ON A FINER TOPOLOGICAL SPACE THAN τ_θ AND SOME MAPS

E. Ekici, S. Jafari and R. M. Latif

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Abstract

In 1943, Fomin [7] introduced the notion of θ-continuity. In 1966, the notions of θ-open subsets, θ-closed subsets and θ-closure were introduced by Veličko [18] for the purpose of studying the important class of H-closed spaces in terms of arbitrary filterbases. He also showed that the collection of θ-open sets in a topological space (X, τ) forms a topology on X denoted by τ_θ (see also [12]). Dickman and Porter [4], [5], Joseph [11] continued the work of Veličko. Noiri and Jafari [15], Caldas et al. [1] and [2], Steiner [16] and Cao et al [3] have also obtained several new and interesting results related to these sets.
In this paper, we will offer a finer topology on X than τ_θ by utilizing the new notions of ω_θ-open and ω_θ-closed sets. We will also discuss some of the fundamental properties of such sets and some related maps.
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1 Introduction

In 1982, Hdeib [8] introduced the notion of ω-closedness by which he introduced and investigated the notion of ω-continuity. In 1943, Fomin [7] introduced the notion of θ-continuity. In 1966, the notions of θ-open subsets, θ-closed subsets and θ-closure were introduced by Veličko [18] for the purpose of studying the important class of H-closed spaces in terms of arbitrary filterbases. He also showed that the collection of θ-open sets in a topological space (X, τ) forms a topology on X denoted by τ_θ (see also [12]). Dickman and Porter [4], [5], Joseph [11] continued the work of Veličko. Noiri and Jafari [15], Caldas et al. [1] and [2], Steiner [16] and Cao et al [3] have also obtained several new and interesting results related to these sets. In this paper, we will offer a finer topology on X than τ_θ by utilizing the new notions of ω_θ-open and ω_θ-closed sets. We will also discuss some of the fundamental properties of such sets and some related maps.

Throughout this paper, by a space we will always mean a topological space. For a subset A of a space X, the closure and the interior of A will be denoted by $cl(A)$ and $int(A)$, respectively. A subset A of a space X is said to be α-open [14] (resp. preopen [13], regular open [17], regular closed [17]) if $A \subset int(cl(int(A)))$ (resp. $A \subset int(cl(A))$, $A = int(cl(A))$, $A = cl(int(A))$)

A point $x \in X$ is said to be in the θ-closure [18] of a subset A of X, denoted by θ-$cl(A)$, if $cl(U) \cap A \neq \varnothing$ for each open set U of X containing x. A subset A of a space X is called θ-closed if $A = \theta$-$cl(A)$. The complement of a θ-closed set is called θ-open. The θ-interior of a subset A of X is the union of all open sets of X whose closures are contained in A and is denoted by θ-$int(A)$. Recall that a point p is a condensation point of A if every open set containing p must contain uncountably many points of A. A subset A of a space X is ω-closed [8] if it contains all of its condensation points. The complement of an ω-closed subset is called ω-open. It was shown that the collection of all ω-open subsets forms a topology that is finer than the original topology on X. The union of all ω-open sets of X contained in a subset A is called the ω-interior of A and is denoted by ω-$int(A)$.

The family of all ω-open (resp. θ-open, α-open) subsets of a space (X, τ) is denoted by $\omega O(X)$ (resp, $\tau_\theta = \theta O(X)$, $\alpha O(X)$).

A function $f : X \to Y$ is said to be ω-continuous [9] (resp. θ-continuous [7]) if $f^{-1}(V)$ is ω-open (resp. θ-open) in X for every open subset V of Y. A function $f : X \to Y$ is called weakly ω-continuous [6] if for each $x \in X$ and each open subset V in Y containing $f(x)$, there exists an ω-open subset U in X containing x such that $f(U) \subset cl(V)$.

2 A finer topology than τ_θ

Definition 1 *A subset A of a space (X, τ) is called ω_θ-open if for every $x \in A$, there exists an open subset $B \subset X$ containing x such that $B \backslash \theta$-$int(A)$ is countable. The complement of an ω_θ-open subset is called ω_θ-closed.*

The family of all ω_θ-open subsets of a space (X, τ) is denoted by $\omega_\theta O(X)$.

