphi \le 2\pi$) the solid angle of the object is the double integral

$$\Omega = \int \sin\theta \, d\theta \, d\varphi, \tag{3}$$

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where the two angular coordinates scan the surface of the object. We transform the Cartesian coordinates (1) to spherical coordinates and perform the double integral, using the generic

$$x = r\sin\theta\cos\varphi, \tag{4}$$
$$y = r\sin\theta\sin\varphi, \tag{5}$$
$$z = r\cos\theta, \tag{6}$$

and the inverse transformation

$$\varphi = \arctan\frac{y}{x}, \quad \theta = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}. \tag{7}$$

The symmetry of the problem allows us to consider only one octant of the double hemisphere region seen by the observer, which reduces to a quadrant of the plate,

$$\Omega = 4 \int\limits_{\substack{0\le x\le a/2\\ 0\le y\le b/2}} \sin\theta\, d\theta\, d\varphi. \tag{8}$$

This region of area $ab/4$ is split into two triangular regions along the diagonal of the plate, separated by a dotted line in Fig. 1:

- One triangular region (I) covers the interval

  $$0\le\varphi\le\arctan(b/a). \tag{9}$$

  A point on the dashed straight line in Fig. 1 from the center of the plate up to the plate circumference has the coordinates x, $y = x\tan\varphi$, $z = d$. The line leaves the plate at $x = a/2$, $y = (a/2)\tan\varphi$, $z = d$. Therefore region I covers

  $$0\le\theta\le\arccos\frac{d}{\sqrt{d^2+\frac{a^2}{4}+\frac{a^2}{4}\tan^2\varphi}} \tag{10}$$

  according to (7).

- The complementary triangular region (II) covers

  $$\arctan(b/a)\le\varphi\le\pi/2 \tag{11}$$

  and

  $$0\le\theta\le\arccos\frac{d}{\sqrt{d^2+\frac{b^2}{4}+\frac{b^2}{4\tan^2\varphi}}}. \tag{12}$$

III. EVALUATION OF INTEGRALS

The previous four equations turn Eq. (3) into

$$\Omega = 4\left[\int_0^{\arctan\frac{b}{a}} d\varphi \int_0^{\arccos\frac{d}{\sqrt{d^2+\frac{a^2}{4}(1+\tan^2\varphi)}}} \sin\theta\, d\theta + \int_{\arctan\frac{b}{a}}^{\pi/2} d\varphi \int_0^{\arccos\frac{d}{\sqrt{d^2+\frac{b^2}{4}(1+\cot^2\varphi)}}} \sin\theta\, d\theta\right]. \tag{13}$$

In the θ-integrals we substitute $t=\cos\theta$, $dt = -\sin\theta d\theta$ with

$$\int_{\theta=0}^{\theta=\arccos\ldots} \sin\theta d\theta = -\int_1^{\ldots} dx = 1-\ldots \tag{14}$$

to get

$$\Omega = 4\left[\int_0^{\arctan\frac{b}{a}} \left(1-\frac{d}{\sqrt{d^2+\frac{a^2}{4}(1+\tan^2\varphi)}}\right) d\varphi + \int_{\arctan\frac{b}{a}}^{\pi/2} \left(1-\frac{d}{\sqrt{d^2+\frac{b^2}{4}(1+\cot^2\varphi)}}\right) d\varphi\right]. \tag{15}$$

We integrate over the two constant parts and condense the trigonometric functions $1+\ldots^2\varphi$ underneath the square roots,

$$\Omega = 4\left[\frac{\pi}{2} - \int_0^{\arctan\frac{b}{a}} \frac{d}{\sqrt{d^2+\frac{a^2}{4\cos^2\varphi}}}d\varphi - \int_{\arctan\frac{b}{a}}^{\pi/2} \frac{d}{\sqrt{d^2+\frac{b^2}{4\sin^2\varphi}}}d\varphi\right]. \tag{16}$$

In the second integral we substitute $\varphi\prime = \pi/2-\varphi$,

$$\Omega = 4\left[\frac{\pi}{2} - \int_0^{\arctan\frac{b}{a}} \frac{1}{\sqrt{1+\frac{a^2}{4d^2\cos^2\varphi}}}d\varphi - \int_0^{\arctan\frac{a}{b}} \frac{1}{\sqrt{1+\frac{b^2}{4d^2\cos^2\varphi\prime}}}d\varphi\prime\right]. \tag{17}$$

The two indefinite integrals of the type

$$I(\alpha,\varphi) \equiv \int \frac{d\varphi}{\sqrt{1+\alpha^2\frac{1}{\cos^2\varphi}}} = \int \frac{\cos\varphi d\varphi}{\sqrt{\cos^2\varphi+\alpha^2}} \tag{18}$$

are solved with the substitution [1, 2.584.3]

$$v = \frac{\sin\varphi}{\sqrt{1+\alpha^2}}, \quad dv = d\varphi\frac{\cos\varphi}{\sqrt{1+\alpha^2}}, \quad v^2(1+\alpha^2) = \sin^2\varphi = 1-\cos^2\varphi, \tag{19}$$

$$I(\alpha,\varphi) = \int \frac{\sqrt{1+\alpha^2}dv}{\sqrt{1+\alpha^2-v^2(1+\alpha^2)}} = \int \frac{dv}{\sqrt{1-v^2}} = \arcsin v = \arcsin\frac{\sin\varphi}{\sqrt{1+\alpha^2}}. \tag{20}$$

