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On A Weaker Form Of Complete Irresoluteness

Erdal Ekici and Saeid Jafari

ABSTRACT: The aim of this paper is to present a new class of complete irresoluteness. The notion of completely δ-semi-irresolute functions is introduced and studied.

Key Words: irresolute function, δ-semiopen set, regular open set.

Contents

1. Introduction

In the literature, many authors investigated and studied various types of continuities. Continuity is one of the most basic and important notion of the whole mathematics an in special General Topology. Navalagi [10] introduced the notion of complete-α-irresoluteness as a new form of continuity. In this paper, we offer a weaker form of complete irresoluteness called complete δ-semi-irresoluteness. We also study some of the properties of completely δ-semi-irresolute functions.

For a subset A of a topological space X, the closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

Definition 1.1. *A subset A of a space X is said to be*
(1) regular open [20] if $A = int(cl(A))$,
(2) semi-open [7] if $A \subset cl(int(A))$,
(3) α-open [13] if $A \subset int(cl(int(A)))$.

The complement of a regular open (resp. α-open) set is said to be regular closed (resp. α-closed). The collection of all regular open sets forms a base for a topology τ_s. When $\tau = \tau_s$, the space (X, τ) is called semi-regular [20]. The δ-interior of a subset A of X is the union of all regular open sets of X contained in A and is denoted by δ-$int(A)$ [21]. A subset A is called δ-open [21] if $A = \delta$-$int(A)$, i. e., a set is δ-open if it is the union of regular open sets. The complement of δ-open set is called δ-closed. δ-closure of a set A is denoted by δ-$cl(A)$ and defined by δ-$cl(A) = \{x \in X : A \cap int(cl(U)) \neq \emptyset, U \in \tau \text{ and } x \in U\}$.

A subset S of a topological space X is said to be δ-semiopen [16] if $S \subset cl(\delta$-$int(S))$. The complement of a δ-semiopen set is called a δ-semiclosed set [16].

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The union (resp. intersection) of all δ-semiopen (resp. δ-semiclosed) sets, each contained in (resp. containing) a set S in a topological space X is called the δ-semi-interior (resp. δ-semiclosure) of S and it is denoted by δ-$sint(S)$ (resp. δ-$scl(S)$) [16].

For a function $f : X \to Y$, a function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is called the graph function of f. The subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$. The family of all regular open (resp. δ-semiopen) sets of a space (X, τ) is denoted by $RO(X)$ (resp. $\delta SO(X)$).

2. Completely δ-semi-irresolute functions

Definition 2.1. *A function $f : X \to Y$ is said to be completely δ-semi-irresolute if the inverse image of each δ-semiopen set V of Y is regular open set in X.*

Theorem 2.1. *The following are equivalent for a function $f : X \to Y$:*
(1) f is completely δ-semi-irresolute,
(2) the invese image of each δ-semiclosed set V of Y is regular closed set in X.

Proof. Obvious. □

Definition 2.2. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be completely irresolute [10] if $f^{-1}(V)$ is regular open in (X, τ) for every semi-open set V in Y.*

Remark 2.1. *The following implication hold for a function $f : X \to Y$:*

completely irresolute $\Rightarrow$ completely δ-semi-irresolute

Observe that the converse is not true as shown by the following example.

Example 2.1. *Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be the identity function. Then f is completely δ-semi-irresolute but it is not completely irresolute.*

Theorem 2.2. *The following are equivalent for a function $f : X \to Y$, where Y is a semi-regular space:*
(1) f is completely δ-semi-irresolute,
(2) f is completely irresolute.

Definition 2.3. *A subset A of a topological space (X, τ) is called simply open [12] if $A = U \cup N$, where U is open and N is nowhere dense.*

Theorem 2.3. *For a topological space (Y, σ), suppose that one of the following conditions satisfy:*
(1) every simple open set is semi-closed,
(2) every open set is regular open,
(3) Y is locally indiscrete,
(4) every simple open set is α-closed,
Then a function $f : (X, \tau) \to (Y, \sigma)$ is completely δ-semi-irresolute if and only if it is completely irresolute.