Theorem 2 *$(X, \omega_\theta O(X))$ is a topological space for a topological space (X, τ).*

Proof. It is obvious that $\varnothing, X \in \omega_\theta O(X)$. Let A, $B \in \omega_\theta O(X)$ and $x \in A \cap B$. There exist open sets U, $V \subset X$ containing x such that $U \backslash \theta\text{-}int(A)$ and $V \backslash \theta\text{-}int(B)$ are countable. Then $(U \cap V) \backslash \theta\text{-}int(A \cap B) = (U \cap V) \backslash [\theta\text{-}int(A) \cap \theta\text{-}int(B)] \subset [(U \backslash \theta\text{-}int(A)) \cup (V \backslash \theta\text{-}int(B))]$. Thus, $(U \cap V) \backslash \theta\text{-}int(A \cap B)$ is countable and hence $A \cap B \in \omega_\theta O(X)$. Let $\{A_i : i \in I\}$ be a family of ω_θ-open subsets of X and $x \in \cup_{i \in I} A_i$. Then $x \in A_j$ for some $j \in I$. This implies that there exists an open subset B of X containing x such that $B \backslash \theta\text{-}int(A_j)$ is countable. Since $B \backslash \theta\text{-}int(\cup_{i \in I} A_i) \subset B \backslash \cup_{i \in I} \theta\text{-}int(A_i) \subset B \backslash \theta\text{-}int(A_j)$, then $B \backslash \theta\text{-}int(\cup_{i \in I} A_i)$ is countable. Hence, $\cup_{i \in I} A_i \in \omega_\theta O(X)$. ■

Theorem 3 *Let A be a subset of a space (X, τ). Then A is ω_θ-open if and only if for every $x \in A$, there exists an open subset U containing x and a countable subset V such that $U \backslash V \subset \theta\text{-}int(A)$.*

Proof. Let $A \in \omega_\theta O(X)$ and $x \in A$. Then there exists an open subset U containing x such that $U \backslash \theta\text{-}int(A)$ is countable. Take $V = U \backslash \theta\text{-}int(A) = U \cap (X \backslash \theta\text{-}int(A))$. Thus, $U \backslash V \subset \theta\text{-}int(A)$.

Conversely, let $x \in A$. There exists an open subset U containing x and a countable subset V such that $U \backslash V \subset \theta\text{-}int(A)$. Hence, $U \backslash \theta\text{-}int(A)$ is countable. ■

Remark 4 *The following diagram holds for a subset A of a space X:*

$$\begin{array}{ccc} \omega_\theta\text{-open} & \longrightarrow & \omega\text{-open} \\ \uparrow & & \uparrow \\ \theta\text{-open} & \longrightarrow & \text{open} \end{array}$$

The following examples show that these implications are not reversible.

Example 5 *(1) Let R be the real line with the topology $\tau = \{\varnothing, R, R \backslash (0,1)\}$. Then the set $R \backslash (0,1)$ is open but it is not ω_θ-open.*

(2) Let R be the real line with the topology $\tau = \{\varnothing, R, Q'\}$ where Q' is the set of irrational numbers. Then the set $A = Q' \cup \{1\}$ is ω-open but it is not ω_θ-open.

Example 6 *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the set $A = \{a, b, d\}$ is ω_θ-open but it is not open.*

Theorem 7 *Let A be an ω_θ-closed subset of a space X. Then $\theta\text{-}cl(A) \subset K \cup V$ for a closed subset K and a countable subset V.*

Proof. Since A is ω_θ-closed, then $X\backslash A$ is ω_θ-open. For every $x \in X\backslash A$, there exists an open set U containing x and a countable set V such that $U\backslash V \subset \theta\text{-}int(X\backslash A) = X\backslash\theta\text{-}cl(A)$. Hence, $\theta\text{-}cl\,(A) \subset X\backslash\,(U\backslash V) = X \cap ((X\backslash U) \cup V) = (X\backslash U) \cup V$. Take $K = X\backslash U$. Thus, K is closed and $\theta\text{-}cl(A) \subset K \cup V$. ■

Definition 8 *The intersection of all ω_θ-closed sets of X containing a subset A is called the ω_θ-closure of A and is denoted by $\omega_\theta\text{-}cl(A)$. The union of all ω_θ-open sets of X contained in a subset A is called the ω_θ-interior of A and is denoted by $\omega_\theta\text{-}int(A)$.*

Lemma 9 *Let A be a subset of a space X. Then*

(1) $\omega_\theta\text{-}cl(A)$ is ω_θ-closed in X.