Applied to (17) we have [2, 3]

$$\Omega = 4\left[\frac{\pi}{2} - \arcsin\frac{\sin\arctan\frac{b}{a}}{\sqrt{1+\frac{a^2}{4d^2}}} - \arcsin\frac{\sin\arctan\frac{a}{b}}{\sqrt{1+\frac{b^2}{4d^2}}}\right]. \tag{21}$$

The sines simplify according to the column `tan x=a` of [4, 4.3.45]

$$\Omega = 4\left[\frac{\pi}{2} - \arcsin\frac{\frac{b}{a}}{\sqrt{1+\frac{a^2}{4d^2}}\sqrt{1+(\frac{b}{a})^2}} - \{a\leftrightarrow b\}\right] \tag{22}$$

$$= 4\left[\frac{\pi}{2} - \arcsin\frac{b}{\sqrt{1+\frac{a^2}{4d^2}}\sqrt{a^2+b^2}} - \{a\leftrightarrow b\}\right], \tag{23}$$

where $a\leftrightarrow b$ means the previous term is to be repeated with roles of a and b interchanged. With [4, 4.4.32], we combine the two arc-sines into one,

$$\Omega = 4\left[\frac{\pi}{2} - \arcsin\left(\frac{b}{\sqrt{1+\frac{a^2}{4d^2}}\sqrt{a^2+b^2}}\sqrt{1-\frac{a^2}{[1+(\frac{b}{2d})^2][a^2+b^2]}} + \{a\leftrightarrow b\}\right)\right] \tag{24}$$

$$= 4\left[\frac{\pi}{2} - \arcsin\left(\frac{b^2\sqrt{1+(\frac{a}{2d})^2+(\frac{b}{2d})^2}}{\sqrt{1+(\frac{a}{2d})^2}\sqrt{1+(\frac{b}{2d})^2}(a^2+b^2)} + \{a\leftrightarrow b\}\right)\right] \tag{25}$$

$$= 4\left[\frac{\pi}{2} - \arcsin\frac{\sqrt{1+(\frac{a}{2d})^2+(\frac{b}{2d})^2}}{\sqrt{1+(\frac{a}{2d})^2}\sqrt{1+(\frac{b}{2d})^2}}\right]. \tag{26}$$

With the definition of two cone parameters $\alpha\equiv a/(2d)$ and $\beta\equiv b/(2d)$, the previous equation yields the result

$$\Omega(a,b,d) = 4\arccos\sqrt{\frac{1+\alpha^2+\beta^2}{(1+\alpha^2)(1+\beta^2)}} = 4\arcsin\frac{\alpha\beta}{\sqrt{(1+\alpha^2)(1+\beta^2)}} = 4\arctan\frac{\alpha\beta}{\sqrt{1+\alpha^2+\beta^2}} \tag{27}$$

which is known for more than 50 years [5]. Power series for small values of α and β are

$$\frac{1+\alpha^2+\beta^2}{(1+\alpha^2)(1+\beta^2)} \approx 1-(\alpha\beta)^2+\alpha^2\beta^2(\alpha^2+\beta^2), \tag{28}$$

$$\sqrt{\frac{1+\alpha^2+\beta^2}{(1+\alpha^2)(1+\beta^2)}} \approx 1-\frac{(\alpha\beta)^2}{2}+\frac{(\alpha\beta)^2}{2}(\alpha^2+\beta^2)-\frac{(\alpha\beta)^2}{8}(4\alpha^4+5\alpha^2\beta^2+4\beta^4)+\frac{(\alpha\beta)^2}{4}(\alpha^2+\beta^2)(2\alpha^4+\alpha^2\beta^2+2\beta^4), \tag{29}$$

$$\Omega \approx 4\alpha\beta - 2\alpha\beta(\alpha^2+\beta^2)+\frac{\alpha\beta}{6}(9\alpha^4+10\alpha^2\beta^2+9\beta^4)-\frac{\alpha\beta}{4}(\alpha^2+\beta^2)(5\alpha^4+2\alpha^2\beta^2+5\beta^4). \tag{30}$$

The leading term $\Omega \approx 4\alpha\beta = ab/d^2$ is the ratio of the plate area over the squared distance.

IV. OFF-AXIS GEOMETRIES

The more generic case occurs when the shortest distance d from the observer to the plane which contains the plate leads to a point different from the plate center. Figure 2 shows the case when this pointing direction would not meet the plate at all but miss it by distances A and B relative to the closest edge of the plate. The line of sight hits the

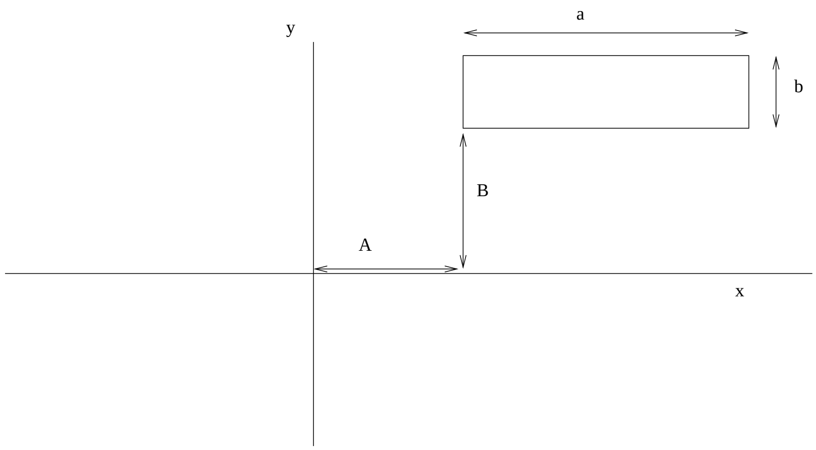

FIG. 2. Lengths in the off-axis 1-quadrant geometry, Eq. (31).

plane of the plate at the Cartesian coordinates $(0,0,d)$; the center of the plate is at $(A+a/2, B+b/2, d)$.

