ON $\mathcal{DS}^*$-SETS AND DECOMPOSITIONS OF CONTINUOUS FUNCTIONS

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Abstract

In this paper, the notions of $\mathcal{DS}^*$-sets and $\mathcal{DS}^*$-continuous functions are introduced and their properties and their relationships with some other types of sets are investigated. Moreover, some new decompositions of continuous functions are obtained by using $\mathcal{DS}^*$-continuous functions, $\mathcal{DS}$-continuous functions and $\mathcal{D}$-continuous functions.

Key words and phrases: $\mathcal{DS}^*$-sets, $\mathcal{DS}^*$-continuous function, decomposition.

MSC: 54C08.

1 Introduction

In a recent paper, Ekici and Jafari [12] have studied $\mathcal{DS}$-sets and $\mathcal{D}$-sets and obtained some decompositions of continuous functions via $\mathcal{DS}$-continuous functions and $\mathcal{D}$-continuous functions. In this paper, we introduce a new class of sets called $\mathcal{DS}^*$-sets. Properties of this class are investigated. Furthermore, the notion of $\mathcal{DS}^*$-continuous functions is introduced via $\mathcal{DS}^*$-sets to establish some new decompositions of continuous functions. On the other hand, by using $\mathcal{DS}$-sets and $\mathcal{D}$-sets, other new decompositions of continuous functions are obtained.

In this paper (X, τ) and (Y, σ) represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. A subset A of a space X is called regular open (resp regular closed) [22] if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A is called δ-open [24] if for each $x \in A$, there exists a regular open set U such that $x \in U \subset A$. A is called δ-closed if its complement is δ-open. A point $x \in X$ is called a δ-cluster

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point of A if $A \cap int(cl(U)) \neq \varnothing$ for each open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by δ-$cl(A)$. The union of all regular open sets, each contained in A called the δ-interior of A and is denoted by δ-$int(A)$. A subset A of a space (X, τ) is called semiopen [15] (resp. semi-regular [7], α-open [19], preopen [16] or locally dense [6], b-open [4] or γ-open [13] or sp-open [8], β-open [1] or semi-preopen [3], δ-semiopen [20], δ-preopen [21]) if $A \subset cl(int(A))$ (resp. semiopen and semiclosed, $A \subset int(cl(int(A)))$, $A \subset int(cl(A))$, $A \subset int(cl(A)) \cup cl(int(A))$, $A \subset cl(int(cl(A)))$, $A \subset cl(\delta\text{-}int(A))$, $A \subset int(\delta\text{-}cl(A))$). The complement of a δ-semiopen (resp. semiopen) set is called a δ-semiclosed (resp. semiclosed) set. The union (resp. intersection) of all δ-preopen (resp. δ-semiclosed) sets, each contained in (resp. containing) a set A in a topological space X is called the δ-preinterior (resp. δ-semiclosure) of A and it is denoted by δ-$pint(A)$ (resp. δ-$scl(A)$).

Definition 1 *A subset A of a space (X, τ) is called*

(1) a $\mathcal{D}$-set [12] if $A = U \cap V$, where U is open and V is δ-closed,

(2) a $\mathcal{DS}$-set [12] if $A = U \cap V$, where U is open and V is δ-semiclosed,

(3) a $\mathcal{B}$-set [23] if $A \in \mathcal{B}(X) = \{U \cap V : U \in \tau,\ int(cl(V)) \subset V\}$,

(4) an $\mathcal{AB}$-set [9] if $A \in \mathcal{AB}(X) = \{U \cap V : U \in \tau$ and V is semi-regular$\}$.

The family of all $\mathcal{DS}$-sets (resp. $\mathcal{D}$-sets) of a topological space X will be denoted by $\mathcal{DS}(X)$ (resp. $\mathcal{D}(X)$). A topological space X is called a locally indiscrete [10] if every open subset of X is closed and called submaximal [5] if every dense subset of X is open.

Definition 2 *A function $f : X \rightarrow Y$ is called*

(1) β-continuous [1] if $f^{-1}(A)$ is β-open for each $A \in \sigma$.

(2) α-continuous [17] if $f^{-1}(A)$ is α-open for each $A \in \sigma$.

(3) γ-continuous [13] if $f^{-1}(A)$ is γ-open for each $A \in \sigma$.

(4) quasi-continuous [14] if $f^{-1}(A)$ is semiopen for each $A \in \sigma$.