(2) $\omega_\theta\text{-}cl(X\backslash A) = X\backslash\omega_\theta\text{-}int(A)$.

(3) $x \in \omega_\theta\text{-}cl(A)$ if and only if $A \cap G \neq \emptyset$ for each ω_θ-open set G containing x.

(4) A is ω_θ-closed in X if and only if $A = \omega_\theta\text{-}cl(A)$.

Definition 10 *A subset A of a topological space (X, τ) is said to be an (ω_θ, ω)-set if $\omega_\theta\text{-}int(A) = \omega\text{-}int(A)$.*

Definition 11 *A subset A of a topological space (X, τ) is said to be an (ω_θ, θ)-set if $\omega_\theta\text{-}int(A) = \theta\text{-}int(A)$.*

Remark 12 *Every ω_θ-open set is an (ω_θ, ω)-set and every θ-open set is an (ω_θ, θ)-set but not conversely.*

Example 13 *(1) Let R be the real line with the topology $\tau = \{\varnothing, R, Q'\}$ where Q' is the set of irrational numbers. Then the natural number set N is an (ω_θ, ω)-set but it is not ω_θ-open.*

(2) Let R be the real line with the topology $\tau = \{\varnothing, R, (2, 3)\}$. Then the set $A = (1, \frac{3}{2})$ is an (ω_θ, θ)-set but it is not θ-open.

Theorem 14 *Let A be a subset of a space X. Then A is ω_θ-open if and only if A is ω-open and an (ω_θ, ω)-set.*

Proof. Since every ω_θ-open is ω-open and an (ω_θ, ω)-set, it is obvious.

Conversely, let A be an ω-open and (ω_θ, ω)-set. Then $A = \omega\text{-}int(A) = \omega_\theta\text{-}int(A)$. Thus, A is ω_θ-open. ■

Theorem 15 *Let A be a subset of a space X. Then A is θ-open if and only if A is ω_θ-open and an (ω_θ, θ)-set.*

Proof. Necessity. It follows from the fact that every θ-open set is ω_θ-open and an (ω_θ, θ)-set.

Sufficiency. Let A be an ω_θ-open and (ω_θ, θ)-set. Then $A = \omega_\theta\text{-}int(A) = \theta\text{-}int(A)$. Thus, A is θ-open. ■

Recall that a space X is called locally countable if each $x \in X$ has a countable neighborhood.

Theorem 16 *Let (X, τ) be a locally countable space and $A \subset X$.*

(1) $\omega_\theta O(X)$ is the discrete topology.

(2) A is ω_θ-open if and only if A is ω-open.

Proof. (1) : Let $A \subset X$ and $x \in A$. Then there exists a countable neighborhood B of x and there exists an open set U containing x such that $U \subset B$. We have $U \backslash \theta\text{-}int(A) \subset B \backslash\ \theta\text{-}int(A) \subset B$. Thus $U \backslash \theta\text{-}int(A)$ is countable and A is ω_θ-open. Hence, $\omega_\theta O(X)$ is the discrete topology.

(2) : Necessity. It follows from the fact that every ω_θ-open set is ω-open.

Sufficiency. Let A be an ω-open subset of X. Since X is a locally countable space, then A is ω_θ-open. ■

Corollary 17 *If (X, τ) is a countable space, then $\omega_\theta O(X)$ is the discrete topology.*

A space X is called anti locally countable if nonempty open subsets are uncountable. As an example, observe that in Example 5 (1), the topological space (R, τ) is anti locally countable.

Theorem 18 *Let (X, τ) be a topological space and $A \subset X$. The following hold:*

(1) If X is an anti locally countable space, then $(X, \omega_\theta O(X))$ is anti locally countable.

(2) If X is anti locally countable regular space and A is θ-open, then $\theta\text{-}cl(A) = \omega_\theta\text{-}cl(A)$.

Proof. (1) : Let $A \in \omega_\theta O(X)$ and $x \in A$. There exists an open subset $U \subset X$ containing x and a countable set V such that $U \backslash V \subset \theta\text{-}int(A)$. Thus, $\theta\text{-}int(A)$ is uncountable and A is uncountable.

(2) : It is obvious that ω_θ-$cl(A) \subset \theta$-$cl(A)$.