Suitable symmetric decomposition of the area ab into more areas which individually meet the requirement of formula (27) for $\Omega(a,b,d)$ is shown in Figure 3 [2]. There is the virtual plate of size $2(A+a)\times 2(B+b)$ of which encompasses all the quadrangles, a horizontal middle strip of size $2(A+a)\times 2B$, a vertical middle strip of size $2A\times 2(B+b)$, and the center quadrangle of size $2A\times 2B$. Superposition of the solid angles of these with suitable corrections for multiply counted areas yields (according to the inclusion-exclusion principle) for the solid angle of the $a\times b$ rectangle in Fig. 2

$$\Omega^{(I)}(A,B,a,b,d) = \frac{\Omega(2(A+a),2(B+b),d)-\Omega(2A,2(B+b),d)-\Omega(2(A+a),2B,d)+\Omega(2A,2B,d)}{4}. \tag{31}$$

$$\Omega^{(I)}(0,0,a,b,d) = \frac{1}{4}\Omega(2a,2b,d). \tag{32}$$

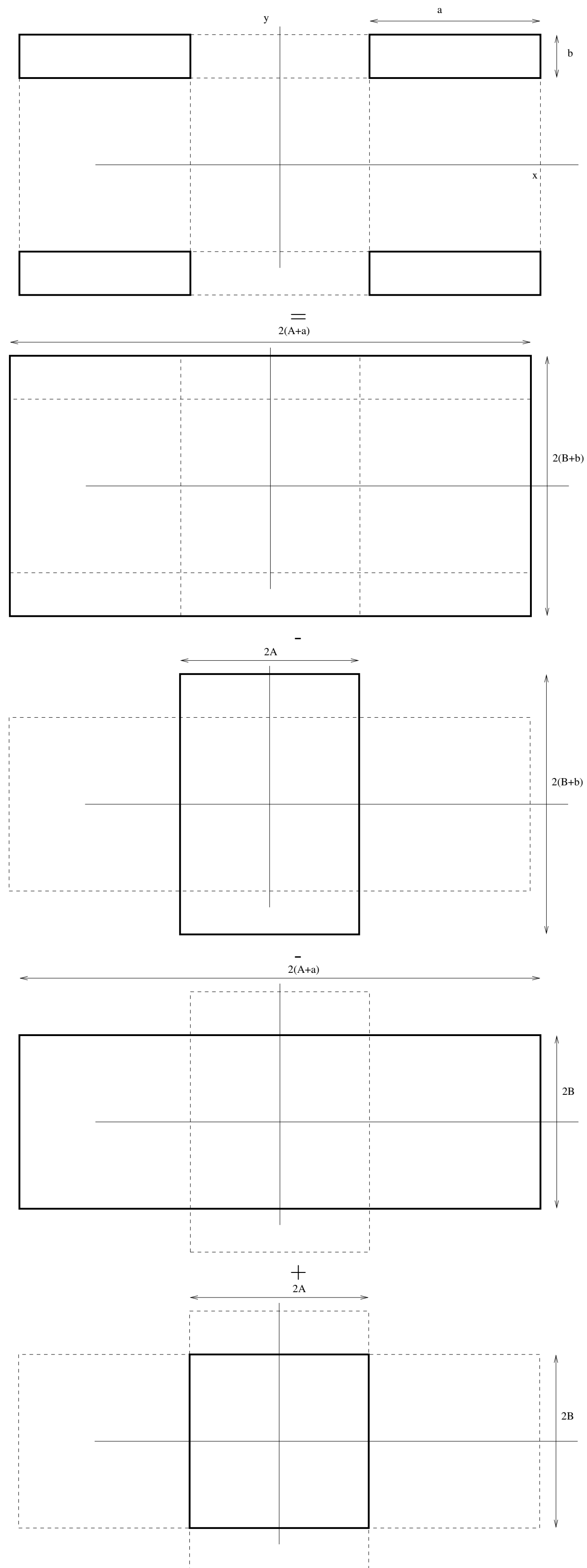

FIG. 3. Decomposition of the off-axis 1-quadrant geometry into four symmetric cases.