Proof. It follows from Proposition 2.6 in [3] and Theorem 3.3 in [6]. □

Definition 2.4. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be*

(1) R-map [2] if $f^{-1}(V)$ is regular open in (X, τ) for every regular open set V in Y.

(2) completely continuous [1] if $f^{-1}(V)$ is regular open in (X, τ) for every open set V in Y.

(3) δ-semicontinuous [5] if $f^{-1}(V)$ is δ-semiopen in X for every open set V of Y.

Theorem 2.4. *Let $f : X \to Y$ and $g : Y \to Z$ be functions.The following properties hold:*

(1) If f is R-map and g is completely δ-semi-irresolute, then $g \circ f : X \to Z$ is completely δ-semi-irresolute.

(2) If f is completely continuous and g is completely δ-semi-irresolute, then $g \circ f : X \to Z$ is completely δ-semi-irresolute.

(3) If f is R-map and g is completely irresolute, then $g \circ f : X \to Z$ is completely δ-semi-irresolute.

(4) If f is completely continuous and g is completely irresolute, then $g \circ f : X \to Z$ is completely δ-semi-irresolute.

(5) If f is completely δ-semi-irresolute and g is δ-semicontinuous, then $g \circ f : X \to Z$ is completely continuous.

Lemma 2.1. *([15]) Let S be an open subset of a space (X, τ).*

(1) If U is regular open set in X, then so is $U \cap S$ in the subspace (S, τ_S).

(2) If B $(\subset S)$ is regular open in (S, τ_S), then there is a regular open set U in (X, τ) such that $B = U \cap S$.

Theorem 2.5. *If a function $f : (X, \tau) \to (Y, \sigma)$ is completely δ-semi-irresolute and A is open in (X, τ), then the restriction $f\,|_A : (A, \tau_A) \to (Y, \sigma)$ is completely δ-semi-irresolute.*

Proof. Let V be any δ-semiopen set of (Y, σ). Since f is completely δ-semi-irresolute $f^{-1}(V)$ is regular open in X. By Lemma 2.1, $f^{-1}(V) \cap A$ is regular open in the subspace (A, τ_A) and $f^{-1}(V) \cap A = (f\,|_A)^{-1}(V)$. This shows that $f\,|_A$ is completely δ-semi-irresolute. □

Theorem 2.6. *A function $f : X \to Y$ is completely δ-semi-irresolute if the graph function $g : X \to X \times Y$ is completely δ-semi-irresolute.*

Proof. Let $x \in X$ and V be a δ-semiopen set containing $f(x)$. Then $X \times V$ is a δ-semiopen set of $X \times Y$ containing $g(x)$. Therefore, $g^{-1}(X \times V) = f^{-1}(V)$ is a regular open set containing x. This shows that f is completely δ-semi-irresolute. □

3. Further properties

Let $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ be two families of topological spaces with the same index set I. The product space of $\{X_i : i \in I\}$ is denoted by $\prod_{i\in I} X_i$. Let $f_i : X_i \to Y_i$ be a function for each $i \in I$. The product function $f : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ is defined by $f((x_i)) = (f(x_i))$ for each $(x_i) \in \prod_{i\in I} X_i$.