Let $x \in \theta$-$cl(A)$ and B be an ω_θ-open subset containing x. There exists an open subset V containing x and a countable set U such that $V \backslash U \subset \theta$-$int(B)$. Then $(V \backslash U) \cap A \subset \theta$-$int(B) \cap A$ and $(V \cap A) \backslash U \subset \theta$-$int(B) \cap A$. Since X is regular, $x \in V$ and $x \in \theta$-$cl(A)$, then $V \cap A \neq \varnothing$. Since X is regular and V and A are ω_θ-open, then $V \cap A$ is ω_θ-open. This implies that $V \cap A$ is uncountable and hence $(V \cap A) \backslash U$ is uncountable. Since $B \cap A$ contains the uncountable set θ-$int(B) \cap A$, then $B \cap A$ is uncountable. Thus, $B \cap A \neq \varnothing$ and $x \in \omega_\theta$-$cl(A)$. ■

Corollary 19 *Let (X, τ) be an anti locally countable regular space and $A \subset X$. The following hold:*

(1) If A is θ-closed, then θ-$int(A) = \omega_\theta$-$int(A)$.

(2) The family of (ω_θ, θ)-sets contains all θ-closed subsets of X.

Theorem 20 *If X is a Lindelof space, then $A \backslash \theta$-$int(A)$ is countable for every closed subset $A \in \omega_\theta O(X)$.*

Proof. Let $A \in \omega_\theta O(X)$ be a closed set. For every $x \in A$, there exists an open set U_x containing x such that $U_x \backslash \theta$-$int(A)$ is countable. Thus, $\{U_x : x \in A\}$ is an open cover for A. Since A is Lindelof, it has a countable subcover $\{U_n : n \in N\}$. Since $A \backslash \theta$-$int(A) = \cup_{n \in N}(U_n \backslash \theta$-$int(A))$, then $A \backslash \theta$-$int(A)$ is countable. ■

Theorem 21 *If A is ω_θ-open subset of (X, τ), then $\omega_\theta O(X)|_A \subset \omega_\theta O(A)$.*

Proof. Let $G \in \omega_\theta O(X)|_A$. We have $G = V \cap A$ for some ω_θ-open subset V. Then for every $x \in G$, there exist U, $W \in \tau$ containing x and countable sets K and L such that $U \backslash K \subset \theta$-$int(V)$ and $W \backslash L \subset \theta$-$int(A)$. We have $x \in A \cap (U \cap W) \in \tau|_A$. Thus, $K \cup L$ is countable and $A \cap (U \cap W) \backslash (K \cup L)$ $\subset (U \cap W) \cap (X \backslash K) \cap (X \backslash L) = (U \backslash K) \cap (W \backslash L) \subset \theta$-$int(V) \cap \theta$-$int(A) \cap A$ $= \theta$-$int(V \cap A) \cap A = \theta$-$int(G) \cap A \subset \theta$-$int_A(G)$. Hence, $G \in \omega_\theta O(A)$. ■

3 Continuities via ω_θ-open sets

Definition 22 *A function $f : X \to Y$ is said to be ω_θ-continuous if for every $x \in X$ and every open subset V in Y containing $f(x)$, there exists an ω_θ-open subset U in X containing x such that $f(U) \subset V$.*

Theorem 23 *For a function $f : X \to Y$, the following are equivalent:*
(1) f is ω_θ-continuous.
(2) $f^{-1}(A)$ is ω_θ-open in X for every open subset A of Y,
(3) $f^{-1}(K)$ is ω_θ-closed in X for every closed subset K of Y.

Proof. (1) $\Rightarrow$ (2) : Let A be an open subset of Y and $x \in f^{-1}(A)$. By (1), there exists an ω_θ-open set B in X containing x such that $B \subset f^{-1}(A)$. Hence, $f^{-1}(A)$ is ω_θ-open.

(2) $\Rightarrow$ (1) : Let A be an open subset in Y containing $f(x)$. By (2), $f^{-1}(A)$ is ω_θ-open. Take $B = f^{-1}(A)$. Hence, f is ω_θ-continuous.

(2) $\Leftrightarrow$ (3) : Let K be a closed subset of Y. By (2), $f^{-1}(Y \backslash K) = X \backslash f^{-1}(K)$ is ω_θ-open. Hence, $f^{-1}(K)$ is ω_θ-closed. ■

Theorem 24 *The following are equivalent for a function $f : X \to Y$:*
(1) f is ω_θ-continuous.
(2) $f : (X, \omega_\theta O(X)) \to (Y, \sigma)$ is continuous.