Two variants of this geometry exist. Figure 4 shows the case where the signs of the Cartesian coordinates of the four corners of the plate differ in either the x-value or the y-value but not both. We still adopt the sign convention

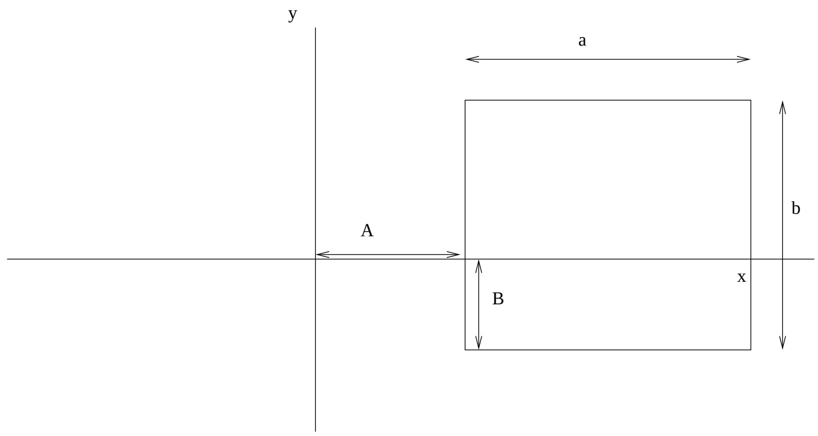

FIG. 4. Off-axis 2-quadrant case, Eq. (33).

that $0 < B \le b/2$ are positive. A similar calculation as above leads to the formula

$$\Omega^{(II)}(A,B,a,b,d) = \frac{\Omega(2(A+a),2(b-B),d) - \Omega(2A,2(b-B),d) + \Omega(2(A+a),2B,d) - \Omega(2A,2B,d)}{4} \tag{33}$$

for the solid angle of the $a \times b$ rectangle in Fig. 4. This may for example be derived by splitting the rectangle into a $a \times (b-B)$ rectangle above the horizontal axis and a $a \times B$ rectangle below the horizontal axis and adding two associated solid angles computed via Eq. (31).

In the second variant, the line of sight hits the plate off-center (Fig. 5) with $0 \le A \le a/2$ and $0 \le B \le b/2$. The solid angle covered by the rectangle $a \times b$ becomes

$$\Omega^{(IV)}(A,B,a,b,d) = \frac{\Omega(2(a-A),2(b-B),d) + \Omega(2A,2(b-B),d) + \Omega(2(a-A),2B,d) + \Omega(2A,2B,d)}{4}. \tag{34}$$

This formula is for example derived by considering the sum of the 4 sub-rectangles in the 4 quadrants: $(a-A)\times(b-B)$

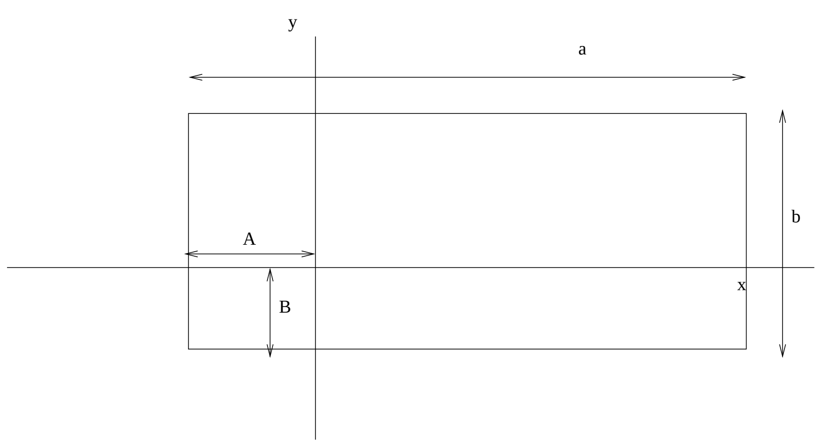

FIG. 5. Off-axis 4-quadrant case, Eq. (34).

where $x, y \ge 0$, $A \times (b-B)$ where $x \le 0, y > 0$, $A \times B$ where $x, y \le 0$, and $(a-A) \times B$ where $x \ge 0, y < 0$: $\Omega^{(IV)}(A,B,a,b,d) = \Omega^{(I)}(0,0,a-A,b-B,d) + \Omega^{(I)}(0,0,A,b-B,d) + \Omega^{(I)}(0,0,A,B,d) + \Omega^{(I)}(0,0,a-A,B,d)$.

V. LAMBERTIAN POWER LAW

The integral (18) is a special case $I^{(0)}(\alpha,\varphi)$ of the integrals family

$$I^{(m)}(\alpha,\varphi) \equiv \int \left(\frac{\cos\varphi}{\sqrt{\cos^2\varphi+\alpha^2}}\right)^{m+1} d\varphi. \tag{35}$$

The two possible parities of $m+1$ are considered separately in the next two sections.

A. Even Power

If $m+1=2m'$ is an even positive integer, $I^{(m)}(\alpha,\varphi)$ shall be computed recursively. Define the auxiliary integrals

$$J^{(l)}(\alpha,\varphi) \equiv \int \frac{1}{(\cos^2\varphi+\alpha^2)^l} d\varphi \tag{36}$$

with the initial values

$$J^{(0)}(\alpha,\varphi) = \varphi \tag{37}$$

and [1, 2.562.2][6, 3.6.1]

$$J^{(1)}(\alpha,\varphi) = \int \frac{1}{\cos^2\varphi+\alpha^2} d\varphi = \frac{1}{\alpha\sqrt{1+\alpha^2}} \arctan \frac{\alpha\tan\varphi}{\sqrt{1+\alpha^2}}. \tag{38}$$