Theorem 3.1. *If the function $f : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ is completely δ-semi-irresolute, then $f_i : X_i \to Y_i$ is completely δ-semi-irresolute for each $i \in I$.*

Proof. Let k be an arbitrarily fixed index and V_k any δ-semiopen set of Y_k. Then $\prod Y_j \times V_k$ is δ-semiopen in $\prod Y_i$, where $j \in I$ and $j \neq k$, and hence $f^{-1}(\prod Y_j \times V_k) = \prod Y_j \times f_k^{-1}(V_k)$ is regular open in $\prod X_i$. Thus, $f_k^{-1}(V_k)$ is regular open in X_k and hence f_k is completely δ-semi-irresolute. □

Theorem 3.2. *A space X is connected if every completely δ-semi-irresolute function from a space X into any T_0-space Y is constant.*

Proof. Suppose that X is not connected and that every completely δ-semi-irresolute function from X into Y is constant. Since X is not connected, there exists a proper nonempty clopen subset A of X. Let $Y = \{a, b\}$ and $\tau = \{Y, \varnothing, \{a\}, \{b\}\}$ be a topology for Y. Let $f : X \to Y$ be a function such that $f(A) = \{a\}$ and $f(X\backslash A) = \{b\}$. Then f is non-constant and completely δ-semi-irresolute such that Y is T_0, which is a contradiction. Hence, X must be connected. □

Definition 3.1. *A space X is said to be s-normal [9] if every pair of nonempty disjoint closed sets can be separeted by disjoint semiopen sets.*

Lemma 3.1. *([14]) For a space X, the following properties are equivalent:*

(1) X is s-normal,

(2) every pair of nonempty disjoint closed sets can be separeted by disjoint δ-semiopen sets.

Theorem 3.3. *If Y is s-normal and $f : X \to Y$ is a completely δ-semi-irresolute closed injection, then X is normal.*

Proof. Let K and L be disjoint nonempty closed sets of X. Since f is injective and closed, $f(K)$ and $f(L)$ are disjoint closed sets of Y. Since Y is s-normal, there exist δ-semiopen sets U and V such that $f(K) \subset U$ and $f(L) \subset V$ and $U \cap V = \varnothing$. Then, f is completely δ-semi-irresolute, $f^{-1}(U)$, $f^{-1}(V) \in RO(X)$. Since $K \subset f^{-1}(U)$, $L \subset f^{-1}(V)$, and $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint, X is normal. □

Definition 3.2. *A topological space (X, τ) is said to be semi-T_2 [8] if for each distinct points x, $y \in X$, there exist semiopen sets U and V containing x and y, respectively, such that $U \cap V = \emptyset$.*

Lemma 3.2. *([14]) For a space X, the following properties are equivalent:*
(1) X is semi-T_2,
(2) for each distinct points x, $y \in X$, there exist U, $V \in \delta SO(X)$ containing x and y, respectively, such that $U \cap V = \emptyset$

Theorem 3.4. *If $f : (X, \tau) \to (Y, \sigma)$ is a completely δ-semi-irresolute injection and (Y, σ) is semi-T_2, then (X, τ) is T_2.*

Proof. Let x, y be any distinct points of X. Then $f(x) \neq f(y)$. Since (Y, σ) is semi-T_2, there exist δ-semiopen sets U, V in Y containing $f(x)$ and $f(y)$, respectively, such that $U \cap V = \emptyset$. Since f is completely δ-semi-irresolute, $f^{-1}(U)$, $f^{-1}(V) \in RO(X)$ containing x and y, respectively, such that $f^{-1}(U) \cap f^{-1}(V) = \varnothing$. Hence, (X, τ) is T_2. □

Definition 3.3. *A space X is said to be semi-connected [17] (resp. δ-semiconnected [5], r-connected [11]) if it cannot be expressed as the union of two non-empty disjoint semiopen (resp. δ-semiopen, regular open) sets.*

Lemma 3.3. *([14]) For a space X, the following properties are equivalent:*
(1) X is semi-connected,
(2) X cannot be expressed as the union of two nonempty disjoint δ-semiopen sets.

The Lemma 2.3 shows that a space X is semi-connected if and only if it is δ-semiconnected.