Definition 25 *A function $f : X \to Y$ is called weakly ω_θ-continuous at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an ω_θ-open subset U in X containing x such that $f(U) \subset cl(V)$. If f is weakly ω_θ-continuous at every $x \in X$, it is said to be weakly ω_θ-continuous.*

Remark 26 *The following diagram holds for a function $f : X \to Y$:*

$$\begin{array}{ccc}
\textit{weakly } \omega_\theta\textit{-continuous} & \longrightarrow & \textit{weakly } \omega\textit{-continuous} \\
\uparrow & & \uparrow \\
\omega_\theta\textit{-continuous} & \longrightarrow & \omega\textit{-continuous} \\
\uparrow & & \uparrow \\
\theta\textit{-continuous} & \longrightarrow & \textit{continuous}
\end{array}$$

The following examples show that these implications are not reversible.

Example 27 *Let R be the real line with the topology $\tau = \{\varnothing, R, (2,3)\}$. Let $Y = \{a, b, c\}$ and $\sigma = \{Y, \varnothing, \{a\}, \{c\}, \{a, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(x) = \begin{cases} a & , \text{if } x \in (0,1) \\ b & , \text{if } x \notin (0,1) \end{cases}$. Then f is weakly ω_θ-continuous but it is not ω_θ-continuous.*

Example 28 *Let R be the real line with the topology $\tau = \{\varnothing, R, Q'\}$ where Q' is the set of irrational numbers. Let $Y = \{a, b, c, d\}$ and $\sigma = \{Y, \varnothing, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Define a function $f : (R, \tau) \to (Y, \sigma)$ as follows: $f(x) = \begin{cases} a & , \text{if } x \in Q' \cup \{1\} \\ b & , \text{if } x \notin Q' \cup \{1\} \end{cases}$. Then f is ω-continuous but it is not weakly ω_θ-continuous.*

Example 29 *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = a$, $f(b) = a$, $f(c) = c$, $f(d) = a$. Then f is ω_θ-continuous but it is not θ-continuous.*

For the other implications, the contra examples are as shown in [6, 9].

Definition 30 *A function $f : X \to Y$ is said to be (ω_θ, ω)-continuous if $f^{-1}(A)$ is an (ω_θ, ω)-set for every open subset A of Y.*

Definition 31 *A function $f : X \to Y$ is said to be (ω_θ, θ)-continuous if $f^{-1}(A)$ is an (ω_θ, θ)-set for every open subset A of Y.*

Remark 32 *Every θ-continuous function is (ω_θ, θ)-continuous and every ω_θ-continuous function is (ω_θ, ω)-continuous but not conversely.*

Example 33 *Let R be the real line with the topology $\tau = \{\varnothing, R, Q'\}$ where Q' is the set of irrational numbers. Define a function $f : (R, \tau) \to (R, \tau)$ as follows: $f(x) = \begin{cases} \pi & , \text{if } x \in N \\ 1 & , \text{if } x \notin N \end{cases}$. Then f is (ω_θ, ω)-continuous but it is not ω_θ-continuous.*

Example 34 *Let R be the real line with the topology $\tau = \{\varnothing, R, (2, 3)\}$. Let $A = (1, \frac{3}{2})$ and $\sigma = \{R, \varnothing, A, R\backslash A\}$. Define a function $f : (R, \tau) \to (R, \sigma)$ as follows: $f(x) = \begin{cases} \frac{5}{4} & , \text{if } x \in (1, 2) \\ 4 & , \text{if } x \notin (1, 2) \end{cases}$. Then f is (ω_θ, θ)-continuous but it is not θ-continuous.*

Definition 35 *A function $f : X \to Y$ is coweakly ω_θ-continuous if for every open subset A in Y, $f^{-1}(fr(A))$ is ω_θ-closed in X, where $fr(A) = cl(A)\backslash int(A)$.*

Theorem 36 *Let $f : X \to Y$ be a function. The following are equivalent:*
(1) f is ω_θ-continuous,
(2) f is ω-continuous and (ω_θ, ω)-continuous,
(3) f is weakly ω_θ-continuous and coweakly ω_θ-continuous.