A partial fraction decomposition of (35) computes $I^{(m)}$ via a set of $J^{(l)}$:

$$\begin{aligned} I^{(m)}(\alpha,\varphi) &= \int \left(\frac{\cos\varphi}{\sqrt{\cos^2\varphi+\alpha^2}}\right)^{2m'} d\varphi = \int \frac{\cos^{2m'}\varphi}{(\cos^2\varphi+\alpha^2)^{m'}} d\varphi \\ &= \sum_{l=0}^{m'} \binom{m'}{l} (-\alpha^2)^l \int \frac{1}{(\cos^2\varphi+\alpha^2)^l} d\varphi = \sum_{l=0}^{m'} \binom{m'}{l} (-\alpha^2)^l J^{(l)}(\alpha,\varphi). \end{aligned} \tag{39}$$

The substitution (19) rephrases $J^{(l)}$:

$$J^{(l)}(\alpha,\varphi) = \frac{1}{(1+\alpha^2)^{l-1/2}} \int \frac{1}{(1-v^2)^l} \frac{dv}{\sqrt{1-(1+\alpha^2)v^2}}. \tag{40}$$

That family of integrals is solved recursively by down-stepping the integer power of $1-v^2$ in the denominator:

$$\begin{aligned} 2l\alpha^2 \int \frac{dv}{(1-v^2)^{l+1}\sqrt{1-(1+\alpha^2)v^2}} &= \frac{\sqrt{1-(1+\alpha^2)v^2}}{v}\left[1-\frac{1}{(1-v^2)^l}\right] - \sum_{s=1}^{l} \int \frac{dv}{(1-v^2)^s\sqrt{1-(1+\alpha^2)v^2}} \\ &+2l(1+\alpha^2)\int \frac{dv}{(1-v^2)^l\sqrt{1-(1+\alpha^2)v^2}}. \end{aligned} \tag{41}$$

The limit of the first term on the right hand side as $\varphi\to 0$ and $v\to 0$ is zero. Since the left hand side with the factor $2l\alpha^2$ is zero for $l=0$ we employ also [1, 2.281.3,2.266]

$$\int \frac{dv}{(1-v^2)\sqrt{1-(1+\alpha^2)v^2}} = \sqrt{1+\alpha^2} J^{(1)}(\alpha,\varphi) = \frac{1}{\alpha} \arctan \frac{\alpha v}{\sqrt{1-(1+\alpha^2)v^2}}. \tag{42}$$

B. Odd Power

Complementary to the previous section let m be an even non-negative integer. We define an integer $\bar m$ via $2\bar m = m$, $\bar m = \lfloor (m+1)/2 \rfloor$, and employ a partial fraction decomposition:

$$\begin{aligned} I^{(m)}(\alpha,\varphi) &= \int \frac{\cos^{2\bar m}\varphi}{(\cos^2\varphi+\alpha^2)^{\bar m}} \frac{\cos\varphi}{\sqrt{\cos^2\varphi+\alpha^2}} d\varphi \\ &= \sum_{l=0}^{\bar m} \binom{\bar m}{l} (-\alpha^2)^l \int \frac{1}{(\cos^2\varphi+\alpha^2)^l} \frac{\cos\varphi}{\sqrt{\cos^2\varphi+\alpha^2}} d\varphi. \end{aligned} \tag{43}$$

The substitution (19) turns these integrals into Gaussian Hypergeometric functions:

$$\int \frac{1}{(\cos^2\varphi+\alpha^2)^l}\frac{\cos\varphi}{\sqrt{\cos^2\varphi+\alpha^2}}d\varphi = \frac{1}{(1+\alpha^2)^l}\int\frac{1}{(1-v^2)^{l+1/2}}dv$$
$$= \frac{1}{(1+\alpha^2)^l}v\,{}_2F_1\left(\begin{matrix}1/2, l+1/2\\ 3/2\end{matrix}\mid v^2\right). \tag{44}$$

These are computed recursively starting from [4, 15.1.6]

$$ {}_2F_1\left(\begin{matrix}1/2,1/2\\ 3/2\end{matrix}|z\right) = \frac{1}{\sqrt{z}}\arcsin\sqrt{z}, \tag{45}$$

then for $l \geq 1$ [4, 15.2.24]

$$ {}_2F_1\left(\begin{matrix}1/2,l+1/2\\ 3/2\end{matrix}\mid z\right) = \frac{1}{2l-1}\left[\frac{1}{(1-z)^{l-1/2}} + 2(l-1){}_2F_1\left(\begin{matrix}1/2,l-1/2\\ 3/2\end{matrix}\mid z\right)\right]$$
$$= \frac{1}{(2l-1)(1-z)^{l-1/2}} + \frac{1}{2}\sum_{t=1}^{l-1}\frac{t\cdot(t+1)\cdot(t+2)\cdot(l-1)}{(t-1/2)(t+1/2)(t+3/2)\cdots(l-1/2)}\frac{1}{(1-z)^{t-1/2}}. \tag{46}$$

Appendix A: Solid Angle of the Sphere Cap

The equivalent calculation for a on-center plane circle of radius R (area πR^2) at distance d (and of the solid angle of a sphere cap at distance $\sqrt{d^2+R^2}$) is

$$\Omega = 2\pi\int_0^{\arctan\frac{R}{d}}\sin\theta\,d\theta = -2\pi\int_1^{\cos\arctan\frac{R}{d}}dt = 2\pi\left(1-\frac{1}{\sqrt{1+(\frac{R}{d})^2}}\right) = 2\pi(1-\cos\phi), \tag{A1}$$

where ϕ is half of the cone angle (i.e., the angle between the circle center and circle rim seen by the observer). For small ϕ (small R/d), the Taylor expansions are

$$\Omega \approx \pi(\frac{R}{d})^2 - \frac{3\pi}{4}(\frac{R}{d})^4 + \frac{5\pi}{8}(\frac{R}{d})^6 \tag{A2}$$
$$\approx \pi\phi^2 - \frac{\pi}{12}\phi^4 + \frac{\pi}{360}\phi^6. \tag{A3}$$

Circles from more general viewing angles are discussed elsewhere [7–12].