(5) precontinuous [16] if $f^{-1}(A)$ is preopen for each $A \in \sigma$.

(6) δ-almost continuous [21] if $f^{-1}(A)$ is δ-preopen for each $A \in \sigma$.

(7) δ-semicontinuous [11] if $f^{-1}(A)$ is δ-semiopen for each $A \in \sigma$.

(8) super-continuous [18] if $f^{-1}(A)$ is δ-open for each $A \in \sigma$.

2 $\mathcal{DS}^*$-sets in topological spaces

Definition 3 *A subset A of a topological space X is called a $\mathcal{DS}^*$-set if $A = U \cap V$, where U is open and V is δ-semiclosed and $int(\delta\text{-}cl(V)) = cl(\delta\text{-}int(V))$.*

The family of all $\mathcal{DS}^*$-sets of a topological space X will be denoted by $\mathcal{DS}^*(X)$.

Remark 4 *The following diagram holds for a subset of a space X:*

$$\begin{array}{c} \mathcal{B}\text{-set} \\ \uparrow \\ \mathcal{DS}\text{-set} \\ \uparrow \\ \mathcal{DS}^*\text{-set} \end{array}$$

The following example shows that first implication is not reversible. The other example is as in [12].

Example 5 *Let $X = \{a, b, c, d\}$ and let $\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$. The set $\{a, c, d\}$ is a $\mathcal{DS}$-set but it is not a $\mathcal{DS}^*$-set.*

Remark 6 *Every open set is a $\mathcal{DS}^*$-set. The converse is not true.*

Example 7 *Let $X = \{a, b, c, d\}$ and let $\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$. The set $\{a, c\}$ is a $\mathcal{DS}^*$-set but it is not open.*

Theorem 8 *The following are equivalent for a subset A of a space X:*
(1) A is open,
(2) A is α-open and a $\mathcal{DS}^*$-set,
(3) A is semiopen and a $\mathcal{DS}^*$-set,
(4) A is preopen and a $\mathcal{DS}^*$-set,
(5) A is γ-open and a $\mathcal{DS}^*$-set.
(6) A is β-open and a $\mathcal{DS}^*$-set.

Proof. $(1) \Rightarrow (2)$: It follows from the fact that every open set is α-open and a $\mathcal{DS}^*$-set.
$(2) \Rightarrow (3) \Rightarrow (5)$: Obvious.
$(2) \Rightarrow (4) \Rightarrow (5)$: Obvious.

(5) $\Rightarrow$ (6) : Obvious.

(6) $\Rightarrow$ (1) : Let A be β-open and a $\mathcal{DS}^*$-set. Since A is β-open, $A \subset cl(int(cl(A)))$. Since A is a $\mathcal{DS}^*$-set, then $A = U \cap V$, where U is open and V is δ-semiclosed and $int(\delta\text{-}cl(V)) = cl(\delta\text{-}int(V))$. Also, by δ-semiclosedness of V, we have $\delta\text{-}int(V) = \delta\text{-}int(\delta\text{-}cl(V))$. Furthermore, we obtain

$$\begin{aligned} A = A \cap U \subset cl(int(cl(A))) \cap U \quad &= cl(int(cl(U \cap V))) \cap U \\ &\subset cl(int(cl(U))) \cap cl(int(cl(V))) \cap U \\ &= cl(int(cl(V))) \cap U \\ &\subset cl(int(\delta\text{-}cl(V))) \cap U \\ &= cl(\delta\text{-}int(V)) \cap U \\ &= int(\delta\text{-}cl(V)) \cap U \\ &= \delta\text{-}int(V) \cap U. \end{aligned}$$

Thus, $A = \delta\text{-}int(V) \cap U$ and hence A is open. ■

Theorem 9 *The following are equivalent for a space X:*

(1) X is indiscrete,

(2) the $\mathcal{DS}^*$-sets in X are the trivial ones.