Proof. $(1) \Leftrightarrow (2)$: It is an immediate consequence of Theorem 14.
$(1) \Rightarrow (3)$: Obvious.
$(3) \Rightarrow (1)$: Let f be weakly ω_θ-continuous and coweakly ω_θ-continuous. Let $x \in X$ and V be an open subset of Y such that $f(x) \in V$. Since f is weakly ω_θ-continuous, then there exists an ω_θ-open subset U of X containing x such that $f(U) \subset cl(V)$. We have $fr(V) = cl(V) \backslash V$ and $f(x) \notin fr(V)$. Since f is coweakly ω_θ-continuous, then $x \in U \backslash f^{-1}(fr(V))$ is ω_θ-open in X. For every $y \in f(U \backslash f^{-1}(fr(V)))$, $y = f(x_1)$ for a point $x_1 \in U \backslash f^{-1}(fr(V))$. We have $f(x_1) = y \in f(U) \subset cl(V)$ and $y \notin fr(V)$. Hence, $f(x_1) = y \notin fr(V)$ and $f(x_1) \in V$. Thus, $f(U \backslash f^{-1}(fr(V))) \subset V$ and f is ω_θ-continuous. ■

Theorem 37 *The following are equivalent for a function $f : X \to Y$:*
(1) f is θ-continuous,
(2) f is ω_θ-continuous and (ω_θ, θ)-continuous.

Proof. It is an immediate consequence of Theorem 15. ■

Theorem 38 *Let $f : X \to Y$ be a function. The following are equivalent:*
(1) f is weakly ω_θ-continuous,
(2) ω_θ-$cl(f^{-1}(int(cl(K)))) \subset f^{-1}(cl(K))$ for every subset K of Y,
(3) ω_θ-$cl(f^{-1}(int(A))) \subset f^{-1}(A)$ for every regular closed set A of Y,
(4) ω_θ-$cl(f^{-1}(A)) \subset f^{-1}(cl(A))$ for every open set A of Y,
(5) $f^{-1}(A) \subset \omega_\theta$-$int(f^{-1}(cl(A)))$ for every open set A of Y,
(6) ω_θ-$cl(f^{-1}(A)) \subset f^{-1}(cl(A))$ for each preopen set A of Y,
(7) $f^{-1}(A) \subset \omega_\theta$-$int(f^{-1}(cl(A)))$ for each preopen set A of Y.

Proof. $(1) \Rightarrow (2)$: Let $K \subset Y$ and $x \in X \backslash f^{-1}(cl(K))$. Then $f(x) \in Y \backslash cl(K)$. This implies that there exists an open set A containing $f(x)$ such that $A \cap K = \emptyset$. We have, $cl(A) \cap int(cl(K)) = \emptyset$. Since f is weakly ω_θ-continuous, then there exists an ω_θ-open set B containing x such that $f(B) \subset cl(A)$. We have $B \cap f^{-1}(int(cl(K))) = \emptyset$. Thus, $x \in X \backslash \omega_\theta$-$cl(f^{-1}(int(cl(K))))$ and ω_θ-$cl(f^{-1}(int(cl(K)))) \subset f^{-1}(cl(K))$.
$(2) \Rightarrow (3)$: Let A be any regular closed set in Y. Thus, ω_θ-$cl(f^{-1}(int(A))) = \omega_\theta$-$cl(f^{-1}(int(cl(int(A))))) \subset f^{-1}(cl(int(A))) = f^{-1}(A)$.

(3) $\Rightarrow$ (4) : Let A be an open subset of Y. Since $cl(A)$ is regular closed in Y, ω_θ-$cl(f^{-1}(A)) \subset \omega_\theta$-$cl(f^{-1}(int(cl(A)))) \subset f^{-1}(cl(A))$.

(4) $\Rightarrow$ (5) : Let A be any open set of Y. Since $Y \backslash cl(A)$ is open in Y, then $X \setminus \omega_\theta$-$int(f^{-1}(cl(A))) = \omega_\theta$-$cl(f^{-1}(Y \setminus cl(A))) \subset f^{-1}(cl(Y \setminus cl(A))) \subset X \setminus f^{-1}(A)$. Thus, $f^{-1}(A) \subset \omega_\theta$-$int(f^{-1}(cl(A)))$.

(5) $\Rightarrow$ (1) : Let $x \in X$ and A be any open subset of Y containing $f(x)$. Then $x \in f^{-1}(A) \subset \omega_\theta$-$int(f^{-1}(cl(A)))$. Take $B = \omega_\theta$-$int(f^{-1}(cl(A)))$. Thus $f(B) \subset cl(A)$ and f is weakly ω_θ-continuous at x in X.