Triangulation of more complicated surfaces to finite elements leads to the solid angle of triangles [13–16].

Appendix B: Solid Angle of the Planar Ellipse

The solid angle of an on-axis planar ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1;\quad z=d \tag{B1}$$

with semiaxes $a \geq b$ at distance d to the observer is 4 times the solid angle of the face in one of the octants:

$$\Omega = 4\iint_{x,y,\geq 0, x^2/a^2+y^2/b^2\leq 1}\sin\theta d\theta d\varphi. \tag{B2}$$

As the outer integral we select $d\varphi$, so we need to find the upper limit of the θ-coordinate for a fixed φ. The ratio of (5) and (4) produces

$$y = x\tan\varphi \tag{B3}$$

and inserting this into (B1) the rim of the ellipse is characterized by

$$x^2\left(\frac{1}{a^2}+\frac{\tan^2\varphi}{b^2}\right)=1. \tag{B4}$$

Solving for x^2,

$$x^2+y^2+z^2=x^2(1+\tan^2\varphi)+z^2=\frac{x^2}{\cos^2\varphi}+z^2=\frac{1}{\cos^2\varphi(\frac{1}{a^2}+\frac{\tan^2\varphi}{b^2})}+d^2=\frac{1}{\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}}+d^2. \tag{B5}$$

The θ-value on the rim is with (7), introducing the dimensionless

$$\alpha\equiv a/d;\quad \beta\equiv b/d,\quad \xi=b/a\le 1, \tag{B6}$$

$$\cos\theta=\frac{d}{\sqrt{\frac{1}{\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}}+d^2}}=\frac{1}{\sqrt{\frac{1}{\frac{\cos^2\varphi}{\alpha^2}+\frac{\sin^2\varphi}{\beta^2}}+1}}=\frac{1}{\sqrt{\frac{\alpha^2\beta^2}{\beta^2\cos^2\varphi+\alpha^2\sin^2\varphi}+1}}=\sqrt{\frac{\beta^2\cos^2\varphi+\alpha^2\sin^2\varphi}{\alpha^2\beta^2+\beta^2\cos^2\varphi+\alpha^2\sin^2\varphi}} \tag{B7}$$

With this upper θ-limit, the solid angle is

$$\Omega=4\int_0^{\pi/2}d\varphi\int_0^{\arccos\sqrt{\frac{\beta^2\cos^2\varphi+\alpha^2\sin^2\varphi}{\alpha^2\beta^2+\beta^2\cos^2\varphi+\alpha^2\sin^2\varphi}}}\sin\theta d\theta. \tag{B8}$$

the substitution $\cos\theta=t$ gives a first integral

$$\begin{aligned}\Omega&=4\int_0^{\pi/2}\left(1-\sqrt{\frac{\beta^2\cos^2\varphi+\alpha^2\sin^2\varphi}{\alpha^2\beta^2+\beta^2\cos^2\varphi+\alpha^2\sin^2\varphi}}\right)d\varphi\\&=2\pi-4\int_0^{\pi/2}\sqrt{\frac{\beta^2\cos^2\varphi+\alpha^2\sin^2\varphi}{\alpha^2\beta^2+\beta^2\cos^2\varphi+\alpha^2\sin^2\varphi}}d\varphi=2\pi-\bar\Omega,\end{aligned} \tag{B9}$$

where we have defined the complementary solid angle

$$\bar\Omega\equiv 4\int_0^{\pi/2}\sqrt{\frac{\beta^2\cos^2\varphi+\alpha^2\sin^2\varphi}{\alpha^2\beta^2+\beta^2\cos^2\varphi+\alpha^2\sin^2\varphi}}d\varphi=4\int_0^{\pi/2}\sqrt{\frac{\xi^2\cos^2\varphi+\sin^2\varphi}{\beta^2+\xi^2\cos^2\varphi+\sin^2\varphi}}d\varphi. \tag{B10}$$

[A square root seems to be missing in Masket's [8, (34)] at this point.] Substituting $\sin\varphi=u$, then $u^2=v$, then [17, 256.02], this is a Complete Elliptic Integral of the Third Kind,

$$\begin{aligned}\bar\Omega&=4\int_0^1\frac{1}{\sqrt{1-u^2}}\sqrt{\frac{\xi^2(1-u^2)+u^2}{\beta^2+\xi^2(1-u^2)+u^2}}du=4\int_0^1\frac{1}{\sqrt{1-u^2}}\sqrt{\frac{\xi^2+(1-\xi^2)u^2}{\beta^2+\xi^2+(1-\xi^2)u^2}}du\\&=2\int_0^1\frac{1}{\sqrt v}\frac{1}{\sqrt{1-v}}\sqrt{\frac{\xi^2+(1-\xi^2)v}{\xi^2+\beta^2+(1-\xi^2)v}}dv=2\int_0^1\frac{1}{\sqrt v}\frac{1}{\sqrt{1-v}}\sqrt{\frac{\frac{\xi^2}{1-\xi^2}+v}{\frac{\xi^2+\beta^2}{1-\xi^2}+v}}dv=4\frac{\xi^2}{\sqrt{\xi^2+\beta^2}}\Pi(\pi/2,1-\xi^2,k).\end{aligned} \tag{B11}$$

The squared modulus is $k^2=\frac{\beta^2(1-\xi^2)}{\beta^2+\xi^2}$. With the aid of [17, 412.01] or [17, 413.01] this could also be written as a linear combination of products of Elliptic Integrals of the First and Second Kind.