Proof. Since every $\mathcal{DS}^*$-set is $\mathcal{DS}$-set, by Theorem 16 [12], the proof is completed. ■

Theorem 10 *Let X be a topological space and $A \subset X$. If $A \in \mathcal{DS}(X)$, then $\delta\text{-}pint(A) = int(A)$.*

Proof. Let $A \in \mathcal{DS}(X)$. Then, $A = U \cap V$, where U is open and V is δ-semiclosed. Since V is δ-semiclosed, then we have $\delta\text{-}int(V) = \delta\text{-}int(\delta\text{-}cl(V))$. Moreover, we obtain

$$\begin{aligned} \delta\text{-}pint(A) = A \cap \delta\text{-}int(\delta\text{-}cl(A)) \quad &\subset U \cap \delta\text{-}int(\delta\text{-}cl(V)) \\ &= U \cap \delta\text{-}int(V) \\ &\subset U \cap int(V) \\ &= int(A). \end{aligned}$$

Thus, $\delta\text{-}pint(A) = int(A)$. ■

Theorem 11 *The following are equivalent for a subset A of a space X:*

(1) A is open,

(2) A is δ-preopen and a $\mathcal{D}$-set,

(3) A is δ-preopen and a $\mathcal{DS}$-set.

Proof. (1) $\Rightarrow$ (2) : Since every open set is δ-preopen and a $\mathcal{D}$-set, it is completed.

(2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Let A be δ-preopen and a $\mathcal{DS}$-set. By Theorem 10, δ-$pint(A) = int(A)$. Also, since A is δ-preopen, $A = \delta$-$pint(A) = int(A)$. Thus, A is open. ■

Theorem 12 *Let X be a topological space and $A \subset X$. If $A \in \mathcal{DS}^*(X)$, then $A = U \cap \delta$-$scl(A)$ for some open set U.*

Proof. Let $A \in \mathcal{DS}^*(X)$. This implies that $A = U \cap V$, where U is open and V is δ-semiclosed and $int(\delta$-$cl(V)) = cl(\delta$-$int(V))$. Since $A \subset V$, δ-$scl(A) \subset \delta$-$scl(V) = V$. Moreover, $U \cap \delta$-$scl(A) \subset U \cap V = A \subset U \cap \delta$-$scl(A)$ and hence $A = U \cap \delta$-$scl(A)$. ■

Theorem 13 *Let X be a topological space and $A \subset X$. If β-open and a $\mathcal{DS}^*$-set, then it is an $\mathcal{AB}$-set.*

Proof. Let A be β-open and a $\mathcal{DS}^*$-set. Since A is a $\mathcal{DS}$-set, by Theorem 11 [12], A is an $\mathcal{AB}$-set. ■

Definition 14 *Let X be a topological space and $A \subset X$. Then A is called a δ^*-set if δ-$int(A)$ is δ-closed.*

Theorem 15 *Let X be a topological space and $A \subset X$. If A is a δ^*-set and δ-semiopen, then it is δ-open.*

Proof. Let A be a δ^*-set and δ-semiopen. Then $A \subset cl(\delta$-$int(A)) = \delta$-$int(A)$. Thus, A is δ-open. ■

Theorem 16 *Let X be a topological space and $A \subset X$. Then A is open if A is a δ-semiopen $\mathcal{DS}^*$-set and A is preopen or a δ^*-set.*

Proof. Let A be a δ-semiopen $\mathcal{DS}^*$-set. Suppose that A is preopen or a δ^*-set. If A is a preopen $\mathcal{DS}^*$-set, then it is a preopen $\mathcal{B}$-set. So, by Proposition 9 [23], A is open. Also, if A is a δ^*-set and δ-semiopen, by Theorem 15, A is open. Thus, the proof is completed. ■

Theorem 17 *The following are equivalent for a space X :*

(1) X is a locally indiscrete space,

(2) every $\mathcal{DS}^*$-set is clopen,

(3) every $\mathcal{DS}^*$-set is closed.

Proof. (1) $\Rightarrow$ (2) : Let A be a $\mathcal{DS}^*$-set. Then there exist an open set U and a δ-semiclosed set V such that $A = U \cap V$ and $int(\delta\text{-}cl(V)) = cl(\delta\text{-}int(V))$. Since U is clopen, then A is semiclosed. By [2], since X is a locally indiscrete space, then A is clopen.

(2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Let $A \subset X$ be an open set. Since A is a $\mathcal{DS}^*$-set, then A is closed. Hence, X is a locally indiscrete space. ■

Theorem 18 *Let X be a topological space. Then X is submaximal if and only if every dense subset of X is a $\mathcal{DS}^*$-set.*

Proof. ($\Rightarrow$) : Let A be a dense subset of X. Since X submaximal, then A is open and so A is a $\mathcal{DS}^*$-set.