Theorem 3.5. *If $f : X \to Y$ is a completely δ-semi-irresolute surjection and X is r-connected (resp. connected), then Y is semi-connected.*

Proof. Suppose that Y is not semi-connected. There exist nonempty δ-semiopen sets V_1 and V_2 of Y such that $Y = V_1 \cup V_2$ and $V_1 \cap V_2 = \varnothing$. Since f is completely δ-semi-irresolute, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ nonempty regular open sets of X such that $X = f^{-1}(V_1) \cup f^{-1}(V_2)$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \varnothing$. Thus, X is not r-connected. This is a contradiction. □

Definition 3.4. *A space X is said to be*
(1) nearly compact [18] if every regular open cover of X has a finite subcover,
(2) δ-semi-compact [4, 5] if every δ-semiopen cover of X has a finite subcover.

Theorem 3.6. *If $f : X \to Y$ is a completely δ-semi-irresolute surjection and X is nearly compact, then Y is δ-semi-compact.*

Proof. Let $\{V_i : \alpha \in I\}$ be a δ-semiopen cover of Y. Since f is completely δ-semi-irresolute $\{f^{-1}(V_i) : i \in I\}$ is a regular open cover of X. Since X is nearly compact, there exists a finite subset I_0 of I such that $X = \{f^{-1}(V_i) : i \in I_0\}$. Therefore, we obtain $Y = \{V_i : i \in I_0\}$ and hence Y is δ-semi-compact. □

A space X is called hyperconnected [19] if every open subset of X is dense.

Theorem 3.7. *Completely δ-semi-irresolute images of hyperconnected spaces are semi-connected.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be completely δ-semi-irresolute such that X is hyperconnected. Assume that B is a proper δ-semi-clopen subspace of Y. Then $A = f^{-1}(B)$ is both regular open and regular closed as f is completely δ-semi-irresolute. This clearly contradicts the fact that X is hyperconnected. Thus, Y is semi-connected. □

Definition 3.5. *A graph $G(f)$ of a function $f : X \to Y$ is said to be r-δ_s-graph if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist a regular open set U in X containing x and an δ-semiopen V containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Proposition 3.1. *The following properties are equivalent for a graph $G(f)$ of a function f:*

(1) $G(f)$ is r-δ_s-graph,

(2) for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist a regular open set U in X containing x and a δ-semiopen V containing y such that $f(U) \cap V = \emptyset$.

Theorem 3.8. *If $f : (X, \tau) \to (Y, \sigma)$ is completely δ-semi-irresolute and Y is semi-T_2, $G(f)$ is r-δ_s-graph in $X \times Y$.*

Proof. Suppose that Y is semi-T_2. Let $(x, y) \in (X \times Y) \backslash G(f)$. It follows that $f(x) \neq y$. Since Y is semi-T_2, there exist δ-semiopen sets U and V containing $f(x)$ and y, respectively, such that $U \cap V = \emptyset$. Since f is completely δ-semi-irresolute, $f^{-1}(U) = G$ is a regular open set containing x. Therefore, $f(G) \cap V = \emptyset$ and $G(f)$ is r-δ_s-graph in $X \times Y$. □

Theorem 3.9. *Let $f : (X, \tau) \to (Y, \sigma)$ has the r-δ_s-graph. If f is injective, then X is T_1.*

Proof. Let x and y be any two distinct points of X. Then, we have $(x, f(y)) \in (X \times Y) \backslash G(f)$. Then, there exist a regular open set U in X containing x and a δ-semiopen set V containing $f(y)$ such that $f(U) \cap V = \emptyset$; hence $U \cap f^{-1}(V) = \emptyset$. Therefore, we have $y \notin U$. This implies that X is T_1. □

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Erdal Ekici
Department of Mathematics,
Canakkale Onsekiz Mart University,
Terzioglu Campus,
17020 Canakkale -TURKEY
eekici@comu.edu.tr

Saeid Jafari
College of Vestsjaelland South,
Herrestraede 11,
4200 Slagelse, Denmark
jafari@stofanet.dk