The circular limit $\xi\to1$, $k\to0$, $\Pi(\pi/2,0,0)\to\pi/2$, is $\bar\Omega\to2\pi/\sqrt{1+\beta^2}$, compatible with (A1).

For small β, ellipses far away, the Taylor expansions of the two factors in the final term of (B11) are

$$\frac{\xi^2}{\sqrt{\xi^2+\beta^2}}=\xi-\frac{1}{\xi}\beta^2+\frac{3}{8\xi^3}\beta^4-\frac{5}{16\xi^5}\beta^6+O(\beta^8) \tag{B12}$$

and [17, 906.00]

$$\Pi(\pi/2,1-\xi^2,k)=\frac{\pi}{2\xi}+\frac{\pi}{4(1-\xi^2)}[\frac{1}{\xi}-1]k^2+\frac{3\pi}{32(1-\xi^2)^2}[\frac{2}{\xi}-2-(1-\xi^2)]k^4+O(k^6) \tag{B13}$$

compatible with (A2) if $\xi\to1$.

Appendix C: Solid Angle of the Segmented Circle

A mixed variant of the geometry occurs if a circle of radius R is d away from the observer and two crossed lanes of width g are chopped off, such that four isolated segments remain in the four quadrants: Figure 6. The rotational

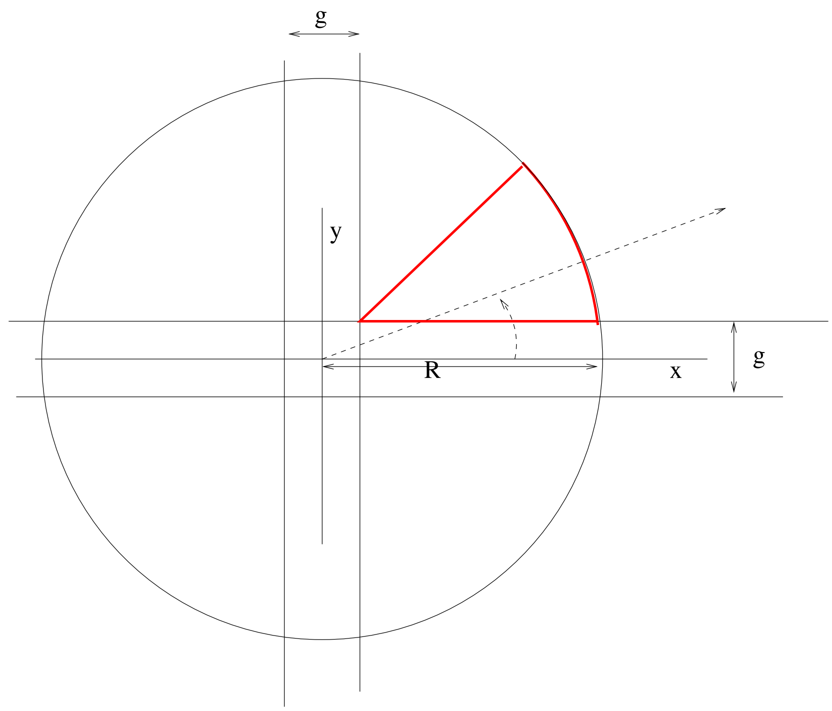

FIG. 6. Segmented circle of radius R and a vertical and horizontal gap of width g.

and mirror symmetry of the figure allows to consider eight times the solid angle of the red pie-shaped subarea of the octant. The angle φ (dashed angle in Figure 6) is kept smaller than 45°:

$$\Omega = 8 \int_{0 \le \varphi \le \pi/4} d\varphi \int_{0 \le \theta} \sin\theta d\theta. \tag{C1}$$

In (7) the gap sets constraints $y \ge g/2$, $x \ge g/2$ in the first quadrant, so the azimuth φ is

$$\varphi \ge \arctan \frac{g/2}{\sqrt{R^2 - (g/2)^2}}. \tag{C2}$$

The upper limit of θ (which is the radial coordinate in the figure) is reached at the rim of the circle where $x^2 + y^2 = R^2$ in (7):

$$\theta \le \arccos \frac{d}{\sqrt{d^2 + R^2}}. \tag{C3}$$

The lower limit of θ is produced where the dashed arrow hits the line $y = g/2$; the distance f between the circle center and the point of intersection is $\sin\varphi = (g/2)/f$, and f^2 is substituted for $x^2 + y^2$ in (7):

$$\theta \ge \arccos \frac{d}{\sqrt{d^2 + (\frac{g}{2\sin\varphi})^2}}. \tag{C4}$$