($\Leftarrow$) : Since every dense subset is a $\mathcal{DS}^*$-set and every $\mathcal{DS}^*$-set is a $\mathcal{DS}$-set, then by Theorem 17 [12], X is submaximal. ■

3 Some new decompositions of continuity

Definition 19 *A function $f : (X, \tau) \to (Y, \sigma)$ is called*

(1) $\mathcal{DS}^*$-continuous if $f^{-1}(V) \in \mathcal{DS}^*(X)$ for each $V \in \sigma$.

(2) δ^*-continuous if $f^{-1}(V)$ is a δ^*-set for each $V \in \sigma$.

Definition 20 *A function $f : (X, \tau) \to (Y, \sigma)$ is called*

(1) $\mathcal{D}$-continuous [12] if $f^{-1}(V) \in \mathcal{D}(X)$ for each $V \in \sigma$.

(2) $\mathcal{DS}$-continuous [12] if $f^{-1}(V) \in \mathcal{DS}(X)$ for each $V \in \sigma$.

(3) $\mathcal{AB}$-continuous [9] if $f^{-1}(V) \in \mathcal{AB}(X)$ for each $V \in \sigma$.

(4) $\mathcal{B}$-continuous [23] if $f^{-1}(V) \in \mathcal{B}(X)$ for each $V \in \sigma$.

Remark 21 *(1) The following diagram holds for a function $f : X \to Y$:*

$\mathcal{B}$-continuous

$\Uparrow$

$\mathcal{DS}$-continuous

$\Uparrow$

$\mathcal{DS}^*$-continuous

None of these implications is reversible as shown in the following example and in [12]:

Example 22 *Let* $X = Y = \{a, b, c, d\}$ *and* $\tau = \{\varnothing,\ X,\ \{a\},\ \{b\},\ \{a, b\}\}$, $\sigma = \{\varnothing,\ X,\ \{a\},\ \{c\},\ \{a, b\},\ \{a, c\},\ \{a, b, c\},\ \{a, c, d\}\}$. *Then the function* $f : (X, \tau) \to (Y, \sigma)$, *defined as:* $f(a) = c$, $f(b) = b$, $f(c) = c$, $f(d) = d$, *is* $\mathcal{DS}$*-continuous but it is not* $\mathcal{DS}^*$*-continuous.*

Remark 23 *Every continuous function is* $\mathcal{DS}^*$*-continuous but not conversely.*

Example 24 *Let* $X = Y = \{a, b, c, d\}$ *and* $\tau = \{\varnothing,\ X,\ \{a\},\ \{b\},\ \{a, b\}\}$, $\sigma = \{\varnothing,\ X,\ \{a\},\ \{c\},\ \{a, b\},\ \{a, c\},\ \{a, b, c\},\ \{a, c, d\}\}$. *Then the function* $f : (X, \tau) \to (Y, \sigma)$, *defined as:* $f(a) = c$, $f(b) = b$, $f(c) = c$, $f(d) = b$, *is* $\mathcal{DS}^*$*-continuous but it is not continuous.*

Theorem 25 *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a function. If* β*-continuous and* $\mathcal{DS}^*$*-continuous, then it is* $\mathcal{AB}$*-continuous.*

Proof. It follows from Theorem 13. ■

Theorem 26 *The following are equivalent for a function* $f : X \to Y$*:*

(1) f *is continuous,*
(2) f *is* α*-continuous and* $\mathcal{DS}^*$*-continuous,*
(3) f *is quasi-continuous and* $\mathcal{DS}^*$*-continuous,*
(4) f *is precontinuous and* $\mathcal{DS}^*$*-continuous,*
(5) f *is* γ*-continuous and* $\mathcal{DS}^*$*-continuous,*
(6) f *is* β*-continuous and* $\mathcal{DS}^*$*-continuous.*

Proof. It is an immediate consequence of Theorem 8. ■

Theorem 27 *The following are equivalent for a function* $f : X \to Y$*:*

(1) f *is continuous,*
(2) f *is* δ*-almost continuous and* $\mathcal{D}$*-continuous,*
(3) f *is* δ*-almost continuous and* $\mathcal{DS}$*-continuous.*

Proof. It follows from Theorem 11. ■

Theorem 28 *Let* $f : X \to Y$ *be a function. Then* f *is continuous if* f *is* δ*-semicontinuous,* $\mathcal{DS}^*$*-continuous and precontinuous or* δ^**-continuous.*

Proof. It is an immediate consequence of Theorem 16. ■

Theorem 29 *Let $f : X \to Y$ be a function. Then f is super-continuous if f is δ^*-continuous and δ-semicontinuous.*

Proof. It is an immediate consequence of Theorem 15. ■

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