(1) $\Rightarrow$ (6) : Let A be any preopen set of Y and $x \in X \backslash f^{-1}(cl(A))$. Then there exists an open set W containing $f(x)$ such that $W \cap A = \emptyset$. We have $cl(W \cap A) = \emptyset$. Since A is preopen, then $A \cap cl(W) \subset int(cl(A)) \cap cl(W) \subset cl(int(cl(A)) \cap W) \subset cl(int(cl(A) \cap W)) \subset cl(int(cl(A \cap W))) \subset cl(A \cap W) = \emptyset$. Since f is weakly ω_θ-continuous and W is an open set containing $f(x)$, there exists an ω_θ-open set B in X containing x such that $f(B) \subset cl(W)$. We have $f(B) \cap A = \emptyset$ and hence $B \cap f^{-1}(A) = \emptyset$. Thus, $x \in X \backslash \omega_\theta$-$cl(f^{-1}(A))$ and ω_θ-$cl(f^{-1}(A)) \subset f^{-1}(cl(A))$.

(6) $\Rightarrow$ (7) : Let A be any preopen set of Y. Since $Y \backslash cl(A)$ is open in Y, then $X \setminus \omega_\theta$-$int(f^{-1}(cl(A))) = \omega_\theta$-$cl(f^{-1}(Y \setminus cl(A))) \subset f^{-1}(cl(Y \setminus cl(A))) \subset X \setminus f^{-1}(A)$. Hence, $f^{-1}(A) \subset \omega_\theta$-$int(f^{-1}(cl(A)))$.

(7) $\Rightarrow$ (1) : Let $x \in X$ and A any open set of Y containing $f(x)$. Then $x \in f^{-1}(A) \subset \omega_\theta$-$int(f^{-1}(cl(A)))$. Take $B = \omega_\theta$-$int(f^{-1}(cl(A)))$. Then $f(B) \subset cl(A)$. Thus, f is weakly ω_θ-continuous at x in X. ■

Theorem 39 *The following properties are equivalent for a function $f : X \to Y$:*

(1) $f : X \to Y$ is weakly ω_θ-continuous at $x \in X$.

(2) $x \in \omega_\theta$-$int(f^{-1}(cl(A)))$ for each neighborhood A of $f(x)$.

Proof. (1) $\Rightarrow$ (2) : Let A be any neighborhood of $f(x)$. There exists an ω_θ-open set B containing x such that $f(B) \subset cl(A)$. Since $B \subset f^{-1}(cl(A))$ and B is ω_θ-open, then $x \in B \subset \omega_\theta$-$int(B))) \subset \omega_\theta$-$int(f^{-1}(cl(A)))$.

(2) $\Rightarrow$ (1) : Let $x \in \omega_\theta$-$int(f^{-1}(cl(A)))$ for each neighborhood A of $f(x)$. Take $U = \omega_\theta$-$int(f^{-1}(cl(A)))$. Thus, $f(U) \subset cl(A)$ and U is ω_θ-open. Hence, f is weakly ω_θ-continuous at $x \in X$. ■

Theorem 40 *Let $f : X \to Y$ be a function. The following are equivalent:*

(1) f is weakly ω_θ-continuous,

(2) $f(\omega_\theta$-$cl(K)) \subset \theta$-$cl(f(K))$ for each subset K of X,

(3) ω_θ-$cl(f^{-1}(A)) \subset f^{-1}(\theta$-$cl(A))$ for each subset A of Y,

(4) ω_θ-$cl(f^{-1}(int(\theta$-$cl(A)))) \subset f^{-1}(\theta$-$cl(A))$ for every subset A of Y.

Proof. (1) $\Rightarrow$ (2) : Let $K \subset X$ and $x \in \omega_\theta\text{-}cl(K)$. Let U be any open set of Y containing $f(x)$. Then there exists an ω_θ-open set B containing x such that $f(B) \subset cl(U)$. Since $x \in \omega_\theta\text{-}cl(K)$, then $B \cap K \neq \emptyset$. Thus, $\emptyset \neq f(B) \cap f(K) \subset cl(U) \cap f(K)$ and $f(x) \in \theta\text{-}cl(f(K))$. Hence, $f(\omega_\theta\text{-}cl(K)) \subset \theta\text{-}cl(f(K))$.