The detailed write-up of (C1) for the geometry of Figure 6, 8 times the solid angle of the red area, becomes

$$\Omega = 8\int_{\arctan\frac{g/2}{\sqrt{R^2-(g/2)^2}}}^{\pi/4} d\varphi \int_{\arccos\frac{d}{\sqrt{d^2+(\frac{g}{2\sin\varphi})^2}}}^{\arccos\frac{d}{\sqrt{d^2+R^2}}} \sin\theta d\theta = -8\int_{\arctan\frac{g/2}{\sqrt{R^2-(g/2)^2}}}^{\pi/4} d\varphi \int_{\frac{d}{\sqrt{d^2+(\frac{g}{2\sin\varphi})^2}}}^{\frac{d}{\sqrt{d^2+R^2}}} dt$$
$$= -8\int_{\arctan\frac{g/2}{\sqrt{R^2-(g/2)^2}}}^{\pi/4} [\frac{d}{\sqrt{d^2+R^2}} - \frac{d}{\sqrt{d^2+(\frac{g}{2\sin\varphi})^2}}]d\varphi$$
$$= -8\left[\frac{d}{\sqrt{d^2+R^2}}\left\{\frac{\pi}{4} - \arctan\frac{g/2}{\sqrt{R^2-(g/2)^2}}\right\} - \int_{\arctan\frac{g/2}{\sqrt{R^2-(g/2)^2}}}^{\pi/4} \frac{d}{\sqrt{d^2+(\frac{g}{2\sin\varphi})^2}}d\varphi\right]. \quad \text{(C5)}$$

We define the two length ratios $0 \le \epsilon \equiv g/(2R) \le 1$ and $0 \le \gamma \equiv R/d$ to simplify the notation. The substitution $\varphi' \equiv \pi/2 - \varphi$ leads with (20) to:

$$\Omega = 8\left[\frac{1}{\sqrt{1+\gamma^2}}\left\{\arctan\frac{\epsilon}{\sqrt{1-\epsilon^2}} - \frac{\pi}{4}\right\} - \int_{\pi/2-\arctan\frac{\epsilon}{\sqrt{1-\epsilon^2}}}^{\pi/4} \frac{d}{\sqrt{d^2+(\frac{g}{2\cos\varphi'})^2}}d\varphi'\right]$$
$$= 8\left[\frac{1}{\sqrt{1+\gamma^2}}\left\{\arcsin\epsilon - \frac{\pi}{4}\right\} - I(\frac{g}{2d},\pi/4) + I(\frac{g}{2d},\pi/2-\arctan\frac{\epsilon}{\sqrt{1-\epsilon^2}})\right]$$
$$= 8\left[\frac{1}{\sqrt{1+\gamma^2}}\left\{\arcsin\epsilon - \frac{\pi}{4}\right\} - \arcsin\frac{\sin\pi/4}{\sqrt{1+(\epsilon\gamma)^2}} + \arcsin\frac{\sin(\pi/2-\arctan\frac{\epsilon}{\sqrt{1-\epsilon^2}})}{\sqrt{1+(\epsilon\gamma)^2}}\right]$$
$$= 8\left[\frac{1}{\sqrt{1+\gamma^2}}\left\{\arcsin\epsilon - \frac{\pi}{4}\right\} - \arcsin\frac{1}{\sqrt{2}\sqrt{1+(\epsilon\gamma)^2}} + \arcsin\frac{\cos\arctan\frac{\epsilon}{\sqrt{1-\epsilon^2}}}{\sqrt{1+(\epsilon\gamma)^2}}\right]$$
$$= 8\left[\frac{1}{\sqrt{1+\gamma^2}}\left\{\arcsin\epsilon - \frac{\pi}{4}\right\} - \arcsin\frac{1}{\sqrt{2}\sqrt{1+(\epsilon\gamma)^2}} + \arcsin\frac{\sqrt{1-\epsilon^2}}{\sqrt{1+(\epsilon\gamma)^2}}\right]$$
$$= -\frac{4}{\sqrt{1+\gamma^2}}\arcsin(1-2\epsilon^2) - 8\arcsin\frac{1}{\sqrt{2}\sqrt{1+(\epsilon\gamma)^2}} + 8\arcsin\frac{\sqrt{1-\epsilon^2}}{\sqrt{1+(\epsilon\gamma)^2}}$$
$$= -\frac{4}{\sqrt{1+\gamma^2}}\arcsin(1-2\epsilon^2) - 8\arctan\frac{1}{\sqrt{1+2(\epsilon\gamma)^2}} + 8\arctan\frac{\sqrt{1-\epsilon^2}}{\epsilon\sqrt{1+\gamma^2}}$$
$$= -\frac{4}{\sqrt{1+\gamma^2}}\arcsin(1-2\epsilon^2) + 8\arctan\frac{\sqrt{1-\epsilon^2}\sqrt{1+2(\epsilon\gamma)^2} - \epsilon\sqrt{1+\gamma^2}}{\sqrt{1-\epsilon^2} + \sqrt{1+2(\epsilon\gamma)^2}\epsilon\sqrt{1+\gamma^2}}. \quad \text{(C6)}$$

The Taylor expansion of this expression for small ϵ is

$$\Omega = 2\pi(1-\frac{1}{\sqrt{1+\gamma^2}}) - 8\frac{\gamma^2}{\sqrt{1+\gamma^2}}\epsilon + 4\gamma^2\epsilon^2 + \frac{4\gamma^2(2\gamma^2+1)}{3\sqrt{1+\gamma^2}}\epsilon^3 - 4\gamma^4\epsilon^4 + \cdots. \quad \text{(C7)}$$

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