(2) $\Rightarrow$ (3) : Let $A \subset Y$. Then $f(\omega_\theta\text{-}cl(f^{-1}(A))) \subset \theta\text{-}cl(A)$. Thus, $\omega_\theta\text{-}cl(f^{-1}(A)) \subset f^{-1}(\theta\text{-}cl(A))$.

(3) $\Rightarrow$ (4) : Let $A \subset Y$. Since $\theta\text{-}cl(A)$ is closed in Y, then $\omega_\theta\text{-}cl(f^{-1}(int(\theta\text{-}cl(A)))) \subset f^{-1}(\theta\text{-}cl(int(\theta\text{-}cl(A)))) = f^{-1}(cl(int(\theta\text{-}cl(A)))) \subset f^{-1}(\theta\text{-}cl(A))$.

(4) $\Rightarrow$ (1) : Let U be any open set of Y. Then $U \subset int(cl(U)) = int(\theta\text{-}cl(U))$. Thus, $\omega_\theta\text{-}cl(f^{-1}(U)) \subset \omega_\theta\text{-}cl(f^{-1}(int(\theta\text{-}cl(U)))) \subset f^{-1}(\theta\text{-}cl(U)) = f^{-1}(cl(U))$. By Theorem 38, f is weakly ω_θ-continuous. ■

Recall that a space is rim-compact [10] if it has a basis of open sets with compact boundaries.

Theorem 41 *Let $f : X \to Y$ be a function with the closed graph. Suppose that X is regular and Y is a rim-compact space. Then f is weakly ω_θ-continuous if and only if f is ω_θ-continuous.*

Proof. Let $x \in X$ and A be any open set of Y containing $f(x)$. Since Y is rim-compact, there exists an open set B of Y such that $f(x) \in B \subset A$ and ∂B is compact. For each $y \in \partial B$, $(x, y) \in X \times Y \backslash G(f)$. Since $G(f)$ is closed, there exist open sets $U_y \subset X$ and $V_y \subset Y$ such that $x \in U_y$, $y \in V_y$ and $f(U_y) \cap V_y = \emptyset$. The family $\{V_y\}_{y \in \partial B}$ is an open cover of ∂B. Then there exist a finite number of points of ∂B, say, y_1, y_2, ..., y_n such that $\partial B \subset \cup\{V_{y_i}\}_{i=1}^n$. Take $K = \cap\{U_{y_i}\}_{i=1}^n$ and $L = \cup\{V_{y_i}\}_{i=1}^n$. Then K and L are open sets such that $x \in K$, $\partial B \subset L$ and $f(K) \cap \partial B \subset f(K) \cap L = \emptyset$. Since f is weakly ω_θ-continuous, there exists an ω_θ-open set G containing x such that $f(G) \subset cl(B)$. Take $U = K \cap G$. Then, U is an ω_θ-open set containing x, $f(U) \subset cl(B)$ and $f(U) \cap \partial B = \emptyset$. Hence, $f(U) \subset B \subset A$ and f is ω_θ-continuous.

The converse is obvious. ■

Definition 42 *If a space X can not be written as the union of two nonempty disjoint ω_θ-open sets, then X is said to be ω_θ-connected.*

Theorem 43 *If $f : X \to Y$ is a weakly ω_θ-continuous surjection and X is ω_θ-connected, then Y is connected.*

Proof. Suppose that Y is not connected. There exist nonempty open sets U and V of Y such that $Y = U \cup V$ and $U \cap V = \emptyset$. This implies that U and V are clopen in Y. By Theorem 38, $f^{-1}(U) \subset \omega_\theta\text{-}int(f^{-1}(cl(U))) = \omega_\theta\text{-}int(f^{-1}(U))$. Hence $f^{-1}(U)$ is ω_θ-open in X. Similarly, $f^{-1}(V)$ is ω_θ-open in X. Hence, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, $X = f^{-1}(U) \cup f^{-1}(V)$ and $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. Thus, X is not ω_θ-connected. ■

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Addresses :

Erdal Ekici
Department of Mathematics
Canakkale Onsekiz Mart University
Terzioglu Campus,
17020 Canakkale, TURKEY

E-mail: eekici@comu.edu.tr

Saeid Jafari
College of Vestsjaelland South
Herrestraede 11,
4200 Slagelse, DENMARK

E-mail: jafari@stofanet.dk

Raja Mohammad Latif
Department of Mathematics and Statistics
King Fahd University of Petroleum and Minerals
Dhahran 31261 SAUDI ARABIA
E-mail: raja@kfupm.edu.sa